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THE RIEMANN—LIOUVILLE INTEGRAL AND PARAMETER SHIFTING IN A CLASS OF LINEAR ABSTRACT CAUCHY PROBLEMS*

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Abstract. Many important partial differential equations arising in the applications involve one or more parameters. A shifting relation for a problem involving such an equation permits expressing the solution corresponding to one value of a parameter in terms of a solution corresponding to a different value of this parameter. The Riemann–Liouville integral and its properties are employed to develop a set of shifting relations for solutions of a class of Cauchy problems involving an abstract version of the generalized hypergeometric equation. The results are applied to two examples, one of which involves the Riemann–Zeta function. They are also useful in developing properties of the hypergeometric functions.

1. Introduction. Let X be a Banach space and let A be a closed linear operator in X independent of t. Further, assume that the domain of A^r, $\mathfrak{D}(A^r)$, is dense in X for r sufficiently large. We shall be concerned with a class of Cauchy problems of the form

$$(1)\qquad \begin{aligned}&\left\{tD_t\left(\prod_{j=1}^{q}(tD_t+\beta_j-1)\right)-At\left(\prod_{i=1}^{p}(tD_t+\alpha_i)\right)\right\}u(t)=0, && t>0,\\ &u(0+)=\varphi,\quad \varphi\in\mathfrak{D}(A^r); && \alpha_i,\ \beta_j \text{ real.}\end{aligned}$$

By $u(0+)=\varphi$, we mean $\|u(t)-\varphi\|\xrightarrow[t\to 0+]{}0$. There are many relationships existing among the solutions of (1). Included in these are transformations between solutions of such problems which affect a shift in some one of the parameters α_i or β_j, $\beta_j\neq 0, -1, -2, \cdots$, while preserving the data. The primary interest in this paper lies in presenting a unified treatment of such shifting relations.

The class of problems (1) was considered by the author in [1] in which A was a partial differential operator $P(x, D_x)$. It was noted there as well as in a more recent paper by Donaldson [3], that an extensive number of equations of mathematical physics can be reduced to the form in (1) through changes in the dependent or independent variables. Among the more notable of these are the wave equation, the Euler–Poisson–Darboux equation, and the equation of generalized axially symmetric potential theory (GASPT). In § 4 of [1], a number of results were given which involved nonpositive values for the parameters or specific types of shifting formulas in the parameters. Most of these results were motivated by general properties of solutions of the generalized hypergeometric equation and one could extend this set of relations by exploiting the large collection of formulas available for hypergeometric functions. There are drawbacks to this approach. Firstly, certain of the formulas which involve specific types of choices for the parameters are actually connected with shifting, but their forms would fail to suggest this fact. Secondly, in such an approach, there would be a problem of determining when two apparently different formulas could lead to the same type of shifting result or to different interpretations of one shifting procedure. Finally, and perhaps most

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important, one could overlook some basic notion connected with shifting. It would seem desirable to have a more systematic and unified treatment of the shifting problem.

In this paper, we present such a treatment based upon a pair of integrals, one of which involves a continuous shift in the β parameter while the other involves a continuous shift in the α parameter. Through the use of properties connected with the Riemann–Liouville integral, these integrals can be extended analytically to handle values of the parameters outside of their usual range for convergence (except for the values of β noted earlier). Many of these extensions lead to results which agree with those obtained in [1]. The α shifting formulas, other than those occurring in [1], appear to be new.

The principal notions and results on the Riemann–Liouville integral needed for this development will be given in § 2. These will be employed in § 3 to examine β-type shifts and in § 4 to examine α-type shifts. In this latter case, analytic extensions can be made to obtain both upward and downward shifting properties. A precise determination of the number r in $\mathfrak{D}(A^r)$ in (1) will not be given except in special cases. As we shall see, a restriction on r imposes a limitation on the range of values of α and β which can be considered. Finally, the notions developed will be applied to two examples in § 5. One of these involves the Riemann zeta function. The shifting properties lead to integral representations for $\zeta(\alpha)$ for $\alpha<0$ (but $\neq -1, -2, -3, \cdots$).

It should be mentioned that not all parameter shifting formulas fit into the pattern considered here. For example, the important Weinstein formulas for the Euler–Poisson–Darboux equation [8], [9] as well as the formula given by Theorem 4.5 of [1] are not in this class. These fail to have the data preserving property. Rather, they transform a solution of one equation into the solution of another equation without regard to fulfilling a specific initial condition.

2. The Riemann–Liouville integral. In this section and the ones to follow, we consider functions $f:[a, b]\to X$. Such a function f is said to be strongly continuous on the interval $[a, b]$ if it is continuous in $\|\cdot\|_X$ at each point of $[a, b]$. We then write $f\in C[a, b]$. Similarly, we write $f\in C^n[a, b]$ if f has strong derivatives through order n and these are strongly continuous on $[a, b]$. When no confusion can arise, we write $f\in C$ or $f\in C^n$.

We now summarize the basic definitions and properties of the Riemann–Liouville integral. This integral and its generalizations by M. Riesz [7] have played an important role in the study of partial differential equations (also see [4]). The reader is referred to the Riesz paper for a more detailed treatment of these notions.

Let $f\in C$ on an interval which includes the point $t=a$. If $\operatorname{Re}\alpha>0$, define

$$I^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t f(\sigma)(t-\sigma)^{\alpha-1}\,dt. \tag{2.1}$$

The operator I^α obtained in this way is called the Riemann–Liouville operator and it enjoys the following properties:

$$I^\alpha(I^\beta f(t))=I^{\alpha+\beta}f(t),\qquad \operatorname{Re}\alpha>0,\quad \operatorname{Re}\beta>0, \tag{2.2a}$$

$$D_t(I^{\alpha+1}f(t))=I^\alpha f(t),\qquad \operatorname{Re}\alpha>0. \tag{2.2b}$$

If $f(t) \in C^n$ on an interval containing the point $t = a$, then the analytic continuation formula for $I^\alpha f(t)$ with $\operatorname{Re} \alpha > -n$ is given by

$$I^\alpha f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(\alpha+k+1)}(t-a)^{\alpha+k} + I^{\alpha+n} f^{(n)}(t). \tag{2.3}$$

Finally, we have the additional properties

$$I^0 f(t) = f(t), \qquad f(t) \in C, \tag{2.4a}$$

$$I^{-n} f(t) = f^{(n)}(t), \qquad f(t) \in C^n. \tag{2.4b}$$

3. Shifts in the $\boldsymbol{\beta}$ parameters. We now discuss shifts in the β parameters for the problem (1). After obtaining a general result from the Riemann–Liouville integral, we particularize to the case of (1) where $p = 0$, $q = 1$, and $\beta_1 = \beta$. Some of the properties holding for this special case are also applicable to (1).

It was shown in [1] (corollary to Theorem 3.2) that if $u^\beta(t)$ is a solution to the problem (1) corresponding to $\beta_q = \beta > 1$, then a solution to problem (1) with β replaced by β^*, $\beta^* > \beta$, is given by

$$\begin{aligned} u^{\beta^*}(t) &= \frac{1}{B(\beta, \beta^*-\beta)} \int_0^1 (1-\sigma)^{\beta^*-\beta-1} \sigma^{\beta-1} u^\beta(t\sigma)\, d\sigma \\ &= \frac{\Gamma(\beta^*)}{\Gamma(\beta)} t^{1-\beta^*} \left\{ \frac{1}{\Gamma(\beta^*-\beta)} \int_0^t (t-\sigma)^{\beta^*-\beta-1} \sigma^{\beta-1} u^\beta(\sigma)\, d\sigma \right\}. \end{aligned} \tag{3.1}$$

Although we shall not prove it here, this relationship holds under the less restrictive condition $\beta^* > \beta > 0$. We will, however, prove the analogous result for the α shifting in § 4.

The upward shifting formula (3.1) clearly shows that the solution $u^{\beta^*}(t)$ is a continuous function of β^* for $\beta^* > \beta$. The bracketed term in the third member of (3.1) has a form suggesting the applicability of results on the Riemann–Liouville integral (if we choose $a = 0$ in § 2).

To apply those results, first select $\beta = 1$ and set $\gamma = \beta^* - 1$. Then (3.1) becomes

$$u^{\beta^*}(t) = \Gamma(\gamma+1) t^{-\gamma} I^\gamma u(t), \tag{3.2}$$

where $u(t)$ is a solution of (1) with $\beta_q = \beta = 1$. In this form, the presence of the factor $\Gamma(\gamma+1)$ shows that (2.4b) cannot be used for obtaining a solution when γ is a negative integer. Indeed, this choice for γ leads to logarithmic solutions which are outside the scope of this paper. In all other cases, the analytic continuation formula (2.3) is applicable for assigning a meaning to $u^{\beta^*}(t)$. According to that formula, we get

$$u^{\beta^*}(t) = \Gamma(\gamma+1) t^{-\gamma} \left\{ \sum_{k=0}^{n-1} \frac{u^{(k)}(0) t^{k+\gamma}}{\Gamma(\gamma+k+1)} + I^{n+\gamma} u^{(n)}(t) \right\}, \tag{3.3}$$

provided that $\gamma > -n$, γ not a negative integer. From this and the definition of $I^{n+\gamma}$ we have the following theorem.

THEOREM 3.1. *If $u(t) \in C^n$ is a solution of problem* (1) *corresponding to $\beta_q = 1$, then a solution of* (1) *corresponding to $\beta_q = \beta^* > -n+1$, β^* not an integer, is given* *by*

$$(3.4)\qquad u^{\beta^*}(t)=\Gamma(\beta^*)\left\{\sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{\Gamma(\beta^*+k)}t^k+\frac{t^n}{\Gamma(n+\beta^*-1)}\int_0^1(1-\sigma)^{\beta^*+n-2}u^{(n)}(t\sigma)\,d\sigma\right\},$$

where $u^{(n)}(\tau)=D_\tau^n u(\tau)$.

Note. The starting choice $\beta=1$ may appear to be somewhat artificial. However, we can arrange shifts involving positive values of these parameters which permit this as we shall see later.

To see how (3.4) ties in with earlier results, we interpret its meaning for the particular problem

$$(3.5)\qquad D_t[tD_t+\beta-1]u(t)=Au(t),\quad u(0)=\varphi.$$

We shall suppose that uniqueness holds for solutions of this problem if $\beta>0$ and that $u(t)\in C^{n+2}$ (see, for example, [1], [10]).

Taking $u(t)$ to be a solution of (3.5) with $\beta=1$, repeated differentiations show that $u^{(k-1)}(t)$ satisfies the equation

$$tD_t^2u^{(k-1)}(t)+kD_tu^{(k-1)}(t)-Au^{(k-1)}(t)=0,\qquad k=1,2,\cdots.$$

From the fact that $ku^{(k)}(0)-Au^{(k-1)}(0)=0$, we obtain inductively that $u^{(k)}(0)=(1/k!)A^k\varphi$. As a consequence, the summed expression in (3.4) takes the form

$$(3.6)\qquad \Gamma(\beta^*)\sum_{k=0}^{n-1}\frac{(tA)^k\varphi}{k!\Gamma(\beta^*+k)}.$$

In order to interpret the integral in (3.4), we must compute $u^{(n)}(t\sigma)$. But by the repeated differentiation procedure employed to obtain (3.6), one can show that $u^{(n)}(t)$ is a solution of the problem

$$tD_t^2u^{(n)}(t)+(n+1)D_tu^{(n)}(t)-Au^{(n)}(t)=0,\qquad u^{(n)}(0)=\frac{1}{n!}A^n\varphi.$$

Define $v(t)$ to be a solution of the problem $tD_t^2v(t)+(n+1)D_tv(t)-Av(t)=0$, $v(0)=\varphi$. Uniqueness shows that

$$u^{(n)}(t)=\frac{1}{n!}v(t)=\frac{1}{n!}A^n\left\{n\int_0^1(1-\xi)^{n-1}u(t\xi)\,d\xi\right\},$$

where we have expressed $v(t)$ in terms of $u(t)$ by means of (3.1) with $\beta=1$ and $\beta^*=n+1$. Inserting the expression for $u^{(n)}(t)$ in the integral in (3.4) with t replaced by $t\sigma$, the integral in (3.4) becomes

$$\frac{\Gamma(\beta^*)t^n}{\Gamma(n+\beta^*-1)}\int_0^1(1-\sigma)^{n+\beta^*-2}\left\{\frac{A^n}{(n-1)!}\int_0^1(1-\xi)^{n-1}u(t\sigma\xi)\,d\xi\right\}d\sigma$$

$$=\frac{\Gamma(\beta^*)(tA)^n}{(n-1)!\Gamma(\beta^*+n-1)}\int_0^1(1-\xi)^{n-1}\left\{\int_0^1(1-\sigma)^{n+\beta^*-2}u(t\xi\sigma)\,d\sigma\right\}d\xi.$$

The strong continuity of $A^nv(t)$ has permitted us to remove A^n from under the sign of integration. Again, from (3.1), the inner integral in this last expression is

$$\{\Gamma(n+\beta^*-1)/\Gamma(n+\beta^*)\}u^{n+\beta^*}(t\xi).$$

From this, we finally see that the integral in (3.4) reduces to

$$\frac{\Gamma(\beta^*)(tA)^n}{(n-1)!\Gamma(n+\beta^*)}\int_0^1 (1-\xi)^{n-1}u^{n+\beta^*}(t\xi)\,d\xi. \tag{3.7}$$

Combining (3.6) and (3.7) we find that

$$u^{\beta^*}(t)=\Gamma(\beta^*)\sum_{k=0}^{n-1}\frac{(tA)^k\varphi}{k!\Gamma(\beta^*+k)} + \frac{\Gamma(\beta^*)}{(n-1)!\Gamma(\beta^*+n)}(tA)^n\int_0^1 (1-\sigma)^{n-1}u^{\beta^*+n}(t\sigma)\,d\sigma. \tag{3.8}$$

This agrees with Theorem 4.2 of [1] for the special problem (3.5) with β replaced by β^*.

It was also shown in [1], by a consideration of the equation in (1), that if $\beta \neq 0$, then

$$u^{\beta}(t)=\frac{1}{\beta}(tD_t+\beta)u^{\beta+1}(t). \tag{3.9}$$

If $\beta>0$, this follows readily from (3.1). For then

$$u^{\beta+1}(t)=\beta t^{-\beta}\int_0^t \sigma^{\beta-1}u^{\beta}(\sigma)\,d\sigma.$$

Hence

$$\begin{aligned} D_tu^{\beta+1}(t)&=-\beta^2t^{-\beta-1}\int_0^t \sigma^{\beta-1}u^{\beta}(\sigma)\,d\sigma+\beta t^{-1}u^{\beta}(t)\\ &=-\frac{\beta}{t}u^{\beta+1}(t)+\frac{\beta}{t}u^{\beta}(t). \end{aligned}$$

Solving this for $u^{\beta}(t)$, we obtain (3.9). This property is also easily shown to be valid for negative values of β^*-1 by using the $u^{\beta^*}(t)$ obtained through the extension (3.4).

4. Shifts in the α parameters. As we have observed, the formula (3.1) served as the basic starting point for building up the shifting relations. In the case of the α parameters, it is desirable to have a similar relating integral. Such an integral will be given below. To simplify details, we center our discussion around the problem

$$u_t(t)-A(tD_t+\alpha)u(t)=0, \qquad u(0)=\varphi, \tag{4.1}$$

obtained by selecting $p=1$, $q=0$, and $\alpha_1=\alpha$ in (1). We shall indicate which results are applicable to the more general problem (1).

THEOREM 4.1. *Let $u^{\alpha}(t)$ be a solution of* (1) *for $\alpha_i=\alpha>0$ for some i and let $\alpha^*>0$ with $\alpha^*<\alpha$. If $Au^{\alpha}(t)$ is strongly continuous, then a solution to* (1) *with α replaced by α^* is given by*

$$u^{\alpha^*}(t)=\frac{\Gamma(\alpha)}{\Gamma(\alpha^*)\Gamma(\alpha-\alpha^*)}\int_0^1 \sigma^{\alpha^*-1}(1-\sigma)^{\alpha-\alpha^*-1}u^{\alpha}(t\sigma)\,d\sigma. \tag{4.2}$$

Note. This formula, which is applicable to shifting down on any one of the p parameters α_i in (1), is analogous to (3.1) except that the unstarred and starred parameters have been interchanged. This turns out to be significant in that we can apply the results of § 2 in two ways to obtain extensions of this integral.

Proof. We prove this result for the problem (4.1) and note that this proof can be extended to apply to (1). The general proof for (1) involves the arranging of more factors and is quite similar to the arranging process carried out in [2] for the generalization of (3.1) to nonhomogeneous problems.

The condition $u^{\alpha^*}(0+)=\varphi$ is easily checked. Substituting (4.2) into (4.1) with α replaced by α^* in (4.1), we can invoke the strong continuity property to apply the derivative operators to u^α under the sign of integration. We can then use the relation $D_t u^\alpha(\sigma t)=\sigma D_{(\sigma t)} u^\alpha(\sigma t)$ to reduce the differentiations to ones involving the variable σt. Using (4.1) with the variable t replaced by σt and the strong continuity property, the operator A can be removed from under the sign of integration. Finally, by a conversion of all differentiations to D_σ under the sign of integration, it follows that the remaining integral has the primative

$$\sigma^{\alpha^*}(1-\sigma)^{\alpha-\alpha^*}u^\alpha(t\sigma),$$

which vanishes at $\sigma=0$ and $\sigma=1$ for $\alpha>\alpha^*>0$.

(A) *Downward shifting.* Although the formula (4.2) permits us to shift down on α, we are restricted to values of $\alpha^*>0$. To extend beyond this range, we rewrite (4.2), through a change of variables, in the form

$$u^{\alpha^*}(t)=\frac{\Gamma(\alpha)t^{1-\alpha}}{\Gamma(\alpha^*)\Gamma(\alpha-\alpha^*)}\int_0^t (t-\sigma)^{\alpha^*-1}\sigma^{\alpha-\alpha^*-1}u^\alpha(t-\sigma)\,d\sigma. \tag{4.3}$$

Strictly speaking, this integral is not in the form required for the application of the results of § 2. This is due to the presence of the t variable in the u^α function in the integrand. We can circumvent this difficulty by first regarding this t variable in u^α as a parameter λ and then replace λ by t after applying the Riemann–Liouville techniques. With this understanding, we observe that (4.3) can be expressed as

$$u^{\alpha^*}(t)=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\alpha^*)}t^{1-\alpha}I^{\alpha^*}\{t^{\alpha-\alpha^*-1}u^\alpha(\lambda-t)\}. \tag{4.4}$$

If we take $\alpha>0$ and replace α^* by $\alpha-(k+1)$, k being a nonnegative integer, then (4.4) becomes

$$u^{\alpha-(k+1)}(t)=\frac{t^{1-\alpha}\Gamma(\alpha)}{k!}I^{\alpha-(k+1)}\{t^k u^\alpha(\lambda-t)\}. \tag{4.5}$$

In this form, the results of § 2 are applicable. We distinguish two cases: (i) $\alpha-(k+1)$ a nonpositive integer and (ii) $\alpha-(k+1)$ a negative noninteger.

Case (i). In this situation, $\alpha-(k+1)=-m$, m a positive integer or zero. Since $\alpha\geqq 1$, we have $m\leqq k$. Then (4.5) along with (2.4) readily proves that the following theorem holds.

THEOREM 4.2. *If α is a positive integer and if $\alpha-(k+1)$ is a nonpositive integer, then*

$$u^{\alpha-(k+1)}(t)=\Gamma(\alpha)\sum_{j=0}^{k+1-\alpha}\binom{m}{j}\frac{(-1)^j t^j}{(\alpha-1+j)!}[D_\sigma^j u^\alpha(\sigma)]_{\sigma=0}. \tag{4.6}$$

In this case, the shift to $u^{\alpha-(k+1)}(t)$ is determined from $u^{\alpha}(t)$ through its value and the values of its derivatives at $t=0$.

We now correlate (4.7), which holds for the general problem (1), with results in [1] by using (4.1). Successive differentiations of the equation in (4.1) will show that $D_t^j u^{\alpha}(t)|_{t=0}=(\alpha+j-1)AD_t^{j-1}u^{\alpha}(t)|_{t=0}$. By induction, we conclude that $D_t^j u^{\alpha}(t)|_{t=0}=(\alpha)_j A^j\varphi$. Using this and the fact that $(k+1-\alpha)!/(k+1-\alpha-j)! =(-1)^j(\alpha-1-k)_j$, we see that (4.6) reduces to

$$u^{\alpha-(k+1)}(t)=\sum_{j=0}^{k+1-\alpha}\frac{1}{j!}(\alpha-k-1)_j(tA)^j\varphi, \tag{4.7}$$

and this is just the solution of problem (4.1) with α there replaced by $(\alpha-k-1)$ (see Theorem 4.1 of [1]). The above discussion requires that $\varphi\in D(A^{k+1-\alpha})$.

Case (ii). We can apply (2.3) directly to (4.5) to get

$$\begin{aligned} u^{\alpha-(k+1)}(t)=\frac{\Gamma(\alpha)t^{1-\alpha}}{k!}\Bigg\{&\sum_{j=0}^{k}\frac{[D_{\sigma}^j\{\sigma^k u^{\alpha}(\lambda-\alpha)\}]_{\sigma=0}}{\Gamma(\alpha-k+j)}t^{\alpha-k-1+j}\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\sigma)^{\alpha-1}D_{\sigma}^{k+1}[\sigma^k u^{\alpha}(\lambda-\sigma)]\,d\sigma\Bigg\}.\end{aligned} \tag{4.8}$$

The first sum clearly reduces to $k!u^{\alpha}(\lambda)t^{\alpha-1}/\Gamma(\alpha)$. Using this fact and replacing λ by t, we obtain

THEOREM 4.3. *Let $u^{\alpha}(t)\in C^{k+1}$ satisfy* (1) *for some $\alpha_j=\alpha>0$. If α is not an integer, then*

$$u^{\alpha-(k+1)}(t)=u^{\alpha}(t)+\frac{t^{1-\alpha}}{k!}\int_0^t(t-\sigma)^{\alpha-1}D_{\sigma}^{k+1}\{\sigma^k u^{\alpha}(t-\sigma)\}\,d\sigma \tag{4.9}$$

satisfies (1) *with $\alpha_j=\alpha-(k+1)$.*

In order to note some interpretations of (4.9), it is useful to rewrite it in the form

$$u^{\alpha-(k+1)}(t)=u^{\alpha}(t)+\frac{(-1)^{k+1}}{k!}\int_0^1\sigma^{\alpha-1}D_{\sigma}^{k+1}[(1-\sigma)^k u^{\alpha}(t\sigma)]\,d\sigma. \tag{4.10}$$

COROLLARY 1. *Let $u^{\alpha}(t)\in C^{k+1}$ satisfy* (4.1) *and let $u^{\alpha-(k+1)}(t)$ be defined by* (4.10). *Then $u^{\alpha-(k+1)}(t)=(1-tA)^{k+1}u^{\alpha}(t)$.*

Proof. If $k=0$, the proof is trivial. If $u^{\alpha}(t)\in C^{k+1}$, it follows by (4.9) that $u^{\alpha-l}(t)\in C^{k+1-l}$ for integral l, $0\leqq l\leqq k$. Then $u^{(\alpha-l)-1}(t)=(1-tA)u^{\alpha-l}(t)$ by the first part of the proof. Successive applications of this formula yield the stated result.

Remark. If A and φ were constants in (4.1), then Corollary 1 would reduce to the trivial relation $(1-tA)^{k+1-\alpha}\varphi=(1-tA)^{k+1}[(1-tA)^{-\alpha}\varphi]$, $|t|<|A|^{-1}$. Corollary 1 shows that what one would intuitively expect to get agrees with the correct result.

By applying Taylor's theorem with remainder to the relationship in Corollary 1, one can readily deduce that

$$(4.11)\qquad \begin{aligned} u^{\alpha-(k+1)}(t) &= \varphi + \sum_{j=1}^{k} \frac{1}{j!}(\alpha-k-1)_j (tA)^j \varphi \\ &\quad + \frac{(\alpha-k-1)_{k+1}}{k!} A^{k+1} \int_0^t (t-\sigma)^k u^{\alpha}(\sigma)\, d\sigma, \end{aligned}$$

and this is just Theorem 4.2 of [1] particularized to the case of problem (4.1).

COROLLARY 2 (*Upward shifting formula*). *Given the conditions of Corollary* 1, *then* $u^{\alpha-k}(t) = (\alpha-k-1)^{-1}[tD_t + (\alpha-k-1)]u^{\alpha-k-1}(t)$.

Proof. This follows readily from the formula $u^{\alpha-(k+1)}(t) = (1-tA)u^{\alpha-k}(t)$ (see Corollary 1) by computing $[tD_t + (\alpha-k-1)]u^{\alpha-k-1}(t)$ in terms of $u^{\alpha-k}(t)$ and noting the equation that $u^{\alpha-k}(t)$ satisfies. This upward shifting holds, in fact, if $\alpha \neq k+1$ (see Theorem 4.4 of [1]) and applies to the general problem (1).

(B) *Upward shifting.* In order to treat this situation, we replace the variable of integration σ in (4.2) by σ/t to get

$$(4.12)\qquad \begin{aligned} u^{\alpha^*}(t) &= \frac{\Gamma(\alpha)}{\Gamma(\alpha^*)} t^{1-\alpha} \left\{ \frac{1}{\Gamma(\alpha-\alpha^*)} \int_0^t (t-\sigma)^{\alpha-\alpha^*-1} \sigma^{\alpha^*-1} u^{\alpha}(\sigma)\, d\sigma \right\} \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha^*)} t^{1-\alpha} I^{\alpha-\alpha^*} \left\{ t^{\alpha^*-1} u^{\alpha}(t) \right\}. \end{aligned}$$

Rather than use (2.3) in its full generality, we shall consider the cases in which (a) $\alpha^* - \alpha$ is a positive integer, and (b) $0 < \alpha^* - \alpha < 1$. Any other shifting relation can be obtained by using a combination of these two cases.

Case (a). $\alpha^* - \alpha = p$. We rewrite (4.12), by means of (2.4b) as

$$(4.13)\qquad u^{\alpha^*}(t) = t^{1-\alpha} D_t^p \{ t^{\alpha+p-1} u^{\alpha}(t) \}.$$

Successive applications of D_t to $t^{\alpha+p-1}u^{\alpha}(t)$ give

$$(4.14)\qquad \begin{aligned} u^{\alpha+p}(t) &= \frac{\Gamma(\alpha)}{\Gamma(\alpha+p)} \{ (tD_t+\alpha)(tD_t+\alpha+1) \cdots (tD_t+\alpha+p-1) \} u^{\alpha}(t) \\ &= \prod_{j=0}^{p-1} \left(\frac{tD_t+\alpha+j}{\alpha+j} \right) u^{\alpha}(t) \quad \text{if } u^{\alpha}(t) \in C^p. \end{aligned}$$

By taking $p = 1$, we see that this gives the same type of upward shifting formula as Corollary 2 to Theorem 4.3.

Case (b). $\alpha^* = \alpha + 1 - \mu$, $0 < \mu < 1$. From case (a), we have that $u^{\alpha+1}(t) = \alpha^{-1}(tD_t + \alpha)u^{\alpha}(t)$. Then (4.2) shows that

$$(4.15)\qquad \begin{aligned} u^{\alpha+1-\mu}(t) &= \frac{\Gamma(\alpha) t^{1-(\alpha+1)}}{\Gamma(\alpha+1-\mu)\Gamma(\mu)} \int_0^t (t-\sigma)^{\mu-1} \sigma^{\alpha-\mu} [\sigma D_\sigma + \alpha] u^{\alpha}(\sigma)\, d\sigma \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha+1-\mu)\Gamma(\mu)} (tD_t+\alpha) \int_0^1 (1-\sigma)^{\mu-1} \sigma^{\alpha-\mu} u^{\alpha}(t\sigma)\, d\sigma. \end{aligned}$$

The results of cases (a) and (b) are applicable to the general problem (1) since their developments did not depend upon using properties of (4.1).

5. Applications of shifting. The shifting relations developed in §§ 3 and 4 are applicable to a variety of problems in operator differential equations. As was noted in the introduction, the Euler–Poisson–Darboux and GASPT equations are ones which involve parameters. The author gave an extensive discussion of the first of these in § 6 of [1] and treated the nonhomogeneous GASPT equation in [2]. In this section, we limit our discussion to two examples, the second of which involves both an α- and a β-type parameter.

(A) *The Riemann zeta function.* Consider the Cauchy problem $D_t u(x, t) - D_x(tD_t+\alpha)u(x, t)=0$, $\alpha>0$, $u(x, 0)=e^x/(e^x+1)$. This problem has the form discussed in § 4 with $A = D_x$. It is easy to verify that a solution to it is

$$u^\alpha(x, t)=\frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{\sigma^{\alpha-1}\, e^{x+(t-1)\sigma}}{e^{x+\sigma t}+1}\, d\sigma. \tag{5.1}$$

From this and [5, p. 20], we find that

$$u^\alpha(0, 1)=\frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{\sigma^{\alpha-1}}{e^{\sigma}+1}\, d\sigma=(1-2^{1-\alpha})\zeta(\alpha), \tag{5.2}$$

Re $\alpha>0$, in which $\zeta(\alpha)$ denotes the Riemann zeta function. Even though $\zeta(\alpha)$ has a pole at $\alpha=1$, $u^\alpha(0, 1)$ is well-defined there. The results on downward shifting given in § 4 are applicable for extending $u^\alpha(x, t)$ for $\alpha\leqq 0$ (we could consider α complex but shall not do so here). In particular, (4.7) and (5.2) together show that

$$\begin{aligned}(1-2^{1-(\alpha-k-1)})\zeta(\alpha-k-1)&=u^{\alpha-(k+1)}(0, 1)\\ &=\sum_{j=0}^{k+1-\alpha}\frac{1}{j!}(\alpha-k-1)_j(tD_x)^j(e^x/(e^x+1))\big|_{x=0,t=1}\end{aligned} \tag{5.3}$$

for α a positive integer, and (4.11) and (5.2) show that

$$\begin{aligned}(1-2^{1-(\alpha-k-1)})\zeta(\alpha-k-1)&=\sum_{j=0}^{k}\frac{1}{j!}(\alpha-k-1)_j D_x^j\{e^x/(e^x+1)\}\big|_{x=0}\\ &\quad+\frac{(\alpha-k-1)_{k-1}}{k!}D_x^{k+1}\int_0^1(1-\sigma)^k u^\alpha(x, \sigma)\, d\sigma\big|_{x=0},\end{aligned} \tag{5.4}$$

if $\alpha-k-1$ is a negative noninteger. An evaluation of (5.3) with $\alpha=1$ and $k\geqq 0$ leads to the familiar results $\zeta(0)=-1/2$, $\zeta(-1)=-1/12$, $\zeta(-2)=0$, $\zeta(-3)=1/120$, etc. (see [5]).

To handle (5.4), we observe that if

$$K=e^{x+(t-1)\sigma}/(e^{x+t\sigma}+1)=e^{(t-1)\sigma}(e^{\sigma t}+e^{-x})^{-1},$$

then

$$\begin{aligned}K_x&=e^{(t-1)\sigma}\{e^{-x}(e^{\sigma t}+e^{-x})^{-2}\},\\ K_{xx}&=e^{(t-1)\sigma}\{-e^{-x}(e^{\sigma t}+e^{-x})^{-2}+2e^{-2x}(e^{\sigma t}+e^{-x})^{-3}\}.\end{aligned}$$

Using these in (5.4), we obtain, for $0<\alpha<1$,

$$\zeta(\alpha-1)=(1-2^{2-\alpha})^{-1}\left\{\frac{1}{2}+\frac{1}{\Gamma(\alpha-1)}\int_0^1\left\{\int_0^\infty\frac{\xi^{\alpha-1}\,e^{(\sigma-1)\xi}}{(e^{\sigma\xi}+1)^2}\,d\xi\right\}\,d\sigma\right\},$$

$$\zeta(\alpha-2)=(1-2^{3-\alpha})^{-1}\left\{(1+\alpha)/4+\frac{1}{2\Gamma(\alpha-2)}\int_0^1(1-\sigma)\right. \tag{5.5}$$

$$\left.\cdot\left[\int_0^\infty\xi^{\alpha-1}\,e^{(\sigma-1)\xi}\left\{\frac{-1}{(e^{\sigma\xi}+1)^2}+\frac{2}{(e^{\sigma\xi}+1)^3}\right\}\,d\xi\right]\,d\sigma\right\}.$$

(B) *A Cauchy problem.* The Cauchy problem

$$\begin{aligned}&U_{tt}(x,t)=t^mU_{xx}(x,t)-\nu t^{m/2-1}U_x(x,t), \qquad t>0,^1 \quad m\geqq 2,\\ &U(x,0)=\varphi(x), \qquad U_t(x,0)=\psi(x)\end{aligned} \tag{5.6}$$

was considered in [6, p. 182] under the assumption $|\nu|\leqq m/2$. This restriction on ν was removed in [1, p. 232] under the same condition on m but with $\psi(x)=0$. We shall here permit m to assume values in $(-2,0)$ (but not of the form $-2n/(n+1)$, n a positive integer, which leads to the logarithmic case), while permitting ν to take on any real value.

As was shown in [1], the problem (5.6), with $\psi(x)=0$, can be reduced, through changes in the independent variables, to

$$\begin{aligned}&[D_z(zD_z+\beta-1)-D_y(zD_z+\alpha)]U(y,z)=0,\\ &\qquad\qquad U(y,0)=\varphi(y),\end{aligned} \tag{5.7}$$

with $\alpha=(m-2\nu)/(2m+4)$ and $\beta=m/(m+2)$. If we take $\varphi(y)\in C^r$ and $\|\varphi\| =\sup|\varphi(y)|$, then this problem is in the form (1) with $A=D_y$ and t replaced by z. To simplify the writing, we denote a solution of (5.7) by $U^{\alpha,\beta}(y,z)$. We shall indicate throughout the corresponding sets of values of m and ν that go with these $U^{\alpha,\beta}(y,z)$.

The procedure we follow is first to obtain a solution of (5.7) for specific positive choices of α and β (and hence for choices of m and ν) and then use the method of § 3 to shift on the β parameter to $\beta=-n+\gamma$, $0<\gamma<1$ (in which case $-2n/(n+1)<m<-2(n-1)/n$). Associated with this choice of β are the corresponding possible alternatives for α (and hence ν), namely (i) $\alpha>0$ or $\nu<m/2$, (ii) $\alpha=-l$, l a nonnegative integer or $\nu=(l+1/2)m+2l$, and (iii) $-l<\alpha<-l+1$, l a positive integer or $(l-1/2)m+2(l-1)<\nu<(l+1/2)m+2l$.

Select $\beta=\gamma$ and $\alpha=\gamma/2$. From [1, p. 332] we find that

$$U^{\gamma/2,\gamma}(y,z)=\frac{\Gamma(\gamma)}{[\Gamma(\gamma/2)]^2}\int_0^1\sigma^{\gamma/2-1}(1-\sigma)^{\gamma/2-1}\varphi(y+\sigma z)\,d\sigma. \tag{5.8}$$

In this particular solution, the choice β corresponds to a positive value of m_1 such that $m_1/(m_1+2)=\gamma$ and α corresponds to this same m_1 with $\nu=0$. Then

[1] There is a misprint in [1]. The minus sign rather than the plus sign should proceed ν in the equation corresponding to (5.6).

according to (3.9),

$$U^{\gamma/2,-n+\gamma}(y,z)=\left\{\prod_{j=1}^{n}\left(\frac{zD_z+\gamma-j}{\gamma-j}\right)\right\}U^{\gamma/2,\gamma}(y,z). \tag{5.9}$$

Note that this shift on β has forced a shift in ν so that the parameter α retains the value $\gamma/2$. Here, ν was shifted from 0 to $-n/(n+1-\gamma)$.

(i) $\alpha>0$. With m as chosen to obtain the shift in β in (5.9), we observe that $\alpha>0$ if we choose $\nu<-(n-\gamma)/(n+1-\gamma)$. Then $\alpha=-\frac{1}{2}\{(n-\gamma)+(n+1-\gamma)\nu\}$. According as $\alpha<\gamma/2$ or $\alpha>\gamma/2$, we must consider different types of shifts.

(a) $\alpha<\gamma/2$ (or $-n/(n+1-\gamma)<\nu<-(n-\gamma)/(n+1-\gamma)$). Under these circumstances (4.2) shows that

$$U^{\alpha,-n+\gamma}(y,z)=\frac{\Gamma(\gamma/2)}{\Gamma(\alpha)\Gamma(\gamma/2-\alpha)}\int_0^1\sigma^{\alpha-1}(1-\sigma)^{\gamma/2-\alpha-1}U^{\gamma/2,-n+\gamma}(y,z\sigma)\,d\sigma. \tag{5.10}$$

(b) $\alpha>\gamma/2$ (or $\nu<-n/(n+1-\gamma)$). Here we have the possibility that $\alpha-\gamma/2$ may or may not be a positive integer. In the first situation, $\alpha-\gamma/2=p$, a positive integer, and $\nu=-(2p+n)/(n+1-\gamma)$. Then by (4.14)

$$U^{\gamma/2+p,-n+\gamma}(y,z)=\prod_{j=0}^{p-1}\left(\frac{zD_z+\gamma/2+j}{\gamma/2+j}\right)U^{\gamma/2,-n+\gamma}(y,z). \tag{5.11}$$

In the second situation, $\alpha-\gamma/2=p-\mu$, $0<\mu<1$. Then $\nu=-(2p+n)/(n+1-\gamma)+2\mu/(n+1-\gamma)$ and we obtain, from (5.11) and (4.15),

$$\begin{aligned}U^{\gamma/2+p-\mu,-n+\gamma}(y,z)&=\frac{\Gamma(p+\gamma/2)}{\Gamma(p+\gamma/2-\mu)\Gamma(\mu)}\\&\quad\cdot\int_0^1\sigma^{p+\gamma/2-\mu-1}(1-\sigma)^{\mu-1}U^{\gamma/2+p,-n+\gamma}(y,z\sigma)\,d\sigma.\end{aligned} \tag{5.12}$$

(ii) $\alpha\leqq 0$, *α an integer.* If $\alpha=-l$, then $\nu=(l+1/2)m+2l$. Using (5.12) with $p=1$ and $\mu=-\gamma/2$ we obtain $U^{1,-n+\gamma}(y,z)$. Then (4.6) (with $\alpha=1$ and $m=l$ in that formula) shows that

$$\begin{aligned}U^{-l,-n+\gamma}(y,z)&=\frac{1}{l!}D_s^l\{s^lU^{1,-n+\gamma}(y,z-s)\}_{s=z}\\&=\sum_{j=0}^{l}(-1)^j\binom{l}{j}\frac{1}{j!}z^jD_s^j[U^{1,-n+\gamma}(y,s)]_{s=0}.\end{aligned}$$

From (5.7) satisfied by $U^{1,-n+\gamma}(y,z)$, we compute

$$D_s^jU^{1,-n+\gamma}(y,s)|_{s=0}=\frac{j!\Gamma(\gamma-n)}{\Gamma(\gamma-n+j)}D_y^j\varphi(y).$$

Hence

$$U^{-l,-n+\gamma}(y,z)=\sum_{j=0}^{l}(-1)^j\binom{l}{j}\frac{\Gamma(\gamma-n)}{\Gamma(\gamma-n+j)}(zD_y)^j\varphi(y). \tag{5.13}$$

(iii) $\alpha < 0$ *and not an integer.* In this final case, $-l < \alpha < -l+1$ or $\alpha = -l+\tau$, $0 < \tau < 1$. From the results for case (i) above, we determine $U^{\tau,-n+\gamma}(y, z)$. Then (4.9), with $k = l-1$, shows that

$$U^{\tau-l,-n+\gamma}(y, z) = U^{\tau,-n+\gamma}(y, z) + \frac{z^{1-\tau}}{(l-1)!} \cdot \int_0^z (z-\sigma)^{\tau-1} D_\sigma^l[\sigma^{l-1} U^{\tau,-n+\gamma}(y, z-\sigma)]\, d\sigma. \tag{5.14}$$

The reader can easily write this solution in the form given in (4.13).

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OSCILLATION CRITERIA FOR THIRD ORDER DIFFERENTIAL EQUATIONS*

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Abstract. This paper gives some sufficient conditions for

$$y'''+p(x)y'+q(x)y=0$$

to have oscillatory solutions.

1. Introduction. By an oscillatory solution of

$$y'''+p(x)y'+q(x)y=0, \tag{1}$$

we will mean a solution of (1) that has zeros for arbitrarily large values of x. Other solutions of (1) will be called nonoscillatory. We will say that (1) is oscillatory if it has a nontrivial oscillatory solution. We will study (1) assuming p, q and p' are continuous on $[0, +\infty)$.

In studying (1), we will make use of its adjoint

$$y'''+p(x)y'+(p'(x)-q(x))y=0 \tag{2}$$

and the following easily verified lemma.

LEMMA. *If N is a solution of* (1) *such that $N(x)>0$ for $x>a$, then two independent solutions of* (2) *satisfy*

$$(y'/N)'+[(N''+pN)/N^2]y=0. \tag{3}$$

2. Oscillation theorems. If $2q(x)-p'(x)\equiv 0$, then any solution of (1) is a linear combination of u^2, uv and v^2 where u and v form a fundamental system of solutions for

$$y''+(p/4)y=0. \tag{4}$$

It follows that in this case (1) will be oscillatory *if and only if* (4) is oscillatory.

Our first result shows that there is a connection between (1) and (4) when $2q-p'$ is sign definite.

THEOREM 1. *If $2q-p'\geqq 0$ ($\leqq 0$) with $2q-p'=0$ possible only at isolated points, then* (1) *is oscillatory if* (4) *is oscillatory.*

Proof. Assuming $p'-2q\geqq 0$, let N be a solution of (1) that satisfies $N(a)=N'(a)=0$, $N''(a)=1$. Then $N(x)>0$ for $x>a\geqq 0$ [2]. Now letting

$$F[N(x)]\equiv N'^2(x)-2N(x)N''(x)-p(x)N^2(x),$$

we have $N'^2(x)-2N(x)N''(x)-p(x)N^2(x)=F[N(a)]+\int_a^x(2q-p')N^2$. Thus

$$N'^2(x)-2N(x)N''(x)-p(x)N^2(x)\leqq 0 \quad \text{for } x>a$$

and

$$\frac{2N''(x)}{N(x)}-\frac{N'^2(x)}{N^2(x)}\geqq -p(x). \tag{5}$$

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Letting $y = wN^{1/2}$, we see that (3) becomes

$$w''+\left[p+\frac{3}{4}\left(\frac{2N''}{N}-\frac{N'^2}{N^2}\right)\right]w=0. \tag{6}$$

Now by (5), $p+\frac{3}{4}(2N''/N-N'^2/N^2)\geqq p-3p/4=p/4$. Thus by the Sturm comparison theorem if (4) is oscillatory, then (3) is oscillatory. Thus since oscillation of (3) implies oscillation of (2), we have by [2] oscillation of (1).

If $2q-p'\geqq 0$, apply the first part of the proof to (2).

THEOREM 2. *Suppose* $p\geqq 0$, $q\leqq 0$, $p'-2q\geqq 0$ *with zeros possible only at isolated points. If* $\int_a^\infty (p'-2q)=+\infty$, *then* (1) *is oscillatory.*

Proof. For every nontrivial solution y of (1), let $F[y(x)] \equiv y'^2(x)-2y(x)y''(x)-p(x)y^2(x)=F[y(a)]+\int_a^x (2q-p')y^2$. It is clear that $F[y(x)]$ is decreasing. Suppose (1) is nonoscillatory. Let u_1, u_2, u_3 be a basis for the solution space of (1). Define y_n by $y_n(n)=y_n'(n)=0$ and such that $y_n = C_{n,1}u_1+C_{n,2}u_2+C_{n,3}u_3$ where $C_{n,1}^2+C_{n,2}^2+C_{n,3}^3=1$. Assume, without loss of generality, that $\lim C_{n,i}=C_i$ for $i=1$, 2, 3. Let $N=C_1u_1+C_2u_2+C_3u_3$. The function N is nontrivial since $C_1^2+C_2^2+C_3^2=1$. Now $F[y_n(n)]=0$ and $F[y(x)]$ is decreasing for every nontrivial solution y of (1). Thus, $F[N(x)]>0$ for all x. Since $F[-N(x)]=F[N(x)]$ and (1) is nonoscillatory, we shall assume N is eventually positive, i.e., $N(x)>0$ for $x>a$. Suppose $N'(b)=0$ for $b>a$. Then $(NN''+pN^2/2)(b)=-F[N(b)]/2<0$. Thus $N''(b)<0$ and N' can have at most one zero after a. If $N'(x)<0$ for large x, then $N'''\geqq 0$ with equality possible only at isolated points. In that case N'' is eventually one sign. If $N''>0$ for large x, then N' is eventually positive, which is a contradiction. If $N''<0$ for large x, then since $N'<0$, N is eventually negative. Thus we conclude that $N'(x)>0$ for large x. Thus if $N(C)>0$ and $N'(x)>0$ on $[C, +\infty)$,

$$\begin{aligned} 0<F[N(x)]&=F[N(C)]+\int_C^x (2q-p')N^2 \\ &\leqq F[N(C)]+N^2(C)\int_C^x (2q-p')\to -\infty \quad \text{with } x. \end{aligned}$$

Thus (1) must have an oscillatory solution.

We next generalize a theorem of Lazer [3] which applies to the case where $p\leqq 0$, $q>0$ to apply to the case where p need not be of constant sign. We shall state our hypothesis in terms of a property of a nonoscillatory solution of (2). Later we shall give conditions on the coefficients of (1) to guarantee such a solution of (2).

THEOREM 3. *Suppose* $q>0$ *and* (1) *is* C_I *(see* [2]). *Suppose there is a point* x_0 *such that the nonoscillatory solution of* (2) z *defined by* $z(x_0)=z'(x_0)=0$, $z''(x_0)=1$ *has the property that* $\int_{x_0}^\infty z=+\infty$. *Then a sufficient condition for oscillation of* (1) *is that*

$$\int_N q-\frac{2(-p)^{3/2}}{3\sqrt{3}}+\int_{-N} q=+\infty,$$

where

$$N=\{x : p(x)\leqq 0\}.$$

Proof. First we observe that two independent solutions of (1) satisfy

$$(y'/z)'+[(z''+pz)/z^2]y=0 \tag{7}$$

for $x>x_0$. Assuming (1) is nonoscillatory, we have that (7) is nonoscillatory. Since $\int_{x_0}^{\infty} z=+\infty$, there exists a solution y of (7) and hence of (1) such that $y>0$ and $y'>0$ for large values of x [1]. Let $t=y'/y$. Then

$$t''+3t't=-(t^3+pt+q).$$

Thus if $p(x)\leqq 0$, $-(t^3+pt+q)\leqq -q+2(-p)^{3/2}/3\sqrt{3}$ and if $p(x)>0$, $-(t^3+pt+q)\leqq -q$. Thus $t'(x)\leqq t'(C)+3t^2(C)/2-3t^2(x)/2-\int_{N\cap(C,x)} q-2(-p)^{3/2}/3\sqrt{3}-\int_{-N\cap(C,x)} q$. Since the right-hand side goes to $-\infty$ by hypothesis we have a contradiction. Thus (1) is oscillatory.

See [2] and [3] for various conditions under which (1) is C_I. If p is bounded above, then since $(z''+pz)'=qz>0$ for $x>x_0$ and $(z''+pz)(x_0)=1$, we have $x-x_0<\int_{x_0}^{x} z''+pz\leqq z'(x)-z'(x_0)+B\int_{x_0}^{x} z$, where $p(x)<B$ for all x. Thus if $\int_{x_0}^{\infty} z<\infty$, we have $z'(x)\to+\infty$ which implies $z\to+\infty$. Thus we state the following corollary to Theorem 3.

COROLLARY. *If $q>0$, $p<B$ for some real number B and $y''+py=0$ is nonoscillatory, a sufficient condition for oscillation of* (2) *is that*

$$\int_N q-2(-p)^{3/2}/3\sqrt{3}+\int_{-N} q=+\infty,$$

where

$$N=\{x:p(x)\leqq 0\}.$$

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CONVOLUTIONS OF ORTHONORMAL POLYNOMIALS*

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Abstract. In this paper, we determine all pairs of orthogonal polynomial sequences $\{p_n(x)\}$ and $\{q_n(x)\}$, such that their convolution,

$$Q_n(x, y) = \sum_{k=0}^{n} p_k(x)q_{n-k}(y), \qquad n \geqq 0,$$

defines $\{Q_n(x, y)\}$ as an orthogonal polynomial sequence in x for all y. All such triples are determined explicitly in terms of their three-term recurrence formulas. Generating functions and "explicit" representation formulas are obtained. The resulting sequences are found to consist of a class of orthogonal polynomials characterized by J. Meixner (which class includes the Laguerre and Hermite polynomials) together with a new class of orthogonal polynomials which includes the orthogonal q-polynomials of Al-Salam and Carlitz. Explicit orthogonality relations are found for one new special case of this latter class.

1. Introduction. Two rather pretty identities involving the Hermite and Laguerre polynomials are [6, § 10.12 (41), 10.13 (38)]:

$$2^{-n/2}H_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} H_k(2^{1/2}x)H_{n-k}(2^{1/2}y),$$

$$L_n^{\alpha+\beta+1}(x+y) = \sum_{k=0}^{n} L_k^{\alpha}(x)L_{n-k}^{\beta}(y).$$

Notice that the first expresses an orthogonal polynomial as a convolution of members of the same set, while the second expresses an orthogonal polynomial as a convolution of members of two different sets of orthogonal polynomials.

In looking for a common basis for such identities, one is led rather naturally to consider their generating functions. One quickly observes that the class of orthogonal polynomials characterized by Meixner [7] satisfy such identities. Meixner's class consists of the orthogonal polynomial sequences $\{P_n(x)\}$ having generating functions of the form

$$A(w)\, e^{xB(w)} = \sum_{n=0}^{\infty} P_n(x)w^n,$$

where $A(w)$ and $B(w)$ are formal power series such that $A(0) \neq 0$, $B(0) = 0$, and $B'(0) \neq 0$.

In addition to the Hermite and Laguerre polynomials, Meixner's class includes the Charlier polynomials $c_n(x; a)$, the Meixner polynomials $m_n(x; \beta, c)$, and the Pollaczek polynomials $p_n^{\lambda}(x, \phi)$; apart from trivial transformations of these, no others are included. For these polynomials, see [6, §§ 10.21, 10.24, 10.25].

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From their generating functions, the corresponding "convolution formulas" are easily found:

$$(a+b)^n c_n(x+y;a+b)=\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}c_k(x;a)c_{n-k}(y;b),$$

$$m_n(x+y;\alpha+\beta,c)=\sum_{k=0}^{n}\binom{n}{k}m_k(x;\alpha,c)m_{n-k}(y;\beta,c),$$

$$P_n^{\lambda+\mu}(x+y,\phi)=\sum_{k=0}^{n}P_k^{\lambda}(x;\phi)P_{n-k}^{\mu}(y,\phi).$$

The question then naturally arises as to whether there are other polynomials for which such identities exist. Specifically, are there orthogonal polynomial sequences (OPS) $\{p_n(x)\}$ and $\{q_n(x)\}$, such that if we define $Q_n(x,y) =\sum_{k=0}^{n}p_n(x)q_{n-k}(y)$, then $\{Q_n(x,y)\}$ is also an OPS in x for infinitely many values of y?

We shall answer this question by determining explicitly all such triples. The resulting class of OPS will include previously studied polynomials as well as new ones. Although we have been unable to determine orthogonality relations for these in general, we shall obtain them for one new special case.

2. Necessary conditions. Let $\{p_n(x)\}$ and $\{q_n(y)\}$ be OPS, and let their three-term recurrence relations be

(2.1) $$p_{n+1}(x)=(A_nx+B_n)p_n(x)-C_np_{n-1}(x),$$

(2.2) $$q_{n+1}(y)=(\alpha_ny+\beta_n)q_n(y)-\gamma_nq_{n-1}(y).$$

Next, let $\{Q_n(x,y)\}$ be defined by

(2.3) $$Q_n(x,y)=\sum_{k=0}^{n}p_k(x)q_{n-k}(y).$$

Since the coefficients of x^n in $Q_n(x,y)$ and $p_n(x)$ are identical, $\{Q_n(x,y)\}$ will be an OPS in x if and only if it satisfies a recurrence relation of the form

(2.4) $$Q_{n+1}w_1(x,y)=(A_nx+D_n)Q_n(x,y)-E_nQ_{n-1}(x,y).$$

In all three recurrence relations, $n\geqq 0$ and the initial conditions are

$$p_{-1}(x)=q_{-1}(y)=Q_{-1}(x,y)=0,\qquad p_0(x)=q_0(y)=Q_0(x,y)=1.$$

Also, $A_n, B_n, C_n, \alpha_n, \beta_n, \gamma_n$ are independent of x and y; D_n and E_n are independent of x, $n\geqq 0$; and $A_nC_n\neq 0$, $\alpha_n\gamma_n\neq 0$, $E_n\neq 0$, $n\geqq 1$. The polynomials are orthogonal with respect to a real distribution on the real line if and only if all coefficients are real and $C_nA_nA_{n-1}>0$, $\gamma_n\alpha_n\alpha_{n-1}>0$, $E_nA_nA_{n-1}>0$, $n\geqq 1$.

To begin with, we use (2.3) in (2.4) to obtain

$$\sum_{k=0}^{n+1}p_k(x)q_{n+1-k}(y)=D_n\sum_{k=0}^{n}p_k(x)q_{n-k}(y)-E_n\sum_{k=0}^{n}p_k(x)q_{n-1-k}(y)$$
$$+A_n\sum_{k=0}^{n}A_k^{-1}q_{n-k}(y)[p_{k+1}(x)-B_kp_k(x)+C_kp_{k-1}(x)].$$

Now, $p_k(x)$ is a polynomial of precise degree k. In the preceding identity, the coefficients of all $p_i(x)$ are independent of x, so we can equate the coefficients of $p_k(x)$ on both sides of the identity to obtain

$$q_{n+1-k}(y)=D_nq_{n-k}(y)-E_nq_{n-1-k}(y)$$
$$+A_n\left[\frac{1}{A_{k-1}}q_{n+1-k}(y)-\frac{B_k}{A_k}q_{n-k}(y)+\frac{C_{k+1}}{A_{k+1}}q_{n-k-1}(y)\right].$$

This identity remains valid for $k=0$ if we interpret $1/A_{-1}=0$.

Collecting terms, we then obtain

(2.5)
$$\left(1-\frac{A_n}{A_{k-1}}\right)q_{n+1-k}(y)=\left[D_n-A_n\frac{B_k}{A_k}\right]q_{n-k}(y)+\left[A_n\frac{C_{k+1}}{A_{k+1}}-E_n\right]q_{n-1-k}(y),$$
$$k=0,1,\cdots,n,\quad 1/A_{-1}=0.$$

Setting $k=n$ in (2.5) then yields the necessary condition

$$\left(1-\frac{A_n}{A_{n-1}}\right)q_1(y)=(D_n-B_n)q_0,$$

whence

(2.6)
$$D_n=B_n+\left(1-\frac{A_n}{A_{n-1}}\right)q_1(y),\qquad n\geqq 0.$$

Thus D_n is at most linear in y.

Next, take $k=0$ in (2.5) to obtain

$$q_{n+1}(y)=\left[D_n-\frac{B_0}{A_0}A_n\right]q_n(y)-\left[E_n-A_n\frac{C_1}{A_1}\right]q_{n-1}(y).$$

Comparing the latter with (2.2), we obtain

(2.7)
$$\left[D_n-A_n\frac{B_0}{A_0}-\alpha_ny-\beta_n\right]q_n(y)=\left[E_n-A_n\frac{C_1}{A_1}-\gamma_n\right]q_{n-1}(y).$$

Now, D_n is at most linear in y, and hence E_n is at most quadratic in y. Since $q_n(y)$ and $q_{n-1}(y)$ have no zeros in common (a well-known consequence of the recurrence relation), it follows that, at least for $n\geqq 3$, the coefficients of $q_n(y)$ and $q_{n-1}(y)$ in (2.7) must both vanish identically. That is, at least for $n\geqq 3$, we must have

(2.8)
$$D_n=\alpha_ny+\beta_n+A_n\frac{B_0}{A_0},$$

(2.9)
$$E_n=A_n\frac{C_1}{A_1}+\gamma_n.$$

Comparison of (2.8) with (2.6) then yields (for $n\geqq 3$):

(2.10)
$$1-\frac{A_n}{A_{n-1}}=\frac{\alpha_n}{\alpha_0},$$

(2.11)
$$\left(\frac{B_n}{A_n}-\frac{B_0}{A_0}\right)A_n=\left(\frac{\beta_n}{\alpha_n}-\frac{\beta_0}{\alpha_0}\right)\alpha_n.$$

Returning to (2.7), we note it reduces to an identity for $n=0$ if (2.6) is invoked. For $n=1$, (2.7) requires

$$E_1=[D_1-\alpha_1 y-\beta_1-A_1B_0/A_0]q_1(y)+C_1+\gamma_1, \tag{2.12}$$

while for $n=2$, it requires

$$D_2=kq_1(y)+\alpha_2 y+\beta_2+A_2B_0/A_0, \tag{2.13}$$

$$E_2=kq_2(y)+\gamma_2+A_2C_1/A_1, \tag{2.14}$$

with $k=1-A_2/A_1-\alpha_2/\alpha_0$.

If we compare (2.13) with (2.6), we then find that (2.11) must hold for $n=2$. Further, it is clear that if (2.10) is satisfied for $n=1$ and 2, then D_2 and E_2 reduce to (2.8) and (2.9), while E_1 is at most linear in y. But if E_1 is at most linear, (2.7) would then require $D_1-A_1B_0/A_0-\alpha_1 y-\beta_1=0$ and $E_1=C_1+\gamma_1$. Then reference to (2.6) would show that (2.11) is valid for $n=1$ also.

Thus our next task is to show that (2.10) *is* satisfied for $n=1$, 2. Then (2.8)–(2.11) are satisfied for $n\geqq 1$ and, in particular, E_n is independent of y for $n\geqq 1$.

3. Independence of E_n from y; a characterization of Meixner's class. Since we do know E_n is independent of y for $n\geqq 3$, we can equate coefficients of y^{n+1-k} in (2.5). This yields

$$1-\frac{A_n}{A_{k-1}}=\frac{\alpha_n}{\alpha_{n-k}}, \qquad 0\leqq k\leqq n, \quad n\geqq 3. \tag{3.1}$$

Taking $k=n-1$ and $k=n$ in (3.1), we eliminate α_n and get

$$(\alpha_0-\alpha_1)\rho_n-\alpha_0\rho_{n-1}+\alpha_1\rho_{n-2}=0, \qquad n\geqq 3, \tag{3.2}$$

where $\rho_n=A_n^{-1}$.

In this difference equation, $\alpha_0\neq\alpha_1$, since otherwise $A_n=A_1$ and (3.1) would require $\alpha_n=0$. Thus, taking $\alpha_1\neq\alpha_0$, we obtain the solution,

$$\rho_n=\frac{1}{A_1}+\frac{A_1-A_2}{A_1A_2}\frac{1-q^{n-1}}{1-q}, \qquad n\geqq 1, \tag{3.3}$$

where

$$q=\frac{\alpha_1}{\alpha_0-\alpha_1}\neq 1.$$

The case $q=1(\alpha_0=2\alpha_1)$ corresponds to repeated roots of the characteristic equation and can be treated as the limiting case $q\to 1$ (or, equivalently, $\alpha_1\to\alpha_0/2$).

Thus we have

$$A_n=\frac{A_1A_2(1-q)}{A_1-qA_2-(A_1-A_2)q^{n-1}}, \qquad n\geqq 1. \tag{3.4}$$

Next, use (2.10) to eliminate α_n from (3.1). This gives, for $n\geqq 3$, $0\leqq m\leqq n$,

$$\alpha_m=\alpha_0\frac{(1-A_n/A_{n-1}}{(1-A_n/A_{n-m-1})}.$$

Using (3.4), we then find

$$\alpha_m = \frac{\alpha_0(1-q)q^m}{1-q^{m+1}}, \qquad m \geqq 0. \tag{3.5}$$

On the other hand, if we use (2.10) and (3.4) to determine α_n for $n \geqq 3$ and then compare the result with (3.5), we find that consistency requires $A_1(1-q^2) = A_2(1-q^3)$. But this means (2.10) is valid for $n = 2$.

Finally take $n = 3$ and $k = 1, 2$ in (3.1), and eliminate A_3. This leads to $A_0(1-q) = A_1(1-q^2)$, and this means (2.10) holds for $n = 1$. We can then also rewrite (3.4) as

$$A_n = \frac{A_0(1-q)}{1-q^{n+1}}, \qquad n \geqq 0. \tag{3.6}$$

Now that we have shown that E_n is independent of y, we can characterize the orthogonal polynomials of Meixner's class as those corresponding to $q = 1$. Note that $q = 1$ is equivalent to $\alpha_n = (\alpha_0 A_0^{-1})A_n$.

THEOREM. *If* $\{p_n(x)\}$, $\{q_n(x)\}$ and $\{Q_n(x, y)\}$ *are all OPS in x related by* (2.3), *and if* $\alpha_n = cA_n$, $n \geqq 0$, *in* (2.1) *and* (2.2), *then all three OPS belong to the Meixner class.*

Proof. Introduce the formal generating functions

$$F(x; w) = \sum p_n(x)w^n, \qquad G(y; w) = \sum q_n(y)w^n,$$
$$H(x, y; w) = \sum Q_n(x, y)w^n,$$

so that

$$H(x, y; w) = F(x; w)G(y; w).$$

If $\alpha_n = cA_n$, the recurrence formula (2.4) together with (2.8) shows that $Q_n(x, y)$ is a polynomial in $x + cy$. We can assume without loss of generality that $c = 1$. Thus we can write (with a slight abuse of notation):

$$H(x+y; w) = F(x; w)G(y; w) = F(y; w)G(x; w).$$

It follows that $F(0; w) \neq 0$, and that

$$H(x; w) = G(0; w)F(x; w), \qquad G(y; w) = G(0; w)F(y; w)/F(0; w).$$

That is,

$$F(0; w)F(x+y; w) = F(x; w)F(y; w).$$

Expanding both sides as formal power series in x, y, we compare the coefficients and conclude that $F(x; w)$ is of the form

$$F(x; w) = A(w)\exp\{xB(w)\}$$

(which is essentially Cauchy's theorem). Since $p_n(x)$ is a polynomial of degree n, the conditions $A(0) \neq 0$, $B(0) = 0$, $B'(0) \neq 0$ are automatically satisfied. Thus F is a Meixner-type generating function, and $\{p_n(x)\}$, $\{q_n(x)\}$, $\{Q_n(x, y)\}$ all belong to the Meixner class of OPS.

4. Explicit determination of recurrence formulas. We turn next to the determination of the polynomials in the general case, $q \neq 1(\alpha_0 \neq 2\alpha_1)$. As we have noted previously, we must have $\alpha_0 \neq \alpha_1$.

Since E_n is now known to be independent of y for all n, we can compare (2.5) with (2.2) and conclude (with the aid of (2.8), (2.9) and (2.10) that

$$\left(\frac{\beta_n}{\alpha_n}-\frac{\beta_{n-k}}{\alpha_{n-k}}\right)\alpha_n=\left(\frac{B_k}{A_k}-\frac{B_0}{A_0}\right)A_n, \tag{4.1}$$

$$\left(\frac{\gamma_n}{\alpha_n}-\frac{\gamma_{n-k}}{\alpha_{n-k}}\right)\alpha_n=\left(\frac{C_{k+1}}{A_{k+1}}-\frac{C_1}{A_1}\right)A_n. \tag{4.2}$$

Using (3.5) and (3.6), we then obtain from (4.1)

$$\frac{B_n}{A_n}=\frac{1}{1-q}\left[\frac{B_1}{A_1}-q\frac{B_0}{A_0}-\left(\frac{B_1}{A_1}-\frac{B_0}{A_0}\right)q^n\right], \tag{4.3}$$

$$\frac{\beta_n}{\alpha_n}=\frac{q}{q-1}\left[\frac{\beta_1}{\alpha_1}-\frac{\beta_0}{q\alpha_0}-\left(\frac{\beta_1}{\alpha_1}-\frac{\beta_0}{\alpha_0}\right)q^{-n}\right]. \tag{4.4}$$

Similarly, (4.2) leads to

$$\frac{C_n}{A_nA_{n-1}}=\frac{1}{A_0(1-q)^2}\left[\frac{C_2}{A_2}-q\frac{C_1}{A_1}-\left(\frac{C_2}{A_2}-\frac{C_1}{A_1}\right)q^{n-1}\right](1-q^n), \tag{4.5}$$

$$\frac{\gamma_n}{\alpha_n\alpha_{n-1}}=\frac{q^2}{\alpha_0(1-q)^2}\left[\frac{\gamma^2}{\alpha^2}-\frac{\gamma_1}{q\alpha_1}-\left(\frac{\gamma_2}{\alpha_2}-\frac{\gamma_1}{\alpha_1}\right)q^{1-n}\right](1-q^{-n}). \tag{4.6}$$

Introduce the monic polynomials

$$\hat{p}_n(x)=(A_0A_1\cdots A_{n-1})^{-1}p_n(x)=[A_0(1-q)]^{-n}[q]_np_n(x), \tag{4.7}$$

$$\hat{q}_n(y)=(\alpha_0\alpha_1\cdots\alpha_{n-1})^{-1}q_n(y)=[\alpha_0(1-q)]^{-n}[q]_nq^{-n(n-1)/2}q_n(y), \tag{4.8}$$

where $[a]_0=1$, $[a]_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$, $n>0$.

With the use of (4.3)–(4.6), the recurrence formulas (2.1) and (2.2) can now be written

$$\hat{p}_{n+1}(x)=[x+f-aq^n]\hat{p}_n(x)-(g-bq^{n-1})(1-q^n)\hat{p}_{n-1}(x), \tag{4.9}$$

$$\hat{q}_{n+1}(y)=[y+h-cq^{-n}]\hat{q}_n(y)-(k-dq^{1-n})(1-q^{-n})\hat{q}_{n-1}(y), \tag{4.10}$$

where

$$a=\frac{1}{1-q}\left(\frac{B_1}{A_1}-\frac{B_0}{A_0}\right),\qquad f=\frac{1}{1-q}\left(\frac{B_1}{A_1}-\frac{qB_0}{A_0}\right),$$
$$b=\frac{1}{A_0(1-q)^2}\left(\frac{C_2}{A_2}-\frac{C_1}{A_1}\right),\qquad g=\frac{1}{A_0(1-q)^2}\left(\frac{C_2}{A_2}-\frac{qC_1}{A_1}\right), \tag{4.11}$$

$$c=\frac{q}{q-1}\left(\frac{\beta_1}{\alpha_1}-\frac{\beta_0}{\alpha_0}\right),\qquad h=\frac{q}{1-q}\left(\frac{\beta_1}{\alpha_1}-\frac{\beta_0}{q\alpha_0}\right),$$
$$d=\frac{q^2}{\alpha_0(1-q)^2}\left(\frac{\gamma_2}{\alpha_2}-\frac{\gamma_1}{\alpha_1}\right),\qquad k=\frac{q^2}{\alpha_0(1-q)^2}\left(\frac{\gamma_2}{\alpha_2}-\frac{\gamma_1}{q\alpha_1}\right). \tag{4.12}$$

These coefficients are related by the additional necessary conditions

$$\left(\frac{\beta_1}{\alpha_1}-\frac{\beta_0}{\alpha_0}\right)\alpha_1=\left(\frac{B_1}{A_1}-\frac{B_0}{A_0}\right)A_1, \qquad \left(\frac{\gamma_2}{\alpha_2}-\frac{\gamma_1}{\alpha_1}\right)\alpha_2=\left(\frac{C_2}{A_2}-\frac{C_1}{A_1}\right)A_2,$$

which are equivalent to

$$\alpha_0 c = -A_0 a, \qquad \alpha_0^2 d = A_0^2 b. \tag{4.13}$$

Note that (4.9)–(4.13) show that $\hat{q}_n(x)$ can be obtained formally from $\hat{p}_n(x)$ by replacing q by q^{-1} and A_i, B_i, C_{i+1} by α_i, β_i, γ_{i+1} respectively ($i=0, 1$).

Next, introduce the monic polynomials

$$R_n(x)=(A_0A_1, \cdots, A_{n-1})^{-1}Q_n(x, y), \tag{4.14}$$

$$S_n(y)=(\alpha_0\alpha_1 \cdots \alpha_{n-1})^{-1}Q_n(x, y). \tag{4.15}$$

Use (4.3)–(4.6) together with (2.8) and (2.9) to obtain

$$R_{n+1}(x)=(x+F-Aq^n)R_n(x)-(G-Bq^{n-1})(1-q^n)R_{n-1}(x), \tag{4.16}$$

$$S_{n+1}(y)=(y+H-Cq^{-n})S_n(y)-(K-Dq^{1-n})(1-q^{-n})S_{n-1}(y), \tag{4.17}$$

where

$$A=\frac{\alpha_0}{A_0(1-q)}\left[q\frac{\beta_1}{\alpha_1}-\frac{\beta_0}{\alpha_0}-(1-q)y\right], \qquad F=\frac{1}{1-q}\left(\frac{B_1}{A_1}-q\frac{B_0}{A_0}\right),$$

$$B=\frac{\alpha_0 q}{A_0^2(1-q)^2}\left(q\frac{\gamma^2}{\alpha_2}-\frac{\gamma_1}{\alpha_1}\right), \qquad G=\frac{1}{A_0(1-q)^2}\left(\frac{C_2}{A_1}-q\frac{C_1}{A_1}\right),$$

$$C=\frac{A_0 q}{\alpha_0(q-1)}\left[\frac{B_1}{qA_1}-\frac{B_0}{A_0}-(1-q^{-1})x\right], \qquad H=\frac{q}{q-1}\left(\frac{\beta_1}{\alpha_1}-\frac{\beta_0}{q\alpha_0}\right),$$

$$D=\frac{A_0 q}{\alpha_0^2(1-q)^2}\left(\frac{C_2}{qA_2}-\frac{C_1}{A_1}\right), \qquad K=\frac{q^2}{\alpha_0(1-q)^2}\left(\frac{\gamma_2}{\alpha_2}-\frac{\gamma_1}{q\alpha_1}\right).$$

The latter can be simplified to

$$A=-\frac{\alpha_0}{A_0}(y+h), \qquad F=f, \qquad B=\left(\frac{\alpha_0}{A_0}\right)^2 k, \qquad G=g, \tag{4.18}$$

$$C=-\frac{A_0}{\alpha_0}(x+f), \qquad H=h, \qquad D=\left(\frac{A_0}{\alpha_0}\right)^2 g, \qquad K=k, \tag{4.19}$$

Thus we see that $\{Q_n(x, y)\}$ is, in general, an OPS in y as well as in x. As before, we observe that $\{S_n(y)\}$ can be obtained from $\{R_n(x)\}$ by interchanging x and y and making the same replacements that turn $\{p_n(x)\}$ into $\{q_n(x)\}$. Note further the interesting, and perhaps surprising, fact that $Q_n(x, y)$ is independent of a, b, c and d.

5. Sufficiency; generating functions. It still remains to verify that these polynomials do indeed satisfy the convolution property (2.3). This will now be accomplished by means of generating functions.

We first simplify things a bit by taking (without loss of generality):

$$A_0 = -\alpha_0 = (1-q)^{-1}, \qquad f = h = 0.$$

Referring to (4.7), we put

$$\Phi(x;t) = \sum_{n=0}^{\infty} \hat{p}_n(x)\frac{t^n}{[q]_n} = \sum_{n=0}^{\infty} p_n(x)t^n. \tag{5.1}$$

From the recurrence formula (4.7), we find that

$$\begin{aligned} &\Phi(x;t) - \Phi(x;qt) = tx\Phi(x;t) - at\Phi(x;qt) - t^2[g\Phi(x;t) - b\Phi(x;qt)], \\ &\Phi(x;t) = \frac{1-at+bt^2}{1-xt+gt^2}\Phi(x;qt). \end{aligned} \tag{5.2}$$

We set

$$\begin{aligned} 1-at+bt^2 &= (1-\alpha t)(1-\beta t), \\ 1-xt+gt^2 &= (1-\gamma t)(1-\delta t), \end{aligned} \tag{5.3}$$

with the convention that if $b = 0$ or $g = 0$, we take $\beta = 0$ or $\delta = 0$, respectively. Then for $|q| < 1$, we have

$$\begin{aligned} \Phi(x;t) &= \prod_{k=0}^{\infty} \frac{(1-\alpha tq^k)(1-\beta tq^k)}{(1-\gamma tq^k)(1-\delta tq^k)}, \\ \Phi(x;t) &= \frac{e(\gamma t)\, e(\delta t)}{e(\alpha t)\, e(\beta t)}, \end{aligned} \tag{5.4}$$

where

$$e(w) = \prod_{n=0}^{\infty} (1-wq^n)^{-1} = \prod_{n=0}^{\infty} \frac{w^n}{[q]_n}.$$

Next, referring to (4.8), we set

$$\Psi(y;t) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \hat{q}_n(y)\frac{t^n}{[q]_n} = \sum_{n=0}^{\infty} q^n q_n(y)t^n. \tag{5.5}$$

Now, Ψ can be regarded as being obtained from (5.1) by replacing q by q^{-1}, A_i by α_i, etc. Under this replacement, a, b and g are replaced by c, d and k. Further, according to (4.13), $c = a$, $d = b$. From (5.2) we thus obtain that

$$\Psi(y;qt) = \frac{1-aqt+bq^2t^2}{1-yqt+kq^2t^2}\Psi(y;t),$$

whence, for $|q|<1$,

$$\Psi(y;t)=\frac{e(\alpha qt)\,e(\beta qt)}{e(\gamma' qt)\,e(\delta' qt)}, \tag{5.6}$$

where

$$1-yt+kt^2=(1-\gamma' t)(1-\delta' t) \tag{5.7}$$

and $\delta'=0$ in case $k=0$.

We then have

$$\Phi(x;t)\Psi(y;q^{-1}t)=\frac{e(\gamma t)\,e(\delta t)}{e(\gamma' t)\,e(\delta' t)}.$$

But if we compare (4.16) with (4.9), it becomes clear that the generating function for $\{R_n(x)/[q]_n\}$ must be

$$W(x;t)=\frac{e(\gamma^* t)\,e(\delta^* t)}{e(\alpha^* t)\,e(\beta^* t)},$$

where

$$1-At+Bt^2=(1-\alpha^* t)(1-\beta^* t),$$

$$1-xt+Gt=(1-\gamma^* t)(1-\delta^* t).$$

But according to (4.18), $A=y$, $B=k$, $G=g$, and hence, referring to (5.3) and (5.7), we see that $\alpha^*=\gamma'$, $\beta^*=\delta'$, $\gamma^*=\gamma$, $\delta^*=\delta$. That is,

$$W(x;t)=\Phi(x;t)\Psi(y;q^{-1}t),$$

which shows that (2.3) is satisfied.

Summarizing to this point, we see that if $\{p_n(x)\}$, $\{q_n(y)\}$ and $\{Q_n(x,y)\}$ are all OPS related by the convolution property (2.3), then either all three belong to the Meixner class (and Q_n is a polynomial in $x+y$) or they are essentially the polynomials satisfying, respectively, (4.9), (4.10) (related by (4.13)) and (4.14), (4.15). In the former case, the convolution formula will be essentially one of the five given in § 1.

6. Explicit formulas; special cases. A further study of the polynomials in §§ 4 and 5 will now be made. To fix the notation, we shall standardize our polynomials by taking

$$\hat{p}_n(x)\equiv P_n(x)\equiv P_n(x;q;a,b,g)$$

as the polynomials satisfying

$$P_{n+1}(x)=(x-aq^n)P_n(x)-(g-bq^{n-1})(1-q^n)P_{n-1}(x), \tag{6.1}$$

where $P_{-1}(x)=0$, $P_0(x)=1$, and q, a, b, g are parameters with $q\neq 1$, $g-bq^{n-1}\neq 0$, $n\geqq 1$. With this notation, the remaining polynomials are

$$\begin{aligned}\hat{q}_n(y)&=P_n(y;q^{-1};a,b,k),\\ R_n(x)&=P_n(x;q;y,k,g),\\ S_n(y)&=P_n(y;q^{-1};x,g,k).\end{aligned} \tag{6.2}$$

The convolution property reads

(6.3) $$P_n(x;q;y,k,g)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{k(k-1)/2}P_{n-k}(x;q;a,b,q)P_k(y;q^{-1};a,b,k),$$

where

$$\begin{bmatrix}n\\k\end{bmatrix}=\frac{[q]_n}{[q]_k[q]_{n-k}}.$$

We next note the identities (see [2, (1.4), (1.5)]):

$$e(\gamma t)\,e(\delta t)=\sum H_n(\delta/\gamma)(\gamma t)^n/[q]_n,$$
$$[e(\alpha t)\,e(\beta t)]^{-1}=\sum(-1)^n q^{n(n-1)/2}G_n(\beta/\alpha)(\alpha t)^n/[q]_n,$$

where

$$H_{n+1}(x)=(1+x)H_n(x)-x(1-q^n)H_{n-1}(x),$$
$$G_{n+1}(x)=(1+x)G_n(x)-x(1-q^{-n})G_{n-1}(x).$$

Let

$$h_n(x)=\gamma^n H_n(\delta/\gamma),\qquad g_n=(\gamma')^n G_n(\delta'/\gamma'),$$

so that (cf. (5.3))

(6.4) $$h_{n+1}(x)=xh_n(x)-g(1-q^n)h_{n-1}(x),$$
$$g_{n+1}=ag_n-b(1-q^{-n})g_{n-1}.$$

We see that $h_n(x)$ is a polynomial (in fact orthogonal polynomial) of degree n. The generating function (5.4) now produces the "explicit" formula

(6.5) $$P_n(x;q;a,b,g)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{k(k-1)/2}g_k h_{n-k}(x).$$

With the use of other known series expansions for different rational combinations of $e(w)$, the generating function can be expanded in a variety of ways. For example, use of formula (1.12) of [2], namely,

$$\frac{e(w)}{e(aw)}=\sum_{n=0}^{\infty}[a]_n\frac{w^n}{[q]_n},$$

yields

(6.6) $$P_n(x;q;a,b,g)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\gamma^k[\alpha/\gamma]_k\delta^{n-k}[\beta/\delta]_{n-k}.$$

Turning next to orthogonality questions, we first observe that we shall have orthogonality with respect to a real distribution on the real line in the following cases (only):

All parameters are real, and
(i) $|q|<1, \quad g>b\geqq 0,$
(ii) $0<q<1, \quad g\geqq 0>b,$
(iii) $-1<q<0, \quad g>bq>0,$
(iv) $q>1, \quad b>g, \quad b\geqq 0.$

Referring to (6.1), we write $c_n = aq^{n-1}$, $\lambda_{n+1}=(g-bq^{n-1})(1-q^n)$. If $|q|<1$, then $c_n \to 0$ and $\lambda_n \to g>0$. According to a theorem of Blumenthal [3], the zeros of the $P_n(x)$ are dense in the interval $(-2\sqrt{g}, 2\sqrt{g})$. This conclusion is vacuous if $g=0$, but in this case, a theorem of Krein [1, p. 231] says the corresponding distribution function has a bounded, denumerable spectrum whose only limit point is 0.

When $q>1$, we have $\lambda_n \to \infty$, so the interval of orthogonality is unbounded. If $a\neq 0$, $a>0$, say, then it can be shown (see [4]) that (i) if $4b<a^2$, then the set S of limit points of the zeros is a denumerable set bounded below by $\min(a, ag/b)$ and having no finite limit point, (ii) if $4b=a^2$, the zeros are dense in (σ, ∞) where $\sigma\geqq \min(a, ag/b)$, (iii) if $4b>a^2$, the true interval of orthogonality is $(-\infty, \infty)$.

If $q>1$ and $a=0$, we have the symmetric case. By considering $\{P_{2n}(x^{1/2}; q, 0, b, g)\}$, it can then be shown that S is a denumerable set with no finite limit point.

In a more explicit vein, we note that the case of $g=0$, $ab\neq 0$ can be identified with known OPS. We have

$$P_n(x; q; 1+a, a, 0) = U_n^{(a)}(x),$$
$$P_n(x; q^{-1}; 1+a, a, 0) = V_n^{(a)}(x),$$

where $\{U_n^{(a)}(x)\}$ and $\{V_n^{(a)}(x)\}$ are orthogonal with respect to discrete distributions explicitly found by Al-Salam and Carlitz [2]. (The general case, $g=0$, can be transformed to the above by a linear change of the variable x.)

Finally, for the special case $a=b=0$, $g>0$, $|q|<1$ we have a new class of orthogonal polynomials for which a weight function will now be obtained.

7. Explicit orthogonality relation for $a=b=0$. The case $a=b=0$ corresponds to the recurrence formula (6.4):

$$P_n(x; q; 0, 0, g) = h_n(x).$$

Let

$$h_n^*(x) = g^{-n/2} h_n(2g^{1/2}x), \tag{7.1}$$

and let $x=\cos\theta$, $0\leqq\theta\leqq\pi$. Referring to (5.3), we then observe that we have

$$\gamma = g^{1/2} e^{i\theta}, \qquad \delta=\bar{\gamma}, \qquad \alpha=\beta=0.$$

Thus (6.6) yields

$$h_n^*(x) = g^{-n/2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}\gamma^k\bar{\gamma}^{n-k} = \sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix} e^{i(n-2k)\theta},$$

(7.2)

$$h_n^*(x) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix} T_{n-2k}(x),$$

where $T_m(x)=\cos m\theta$ is the Chebyshev polynomial, and we write $T_{-m}(x) = T_m(x)$.

We shall now find a weight function $w(x)\geqq 0$ such that

$$I_n=\int_{-1}^{1} h_n^*(x)w(x)\,dx=0 \quad \text{for } n>0. \tag{7.3}$$

Since $\{h_n^*(x)\}$ is known to be an OPS, (7.3) will be sufficient to prove $h_n^*(x)$ is orthogonal to $h_m^*(x)$, $m\neq n$, with respect to $w(x)$ for all m, n. (This is a consequence of the recurrence formula; see, e.g., [5].)

First, let

$$\varphi(x)=\sum_{\nu=-\infty}^{\infty}\lambda_\nu T_{2\nu}(x), \tag{7.4}$$

where

$$\lambda_\nu=(-1)^\nu q^{(2\nu+1)^2/8}, \qquad \nu=0, \pm 1, \pm 2, \cdots.$$

Then, putting

$$w(x)=(1-x^2)^{-1/2}\varphi(x), \tag{7.5}$$

we use the well-known orthogonality properties of $\{T_n(x)\}$ to obtain

$$I_{2m}=\begin{bmatrix}2m\\ m\end{bmatrix}\lambda_0\pi+\sum_{\substack{k=0\\ k\neq m}}^{2m}\begin{bmatrix}2m\\ k\end{bmatrix}(\lambda_{m-k}+\lambda_{k-m})\frac{\pi}{2}=\pi\sum_{k=0}^{2m}\begin{bmatrix}2m\\ k\end{bmatrix}\lambda_{m-k},$$

where we have used the fact that

$$\begin{bmatrix}n\\ k\end{bmatrix}=\begin{bmatrix}n\\ n-k\end{bmatrix}, \qquad 0\leqq k\leqq n.$$

Now

$$\lambda_{m-k}=(-1)^m q^{(2m+1)^2/8}\cdot(-1)^k q^{k(k-1)/2}q^{-mk},$$

so that

$$I_{2m}=(-1)^m q^{(2m+1)^2/8}\pi\sum_{k=0}^{2m}\begin{bmatrix}2m\\ k\end{bmatrix}(-1)^k q^{k(k-1)/2}(q^{-m})^k.$$

In view of Gauss' identity,

$$\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}q^{k(k-1)/2}u^k=(1+u)(1+uq)\cdots(1+uq^{n-1}),$$

we see that $I_{2m}=0$ for $m>0$.

Noting that $\varphi(x)(1-x^2)^{-1/2}$ is an even function, we conclude that $I_{2m+1}=0$ also. From the recurrence formula, we then obtain

$$\int_{-1}^{1} x^n h_n^*(x)w(x)\,dx=2^{-n}[q]_n\lambda_0\pi,$$

whence we conclude the orthogonality relations,

$$\int_{-1}^{1} h_m^*(x) h_n^*(x) w(x)\, dx = \pi q^{1/8} [q]_n \delta_{mn}. \tag{7.6}$$

There remains only the task of proving the positivity of $w(x)$. To this end, we note the identity due to Jacobi [6, 17.2.2.(16)],

$$\sum_{k=-\infty}^{\infty} x^{k^2} z^k = \prod_{n=1}^{\infty} \{(1-x^{2n})(1+x^{2n-1}z)(1+x^{2n-1}z^{-1})\}.$$

Taking $x = q^{1/2}$, $z = -q^{1/2} e^{2\theta i}$, we get

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{k(k+1)/2} e^{2k\theta i} = \prod_{n=1}^{\infty} \{(1-q^n)(1-q^n e^{2\theta i})(1-q^{n-1} e^{-2\theta i})\}$$

$$= (1-e^{-2\theta i}) \prod_{n=1}^{\infty} \{(1-q^n)|1-q^n e^{2\theta i}|^2\}.$$

Comparing the latter with (7.4) and (7.5), we conclude that

$$w(x) = 2q^{1/8} \prod_{n=1}^{\infty} (1-q^n)(1-x^2)^{1/2} \prod_{n=1}^{\infty} |1-q^n e^{2\theta i}|^2. \tag{7.7}$$

Note added in proof. The special case discussed in § 7 has been studied previously by W. A. Allaway [8]. Allaway obtained the weight function in the form of a sine series which is equivalent to (7.5). He also obtained the corresponding special cases of the recurrence relation (6.1) and the generating function (5.4) and an explicit formula which is equivalent to a known formula for $H_n(x)$ (see [2, (1, 2)]. However, the closed form (7.7) for the weight function and the consequent proof of the positivity of the weight function are new.

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ON BIRTH-DEATH PROCESSES WITH RATIONAL GROWTH RATES

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Abstract. Birth–death processes can be described by systems of differential equations together with certain probability conditions. Karlin and McGregor have shown that these systems of equations and the probability conditions have solutions which can be expressed by an integral representation formula which uses an associated set of orthogonal polynomials and the distribution function for these polynomials. It is shown that certain general results from the study of orthogonal polynomials can be used to describe this distribution function in most cases where the birth and death rates are rational functions of the population size.

1. Introduction. Karlin and McGregor have shown ([1], [2]) that there is an intimate relationship between birth–death processes and distribution functions with support on a subset of the nonnegative real numbers. They have also shown [3] that in the case where the birth and death rates are linear in the population size, the transition probabilities for the process are determined by a distribution function which is either the distribution function for the Meixner polynomials, the Laguerre polynomials, or a closely related distribution function which can be obtained by a direct computation. Thus they are able to characterize all linear birth–death processes. In this paper we show that a similar characterization can be given for many birth–death processes with nonlinear, in particular, rational, growth rates.

In § 2 we review the work of Karlin and McGregor and state some preliminary results which are needed for the study of rational processes. In § 3 we characterize most rational processes in terms of the associated distribution function. In § 4 we consider some special cases, open questions and related conjectures.

2. Preliminaries. A birth–death process (henceforth called a b–d process) is a special type of Markov chain with states which can be identified with the nonnegative integers and transition probabilities which are subject to certain basic infinitesimal assumptions (the transitions may occur continuously). The probability $P_{ij}(t)$, of a transition from state i at time 0 to state j at time $t \geqq 0$, depends on the time t available for the transition, but it does not depend on how or when the process arrived at state i. The basic infinitesimal assumptions about the quantities $P_{ij}(t)$ which characterize a b–d process are as follows:

$$\text{(S)}\quad \begin{aligned} &P_{ii-1}(\Delta t) = \mu_i \Delta t + o(\Delta t), && \Delta t \to 0,\\ &P_{ii}(\Delta t) = 1 - (\lambda_i + \mu_i)\Delta t + o(\Delta t), && \Delta t \to 0,\\ &P_{ii+1}(\Delta t) = \lambda_i \Delta t + o(\Delta t), && \Delta t \to 0,\\ &P_{ij}(\Delta t) = o(\Delta t), \quad |i - j| > 1, && \Delta t \to 0. \end{aligned}$$

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In (S) the quantities λ_i and μ_i are the birth and death rates, respectively, for the b–d process. These rates depend on the state of the process, but they do not depend on time t. These rates are all positive, except possibly for μ_0, which may be zero. Thus a b–d process is determined by the sequences $\{\lambda_i\}_0^\infty$ and $\{\mu_i\}_0^\infty$, and one seeks the probability functions $P_{ij}(t)$, for $i, j = 0, 1, 2, \cdots$ and $t \geqq 0$. For a general discussion of b–d processes and for applications of this model, see [4, Chap. 7].

In [1], Karlin and McGregor established a correspondence between solutions of the system of differential equations which follow from (S) and the other assumptions of a b–d process and solutions of solvable Stieltjes moment problems. To be specific, let $A = (a_{ij})$ be the matrix with

$$a_{ii} = -(\lambda_i + \mu_i), \quad a_{i,i+1} = \lambda_i, \quad a_{i-1,i} = \mu_i \quad \text{and} \quad a_{ij} = 0, \qquad |i-j| > 1,$$

and let $P(t) = (P_{ij}(t))$. Then the differential equations for a b–d process are summarized as follows:

$$\text{I.} \qquad P'(t) = P(t)A,$$

$$\text{II.} \qquad P'(t) = AP(t),$$

$$\text{III.} \qquad P(0) = I = (\delta_{ij}).$$

In addition to satisfying I, II and III, we also ask that the matrix $P(t)$ satisfy the natural probability assumptions associated with a Markov chain; namely,

$$\text{IV.} \qquad P_{ij}(t) \geqq 0,$$

$$\text{V.} \qquad \sum_{j=0}^{\infty} P_{ij}(t) \leqq 1, \qquad i = 0, 1, 2, \cdots, \quad t \geqq 0,$$

$$\text{VI.} \qquad P(t+s) = P(t)P(s).$$

Karlin and McGregor have shown that matrices $P(t)$ which satisfy I $\to$ VI can be represented by an integral with respect to a special distribution function which solves a related Stieltjes moment problem. The related moment problem is obtained as follows.

Let $\{Q_n(x)\}_0^\infty$ be defined by the recurrence formula

$$Q_0(x) \equiv 1,$$

$$\text{(T)} \qquad -xQ_0(x) = -(\lambda_0 + \mu_0)Q_0(x) + \lambda_0 Q_1(x),$$

$$-xQ_n(x) = \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n)Q_n(x) + \lambda_n Q_{n+1}(x), \qquad n \geqq 1.$$

Thus, since $\lambda_i > 0$, $i \geqq 0$ and $\mu_i > 0$, $i > 0$, it follows that the polynomial $Q_n(x)$ has degree n, and thus these polynomials can be formally used to recursively define a sequence of "moments", $\{M_n\}_0^\infty$. The Stieltjes moment problem (S.M.P.) which corresponds to the sequence $\{M_n\}_0^\infty$ (i.e., the problem of finding a distribution function ψ such that $M_n = \int_0^\infty x^n \, d\psi(x)$, $n = 0, 1, \cdots,$) is called the S.M.P. associated with the sequences $\{\mu_i\}_0^\infty$ and $\{\lambda_i\}_0^\infty$ or the S.M.P. associated with the b–d process determined by these sequences. A discussion of moment problems in general and the S.M.P. in particular can be found in [5].

A summary of the fundamental results of Karlin and McGregor follows. (See [1] for proofs and for many interesting and useful related results.)

DEFINITION. A solution ψ of the S.M.P. associated with the sequences $\{\mu_n\}$ and $\{\lambda_n\}$ is called *extremal* if the Parseval equation

$$\int_0^\infty |f(x)|^2 \, d\psi(x) = \sum_{n=0}^{\infty} \left| \int_0^\infty f(x) Q_n(x) \, d\psi(x) \right|^2 \pi_n$$

is valid for each $f \in L_2(d\psi)$. Here $\{\pi_n\}_0^\infty$ is the sequence defined by $\pi_0 = 1$ and $\pi_n = (\lambda_0 \cdots \lambda_{n-1})/(\mu_1 \cdots \mu_n)$ for $n \geqq 1$.

THEOREM 1. *The S.M.P. associated with the b–d processes determined by the sequences* $\{u_i\}_0^\infty$ *and* $\{\lambda_i\}_0^\infty$ *is solvable. If* ψ *is any solution of this S.M.P. and if the matrix* $P(t) = P(t, \psi)$ *is defined by*

$$\text{VII.} \qquad P_{ij}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x) Q_j(x) \, d\psi(x), \qquad t \geqq 0,$$

$i, j = 0, 1, 2, \cdots$, *then* $P(t)$ *satisfies* I, II, III *and* IV. *In order to satisfy* VI, *it is necessary and sufficient that* ψ *be extremal.*

If $\mu_0 = 0$, *then* $P(t)$ *given by* VII *satisfies* V *whenever* ψ *is extremal. If* $\mu_0 > 0$, *then there is at least one extremal* ψ *such that* $P(t) = P(t, \psi)$ *satisfies* V. *This extremal* ψ *is characterized by the property that the spectrum of* ψ *has no points in the interval* $(-\infty, \xi)$, *where* $\xi = \lim_{n \to \infty} \xi_{1n}$, *and* ξ_{1n} *is the first zero of* Q_n.

THEOREM 2. *Any matrix* $P(t)$ *which satisfies* I $\to$ VI *can be represented in the form* VII, *where* ψ *is an extremal solution of the associated S.M.P.*

THEOREM 3. *In order that there be one and only one matrix* $P(t)$ *with properties* I $\to$ VI, *it is necessary and sufficient that*

$$\sum_{n=0}^{\infty} \left(\pi_n + \frac{1}{\lambda_n \pi_n} \right) = \infty .$$

In [3] the results given above are used to study and classify b–d processes for which λ_n and μ_n are linear in n. In § 3 below we shall consider b–d processes for which λ_n and μ_n are rational functions of n, subject of course to the constraints $\lambda_n > 0, \mu_n > 0, n = 1, 2, \cdots, \lambda_0 > 0, \mu_0 \geqq 0$. In most such cases we shall show how to obtain the distribution function ψ for the representation VII and we shall describe the spectrum of ψ. In order to do this we will need the following results relating the coefficients in the triple recurrence formula for a set of orthogonal polynomials to the distribution function of the orthogonal polynomials.

THEOREM 4. (Favard [6]). *Let the sequence* $\{\varphi_n\}_0^\infty$ *of polynomials be defined by the formulas*

$$\text{(R)} \qquad \begin{aligned} &\varphi_0(x) = 1, \qquad \varphi_1(x) = (x - a_0), \\ &\varphi_{n+1}(x) = (x - a_n)\varphi_n(x) - b_n \varphi_{n-1}(x), \qquad n \geqq 1, \end{aligned}$$

where $\{a_n\}_0^\infty$ *is a real sequence and* $\{b_n\}_1^\infty$ *is a positive sequence. Then there is a* *distribution function* ψ *such that the polynomials* $\{\varphi_n\}$ *are orthogonal with respect to* ψ. *Moreover, if the polynomials* $\{\varphi_n\}$ *are used to define a moment sequence* $\{M_n\}_0^\infty$, *then* ψ *is a solution of the moment problem associated with this sequence. Thus,*

$$\int_{-\infty}^{+\infty} \varphi_n \varphi_m \, d\psi = 0, \qquad n \neq m, \quad n, m = 0, 1, 2, \cdots,$$

$$\int_{-\infty}^{+\infty} \varphi_n^2 \, d\psi \neq 0, \qquad n = 0, 1, 2, \cdots,$$

$$\int_{-\infty}^{+\infty} x^n \, d\psi(x) = M_n, \qquad n = 0, 1, 2, \cdots.$$

Since all sets of orthogonal polynomials obey a triple recurrence formula such as (R), Theorem 4 establishes a correspondence between pairs of sequence $\{a_n\}_0^\infty$ and $\{b_n\}_1^\infty$ with the b_n's positive and distribution functions on $(-\infty, +\infty)$. The correspondence is not one-to-one, as many distribution functions may correspond to the same set of orthogonal polynomials (same pair of sequences $\{a_n\}$ and $\{b_n\}$). We are interested in how properties of the sequences $\{a_n\}$ and $\{b_n\}$ are related to properties of the corresponding distribution functions ψ. In particular we need the following result (see [7, p. 291] for a proof and related results).

THEOREM 5. *Let the sequence* $\{\varphi_n\}_0^\infty$ *of polynomials be defined by* (R) *and suppose the sequences* $\{a_n\}_0^\infty$ *and* $\{b_n\}_1^\infty$ *obey the conditions*

(i) $$a_n \to \infty \quad as \quad n \to \infty$$

(ii) $$\limsup_{n \to \infty} \left| \frac{b_n}{a_n a_{n-1}} \right| = L < \frac{1}{4}.$$

Then the polynomials $\{\varphi_n\}_0^\infty$ *are orthogonal with respect to a distribution function* ψ *which is a step function obtained as follows.*

Form the continued fraction

$$K(x) = \frac{1\,|}{|x - a_0} - \frac{b_1\,|}{|x - a_1} - \frac{b_2\,|}{|x - a_2} - \cdots.$$

This continued fraction converges completely to a meromorphic function which has the representation $K(x) = \sum_{i=1}^\infty A_i/(x - \alpha_i)$, *where* $\alpha_{i+1} > \alpha_i$, $i = 1, 2, \cdots, \alpha_i \to \infty$ *as* $i \to \infty$, $A_i > 0$, $i = 1, 2, \cdots$, *and* $\sum_{i=1}^\infty A_i = 1$. *Then* ψ *is the distribution function which is constant on each interval* $(-\infty, \alpha_1), (\alpha_1, \alpha_2), \cdots, (\alpha_i, \alpha_{i+1}), \cdots$, *and which has jump* A_i *at* α_i, $i = 1, 2, \cdots$. *Moreover* α_i *is the limit of the i-th zero of* φ_n, *as* $n \to \infty$, *and* ψ *is an extremal solution of the associated moment problem.*

Remark. The last sentence of Theorem 5 is not explicitly stated in [7], however an examination of the proof of the result given in [7] shows that this fact has also been established.

We also need the following result about orthogonal polynomials (see [8, pp. 438–440]).

THEOREM 6. *Let the real sequences* $\{a_n\}_0^\infty$ *and* $\{b_n\}_1^\infty$ *of* (R) *satisfy the conditions*

(i) $$a_n \to \alpha \neq \infty \quad as \quad n \to \infty,$$

(ii) $$b_n > 0, \quad n = 1, 2, \cdots, \quad b_n \to 0 \quad as \quad n \to \infty.$$

Then the continued fraction

$$K(x) = \frac{1}{\lfloor x - a_0} - \frac{b_1}{\lfloor x - a_1} - \frac{b_2}{\lfloor x - a_2} - \cdots$$

is meromorphic in the complex plane with the point α *deleted and has the representation*

$$K(x) = \frac{A_0}{x - \alpha} + \sum_{i=1}^{\infty} \frac{A_i}{x - \alpha_i},$$

where $\alpha_i \to \alpha$ *as* $i \to \infty$, *each* A_i *is nonnegative, and* $\sum_0^\infty A_i$ *converges. Moreover the polynomials* $\{\varphi_n\}_0^\infty$ *are orthogonal with respect to a unique distribution function* ψ *which is a step function with jumps of size* A_i *at* α_i, $i = 1, 2, \cdots$, *and a jump size* A_0 *at* α.

Remark. Theorem 6 is a special case of a similar result which holds when the sequence $\{a_n\}$ is bounded and the derived set of $\{a_n\}$ contains a finite number of points. The assumption $b_n \to 0$ is retained. See [8, p. 450].

3. Distribution functions for b–d processes with rational growth rates. In recurrence formula (R) the polynomials $\{\varphi_n\}$ are monic (the leading coefficient is one) while in formula (T) the polynomials $\{Q_n\}$ are not monic. Thus in order to use Theorems 5 and 6 to study b–d processes, we must first normalize the polynomials $\{Q_n\}$ to be monic. The formula for this normalization is quite simple and it is given by the following lemma.

LEMMA 1. *Let the polynomial set* $\{Q_n\}$ *be given by* (T). *Then there is a nonzero sequence* $\{k_n\}$ *such that the polynomials* $\varphi_n = k_n Q_n$ *satisfy formula* (R), *where the sequences* $\{\lambda_n\}_0^\infty$, $\{\mu_n\}_0^\infty$ *and* $\{a_n\}_0^\infty$, $\{b_n\}_1^\infty$ *are related by*

$$a_n = \lambda_n + \mu_n, \qquad n = 0, 1, 2, \cdots,$$

$$b_n = \lambda_{n-1}\mu_n, \qquad n = 1, 2, \cdots.$$

Proof. The proof follows directly by taking $k_0 = 1$, $k_1 = -\lambda_0$ and

$$k_n = (-1)^n \lambda_0 \lambda_1 \cdots \lambda_{n-1}.$$

We now consider b–d processes with rational rates of growth. We adopt the convention that $A_p(x)$, $B_q(x)$, $C_r(x)$ and $D_s(x)$ are polynomials of degree p, q, r, s, respectively. We also assume that these polynomials are such that $A_p(x)/B_q(x)$ and $C_r(x)/D_s(x)$ are positive for $x = 1, 2, 3, \cdots$, $A_p(0)/B_q(0) > 0$, and $C_r(0)/D_s(0) \geqq 0$. Finally, we let a, b, c, d be the coefficient of the highest order term in A_p, B_q, C_r and D_s, respectively. In this setting we have the following results.

THEOREM 7. *Let* $A_p(x)$, $B_q(x)$, $C_r(x)$ *and* $D_s(x)$ *be as described above and consider the b–d process with* $\lambda_n = A_p(n)/B_q(n)$ *and* $\mu_n = C_r(n)/D_s(n)$, $n = 0, 1, 2, \cdots$. *If*

$p \leqq q$ and $r \leqq s$ and $p + r < q + s$, then the S.M.P. associated with this b–d process has a unique solution ψ, the system I $\to$ VI *has a unique solution $P(t) = P(t, \psi)$, and ψ is a step function obtained as in Theorem* 6.

Proof. From Lemma 1, the sequences $\{a_n\}$ and $\{b_n\}$ are given by

$$a_n = \frac{A_p(n)}{B_q(n)} + \frac{C_r(n)}{D_s(n)}, \qquad n \geqq 0,$$

and

$$b_n = \frac{A_p(n-1)}{B_q(n-1)} \cdot \frac{C_r(n)}{D_s(n)}, \qquad n \geqq 1.$$

Thus, since $p \leqq q$, $r \leqq s$ and $p + r < q + s$, we see that as $n \to \infty$, $b_n \to 0$ and $a_n \to \alpha$, where α is finite. Thus, by Theorem 6, the continued fraction $K(x)$ converges and has the expansion

$$K(x) = \frac{A_0}{x - \alpha} + \sum_{i=1}^{\infty} \frac{A_i}{x - \alpha_i},$$

where $A_i \geqq 0$, $i = 0, 1, 2, \cdots$, $\sum_{i=0}^{\infty} A_i$ converges, and $\alpha_i \to \alpha$ as $i \to \infty$. Also the polynomials $\{\varphi_n\}$ (and hence $\{Q_n\}$) are orthogonal with respect to a unique distribution function ψ which is a step function with a jump of size A_i at α_i, $i = 1, 2, \cdots$, and a jump of size A_0 at α. Now, if the set $\{Q_n\}$ is orthogonal with respect to a unique distribution function ψ, then the associated S.M.P. has a unique solution given by ψ, and by Theorems 1 and 2, the system I–VI has a unique solution given by $P(t) = P(t, \psi)$. Q.E.D.

THEOREM 8. *Let the sequences $\{\lambda_n\}_0^\infty$ and $\{\mu_n\}_0^\infty$ of the b–d process be given by $\lambda_n = A_p(n)/B_q(n)$ and $\mu_n = C_r(n)/D_s(n)$, $n = 0, 1, 2, \cdots$, where A_p, B_q, C_r and D_s are polynomials as described above. Then in each of the following cases, the b–d system* I $\to$ VI *has a unique solution $P(t) = P(t, \psi)$, where ψ is given as in Theorem* 5:

1. $p > q, \quad r \leqq s,$
2. $p \leqq q, \quad r > s,$
3. $p > q, \quad r > s, \quad p - q \neq r - s,$
4. $p > q, \quad r > s, \quad p - q = r - s$ *but* $ad \neq bc$.

(*Recall that a, b, c, and d are the leading coefficients of A_p, B_q, C_r and D_s, respectively*).

Proof. Since $a_n = \lambda_n + \mu_n$ and $b_n = \lambda_{n-1}\mu_n$, the sequences $\{a_n\}_0^\infty$ and $\{b_n/(a_n a_{n-1})\}_1^\infty$ are of order $[n^{p-q} + n^{r-s}]$ and $n^{(p-q)+(r-s)}/[n^{p-q} + n^{r-s}]^2$, respectively. Thus in cases 1, 2 and 3 we immediately have $a_n \to \infty$ and $b_n/(a_n a_{n-1}) \to 0$ as $n \to \infty$. Hence, in these cases the hypotheses of Theorem 5 are true and the polynomials $\{\varphi_n\}$ (and $\{Q_n\}$) are orthogonal with respect to a distribution function ψ given as in Theorem 5. As noted in Theorem 5, this distribution function has support on the interval $[\xi, \infty)$, where $\xi = \lim_{n\to\infty} \xi_{1n}$, and ξ_{1n} is the first zero of Q_n. Since the polynomials $\{Q_n\}$ are known to be orthogonal over $(0, \infty)$ (Theorem 1), all zeros of these polynomials are on $(0, \infty)$, and hence $\xi \geqq 0$. Thus, since ψ is an extremal solution of the S.M.P., by Theorem 1, the matrix $P(t) = P(t, \psi)$ satisfies the b–d system I $\to$ VI. In order to show that this is the only solution of this system we use Theorem 3. By the ratio test, in case 1, $\sum_0^\infty \pi_n = \infty$, and in case 2, $\sum_0^\infty 1/(\lambda_n \pi_n) = \infty$. Thus in both these cases the solution $P(t) = P(t, \psi)$ is unique. The same result holds in case 3, where the ratio test shows that if $p - q > r - s$, then $\sum \pi_n = \infty$, while if $p - q < r - s$, then $\sum_0^\infty 1/(\lambda_n \pi_n) = \infty$. Hence in case 3, the solution is also unique.

In case 4, the situation is slightly more complicated. Clearly, $a_n \to \infty$; however in this case we do not have $b_n/(a_n a_{n-1}) \to 0$. Instead, since both b_n and $a_n a_{n-1}$ are of the order of $n^{(p-q)+(r-s)}$, the ratio $b_n/(a_n a_{n-1})$ has the limit $((a/b) \cdot (c/d))/[(a/b) + (c/d)]^2$, where a, b, c and d are the leading coefficients of A_p, B_q, C_r and D_s, respectively. Also, since $\lambda_n > 0$ and $\mu_n > 0$ for $n > 0$, we can assume without loss of generality that a, b, c and d are all positive. Therefore the inequality

$$\frac{(a/b)(c/d)}{[(a/b) + (c/d)]^2} < \frac{1}{4}$$

is true if and only if $[(a/b) - (c/d)]^2 > 0$. Equivalently, if $ad \neq bc$, then

$$\lim_{n \to \infty} \frac{b_n}{a_n a_{n-1}} = \frac{abcd}{[ad + bc]^2} < \frac{1}{4},$$

and hence Theorem 5 can again be used to describe a distribution function ψ which provides a solution $P(t, \psi)$ of system I $\to$ VI. To show that this solution is unique we note that

$$\lim_{n \to \infty} \frac{\pi_{n+1}}{\pi_n} = \lim_{n \to \infty} \frac{\lambda_n}{\mu_n} = \lim_{n \to \infty} \frac{A_p(n)/B_q(n)}{C_r(n)/D_s(n)} = \frac{a/b}{c/d} = \frac{ad}{bc},$$

and similarly,

$$\lim_{n \to \infty} \frac{\lambda_n \pi_n}{\lambda_{n+1} \pi_{n+1}} = \frac{bc}{ad}.$$

Since $ad \neq bc$, either $ad/bc > 1$ or $bc/ad > 1$, and thus by the ratio test, either $\sum \pi_n = \infty$ or $\sum 1/(\lambda_n \pi_n) = \infty$. In either case, $\sum (\pi_n + 1/(\lambda_n \pi_n)) = \infty$, and by Theorem 3 the solution $P(t, \psi)$ is unique for system I $\to$ VI. Q.E.D.

As an immediate corollary of Theorem 7 we have the following result about polynomial growth rates.

COROLLARY. *Let* $\lambda_n = A_p(n)$ *and* $\mu_n = C_r(n)$, $n = 0, 1, 2, \cdots$, *where* $A_p(x)$ *and* $C_r(x)$ *are polynomials of degree p and q, respectively, with leading coefficients a and c, respectively, and satisfying* $A_p(n) > 0$, $C_r(n) > 0$, $n = 1, 2, 3, \ldots$, $A_p(0) > 0$, $C_r(0) \geqq 0$. *Then, if* $p \neq r$ *or if* $p = r$ *and* $a \neq c$, *then the b–d system* I–VI *with birth and death rates* $\{\lambda_n\}_0^\infty$ *and* $\{\mu_n\}_0^\infty$, *respectively, has a unique solution* $P(t, \psi)$ *with* ψ *given as in Theorem* 5.

Proof. This is Theorem 8 with $q = s = 0$ and $b = d = 1$.

4. Special cases, open questions and conjectures. Theorems 7 and 8 do not include b–d processes with rational rates of growth in which either of the following occurs (we use the notational conventions of § 3):

1. $p > q$, $r > s$, $p - q = r - s$, and $ad = bc$.
2. $p = q$ and $r = s$.

In case 1, $a_n \to \infty$; however, $\lim_{n \to \infty} b_n/(a_n a_{n-1}) = 1/4$, and hence Theorem 5 cannot be used. In this setting, if $p = r = 1$ and $q = s = 0$, then λ_n and μ_n are linear in n, say $\lambda_n = an + k$ and $\mu_n = cn + l$, and this is the situation considered in detail by Karlin and McGregor in [3]. They have shown that, as predicted by Theorem 8, for $a \neq c$, there is a unique distribution function ψ such that $P(t, \psi)$ solves the b–d system I → VI and ψ has a discrete spectrum with no finite limit points. In fact Karlin and McGregor recognized from the recurrence formula for the polynomials $\{Q_n\}$, that when $a \neq c$, these polynomials are essentially the Meixner polynomials and thus a complete description of ψ can be given. If $a = c$, then Theorem 8 gives no information about ψ; however, Karlin and McGregor recognized from the recurrence formula for the set $\{Q_n\}$ that these polynomials are essentially the Laguerre polynomials and thus ψ is absolutely continuous and $d\psi(x) = e^{-x/a}x^{\alpha}$ for some $\alpha > 0$. This linear case and general result about the moment problem suggest the following.

CONJECTURE. *If $p = q + 1$, $r = s + 1$ and $ad = bc$, then there is a unique solution to the system* I → VI *given by $P(t, \psi)$, where ψ is absolutely continuous on* $(0, \infty)$. *If $p > q$, $r > s$, $p + r > s + q + 2$ and $ad = bc$, then there are infinitely many solutions of* I → VI.

In case 2 above, both $\{\lambda_n\}$ and $\{\mu_n\}$ (and hence $\{a_n\}$ and $\{b_n\}$) have finite nonzero limits. Thus neither Theorem 7 nor 8 applies. As a special case of this, consider $\lambda_n = \mu_n = 1/2$ for all $n \geqq 0$. Then $b_n = 1/4$ for $n \geqq 1$, and $a_n = 1$ for $n \geqq 0$. The recurrence formula (R) is then a translation of the formula for the Chebyshev polynomials and hence the related distribution function is unique and absolutely continuous on a bounded interval. This leads us to the following.

CONJECTURE. *If $p = q$ and $r = s$, then there is a unique solution $P(t, \psi)$ for the system* I → VI, *where ψ is an absolutely continuous distribution function defined on a bounded interval.*

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ASYMPTOTICALLY VANISHING OSCILLATORY TRAJECTORIES IN SECOND ORDER RETARDED EQUATIONS*

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Abstract. Conditions have been found on $a(t)$, $r(t)$ and $f(t)$ to ensure that all oscillatory solutions of $(r(t)y'(t))'+a(t)y^\alpha(t-\tau(t))=f(t)$ approach zero asymptotically where $0<\alpha\leqq 1$ and α is a ratio of odd integers.

1. Introduction. The literature is very scanty about the asymptotic nature of the solutions of nonhomogeneous retarded equations. Usual techniques for corresponding ordinary differential equations do not often carry over to retarded equations. Recently Hammett [3] studied an equation of the type

$$y''(t)+p(t)y(t)=f(t) \tag{1}$$

and showed via a theorem of Bhatia [1] that, if $p(t)\geqq k>0$, and $f(t)$ is continuous and integrable on some positive half-line, then all nonoscillatory solutions of (1) approach zero asymptotically. This author and Dahiya [6] extended Hammett's results to equations of the type

$$(r(t)y'(t))'+a(t)y(t-\tau(t))=f(t), \tag{2}$$

after observing an example due to Travis [7] in which Bhatia's theorem and consequently Hammett's technique did not apply to (2). In fact the equation (see Travis [7])

$$y''(t)+\frac{\sin t}{2-\sin t}\,y(t-\pi)=0 \tag{3}$$

has $y=2+\sin t$ as a nonoscillatory solution. But, by Bhatia's theorem, all solutions of the equation

$$y''(t)+\frac{\sin t}{2-\sin t}\,y(t)=0 \tag{4}$$

are oscillatory since

$$\int^\infty \frac{\sin t}{2-\sin t}=\infty.$$

Our purpose here is to find conditions on $a(t)$, $r(t)$ and $f(t)$ to ensure that all *oscillatory* solutions of (2) tend to zero asymptotically. As it stands, one of the conditions is

$$\int^\infty \frac{1}{r(t)}\,dt<\infty,$$

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where $r(t)>0$. This excludes a very important class of equations, namely,

(5) $$y''(t)+a(t)y(t-\tau(t))=f(t).$$

The latter part of this paper singles out a significant class of oscillatory solutions not shared by (4) and (5).

In thoroughly searching the literature we find that very little has been said about the asymptotic nature of oscillatory solutions of (2) and (5). The results of S. Londen [5], T. Burton and R. Grimmer [2], R. S. Dahiya and B. Singh [6], R. Terry [8] and other authors only seem to enhance asymptotic results about nonoscillatory solutions of (1) and (2).

High speed mechanisms which are mathematically associated with retarded equations, are very susceptible to oscillations caused by the delay term (see Minorsky [4, p. 518]). Therefore, given the delay term, it is important to know what additional controls are necessary to ensure that oscillations die out.

2. Definitions and assumptions. Throughout this paper it is assumed that R is the real line; $r(t)$, $\tau(t)$, $r'(t)$, $a(t)$ and $f(t)$ are continuous on R. In addition, $0\leqq\tau\leqq M$, and $r(t)>0$ on some positive half-line $[t_0', \infty)$, $t_0'>0$.

We call a function $h(t)\in C[t_0, \infty]$ *oscillatory* if $h(t)$ has arbitrarily large zeros in $[t_0', \infty]$. Otherwise we call $h(t)$ *nonoscillatory*.

The term "solution" below will apply only to continuously extendable solutions of equations under consideration on some positive half-line.

3. Main results. Our first theorem gives conditions when oscillatory solutions of (2) are bounded.

THEOREM 1. *Suppose*:

(i) $\int^\infty [1/r(t)]\,dt<\infty$,

(ii) $\int^\infty |f(t)|\,dt<\infty$,

(iii) $\int^\infty |a(t)|\,dt<\infty$.

Then oscillatory solutions of (2) *are bounded.*

Remark 1. The condition $\int^\infty[1/r(t)]\,dt<\infty$ is severe and eliminates obvious cases of application, such as variable mass problems where $r(t)$ is usually bounded. This specialized case gives a set of preliminary results for unbounded $r(t)$, which we believe can be extended to a more practical situation where $r(t)$ is bounded. Section 4 presents a partial extension of a situation where $r(t)$ is bounded.

Proof. Let $T\geqq t_0'$ be sufficiently large that for $t\geqq T$, $r(t)>0$. Let $y(t)$ be an oscillatory solution of (2). Let $t_2>t_1>T$ be two consecutive zeros of $y(t)$ and, without any loss of generality, suppose $y(t)>0$ in (t_1, t_2). We can also assume, without any loss, that T is sufficiently large that

(6) $$\int_T^\infty |a(t)|<1,$$

(7) $$\int_T^\infty |f(t)|<1,$$

(8) $$\int_T^\infty [1/r(t)]\,dt<\frac{1}{2}.$$

Let $M_0 = \max y(t)$, $t \in [t_1, t_2]$. Also, let $t_0 \in (t_1, t_2)$ be such that

$$M_0 = y(t_0). \tag{9}$$

Now $M_0 = \int_{t_1}^{t_0} y'(t)\,dt$, which implies that

$$M_0 \leqq \int_{t_1}^{t_0} |y'(t)|\,dt. \tag{10}$$

Also $M_0 = -\int_{t_0}^{t_2} y'(t)\,dt$ so that

$$M_0 \leqq \int_{t_0}^{t_2} |y'(t)|\,dt. \tag{11}$$

From (10) and (11),

$$2M_0 \leqq \int_{t_1}^{t_2} |y'(t)|\,dt, \tag{12}$$

from which it follows that

$$2M_0 \leqq \int_{t_1}^{t_2} [r(t)]^{-1/2}[r(t)]^{1/2}|y'(t)|^{1/2} \cdot |y'(t)|^{1/2}\,dt. \tag{13}$$

Squaring (13) and applying Schwarz's inequality, we have

$$4M_0^2 \leqq \int_{t_1}^{t_2} [1/r(t)]\,dt \int_{t_1}^{t_2} r(t)y'(t) \cdot y'(t)\,dt. \tag{14}$$

Integrating the second integral by parts, we have from (14) that

$$4M_0^2 \leqq \left[\int_{t_1}^{t_2} [1/r(t)]\,dt\right]\left[-\int_{t_1}^{t_2} y(t)(r(t)y'(t))'\,dt\right], \tag{15}$$

since $y(t_1) = y(t_2) = 0$. Making use of (2) in (15), we have

$$\begin{aligned} 4M_0^2\left[\int_{t_1}^{t_2} [1/r(t)]\,dt\right]^{-1} &\leqq \int_{t_1}^{t_2} y(t)a(t)y(t-\tau(t))\,dt - \int_{t_1}^{t_2} y(t)f(t)\,dt \\ &\leqq \int_{t_1}^{t_2} y(t)|a(t)||y(t-\tau(t))|\,dt + \int_{t_1}^{t_2} y(t)|f(t)|\,dt. \end{aligned}$$

This yields the inequality

$$\frac{4M_0}{\int_{t_1}^{t_2} [1/r(t)]\,dt} \leqq \int_{t_1}^{t_2} |a(t)||y(t-\tau(t))|\,dt + \int_{t_1}^{t_2} |f(t)|\,dt \tag{16}$$

since $y(t) \leqq M_0$ for $t \in [t_1, t_2]$. Let $q > p$ be large enough consecutive zeros of $y(t)$ such that

$$p - M > T.$$

Suppose

$$M_q = \max |y(t)|, \qquad t \in [T, q]. \tag{17}$$

Now if $y(t)$ is not bounded, then $\limsup_{t\to\infty}|y(t)|=\infty$. Let $r>q$ be the *smallest* number such that

(18) $$|y(r)|=M_q+1.$$

Let T_1 be the greatest zero of $y(t)$ less than r, and let T_2 be the smallest zero of $y(t)$ greater than r. Then

(19) $$q\leqq T_1<r<T_2,$$

and $T_1<T_2$ are consecutive zeros of $y(t)$. Let $M_2=\max|y(t)|, t\in[T_1, T_2]$, and $M_2=|y(t_{M_2})|$, $t_{M_2}\in[T_1, T_2]$. We shall show that

$$M_2=\max|y(t)|, \qquad t\in[T, T_2].$$

To see this, let

$$L=|y(s)|=\max|y(t)|, \qquad t\in[T, T_2].$$

By definition, $M_2\geqq|y(t)|$, $t\in[T_1, T_2]$. In particular, $r\in[T_1, T_2]$, so that

$$\begin{aligned} M_2\geqq|y(r)|&=M_q+1\\ &>M_q=\max_{[T,q]}|y(t)|\\ &\geqq|y(t)|, \qquad t\in[T, q]. \end{aligned}$$

Moreover, $\max_{[q,T_1]}|y(t)|\leqq M_q+1$, for otherwise the definition of r would be contradicted, since $T_1<r$. Thus $M_2\geqq\max_{[T,T_2]}|y(t)|$. But $M_2=|y(t_{M_2})|$, where $t_{M_2}\in[T_1, T_2]\subset[T, T_2]$. Thus $M_2=L$.

In the inequality (16), we replace t_1 and t_2 by T_1 and T_2, respectively. We have $t-\tau(t)\geqq t-M$, since $M\geqq\tau(t)\geqq 0$. Thus, for $t\in[T_1, T_2]$, $t-\tau(t)\in[T, q]$ by virtue of (17). Hence

(20) $$|y(t-\tau(t)|\leqq M_2$$

for $t\in[T_1, T_2]$. From (16) and (20) we obtain

(21) $$\frac{4M_2}{\int_{T_1}^{T_2}[1/r(t)]\,dt}\leqq M_2\int_{T_1}^{T_2}|a(t)|\,dt+\int_{T_1}^{T_2}|f(t)|\,dt.$$

Dividing (21) by M_2, we have

(22) $$\frac{4}{\int_{T_1}^{T_2}[1/r(t)]\,dt}\leqq\int_{T_1}^{T_2}|a(t)|\,dt+\frac{\int_{T_1}^{T_2}|f(t)|dt}{M_2}.$$

Since $M_2>1$, (6), (7) and (8) contradict (22). The proof is complete.

THEOREM 2. *Suppose conditions* (i)–(iii) *of Theorem* 1 *hold. Let* $y(t)$ *be an oscillatory solution of* (2). *Then* $y(t)\to 0$ *as* $t\to\infty$.

Proof. Let T be the same as in the proof of Theorem 1. If $\lim_{t\to\infty}y(t)\neq 0$, then

(23) $$\liminf_{t\to\infty}|y(t)|=0,$$

and

$$\limsup_{t\to\infty} |y(t)| > 2d > 0. \tag{24}$$

Owing to (5), (6) and the oscillatory nature of $y(t)$, there exist arbitrarily large consecutive zeros T_3 and T_4 such that

$$d < D = \max |y(t)|, \qquad t \in [T_3, T_4], \quad T_3 > T, \tag{25}$$

$$\int_{T_3}^{T_4} |f(t)|\, dt < \frac{1}{2}, \tag{26}$$

$$\int_{T_3}^{T_4} |a(t)||y(t-\tau(t))|\, dt < \frac{1}{2} \tag{27}$$

and

$$\int_{T_3}^{T_4} \frac{1}{r(t)}\, dt < d. \tag{28}$$

We note that (27) is made possible by the boundedness of $y(t)$ from Theorem 1 and condition (iii).

Replacing t_1 and t_2 by T_3 and T_4 respectively, and M_0 by D, we obtain

$$\frac{4D}{\int_{T_3}^{T_4} dt/r(t)} \leqq \int_{T_3}^{T_4} |a(t)||y(t-\tau(t))|\, dt + \int_{T_3}^{T_4} |f(t)|\, dt. \tag{29}$$

Substituting in (29) from (26), (27) and (28), we have

$$\frac{4d}{d} \leqq \frac{1}{2} + \frac{1}{2}.$$

This contradiction proves the theorem.

Example 1. Consider the equation

$$(e^t y'(t))' + e^{-t-2\pi} y(t-\pi) = -e^{-3t} \sin t + e^{-t} \sin t - 3e^{-t} \cos t. \tag{30}$$

All the conditions of Theorem 1 are satisfied. In fact, (30) has $y(t) = e^{-2t} \sin t$ as a solution that satisfies the conclusion of Theorem 2.

Remark 2. The following example shows that it may not be possible to weaken the condition

$$\int^{\infty} \frac{1}{r(t)}\, dt < \infty$$

if all other conditions of Theorem 1 are satisfied.

Example 2. Consider the equation

$$y''(t) + \frac{1}{t^4} y(t) = -\frac{\sin(\log t)}{t^2} - \frac{\cos(\log t)}{t^2} - \frac{\sin(\log t)}{t^4}, \qquad t > 0. \tag{31}$$

Here, all the conditions of Theorem 1 are satisfied except condition (i) on $r(t)$. Equation (31) has $y=\sin(\log t)$ as an oscillatory solution that does not approach zero. However, this solution is bounded. The next example shows that a solution need not be bounded if the condition on $r(t)$ is violated.

Example 3. Consider the equation

$$y''(t)+\frac{1}{t^3}y(t)=\frac{5\sin(\log t)}{4t^{3/2}}, \qquad t>0. \tag{32}$$

Except for the condition on $r(t)$, all other conditions are satisfied. Equation (32) has $y=\sqrt{t}\sin(\log t)$ as an unbounded solution.

Remark 3. The technique of the proof is applicable to a more general equation.

THEOREM 3. *Suppose conditions* (ii) *and* (iii) *of Theorem* 1 *hold. Let* $y(t)$ *be an oscillatory solution of*

$$(r(t)y'(t))'+a(t)y^{\alpha}(t-\tau(t))=f(t), \tag{33}$$

where $0<\alpha\leqq 1$ *and* α *is a ratio of odd integers. Then* $y(t)\to 0$ *as* $t\to\infty$.

Proof. We proceed as in the proof of Theorem 1. The inequality (16) becomes

$$\frac{4M_0}{\int_{t_1}^{t_2}[1/r(t)]\,dt}\leqq\int_{t_1}^{t_2}|a(t)||y(t-\tau(t))|^{\alpha}\,dt+\int_{t_1}^{t_2}|f(t)|\,dt. \tag{34}$$

Proceeding further as in the proof of Theorem 1, the inequality (21) becomes

$$\frac{4M_2}{\int_{T_1}^{T_2}[1/r(t)]\,dt}\leqq M_2^{\alpha}\int_{T_1}^{T_2}|a(t)|\,dt+\int_{T_1}^{T_2}|f(t)|\,dt, \tag{35}$$

which yields

$$\frac{4}{M_2^{\alpha-1}\int_{T_1}^{T_2}[1/r(t)]\,dt}\leqq\int_{T_1}^{T_2}|a(t)|\,dt+\frac{\int_{T_1}^{T_2}|f(t)|\,dt}{M_2^{\alpha}}. \tag{36}$$

Since $\alpha\leqq 1$, (36) gives the right contradiction, and the boundedness of $y(t)$ is proved.

From here on, the proof of Theorem 2 applies verbatim if we replace (27) by

$$\int_{T_3}^{T_4}|a(t)||y(t-\tau(t))|^{\alpha}\,dt<\frac{1}{2}. \tag{37}$$

The proof is now complete.

The following example justifies Theorem 3.

Example 4. Consider the equation

$$(e^{t}y'(t))'+e^{-t-10\pi/7}(y(t-\pi))^{5/7}=-e^{-17t/7}\sin^{5/7}(t)+e^{-t}\sin t-3e^{-t}\cos t, \tag{38}$$

which has $y(t)=e^{-2t}\sin t$ as a solution. All the conditions of Theorem 3 are satisfied.

4. On $y''(t)+a(t)y^{\alpha}(t-\tau(t))=f(t)$. In this section, we shall prove that as long as the distances remain finite between consecutive zeros of an oscillatory solution of

$$y''(t)+a(t)y^{\alpha}(t-\tau(t))=f(t), \tag{39}$$

where $0<\alpha\leqq 1$ and α is a ratio of odd integers, then such a solution approaches zero. Let $y(t)$ be an oscillatory solution of (39). We define a set Z_y by

$$Z_y=\{y_0-x_0 \mid y_0>x_0 \text{ and } y_0 \text{ and } x_0 \text{ are consecutive zeros of } y(t)\}.$$

THEOREM 4. *Let $y(t)$ be an oscillatory solution of* (39). *Suppose conditions* (ii)–(iii) *of Theorem* 1 *hold. Then $y(t)\to 0$ as $t\to\infty$ if the associated set Z_y is bounded.*

Proof. We proceed as in Theorem 1. The inequality (16) becomes

$$\frac{M_0}{t_2-t_1}\leqq\int_{t_1}^{t_2}|a(t)||y(t-\tau(t))|^{\alpha}\,dt+\int_{t_1}^{t_2}|f(t)|\,dt. \tag{40}$$

Let $K_y\leqq C$ for any $K_y\in Z_y$. We now replace (6) and (7) by

$$\int_T^{\infty}|a(t)|<\frac{1}{C} \tag{41}$$

and

$$\int_T^{\infty}|f(t)|<\frac{1}{C}. \tag{42}$$

Inequality (21) now yields

$$\frac{4}{M_2^{\alpha-1}(T_2-T_1)}\leqq\int_{T_1}^{T_2}|a(t)|\,dt+\frac{\int_{T_1}^{T_2}|f(t)|\,dt}{M_2^{\alpha}}, \tag{43}$$

which gives

$$4(M_2^{\alpha-1}C)^{-1}\leqq\frac{1}{C}+\frac{1}{M_2^{\alpha}C}\leqq\frac{2}{C}. \tag{44}$$

Since $M_2>1$ and $\alpha<1$, (44) gives a contradiction. The proof is now complete.

Remark 4. Returning to Example 3 and Example 4, the solutions $y=\sin(\log t)$ and $y=\sqrt{t}\sin(\log t)$, respectively, are such that the distance between their consecutive zeros tends to infinity. In fact, $\sin(\log t)$ vanishes at

$$t_n=\exp(n\pi), \qquad n=0,1,2,\cdots,$$

and

$$\lim_{n\to\infty}[t_{n+1}-t_n]=\infty.$$

Examples 3 and 4 justify Theorem 4 on the boundedness of the set Z_y.

Example 5. Consider the equation

$$y''(t)+e^{-t-\pi}y(t-\pi)=-2e^{-t}\cos t-e^{-2t}\cos t. \tag{45}$$

All conditions of Theorem 4 are satisfied. Thus all oscillatory solutions of (45) of which the associated sets Z_y are bounded, approach zero. The solution $y=e^{-t}\sin t$ is one such solution satisfying the required conditions.

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CONTINUOUS PARAMETER DEPENDENCE IN A CLASS OF VOLTERRA EQUATIONS*

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Abstract. Conditions are found under which the solution of the Volterra integral equation

$$x'(t)+\int_0^t a(t-s,\lambda)x(s)\,ds=k,\qquad x(0)=x_0,$$

is continuous in λ, uniformly in $\{0\leqq t<\infty\}$, when $a(t,\lambda)$ is nonnegative, nonincreasing, and convex as a function of t, for each λ. The main theorem concerns the case where the kernel has a special piecewise linear form and solutions are asymptotic ($t\to\infty$) to nondegenerate periodic functions. This is the case excluded in similar earlier results of the author.

The significance of these results for certain related Volterra equations in Hilbert space is summarized.

1. Introduction. Suppose

(H1) $a(t)$ is nonnegative, nonincreasing and convex on $(0,\infty)$, $a(t)\not\equiv a(\infty)$, $0<a(0+)\leqq\infty$, and $\int_0^1 a(t)\,dt<\infty$,

and consider the equation

$$x'(t)+\int_0^t a(t-s)x(s)\,ds=k,\qquad x(0)=x_0, \tag{1.1}$$

($'=d/dt$), where k and x_0 are prescribed constants. In [1] we showed that $x(t)\to k/\int_0^\infty a(t)\,dt$ ($\equiv 0$ if the integral is infinite) as $t\to\infty$, except in the special cases where

(H2) $a(0)=a(0+)<\infty$ and $a(t)$ is piecewise linear with changes of slope only at integral multiples of $t_0=2\pi/\sqrt{a}(0)$.

If (H1) and (H2) hold, then

$$x_1(t)\to k\Big/\int_0^\infty a(t)\,dt\qquad (t\to\infty), \tag{1.2}$$

where $x_1=x-x_0\Omega_1-k\Omega_2$ with

$$\Omega_1(t)=2\gamma^{-1}\cos\omega t,\qquad \Omega_2(t)=2(\gamma\omega)^{-1}\sin\omega t, \tag{1.3}$$

$$\gamma=3-(a(\infty)/a(0)),\qquad \omega=\sqrt{a}(0). \tag{1.4}$$

In this paper, we permit $a(t)=a(t,\lambda)$ to depend on a real parameter $\lambda\in\Lambda\subset R^1$. Then $x(t)=x(t,\lambda)$. It is clear from (1.2) that $x_1(\cdot,\lambda)$ belongs to the Banach space BC of bounded, continuous functions on $[0,\infty)$ with supremum norm. Our main result is the following.

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THEOREM 1. *For each* $\lambda \in \Lambda$, *suppose*

(1.5) (H1) *and* (H2) *hold for* $a(t) = a(t, \lambda)$ *and*

(1.6) $$a(\infty, \lambda) = 0.$$

Suppose in addition that

(1.7) $A(\lambda) \equiv 1 \Big/ \int_0^\infty a(t, \lambda)\, dt$ *is continuous on* Λ *and*

(1.8) $\lambda \to a(\cdot, \lambda)$ *is continuous as a map from* Λ *to* $L^1(0, R)$ *for each finite* $R > 0$.

(1.9) *Then the map* $\lambda \to x_1(\cdot, \lambda)$ *from* Λ *to BC is continuous.*

The analogous result when (H2) does not hold is as follows.

THEOREM 2. For each $\lambda \in \Lambda$, suppose (H1) *and* (1.6) *hold, but not* (H2). *Assume* (1.7) *and* (1.8). *Then the map* $\lambda \to x(\cdot, \lambda)$ *from* Λ *to BC is continuous.*

Theorem 2 differs from Theorem 4 of [2] only because the latter requires $a'(t)$ to be continuous. The proof here is virtually the same (the necessary alteration is indicated in [3]), so we shall prove Theorem 1 only.

We have not determined whether (1.6) is necessary in Theorems 1 and 2, or whether it could be replaced by

(1.6a) $$a(\infty, \lambda) \quad \text{is continuous.}$$

A simple modification of our proof below shows that (1.6a) is sufficient for continuity at a point $\lambda = \lambda_0$, where $a(\infty, \lambda_0) > 0$, and that (1.6a) is always sufficient if $k = 0$. For this reason we shall write γ, even though $\gamma = 3$ when (1.6) holds.

In § 2, we discuss a Volterra equation in Hilbert space, the study of which led us to consider the parameter dependence of (1.1). We prove Theorem 1 in § 3. The proof contains much in common with the proof of Theorem 4 of [2] and with parts of [1]. In the remark at the end of § 3, we indicate a correction for [1].

2. Consequences for equations in Hilbert space. Consider the equation

$$\mathbf{x}(t) + \int_0^t \mathbf{L}(t-s)\mathbf{x}(s)\, ds = t\boldsymbol{\eta} + \boldsymbol{\xi},$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are prescribed elements of a Hilbert space $\mathcal{H}$, and $\mathbf{L}$ is a densely defined self-adjoint operator on $\mathcal{H}$ with spectral decomposition

$$\mathbf{L}(t) = \int_{-\infty}^\infty \left\{\int_0^t a(s, \lambda)\, ds\right\} d\mathbf{E}_\lambda,$$

and the spectral family $\{\mathbf{E}_\lambda\}$ corresponds to a fixed self-adjoint operator $\mathbf{L}_0$ with spectrum Λ.

Assume (H1), (1.8), and (1.6) ($\lambda \in \Lambda$), and let Λ^* and Λ_0 denote respectively the subsets of Λ where (H2) does and does not hold.

We write $x = x_0 u + kw$, where $u(t)[w(t)]$ is the solution of (1.1), with $x_0 = 1$, $k = 0[x_0 = 0,\ k = 1]$. We showed in [3] that $\mathbf{x}(t)$ has the representation

$$\mathbf{x}(t) = \int_{-\infty}^{\infty} [u(t, \lambda)\, d\mathbf{E}_\lambda\, \boldsymbol{\xi} + w(t, \lambda)\, d\mathbf{E}_\lambda\, \boldsymbol{\eta}],$$

provided $a(t, \lambda)$ satisfies a certain growth condition in λ and

$$(2.1)\qquad |u(t, \lambda)| + |w(t, \lambda)| \leqq B < \infty, \qquad 0 \leqq t < \infty, \quad \lambda \in \Lambda.$$

If, moreover, the conclusions of Theorems 1 and 2 above hold on $\Lambda = \Lambda^*$ and $\Lambda = \Lambda_0$ respectively, we have (see note following the proof of Lemma 3.3 below)

$$(2.2)\qquad \|\mathbf{x}(t) - \boldsymbol{\Omega}(t)\|_H \to 0, \qquad t \to \infty,$$

where

$$\boldsymbol{\Omega}(t) = \int_{\Lambda^*} [\Omega_1(t, \lambda)\, d\mathbf{E}_\lambda\, \boldsymbol{\xi} + \Omega_2(t, \lambda)\, d\mathbf{E}_\lambda\, \boldsymbol{\eta}] + \mathbf{L}^{-1}(\infty)\, \boldsymbol{\eta}.$$

We gave sufficient conditions for (2.1) in [3]. Professor Robert E. L. Turner has pointed out to the author that the conclusions of Theorems 1 and 2 are not necessary for these Hilbert space results. In light of the boundedness theorems of [3], one may simply apply Lebesgue's dominated convergence theorem to

$$\int_\Lambda [u(t, \lambda) - \Omega_1(t, \lambda)]\, d\mathbf{E}_\lambda\, \boldsymbol{\xi} + [w(t, \lambda) - \Omega_2(t, \lambda) - A(\lambda)]\, d\mathbf{E}_\lambda\, \boldsymbol{\eta}$$

($\Omega_j(t, \lambda) = 0$ if $\lambda \in \Lambda_0$) to obtain the following general result.

THEOREM 3. *Suppose* $a(t) = a(t, \lambda)$ *satisfies* (H1) *for* $\lambda \in \Lambda$. *Assume* (1.8), *and suppose there are positive numbers* T *and* M *such that* $\int_0^T a(t, \lambda)\, dt \geqq M (\lambda \in \Lambda)$. *Then with* $\mathbf{L}(t)$ *and* $\boldsymbol{\Omega}(t)$ *as above, the function* $\mathbf{x}(t)$ *given by our representation formula satisfies* (2.2).

The case of (2.2) where $\Lambda = \Lambda_0$ is essentially the result of [2]; our present results deal with a limiting case. To illustrate (2.2) with nonempty Λ^*, we let

$$b(t) = \sum_{k=1}^{\infty} \delta_k \left(1 - \frac{\min\{t, k\}}{k}\right),$$

with $\delta_k > 0$, $\sum_{k=1}^{\infty} \delta_k = 4\pi^2$, $\sum_{k=1}^{\infty} k\delta_k = \infty$. Then $b(t)$ satisfies (H1) and (H2) with $t_0 = 1$, and $\int_0^\infty b(t)\, dt = \infty$. Set

$$a_1(t, \lambda) = \lambda b(\sqrt{\lambda} t).$$

The hypotheses of Theorem 1 hold with $a = a_1$, $\Lambda = [1, \infty)$.

As a first example, let $\mathcal{H}_1 = L^2(0, \pi)$, $\mathbf{L}_1 f(y) = f''(y)$, with boundary conditions $f(0) = f(\pi) = 0$. Then with $a = a_1$, and (for simplicity) $\boldsymbol{\xi} = \mathbf{0}$, we get

$$\boldsymbol{\Omega}(t) = \boldsymbol{\Omega}^1(t) = \frac{2}{3} \sum_{n=1}^{\infty} (\sin ny) \eta_n \sin 2\pi nt / 2\pi n,$$

where the η_n are the Fourier sine coefficients of $\boldsymbol{\eta}$.

For an example with continuous spectrum, take $\mathcal{H}_2 = L^2(1,\infty)$, $\mathbf{L}_2 f(y) = yf(y)$. With $a = a_1$, we get (2.2), with $\mathbf{\Omega} = \mathbf{\Omega}^2$ and

$$\mathbf{\Omega}^2(t) = \tfrac{2}{3}\cos(2\pi t\sqrt{y})\xi(y) + \sin(2\pi t\sqrt{y})\eta(y)/3\pi\sqrt{y}.$$

If a_1 were modified slightly so that the special condition (H2) held on a proper subset Λ^* of Λ(Λ^* necessarily closed, by [3, Lemma 2.1]), the sum in $\mathbf{\Omega}^1$ would include only those n with $n^2 \in \Lambda^*$. $\mathbf{\Omega}^2$ would be multiplied by the characteristic function $\chi_{\Lambda^*}(y)$ of Λ^* (which would make $\mathbf{\Omega}^2(t)$ zero as an element of $L^2(1,\infty)$ if Λ^* were discrete).

Further discussion, examples, and references to related work on problems in Hilbert space will be found in [2] and [3].

3. Proof of Theorem 1. The parameter dependence of a, γ, ω, t_0 and others will often be suppressed in formulas below.

Fix $\lambda \in \Lambda$, and set $u_1 = u - \Omega_1$, $w_1 = w - \Omega_2$. In all statements below, $\mu \in \Lambda$ is understood. By superposition, it is enough to show that $u_1(t, \mu) \to u_1(t, \lambda)$ and $w_1(t, \mu) \to w_1(t, \lambda)$ $(\mu \to \lambda)$ uniformly in $\{0 \leqq t < \infty\}$.

We recall from [1] that when (H1) and (H2) hold, $a(t, \mu)$ may be expressed

$$a(t, \mu) = \sum_{k=1}^{\infty} a_k(\mu)\left(1 - \frac{\min\{t, kt_0(\mu)\}}{kt_0(\mu)}\right), \tag{3.1}$$

where $a_k(\mu) \geqq 0$ and $a(0, \mu) = \sum_{k=1}^{\infty} a_k(\mu) < \infty$.

We begin with the representation

$$\pi u_1(t, \mu) = \lim_{\varepsilon \to 0^+}\left[\int_0^{\omega(\mu)-\varepsilon} + \int_{\omega(\mu)+\varepsilon}^{\infty}\right] \operatorname{Re}\{e^{i\tau t} U_1(i\tau, \mu)\}\, d\tau, \tag{3.2}$$

(valid for $t > 0$) which we developed in [1, p. 549]. Here $U_1(s)$ is the Laplace transform of $u_1(t)$, which one computes as

$$U_1(s) = \frac{1}{s + \hat{a}(s)} - \frac{2s}{\gamma(s^2 + \omega^2)}, \tag{3.3}$$

with $\hat{a}(s) = \int_0^\infty a(t)\exp(-st)\, dt$. $\hat{a}(s)$ is analytic in $\{\operatorname{Re} s > 0\}$ and continuous in $\{\operatorname{Re} s \geqq 0, s \neq 0\}$ and

$$s + \hat{a}(s) \neq 0, \qquad \operatorname{Re} s \geqq 0, \quad s \neq \pm i\omega. \tag{3.4}$$

The representation (3.2) follows from the complex inversion formula for Laplace transforms and a contour shift.

Fix $\theta > 0$. We shall find finite numbers Δ, T such that (3.2) may be decomposed as

$$\pi u_1(t, \mu) = g(t, \mu) + \textstyle\sum_{j=1}^{5} f_j(t, \mu), \tag{3.5}$$

where

$$|g(t, \mu)| \leqq \theta, \qquad T \leqq t < \infty, \tag{3.6}$$

whenever

$$\mu \in N_\lambda = N_\lambda(\theta) = \{|\mu - \lambda| < \Delta, \mu \in \Lambda\},$$

and

$$f_j(t,\mu)\to f_j(t,\lambda)(\mu\to\lambda)\quad \text{uniformly in } \{0^+\leqq t<\infty\}. \tag{3.7}$$

Then choose $\delta<\Delta$ so that

$$|u_1(t,\mu)-u_1(t,\lambda)|<\theta,\qquad 0\leqq t\leqq T, \tag{3.8}$$

and

$$\sum_{j=1}^{5}|f_j(t,\mu)-f_j(t,\lambda)|<\theta\quad \text{if } |\mu-\lambda|<\delta.$$

((3.8) can be ensured: the uniform continuity in μ of u_1 and w_1 on compact t intervals follows from (1.8), by means of a standard argument for Volterra integral equations.) Then $\frac{1}{3}|u_1(t,\mu)-u_1(t,\lambda)|<\theta(0\leqq t<\infty,\ |\mu-\lambda|<\delta,\ \mu\in\Lambda)$; and $\mu\to u_1(\cdot,\mu)$ is continuous at $\mu=\lambda$ as asserted, since θ is arbitrary.

It will be convenient to develop conditions on Δ, T, and a third positive number η (all three depend on θ) as we proceed. These conditions give positive a priori upper bounds on Δ, η, and a lower bound on T. These conditions could be taken in the following logical order:

(i) (3.12), (3.14), and

$$w(\lambda)-\eta>0. \tag{3.9}$$

(ii) (3.25), (3.29), and (3.39) ($\mu\in N_\lambda$).

(iii) (3.13), (3.15), (3.16) ($\mu\in N_\lambda$), and (3.40).

The following two convergence principles will be used in establishing (3.7).

LEMMA 3.1. *Suppose* $0\leqq\alpha\leqq\beta\leqq\infty$ *and*

$$h(\cdot,\mu)\to h(\cdot,\lambda)\quad \text{in } L_1(\alpha,\beta) \text{ as } \mu\to\lambda. \tag{3.10}$$

If $f_j(t,\mu)=\int_\alpha^\beta h(\tau,\lambda)\,e^{i\tau t}\,d\tau$ *(or if* f_j *is the real or imaginary part of this integral), then* (3.7) *holds.*

Proof. $|f_j(t,\mu)-f_j(t,\lambda)|\leqq\|h(\cdot,\mu)-h(\cdot,\lambda)\|_{L^1}$.

LEMMA 3.2. *Let* $0\leqq\alpha\leqq\nu\leqq\beta<\infty$, $h(\tau,\mu)\geqq0$, *and suppose*

(i) $\int_\alpha^\beta h(\tau,\mu)\to\int_\alpha^\beta h(\tau,\lambda)\,d\tau\ (\mu\to\lambda)$.

(ii) *For* $0<\delta<\beta-\alpha$, $h(\tau,\mu)\to h(\tau,\lambda)\,(\mu\to\lambda)$ *uniformly on* $\{\alpha\leqq\tau\leqq\beta,\ |\tau-\nu|\geqq\delta\}$.

Then (3.10) *holds.*

If $\nu=\alpha$, this is Lemma 6.1 of [2]. The general proof is similar (and straightforward) and we omit it.

In the next lemma we collect several facts about a and $\hat{a}$ and their dependence on μ.

LEMMA 3.3. (i) $a(0)$, t_0,ω,γ, *and* a_k $(1\leqq k<\infty)$ *are continuous functions of* μ *at* $\mu=\lambda$. (ii) $\hat{a}(i\tau,\mu)$ *is continuous in* μ *at* $\mu=\lambda$, *uniformly on compact subsets of* $\{0<\tau<\infty\}$. (iii) $\hat{a}(i\tau,\mu)=O(\tau^{-1})$ $(\tau\to\infty)$ *uniformly for* $\mu\in N_\lambda$.

Proof. By (H1), (H2), and (1.8),

$$\min\{a(0,\mu)/2, \pi\sqrt{a}(0,\mu)\} = \tfrac{1}{2}a(0,\mu)\min\{1, t_0(\mu)\} \leqq \int_0^1 a(t,\mu)\,dt$$
$$\to \int_0^1 a(t,\lambda)\,dt \quad (\mu \to \lambda).$$

Then $t_0(\mu) = 2\pi/\sqrt{a}(0,\mu)$ is bounded away from zero near $\mu = \lambda$, and it follows easily from (1.8) that for $|\mu - \lambda| < \Delta_1$, say, we have

$$a(0,\mu) < 4\,a(0,\lambda). \tag{3.11}$$

We require

$$\Delta < \Delta_1, \tag{3.12}$$

so that (3.11) holds in N_λ. Then $t_0(\mu) > t_0(\lambda)/2$ $(\mu \in N_\lambda)$. It is then clear from (1.8) $(R = t_0(\lambda)/2)$ and (H2) that $a(0,\mu) \to a(0,\lambda)$ $(\mu \to \lambda)$. Using (1.4) and (1.6), we see that t_0, ω and γ are also continuous at $\mu = \lambda$. A simple induction argument using (1.8) now shows that $a(kt_0(\mu),\mu) \to a(kt_0(\lambda),\lambda)$ $(\mu \to \lambda, k = 1, 2, \cdots)$, from which we easily deduce the continuity of the a_k. This proves (i).

The uniform continuity of $\hat{a}(it,\mu)$ in μ follows from (H1), (1.6), and (1.8) by taking real and imaginary parts, since, for instance,

$$\left|\operatorname{Im}\hat{a}(i\tau) - \int_0^{k\pi/\tau} a(t)\sin\tau t\,dt\right|$$
$$\leqq \left|\int_0^{\pi/\tau} a(t + k\pi/\tau)\sin\tau t\,dt\right|$$
$$\leqq \pi a(k\pi/\tau)/\tau.$$

For more details, see [2, Lemma 6.2]. This proves (ii).

Finally, by (H1) and (3.11), $|\operatorname{Im}\hat{a}(i\tau,\mu)| \leqq 4\pi a(0,\lambda)/\tau$ $(\mu \in N_\lambda)$, and similarly for $\operatorname{Re}\hat{a}(i\tau,\mu)$. This gives (iii) and completes the proof of Lemma 3.3.

Note. R. K. Miller (private communication) has kindly pointed out that in the proof of Lemma 2.1 of [3], the assertion that $\{a_n(0)\}$ is bounded is not justified by the argument given there. In proving Lemma 3.3 above, we began by showing that $a(0,\mu)$ is bounded near $\mu = \lambda$. If (H2) is not assumed to hold at $\mu = \lambda$, this estimate still shows that $a(0,\mu)$ is bounded in a deleted neighborhood of $\mu = \lambda$; this argument suffices to close the cited gap in [3].

By Lemma 3.3, we can choose Δ so that

$$\omega(\lambda) - \eta/2 < \omega(\mu) < \omega(\lambda) + \eta/2, \tag{3.13}$$

$$.9 < \frac{\gamma(\mu)}{\gamma(\lambda)}, \frac{t_0(\mu)}{t_0(\lambda)}, \frac{\omega(\mu)}{\omega(\lambda)} < 1.1 \qquad (\mu \in N_\lambda). \tag{3.14}$$

Let

$$f_1(t,\mu) = \int_{\omega(\lambda)+\eta}^{\infty} \operatorname{Re}\{e^{i\tau t}\,U_1(i\tau,\mu)\}\,dt.$$

After a little calculation, (3.3), Lemma 3.3 and Lebesgue's dominated convergence theorem show that

$$U_1(i\tau, \mu) = [2\gamma^{-1}(\mu) - 1]i/\tau + P_1(\tau, \mu),$$

where (3.10) holds with $\alpha = \omega(\lambda) + \eta$, $\beta = \infty$, $h = P_1$. Since

$$-\int_{\omega(\lambda)+\eta}^{\infty} \operatorname{Re}\{e^{i\tau t}\, i/\tau\}\, d\tau = \int_{t[\omega(\lambda)+\eta]}^{\infty} x^{-1} \sin x\, dx$$

is bounded on $\{0 \leqq t < \infty\}$ and $\gamma(\mu) \to \gamma(\lambda) > 0$ $(\mu \to \lambda)$, we conclude, with the help of Lemma 3.1, that (3.7) holds, $j = 1$.

Note that by (H1),

$$\begin{aligned}\sqrt{2}|i\tau + \hat{a}(i\tau)| &\geqq -\tau - \operatorname{Im} \hat{a}(i\tau) + \operatorname{Re} \hat{a}(i\tau)\\ &\geqq -\tau + \int_0^{\pi/2\tau} a(t) \cos \tau t\, dt\\ &\geqq -\tau + \frac{1}{2}\int_0^{\pi/3\tau} a(t)\, dt.\end{aligned}$$

Choose τ_0, $0 < \tau_0 \leqq \omega(\lambda)$, such that

$$-\tau_0 + \frac{1}{2}\int_0^{\pi/3\tau_0} a(t, \lambda)\, dt > \frac{1}{4}\int_0^1 a(t, \lambda)\, dt.$$

Then by (1.8), Δ may be chosen so that $\mu \in N_\lambda$ implies

$$|i\tau + \hat{a}(i\tau, \mu)| > \int_0^1 a(t, \lambda)\, dt/10, \qquad 0 < \tau \leqq \tau_0. \tag{3.15}$$

By (3.4) and Lemma 3.3,

$$|i\tau + \hat{a}(i\tau, \mu)|^{-1} \to |i\tau + \hat{a}(i\tau, \lambda)|^{-1}, \qquad \mu \to \lambda,$$

uniformly on $\{\tau_0 \leqq \tau \leqq \omega(\lambda) - \eta\}$. Thus in choosing Δ, we may ensure that there is a constant $M_1 < \infty$ such that

$$D^{-1}(\tau, \mu) \equiv |i\tau + \hat{a}(i\tau, \mu)|^{-1} \leqq M_1, \qquad 0 < \tau \leqq \omega(\lambda) - \eta. \tag{3.16}$$

Now it follows easily that

$$\int_0^{\omega(\lambda)-\eta} |U_1(i\tau, \mu) - U_1(i\tau, \lambda)|\, d\tau \to 0, \qquad \mu \to \lambda.$$

Then

$$f_2(t, \mu) \equiv \int_0^{\omega(\lambda)-\eta} \operatorname{Re}\{e^{i\tau t}\, U_1(i\tau, \mu)\}\, d\tau$$

satisfies (3.7).

Taking transforms in (3.1), we see that

$$\hat{a}(s) = \frac{\omega^2}{s} + \frac{1}{s^2}\sum_{k=1}^{\infty} \frac{a_k}{kt_0}(e^{-skt_0} - 1).$$

A little calculation shows that

$$i\tau+\hat{a}(i\tau)=i\gamma(\tau-\omega)-(\tau-\omega)^2 i[\tau^{-1}+(\tau+\omega)/\tau^2] + \tau^{-2}[iS(\tau-\omega)-C(\tau-\omega)], \tag{3.17}$$

where

$$S(\sigma)=\sum_{k=1}^{\infty} a_k(\sin kt_0\sigma - kt_0\sigma)/kt_0,$$

$$C(\sigma)=\sum_{k=1}^{\infty} a_k(\cos kt_0\sigma - 1)/kt_0.$$

It is clear that

$$C(\sigma,\mu)=C(-\sigma,\mu)=C(\sigma+\omega(\mu),\mu)\leqq 0, \tag{3.18}$$

$$\sigma S(\sigma,\mu)=-\sigma S(-\sigma,\mu)\leqq 0. \tag{3.19}$$

Term-by-term differentiation and (3.11) show that

$$\left|\frac{\partial S}{\partial\sigma}(\sigma,\mu)\right|+\left|\frac{\partial C}{\partial\sigma}(\sigma,\mu)\right|\leqq 12a(0,\lambda), \tag{3.20}$$

when $\mu\in N_\lambda$, $-\infty<\sigma<\infty$.

The following property is more difficult to prove.

LEMMA 3.4. *Given* $\varepsilon>0$, *there exist positive numbers* $\eta(\varepsilon)$, $\Delta(\varepsilon)$ *such that*

$$|S(\sigma,\mu)|+|C(\sigma,\mu|<\varepsilon\sigma \tag{3.21}$$

if $|\mu-\lambda|<\Delta(\varepsilon)$, $|\sigma|\leqq\eta(\varepsilon)$.

Proof. By symmetry, we need only consider $\sigma>0$. Let $n(\sigma)$ denote the greatest integer such that $\sigma n(\sigma)\leqq 1$. Then

$$C(\sigma)+iS(\sigma)=\left[\sum_{k=1}^{n(\sigma)}+\sum_{k=n(\sigma)+1}^{\infty}\right]\frac{a_k}{kt_0}(e^{ikt_0\sigma}-1-ikt_0\sigma)=\textstyle\sum_1+\sum_2. \tag{3.22}$$

Then by Taylor's formula,

$$\begin{aligned}|\textstyle\sum_1|&\leqq\sum_{k=1}^{n(\sigma)}\frac{a_k}{kt_0}(kt_0\sigma)^2\,e^{kt_0\sigma}\leqq e^{t_0}t_0\sigma^2\sum_{k=1}^{n(\sigma)}ka_k\\ &\leqq e^{t_0}\sigma\,\frac{2}{t_0 n(\sigma)}\int_0^{t_0 n(\sigma)}a(t)\,dt.\end{aligned} \tag{3.23}$$

(The last inequality follows from (3.1), since

$$\int_0^{t_0 n(\sigma)}a(t)\,dt\geqq\int_0^{t_0 n(\sigma)}\sum_{k=1}^{n(\sigma)}a_k\left(1-\frac{\min\{t,kt_0\}}{kt_0}\right)dt=\sum_{k=1}^{n(\sigma)}a_k kt_0/2.$$

Term-by-term estimation of the summand shows that

$$|\textstyle\sum_2|\leqq\sum_{k=n(\sigma)+1}^{\infty}[(2a_k/kt_0)+\sigma a_k]\leqq\sigma(1+2/t_0)\sum_{k=n(\sigma)+1}^{\infty}a_k. \tag{3.24}$$

Given ε, choose $\eta(\varepsilon)$ so that $|\sigma|<\eta(\varepsilon)$ implies

$$\left[2e^{t_0(\lambda)}(t_0(\lambda)\eta(\sigma))^{-1}\int_0^{t_0(\lambda)\eta(\sigma)} a(t,\lambda)\,dt+(1+2/t_0(\lambda))\sum_{k=\eta(\sigma)+1}^{\infty} a_k(\lambda)\right]<\varepsilon/3.$$

By continuity, we can find $\Delta(\varepsilon)$, so that this inequality holds with λ replaced by μ, $\varepsilon/3$ replaced by $\varepsilon/2$ and $|\mu-\lambda|<\Delta(\varepsilon)$. Our result then follows from (3.23) and (3.24).

Using Lemma 3.4, we shall select our numbers η and Δ so that

$$|S(\sigma,\mu)|+|C(\sigma,\mu)|<\sigma, \qquad |\sigma|\leqq\eta, \tag{3.25}$$

when $\mu\in N_\lambda$. Then we see also that

$$\sigma^{-1}S(\sigma,\mu)\to\sigma^{-1}S(\sigma,\lambda), \quad \sigma^{-1}C(\sigma,\mu)\to\sigma^{-1}C(\sigma,\lambda), \quad \mu\to\lambda, \tag{3.26}$$

in the L^1-norm on $\{|\sigma|\leqq\eta\}$. This follows from (3.11), the continuity of $a_k(\mu)$ and $t_0(\mu)$ and the dominated convergence theorem.

Using (3.17), one finds after simplifying that

$$\operatorname{Re} U_1(i\tau)=-C(\tau-\omega)(\tau-\omega)^2\gamma^2/\tau^2[P(\tau-\omega)+Q_1(\tau)],$$

where

$$P(\sigma,\mu)=\gamma^4(\mu)\sigma^4+\gamma^2(\mu)\omega^{-4}(\mu)\sigma^2[C^2(\sigma,\mu)+S^2(\sigma,\mu)] \\ +2\gamma^3(\mu)\omega^{-2}(\mu)\sigma^3S(\sigma,\mu), \tag{3.27}$$

and $Q_1(\tau,\mu)$ is a certain function continuous in μ at $\mu=\lambda$, and such that

$$Q_1(\tau,\mu)[\tau-\omega(\mu)]^{-5} \quad \text{is uniformly bounded on } \{\mu\in N_\lambda, |\tau-\omega(\lambda)|\leqq\eta\}. \tag{3.28}$$

Lemma 3.4 and the continuity of γ and ω show that we may choose η and Δ in such a way that

$$\tfrac{1}{2}\sigma^4\gamma^4(\lambda)\leqq P(\sigma,\mu)\leqq 2\sigma^4\gamma^4(\lambda), \qquad |\sigma|\leqq\eta, \tag{3.29}$$

whenever $\mu\in N_\lambda$. It follows that

$$\operatorname{Re} U_1(i\tau)=-C(\tau-\omega)(\tau-\omega)^2\gamma^2/\omega^2P(\tau-\omega)+Q_2(\tau), \tag{3.30}$$

where

$$Q_2(\cdot,\mu)\to Q_2(\cdot,\lambda) \quad \text{in } L^1(\omega(\lambda)-\eta,\omega(\lambda)+\eta) \quad \text{as } \mu\to\lambda. \tag{3.31}$$

Similarly (with the formulas slightly more complicated),

$$\operatorname{Im} U_1(i\tau)=R(\tau-\omega)+C^2(\tau-\omega)(\tau-\omega)\gamma/\omega^4P(\tau-\omega)+Q_3(\tau), \tag{3.32}$$

where

$$R(\sigma)=\gamma\sigma S(\sigma)[\gamma\sigma+S(\sigma)/\omega^2]/\omega^2P(\sigma), \tag{3.33}$$

and (3.31) holds with Q_3 in place of Q_2.

By Lemma 3.1, (3.7) holds for

$$f_3(t,\mu)\equiv\int_{\omega(\gamma)-\eta}^{\omega(\lambda)+\eta}[Q_2(\tau,\mu)\cos\tau t-Q_3(\tau,\mu)\sin\tau t]\,d\tau.$$

Set

$$g(t, \mu) = -\lim_{\varepsilon \to 0^+} \int_{I(\mu, \varepsilon)} R(\tau - \omega(\mu), \mu) \sin \tau t \, d\tau, \tag{3.34}$$

where $I(\mu, \varepsilon) = I(\mu) \cap \{|\tau - \omega(\mu)| \geqq \varepsilon\}$ and $I(\mu)$ is the largest subinterval of $[\omega(\lambda) - \eta, \omega(\lambda) + \eta]$ which is symmetric about $\omega(\mu)$. Then

$$f_4(t, \mu) \equiv \int_{I(\lambda) \setminus I(\mu)} -R(\tau - \omega(\mu), \mu) \sin \tau t \, d\tau$$

clearly satisfies (3.7). To establish (3.6), we first develop some facts concerning $R(\sigma)$. By (3.18), (3.19), and (3.27),

$$R(\sigma) = -R(-\sigma). \tag{3.35}$$

Next we show that there is a number $M_2 < \infty$ such that

$$|\sigma^2 R'(\sigma)| + |\sigma R(\sigma)| \leqq M_2 |S(\sigma)/\sigma| < M_2, \qquad |\sigma| \leqq \eta, \quad \mu \in N_\lambda. \tag{3.36}$$

In fact, using (3.14), (3.25) and (3.29) we see that

$$|\sigma R(\sigma, \mu)| \leqq (8[\gamma(\lambda) + \omega^{-2}(\lambda)]/\omega^2(\lambda)\gamma^3(\lambda)) |\sigma^{-1} S(\sigma, \mu)|.$$

Differentiating (3.33) and using similar estimates, we arrive at (3.36).

From (3.34) and (3.35) we see that

$$g(t, \mu) = -2 \cos [t\omega(\mu)] \int_0^{\eta(\mu)} R(\sigma, \mu) \sin \sigma t \, d\sigma, \tag{3.37}$$

where $\eta(\mu)$ = half the length of $I(\mu) \geqq \eta/2$. The integral exists, since $|\sin \sigma t| \leqq \sigma t$ and (3.36) holds. Clearly,

$$g(0+, \mu) = 0, \qquad \mu \in N_\lambda. \tag{3.38}$$

The argument establishing (3.6) is essentially the same as that in [1, p. 551]. Choose $\rho > 0$ so that $2M_2 < \rho\theta/6$. Using Lemma 3.4, restrict η and Δ so that

$$|S(\sigma, \mu)/\sigma| < \theta/6\rho M_2, \qquad |\sigma| \leqq \eta, \tag{3.39}$$

when $\mu \in N_\lambda$. Finally, choose T so large that

$$2M_2/\eta T < \theta/6 \quad \text{and} \quad 2\rho/T < \eta. \tag{3.40}$$

Then for $t \geqq T$, integration by parts and (3.36) show that

$$\left| \int_{\rho/t}^{\eta(\mu)} R(\sigma, \mu) \sin \sigma t \, d\sigma \right| \leqq t^{-1} \left[|R(\eta(\mu), \mu)| + |R(\rho/t, \mu)| + M_2 \int_{\rho/t}^{\eta(\mu)} \sigma^{-2} \, d\sigma \right]$$

$$\leqq 2M_2[\rho^{-1} + (T\eta)^{-1}] < \theta/3,$$

while

$$\left| \int_0^{\rho/t} R(\sigma, \mu) \sin \sigma t \, d\sigma \right| \leqq \left| \int_0^{\rho/t} \sigma R(\sigma) \cdot \sigma^{-1} \sin \sigma t \, d\sigma \right|$$

$$\leqq t \int_0^{\rho/t} \theta/6\rho \, d\sigma = \theta/6.$$

This proves (3.6).

Referring to (3.2), (3.30), and (3.22), we see that (3.5) holds with

$$-f_5(t,\mu)=\lim_{\varepsilon\to 0^+}\left[\int_{\omega(\lambda)-\eta}^{\omega(\mu)-\varepsilon}+\int_{\omega(\mu)+\varepsilon}^{\omega(\lambda)+\eta}\right]\{h_1(\tau,\mu)\cos\tau t+h_2(\tau,\mu)\sin\tau t\}\,d\tau,$$
$$h_1=C(\tau-\omega)(\tau-\omega)^2\gamma^2/\omega^2P(\tau-\omega),\qquad h_2=h_1C(\tau-\omega)/\omega^2\gamma(\tau-\omega).$$

Setting $t=t_0(\mu)=2\pi/\omega(\mu)$ in (3.2), we find that

$$\text{(3.41)}\qquad \pi u_1(t_0(\mu),\mu)=g(t_0(\mu),\mu)+\sum_{j=1}^{4}f_j(t_0(\mu),\mu)$$
$$-\lim_{\varepsilon\to 0^+}\left[\int_{\omega(\lambda)-\eta}^{\omega(\mu)-\varepsilon}+\int_{\omega(\mu)+\varepsilon}^{\omega(\lambda)+\eta}\right]h_1(\tau,\mu)\cos\tau t_0(\mu)\,d\tau$$
$$-\int_{I(\lambda)}h_2(\tau,\mu)\sin\left[(\tau-\omega(\mu))t_0(\mu)\right]d\tau,$$

since the last integrand is $O(1)$ $(\tau\to\omega(\mu))$ by (3.25) and (3.29). Since $h_1\leqq 0$ and $\cos\tau t_0\geqq\frac{1}{2}$ near $\tau=\omega(\mu)$, we conclude that $h_1(\tau,\mu)$ (and hence also $h_2(\tau,\mu)$) belongs to $L^1(I(\lambda))$ as a function of τ $(\mu\in N_\lambda)$. Letting $t\to 0^+$ in (3.2), and using (3.38) and (3.7) $(1\leqq j\leqq 4)$, we see that

$$\int_{I(\lambda)}h_1(\tau,\mu)\,d\tau=[2\gamma^{-1}(\mu)-1]\pi+\sum_{j=1}^{4}f_j(0^+,\mu).$$

Then by Lemma (3.2),

$$\text{(3.42)}\qquad h_1(\cdot,\mu)\to h_1(\cdot,\lambda)\quad\text{in } L^1(I(\lambda)),\qquad \mu\to\lambda.$$

Since h_2/h_1 is uniformly bounded, we also have

$$\text{(3.43)}\qquad h_2(\cdot,\mu)\to h_2(\cdot,\lambda)\quad\text{in } L^1(I(\lambda)),\qquad \mu\to\lambda.$$

Thus (3.7) is true for $j=5$. This completes our proof that $u_1(\tau,\mu)\to u_1(t,\lambda)(\mu\to\lambda)$ uniformly in $\{0\leqq t<\infty\}$.

Set $V(s)=W_1(s)-A/s$, with

$$W_1(s)=\frac{1}{s(s+\hat{a}(s))}-\frac{2}{\gamma(s^2+\omega^2)}=\frac{U_1(s)}{s}.$$

A Laplace transform argument, similar to the one establishing (3.2) (cf.[2, Lemma 4.4]. The fact that $V(s)=o(s^{-1})$ $(s\to 0,\ \mathrm{Re}\,s\geqq 0)$ is used) shows that

$$\text{(3.44)}\qquad \pi[w_1(t,\mu)-A(\mu)]=\lim_{\varepsilon,\rho\to 0^+}\left[\int_{\varepsilon}^{\omega(\lambda)-\eta}+\int_{J(\rho)}+\int_{\omega(\lambda)+\eta}^{\infty}\right]\mathrm{Re}\{e^{i\tau t}V(i\tau,\mu)\}\,d\tau$$

for $t>0$, where

$$J(\rho)=[\omega(\lambda)-\eta,\ \omega(\mu)-\rho]\cup[\omega(\mu)+\rho,\ \omega(\lambda)+\eta].$$

Here we again fix λ and θ, and choose positive Δ, η, and T according to appropriate restrictions. The proof that $|w_1(t,\mu)-w_1(t,\lambda)|\leqq\theta$ for $\mu\in N_\lambda$ and all t, will be based on the decomposition

$$\pi[w_1(t,\mu)-A(\mu)]=G(t,\mu)+\sum_{j=6}^{12} f_j(t,\mu), \qquad t\geqq 0^+, \tag{3.45}$$

in place of (3.5).

It is clear that

$$f_6(t,\mu)\equiv\int_{\omega(\lambda)+\eta}^{\infty} \operatorname{Re}\{e^{i\tau t}V(i\tau)\}\,d\tau$$

satisfies (3.7), as does

$$f_7(t,\mu)=-A(\mu)\int_{I(\lambda)} \operatorname{Re}\{e^{i\tau t}/i\tau\}\,d\tau.$$

From (3.30) and (3.32), we see that

$$\operatorname{Re}\{e^{i\tau t}W_1(i\tau)\}=\omega^{-1}\{\cos\tau t[R(\tau-\omega)+h_2(\tau)]-\sin\tau t\quad h_1(\tau)\} + Q_4(t,\tau,\mu), \tag{3.46}$$

where Lemma 3.1 shows that

$$f_8(t,\mu)\equiv\int_{I(\lambda)} Q_4(t,\tau,\mu)\,d\tau$$

satisfies (3.7).

Let

$$f_9(t,\mu)=\omega^{-1}(\mu)\int_{I(\lambda)} [h_2(\tau,\mu)\cos\tau t-h_1(\tau,\mu)\sin\tau t]\,d\tau.$$

By (3.42) and (3.43) and Lemma 3.1, (3.7) holds, $j=9$.

As in (3.34) and (3.37), we write

$$\omega^{-1}(\mu)\int_{I(\lambda)} R(\tau-\omega,\mu)\cos\tau t=G(t,\mu)+f_{10}(\tau,\mu),$$

where (3.7) holds ($j=10$) and

$$G(t,\mu)=-2\omega^{-1}(\mu)\sin[t\omega(\mu)]\int_0^{\eta(\mu)} R(\sigma,\mu)\sin\sigma t\,d\sigma.$$

Then from our analysis of $g(t,\mu)$, we conclude that suitable restrictions on η, Δ, and T will ensure that

$$|G(t,\mu)|\leqq\theta, \qquad t\geqq T, \quad \mu\in N_\lambda. \tag{3.47}$$

Moreover,

$$G(0+,\mu)=0. \tag{3.48}$$

Now write

$$V(s)=\frac{1-A\hat{a}(s)}{s(s+\hat{a}(s))}-\frac{A}{s+\hat{a}(s)}-\frac{2}{\gamma(s^2+\omega^2)},$$

and set $K=\omega(\lambda)-\eta$. Using Lemma 3.3 and (3.16) as with f_2, we see that

$$\lim_{\varepsilon\to 0^+}\int_\varepsilon^K \operatorname{Re}\{e^{i\tau t}V(i\tau,\mu)\}\,d\tau=f_{11}(t,\mu)+f_{12}(t,\mu),$$

where

$$f_{12}(t)=\lim_{\varepsilon\to 0^+}\int_\varepsilon^K \operatorname{Re}\left\{e^{i\tau t}\frac{1-A\hat{a}(i\tau)}{i\tau(i\tau+\hat{a}(i\tau))}\right\}d\tau \tag{3.49}$$

and (3.7) holds with $j=11$. We now have the decomposition (3.45), and we need only show that f_{12} satisfies (3.7).

Taking real and imaginary parts in (3.49), we obtain

$$\begin{aligned} f_{12}(t)=\lim_{\varepsilon\to 0^+}&\int_\varepsilon^K \frac{\psi(\tau)\cos\tau t}{\tau D^2(\tau)}\,d\tau-\int_0^K\frac{\operatorname{Re}\ Q(\tau)\cos\tau t}{D^2(\tau)}\,d\tau\\ &+\int_0^K\frac{\sin\tau t}{\tau D^2(\tau)}[\varphi(\tau)\operatorname{Re}\,Q(\tau)-(\psi(\tau)-\tau)\operatorname{Im}\,Q(\tau)]\,d\tau,\end{aligned} \tag{3.50}$$

with $Q(\tau)=1-A\hat{a}(i\tau)$, $\varphi=\operatorname{Re}\hat{a}$, $\psi=-\operatorname{Im}\hat{a}$, and D from (3.16).

The passage to the limit $\varepsilon=0$ in the second and third integrals in (3.50) is justified as follows. Clearly $|Q(\tau)|\leqq 2$. Since (3.16) holds, the second integral exists. Now $|\tau^{-1}\sin\tau t|\leqq t$, while $\varphi(\tau)$ and $\tau-\psi(\tau)$ are, respectively, the real and imaginary parts of $i\tau+\hat{a}(i\tau)$, so that $(|\varphi(\tau)|+|\tau-\psi(\tau)|)/D(\tau)\leqq 2$. Thus the third integral exists as well.

(3.45) and (3.50) together are essentially the same as (4.9) of [2]. The fact that piecewise linear kernels are excluded in [2] makes no difference in the rest of our argument, which follows that running from (6.9) to the end of § 6 in [2].

By Lemmas 3.1 and 3.3 and (3.16), the second integral in (3.50) is continuous in μ at $\mu=\lambda$, uniformly in $0^+\leqq t<\infty$.

Because of (H1), $\psi(\tau)\geqq 0$; thus the first term on the right in (3.50) may be written as $\int_0^K\tau^{-1}D^{-2}(\tau)\psi(\tau)\cos\tau t\,d\tau$, and the coefficient of $\cos\tau t$ in the integrand must belong to $L^1(0,K)$. Letting $t\to 0^+$ in (3.45) and using (3.48), we see that

$$\int_0^K\frac{\psi(\tau,\mu)\,d\tau}{\tau D^2(\tau,\mu)}=\int_0^K\frac{\operatorname{Re}\,Q(\tau,\mu)\,d\tau}{D^2(\tau,\mu)}-\sum_{j=6}^{11}f_j(0+,\mu).$$

Thus the left-hand side is continuous in μ at $\mu=\lambda$, and Lemmas 3.2 and 3.3 and (3.16) show that

$$\int_0^K\left|\frac{\psi(\tau,\mu)}{\tau D^2(\tau,\mu)}-\frac{\psi(\tau,\lambda)}{\tau D^2(\tau,\lambda)}\right|d\tau\to 0,\qquad \mu\to\lambda. \tag{3.51}$$

Thus the first integral in (3.50) is continuous in μ at $\mu=\lambda$, uniformly in t.

Finally, we sketch the treatment of the third integral in (3.50). See [2] for specific estimates and more details.

Writing

$$\varphi(\tau)=\left[\int_0^{\pi/2\tau}+\int_{\pi/2\tau}^\infty\right]a(t)\cos\tau t\,d\tau\equiv\varphi_1(\tau)+\varphi(\tau),$$

we decompose the third integrand in (3.50) as

$$\Phi(\tau, \mu)\tau^{-1} \sin \tau t + E(\tau, \mu) \sin \tau t,$$

where $\Phi(\tau) = [\varphi_1^{-1}(\tau) - A]$. Using Lemma 3.3 together with (3.51) and several estimates involving φ, ψ, φ_1, φ_2 and D, we find that

$$\int_0^K |E(\tau, \mu) - E(\tau, \lambda)|\, d\tau \to 0, \qquad \mu \to \lambda.$$

Finally, we show that $\Phi(\tau) \downarrow 0$ $(\tau \downarrow 0)$; and a direct argument, using the second law of the mean, shows that

$$\int_0^K \tau^{-1}\Phi(\tau, \mu) \sin \tau t\, d\tau$$

is continuous in μ at $\mu = \lambda$, uniformly in $\{0 < t < \infty\}$. This proves that f_{12} satisfies (3.7) and completes the proof of Theorem 1.

Remark. The formula at the bottom of page 551 in [1] is incorrect; the second term on the right-hand side should have expression (3.46) as the integrand in place of $\omega^{-1}\,\mathrm{Re}\{\cdots\}$. The conclusion that this term is $o(1)$ $(t \to \infty)$ then follows, because the coefficients of the trigonometric terms are known to be in $L^1(\omega - \eta, \omega + \eta)$, except that $R(\tau - \omega) \cos \tau t$ must be treated as shown above in this paper.

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NEW IDENTITIES FOR LEGENDRE ASSOCIATED FUNCTIONS OF INTEGRAL ORDER AND DEGREE. I*

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Abstract. The identities for Legendre associated functions $P_\nu^m(x)$ of nonintegral order ν, known as Dougall's identities, are extended to give several new identities for Legendre associated functions $P_n^m(x)$ of integral order and degree. Such identities are required in the simplification and evaluation of expansions arising from the use of Green's functions. The uniform convergence of each new identity is considered in detail.

1. Introduction. In the solution of the boundary value problems of mathematical physics in a separable 3-dimensional coordinate system, the shape of the boundary of the space may be such that the Green's function of the second order differential operator can be expanded as an infinite series of orthogonal functions. In many coordinate systems (such as the spherical, spheroidal and some cyclidal systems), these expansions are given in terms of Legendre associated functions of integral order and degree.

Starting with Dougall's identities for Legendre associated functions of nonintegral degree [1, 3.10(6)(8)(9)], new identities for infinite series of Legendre associated functions of integral degree are derived. Uniform convergence of each new identity is investigated in detail, so that interchange of summation and integration may be performed when required.

This paper is the first in a projected series. In the second, the results and techniques will be generalized, and a sufficient condition found under which a generalized orthogonal function which satisfies Dougall's identity will also satisfy the new identity. This theorem will be applied to the Legendre associated function, the generalized Legendre associated function [3] and to the Jacobi function.

2. The fundamental identities. In the course of deriving our identities, we have to differentiate infinite series term-by-term. We start therefore with two lemmas on uniform convergence.

LEMMA 1. *For all $\nu \in R \sim I$, and for all $m \in N$, the series*

$$(1)\qquad S_1(\theta) = \sum_{n=m}^{\infty} (-1)^n \gamma_{n,m} P_n^m(\cos\theta) P_n'^m(0)\{1/(\nu - n) - 1/(\nu + n + 1)\}$$

is uniformly convergent for $0 < \theta < \frac{1}{2}\pi$, and

$$(2)\qquad S_2(\theta) = \sum_{n=m}^{\infty} (-1)^n \gamma_{n,m} P_n^m(\cos\theta) P_m^m(0)\{1/(\nu - n) - 1/(\nu + n + 1)\}$$

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is uniformly convergent for $0 < \theta < \pi$, *where* $\gamma_{n,m}$ *is defined by*

(3) $$\gamma_{n,m} = \Gamma(n - m + 1)/\Gamma(n + m + 1),$$

and $P_n'^m(0)$ *is the derivative of* $P_n^m(x)$ *at* $x = 0$, *i.e.*,

(4) $$P_n'^m(0) = \left[\frac{d}{dx} P_n^m(x)\right]_{x=0}.$$

Proof. We know that

(5) $$P_n^{-m}(x) = (-1)^m \gamma_{n,m} P_n^m(x), \qquad -1 < x < 1,$$

and

(6) $$P_n'^m(0) = (m + n) P_{n-1}^m(0).$$

Further, a bound on $P_n^m(\cos\theta)$ is given by [2, p. 303].

(7) $$n^{-m} P_n^m(\cos\theta) = (\tfrac{1}{2} n\pi \sin\theta)^{-1/2} \cos\{(n + \tfrac{1}{2})\theta - \pi/4 + \tfrac{1}{2} m\pi\} + O(n^{-3/2}),$$
$$0 < \varepsilon < \theta < \pi - \varepsilon, \quad n > 1, \quad n \gg m,$$

where $O(n^{-3/2})$ depends on ε. Combining these three results we deduce

(8) $$\gamma_{n,m} P_n^m(\cos\theta) P_n'^m(0) = C \sin\{\tfrac{1}{2}(n + m)\pi\} \cos\{(n + \tfrac{1}{2})\theta - \pi/4 - \tfrac{1}{2} m\pi\} + O(n^{-1}),$$
$$0 < \varepsilon < \theta < \pi - \varepsilon, \quad n > 1, \quad n \gg m,$$

for large n, where C is a constant independent of ε, and where $O(n^{-1})$ depends on ε.

Now define u_n and v_n by

(9) $$u_n(\theta) = (-1)^n \sin\{\tfrac{1}{2}(n + m)\pi\} \cos\{(n + \tfrac{1}{2})\theta - \pi/4 - \tfrac{1}{2} m\pi\},$$

(10) $$v_n = 1/(v - n) - 1/(v + n + 1).$$

By summing the series we can show that

$$\sum_{n=0}^{M-1} (-1)^n \sin(n\psi + \alpha) = \frac{\cos(\tfrac{1}{2}\psi)\{\sin(\alpha - \tfrac{1}{2}\psi) - (-1)^M \sin(\alpha - \tfrac{1}{2}\psi + M\psi)\}}{1 + \cos\psi},$$

and hence obtain the bound

(i) $$\left|\sum_{n=0}^{M-1} u_n(\theta)\right| < K \quad \text{if } 0 < \theta < \tfrac{1}{2}\pi.$$

From definition (10), we see that

(ii) $v_n > 0$ and decreases with n for all $n > |v|$,

(iii) $v_n \to 0$ as $n \to \infty$.

Then by Dirichlet's test, $S_1(\theta)$ of (1) is uniformly convergent for $0 < \theta < \tfrac{1}{2}\pi$ as required.

Now consider $S_2(\theta)$. Using (5) and (7) we derive the result

$$|(-1)^{n+m} \gamma_{n,m} P_n^m(\cos\theta) P_n^m(0) \{1/(v - n) - 1/(v + n + 1)\}| = K/n^2 + O(n^{-3}),$$

where $O(n^{-3})$ depends on ε, but where K is a constant independent of ε.

Hence by Weierstrass' M-test, $S_2(\theta)$ is uniformly convergent for $0 < \theta < \pi$ as stated.

LEMMA 2. *For all* $\nu \in R \sim I$, $m \in N$ *and* $x \in (0, 1)$,

$$(11)\quad P_\nu^m(x) = P_\nu^m(0) \sum_{n=m}^{\infty} (-1)^{n+m}\gamma_{n,m}P_n^m(x)P_n'^m(0)\{1/(\nu - n) - 1/(\nu + n + 1)\}$$

$$(12)\quad = P_\nu'^m(0) \sum_{n=m}^{\infty} (-1)^{n+m}\gamma_{n,m}P_n^m(x)P_n^m(0)\{1/(\nu - n) - 1/(\nu + n + 1)\},$$

and both series are uniformly convergent over their range of validity.

Proof. Consider Dougall's third identity [1, 3.10(9)],

$$(13)\quad \begin{aligned} P_\nu^m(\cos\theta)P_\nu^{-m}(\cos\xi) = \{\sin(\nu\pi)/\pi\} \sum_{n=m}^{\infty} (-1)^n P_n^m(\cos\theta)P_n^{-m}(\cos\xi) \\ \cdot\{1/(\nu - n) - 1/(\nu + n + 1)\}, \qquad -\pi < \theta \pm \xi < \pi. \end{aligned}$$

Substitute for $P_\nu^{-m}(\cos\xi)$ and $P_n^{-m}(\cos\xi)$ from (5) into (13), and differentiate with respect to ξ; set $\xi = \frac{1}{2}\pi$. We obtain (writing $x = \cos\theta$)

$$(14)\quad \begin{aligned} \gamma_{\nu,m}P_\nu^m(x)P_\nu'^m(0) = \{\sin(\nu\pi)/\pi\} \sum_{n=m}^{\infty} (-1)^n\gamma_{n,m} \\ \cdot P_n^m(x)P_n'^m(0)\{1/(\nu - n) - 1/(\nu + n + 1)\}, \qquad 0 < x < 1. \end{aligned}$$

Validity of the term-by-term differentiation of (13) follows from the uniform convergence of (14), which was proved in Lemma 1.

Using definition (3) and results [1, 3.4(20)(22)], namely

$$(15)\quad P_\nu^m(0) = 2^m\pi^{-1/2}\cos\{\tfrac{1}{2}(\nu + m)\pi\}\Gamma(\tfrac{1}{2} + \tfrac{1}{2}\nu + \tfrac{1}{2}m)/\Gamma(1 + \tfrac{1}{2}\nu - \tfrac{1}{2}m),$$

$$(16)\quad P_\nu'^m(0) = 2^{m+1}\pi^{-1/2}\sin\{\tfrac{1}{2}(\nu + m)\pi\}\Gamma(1 + \tfrac{1}{2}\nu + \tfrac{1}{2}m)/\Gamma(\tfrac{1}{2} + \tfrac{1}{2}\nu - \tfrac{1}{2}m),$$

we find

$$(17)\quad \gamma_{\nu,m}P_\nu^m(0)P_\nu'^m(0) = (-1)^m\sin(\nu\pi)/\pi.$$

We have used Legendre's duplication formula for gamma functions,

$$(18)\quad \Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z + \tfrac{1}{2}).$$

Equation (11) is now obtained by multiplying both sides of (14) by $P_\nu^m(0)$ and using (17). Similarly, if we multiply (13) by $P_\nu'^m(0)$ and set $\xi = \frac{1}{2}\pi$, a second application of (17) leads immediately to (12).

Uniform convergence of both (11) and (12) follows from Lemma 1, giving the required result.

We are now in a position to derive our basic new identities, which we do in the following theorem.

THEOREM 3. *For all $m, l \in N$, $l \geqq m$, and for all $x \in (0, 1)$,*

$$P_l^m(0) \sum_{\substack{n=m \\ n \neq l}}^{\infty} (-1)^{n+m} \gamma_{n,m} P_n^m(x) P_n'^m(0) \{1/(l-n) - 1/(l+n+1)\} \tag{19}$$
$$= \tfrac{1}{2}\{1 + (-1)^{l+m}\} P_l^m(x),$$

$$P_l'^m(0) \sum_{\substack{n=m \\ n \neq l}}^{\infty} (-1)^{n+m} \gamma_{n,m} P_n^m(x) P_n^m(0) \{1/(l-n) - 1/(l+n+1)\} \tag{20}$$
$$= \tfrac{1}{2}\{1 - (-1)^{l+m}\} P_l^m(x).$$

The convergence is uniform.

Proof. Consider (11). For some $l \geqq m$, we may write it in the form

$$P_\nu^m(x) = (-1)^{l+m} P_\nu^m(0) \gamma_{l,m} P_l^m(x) P_l'^m(0) \{1/(\nu - l) - 1/(\nu + l + 1)\} \tag{21}$$
$$+ P_\nu^m(0) \sum_{\substack{n=m \\ n \neq l}}^{\infty} (-1)^{n+m} \gamma_{n,m} P_n^m(x) P_n'^m(0) \{1/(\nu - n) - 1/(\nu + n + 1)\}.$$

We now wish to go to the limit $\nu \to l \in N$. The only piece of (21) for which this limit is not smooth is the first term of the dexter. Using (3), (15), (16) and (18), we can show that

$$\lim_{\nu \to l} (-1)^{l+m} P_\nu^m(0) \gamma_{l,m} P_l^m(x) P_l'^m(0)/(\nu - l) \tag{22}$$
$$= -(-1)^{l+m} \sin^2 \{\tfrac{1}{2}(l+m)\pi\} P_l^m(x)$$
$$= \tfrac{1}{2}\{1 - (-1)^{l+m}\} P_l^m(x),$$

whence taking the limit $\nu \to l \in N$ of (21), we see that limit exists, and gives

$$P_l^m(x) = \frac{1}{2}\{1 - (-1)^{l+m}\} P_l^m(x) + P_l^m(0) \sum_{\substack{n=m \\ n \neq l}}^{\infty} (-1)^{n+m} \gamma_{n,m} \tag{23}$$
$$\cdot P_n^m(x) P_n'^m(0) \{1/(l-n) - 1/(l+n+1)\}.$$

Rearrangement of (23) gives (19). Proof of its uniform convergence follows similar lines to that of Lemma 1; we can allow ν of Lemma 1 to be an integer l because the term $n = l$ has been excluded from the summation in (19).

We turn now to (12). Again separate out the term for $n = l$, and take the limit $\nu \to l \in N$. The nonsmooth term is (using (15) and (16) as before)

$$\lim_{\nu \to l} P_\nu'^m(0)(-1)^{l+m} \gamma_{l,m} P_l^m(x) P_l^m(0)/(\nu - l) \tag{24}$$
$$= (-1)^{l+m} \cos^2 \{\tfrac{1}{2}(l+m)\pi\} P_l^m(x)$$
$$= \tfrac{1}{2}\{1 + (-1)^{l+m}\} P_l^m(x).$$

Substitute this limit into the term for $n = l$: equation (12) then gives identity (20) as required.

COROLLARY 4. *From* (15) *and* (16) *we deduce that*

$$P_n^m(0) = 0 \quad \textit{unless } (m + n) \textit{ is even}$$

and

$$P_n'^m(0) = 0 \quad \textit{unless } (m + n) \textit{ is odd},$$

whence (19) *and* (20) *become*

$$\text{(25)} \qquad P_l^m(0) \sum_{\substack{n=m \\ n \neq l}}^{\infty} \gamma_{n,m} P_n^m(x) P_n'^m(0) \{1/(l - n) - 1/(l + n + 1)\} = -\tfrac{1}{2}\{1 + (-1)^{l+m}\} P_l^m(x),$$

$$\text{(26)} \qquad P_l'^m(0) \sum_{\substack{n=m \\ n \neq l}}^{\infty} \gamma_{n,m} P_n^m(x) P_n^m(0) \{1/(l - n) - 1/(l + n + 1)\} = \tfrac{1}{2}\{1 - (-1)^{l+m}\} P_l^m(x), \qquad 0 < x < 1.$$

COROLLARY 5. *Equations* (19), (20), (25) *and* (26) *hold for* $0 < x < 1$. *However, if* $-1 < x < 0$, *then* $0 < -x < 1$, *and* $(-x)$ *may be substituted into these four equations. Using*

$$\text{(27)} \qquad P_n^m(-x) = (-1)^{m+n} P_n^m(x), \qquad 0 < x < 1,$$

we obtain for $-1 < x < 0$,

$$\text{(28)} \qquad P_l^m(0) \sum_{\substack{n=m \\ n \neq l}}^{\infty} \gamma_{n,m} P_n^m(x) P_n'^m(0) \{1/(l - n) - 1/(l + n + 1)\} = \tfrac{1}{2}\{1 + (-1)^{l+m}\} P_l^m(x),$$

$$\text{(29)} \qquad P_l'^m(0) \sum_{\substack{n=m \\ n \neq l}}^{\infty} \gamma_{n,m} P_n^m(x) P_n^m(0) \{1/(l - n) - 1/(l + n + 1)\} = -\tfrac{1}{2}\{1 - (-1)^{l+m}\} P_l^m(x),$$

$$\text{(30)} \qquad P_l^m(0) \sum_{\substack{n=m \\ n \neq l}}^{\infty} (-1)^{n+m} \gamma_{n,m} P_n^m(x) P_n'^m(0) \{1/(l - n) - 1/(l + n + 1)\} = -\tfrac{1}{2}\{1 + (-1)^{l+m}\} P_l^m(x),$$

$$\text{(31)} \qquad P_l'^m(0) \sum_{\substack{n=m \\ n \neq l}}^{\infty} (-1)^{n+m} \gamma_{n,m} P_n^m(x) P_n^m(0) \{1/(l - n) - 1/(l + n + 1)\} = -\tfrac{1}{2}\{1 - (-1)^{l+m}\} P_l^m(x),$$

and the convergence is uniform.

Note that the dexters of (28)–(31) are opposite in sign to those of the corresponding identities for $0 < x < 1$.

3. Extension of range of validity of identities to $(-1, 1)$. Having defined identities uniformly, convergent for $0 < |x| < 1$, we now investigate whether we can extend our results to include the points $x = 0, \pm 1$.

COROLLARY 6. *Equations* (19), (20), (25), (26) *and* (28)–(31) cannot *be extended to include the points* $x = \pm 1$ *unless* $m = 0$, *in which case they reduce to*

$$(32)\quad \begin{aligned} P_l(0) \sum_{\substack{n=0 \\ n \neq l}}^{\infty} (-1)^n P_n'(0)\{1/(l-n) - 1/(l+n+1)\} &= \tfrac{1}{2}\{1 + (-1)^l\}, \\ P_l'(0) \sum_{\substack{n=0 \\ n \neq l}}^{\infty} (-1)^n P_n(0)\{1/(l-n) - 1/(l+n+1)\} &= \tfrac{1}{2}\{1 - (-1)^l\}, \\ P_l(0) \sum_{\substack{n=0 \\ n \neq l}}^{\infty} P_n'(0)\{1/(l-n) - 1/(l+n+1)\} &= -\tfrac{1}{2}\{1 + (-1)^l\}, \\ P_l'(0) \sum_{\substack{n=0 \\ n \neq l}}^{\infty} P_n(0)\{1/(l-n) - 1/(l+n+1)\} &= \tfrac{1}{2}\{1 - (-1)^l\}. \end{aligned}$$

Proof. From the behavior of Legendre associated functions near the singular points [1, 3.9.2], we deduce for $m > 0$,

$$P_n^m(x) \to 0 \quad \text{as } x \to 1,$$

and our identities become trivial at $x = 1$, while

$$\begin{aligned} (-1)^m \gamma_{n,m} P_n^m(x) &= P_n^{-m}(x) \quad \text{by (5)} \\ &\to \infty \quad \text{as } x \to -1, \end{aligned}$$

and we cannot extend our identities to the point $x = -1$ if $m > 0$.

If $m = 0$, we use

$$P_n(1) = 1, \qquad P_n(-1) = (-1)^n$$

to obtain (32); the validity of substituting $x = \pm 1$, in this case, comes from the fact that identities (32) can be derived starting from (13) using the methods of Theorem 3.

COROLLARY 7. *The sinisters of relations* (19), (25), (28) *and* (30) *are identically zero, if we set* $x = 0$.

Proof. From (15) and (16), we immediately obtain the identity

$$(33)\quad P_n^m(0) P_n'^m(0) \equiv 0.$$

This completes the proof.

Since by Corollary 7 the point $x = 0$ appears to be a point of discontinuity of our identities, at this stage we recall the following theorem from the theory of Laplace series for Legendre associated functions, which we label as Theorem 8.

THEOREM 8. *The Laplace series*

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \int_{-\pi}^{\pi} d\phi_2 \int_0^{\pi} d\theta_2 \sin\theta_2 f(\theta_2, \phi_2) P_n(\cos\gamma)$$

(*where* $\cos\gamma = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\{\phi_1 - \phi_2\}$) *in which* $f(\theta_2, \phi_2)$ *has an absolutely convergent integral (Lebesgue) over the spherical surface, will converge at* (θ_1, ϕ_1) *to the value* $f(\theta_1, \phi_1)$, *if* (θ_1, ϕ_1) *is a point of continuity of the function with respect to* (θ_1, ϕ_1), *or to the value*

$$\tfrac{1}{2}\{f_1(\theta_1, \phi_1) + f_2(\theta_1, \phi_1)\},$$

if the point (θ_1, ϕ_1) *is such that there passes through it a line of discontinuity such that* $f_1(\theta_1, \phi_1)$ *and* $f_2(\theta_1, \phi_1)$ *are the limits of the function at the point taken from the two sides of the line, provided that the function* $\psi(\gamma)$, *which is the mean value of the function* $f(\theta_1, \phi_1)$, *for each fixed value of* ψ *over the small circle for which* ψ *has that value, has bounded variation in the whole interval* $(0, \pi)$ *of* γ.

Proof. See, for example, [2].

We are now in a position to combine all our above results for 2 identities valid over the entire range $(-1, 1)$. We do this in the following two theorems.

THEOREM 9. *The series*

$$P_l'^m(0) \sum_{\substack{n=m \\ n\neq l}}^{\infty} \gamma_{n,m} P_n^m(x) P_n^m(0) \{1/(l-n) - 1/(l+n+1)\}$$

is uniformly convergent on the interval $(-1, 1)$ *to*

$$\begin{cases} \tfrac{1}{2}\{1-(-1)^{l+m}\}P_l^m(x), & x > 0, \\ 0, & x = 0, \\ -\tfrac{1}{2}\{1-(-1)^{l+m}\}P_l^m(x), & x < 0. \end{cases}$$

Proof. Define $f(\theta_1, \phi_1)$ by

$$f(\theta_1, \phi_1) = \begin{cases} P_\lambda^m(\cos\theta_1)\cos(m\phi_1), \\ \qquad 0 \leqq \theta_1 \leqq \pi/2, \quad 0 \leqq \phi_1 \leqq 2\pi, \\ 0, \quad \pi/2 < \theta_1 \leqq \pi, \quad 0 \leqq \phi_1 \leqq 2\pi. \end{cases} \tag{34}$$

Choose $\theta_1 = 0$, the line of discontinuity of $f(\theta_1, \phi_1)$. Theorem 8 then gives

$$\tfrac{1}{2}\{0 + P_\lambda^m(0)\cos(m\phi_1)\} = \sum_{n=0}^{\infty} \gamma_{n,m}\tfrac{1}{2}(2n+1) \cdot \left\{\int_0^1 dx\, P_\lambda^m(x) P_n^m(x)\right\} P_n^m(0)\cos(m\phi_1),$$

whence we obtain

$$\begin{aligned} P_\lambda^m(0) &= \gamma_{l,m} P_l^m(0) P_\lambda'^m(0)\{1/(\lambda - l) - 1/(\lambda + l + 1)\} P_l^m(0) \\ &\quad + \sum_{\substack{n=m \\ n\neq l}}^{\infty} \gamma_{n,m} P_n^m(0) P_\lambda'^m(0) \{1/(\lambda - n) - 1/(\lambda + n + 1)\} P_n^m(0). \end{aligned} \tag{35}$$

We have used (6) as well as [1, 3.12(1)], namely,

$$
\begin{aligned}
&(\nu - \lambda)(\nu + \lambda + 1)\int_a^b dz\, M_\nu^m(z) M_\lambda^m(z) \\
(36)\qquad &= [z(\nu - \lambda)M_\nu^m(z)M_\lambda^m(z) + (\lambda + m)M_\nu^m(z)M_{\lambda-1}^m(z) \\
&\quad - (\nu + m)M_{\nu+1}^m(z)M_\lambda^m(z)]_a^b.
\end{aligned}
$$

($M_\nu^m(z)$ and $M_\lambda^m(z)$ are any two solutions of Legendre's equation.)

Taking $\lim_{\lambda \to l \in N}$ of (35) we obtain

$$
\begin{aligned}
&P_l'^m(0) \sum_{\substack{n=m \\ n \neq l}}^{\infty} \gamma_{n,m} P_n^m(0) P_n^m(0)\{1/(l - n) - 1/(l + n + 1)\} \\
(37)\qquad &= \tfrac{1}{2}\{1 - (-1)^{l+m}\} P_l^m(0) \quad \text{(by (24))} \\
&= 0 \quad \text{(by (15))}.
\end{aligned}
$$

Define $S_{lm}(x)$ and $F(x)$ by

$$S_{lm}(x) \equiv P_l'^m(0) \sum_{n=m}^{\infty} \gamma_{n,m} P_n^m(x) P_n^m(0)\{1/(l - n) - 1/(l + n + 1)\},$$

and

$$F(x) \equiv \begin{cases} \tfrac{1}{2}\{1 - (-1)^{l+m}\} P_l^m(x), & x > 0, \\ 0, & x = 0, \\ -\tfrac{1}{2}\{1 - (-1)^{l+m}\} P_l^m(x), & x < 0. \end{cases}$$

By Corollaries 4 and 5, $S_{lm}(x)$ is uniformly convergent to $F(x)$ on (0, 1) and (−1, 0), respectively. From (37) we see that $S_{lm}(0) = F(0)$. Further from (15) it is clear that

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^-} F(x) = F(0) = 0.$$

Hence $S_{lm}(x)$ is uniformly convergent to $F(x)$ on (−1, 1).

THEOREM 10. *The series*

$$P_l^m(0) \sum_{\substack{n=m \\ n \neq l}}^{\infty} \gamma_{n,m} P_n^m(x) P_n'^m(0)\{1/(l - n) - 1/(l + n + 1)\}$$

converges on the interval (−1, 1) *to*

$$\begin{cases} -\tfrac{1}{2}\{1 + (-1)^{l+m}\} P_l^m(x), & x > 0, \\ 0, & x = 0, \\ \tfrac{1}{2}\{1 + (-1)^{l+m}\} P_l^m(x), & x < 0. \end{cases}$$

The convergence is uniform on any interval which excludes the origin.

Proof. Define $T_{lm}(x)$ and $G(x)$ by

$$T_{lm}(x) \equiv P_l^m(0) \sum_{\substack{n=m \\ n \neq l}}^{\infty} \gamma_{n,m} P_n^m(x) P_n'^m(0) \{1/(l-n) - 1/(l+n+1)\}$$

and

$$G(x) \equiv \begin{cases} -\frac{1}{2}\{1 + (-1)^{l+m}\} P_l^m(x), & x > 0, \\ 0, & x = 0, \\ \frac{1}{2}\{1 + (-1)^{l+m}\} P_l^m(x), & x < 0. \end{cases}$$

Uniform convergence of $T_{lm}(x)$ to $G(x)$ on $(-1, 0) \cup (0, 1)$ was proved in Corollaries 4 and 5. From Corollary 7 we see that

$$T_{lm}(0) = 0 = G(0),$$

since each term of T_{lm} is zero.

However, for $(l + m)$ even, using (15) we deduce the limits

$$\lim_{x \to 0^+} G(x) = -P_l^m(0) \neq 0$$

and

$$\lim_{x \to 0^-} G(x) = P_l^m(0) \neq 0$$

(for $(l + m)$ odd both sides are identically zero).

Hence no uniform convergence is possible in any neighborhood of the origin.

4. Summary of results. We can express the results of Theorems 9 and 10 in a compact form, if we define $\varepsilon(x)$ by

$$\varepsilon(x) = \begin{cases} +1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases} \tag{38}$$

Then Theorems 9 and 10 become

$$P_l^m(0) \sum_{\substack{n=m \\ n \neq l}}^{\infty} \gamma_{n,m} P_n^m(x) P_n'^m(0) \{1/(l-n) - 1/(l+n+1)\} = -\tfrac{1}{2}\{1 + (-1)^{l+m}\} P_l^m(x) \varepsilon(x), \qquad -1 < x < 1, \tag{39}$$

and

$$P_l'^m(0) \sum_{\substack{n=m \\ n \neq l}}^{\infty} \gamma_{n,m} P_n^m(x) P_n^m(0) \{1/(l-n) - 1/(l+n+1)\} = \tfrac{1}{2}\{1 - (-1)^{l+m}\} P_l^m(x) \varepsilon(x), \qquad -1 < x < 1. \tag{40}$$

A third, related, identity is found by differentiating (40) with respect to x, giving

$$P_l'^m(0)\sum_{\substack{n=m\\n\neq l}}^{\infty}\gamma_{n,m}P_n^m(0)P_n'^m(x)\{1/(l-n)-1/(l+n+1)\} \tag{41}$$
$$=\tfrac{1}{2}\{1-(-1)^{l+m}\}P_l'^m(x)\varepsilon(x), \qquad -1<x<1.$$

That this step is a valid one follows from the uniform convergence of (41) in $(-1,0)\cup(0,1)$, which is proved analogously to Lemma 1 and Corollary 5. The series vanishes at $x=0$, as can be seen from Corollary 7. Setting $x=-x$ in (39) and (40) gives

$$P_l^m(0)\sum_{\substack{n=m\\n\neq l}}^{\infty}(-1)^{n+m}\gamma_{n,m}P_n^m(x)P_n'^m(0)\{1/(l-n)-1/(l+n+1)\} \tag{42}$$
$$=\tfrac{1}{2}\{1+(-1)^{l+m}\}P_l^m(x)\varepsilon(x), \qquad -1<x<1.$$

$$P_l'^m(0)\sum_{\substack{n=m\\n\neq l}}^{\infty}(-1)^{n+m}\gamma_{n,m}P_n^m(x)P_n^m(0)\{1/(l-n)-1/(l+n+1)\} \tag{43}$$
$$=\tfrac{1}{2}\{1-(-1)^{l+m}\}P_l^m(x)\varepsilon(x), \qquad -1<x<1.$$

A similar result can be derived from (41).

In many applications we are required to evaluate $\sum_{n=m}^{\infty}$ rather than $\sum_{\substack{n=m\\n\neq l}}^{\infty}$. To do this we merely reinsert the $n=l$ term which we have evaluated, using limits (22) and (24). We obtain

$$P_l^m(0)\sum_{n=m}^{\infty}\gamma_{n,m}P_n^m(x)P_n'^m(0)\{1/(l-n)-1/(l+n+1)\} \tag{44}$$
$$=-\tfrac{1}{2}\{1+(-1)^{l+m}\}P_l^m(x)\varepsilon(x)-\tfrac{1}{2}\{1-(-1)^{l+m}\}P_l^m(x), \qquad -1<x<1,$$

$$P_l'^m(0)\sum_{n=m}^{\infty}\gamma_{n,m}P_n^m(x)P_n^m(0)\{1/(l-n)-1(l+n+1)\} \tag{45}$$
$$=\tfrac{1}{2}\{1-(-1)^{l+m}\}P_l^m(x)\varepsilon(x)+\tfrac{1}{2}\{1+(-1)^{l+m}\}P_l^m(x), \qquad -1<x<1,$$

and setting $x=-x$,

$$P_l^m(0)\sum_{n=m}^{\infty}(-1)^{n+m}\gamma_{n,m}P_n^m(x)P_n'^m(0)\{1/(l-n)-1(l+n+1)\} \tag{46}$$
$$=\tfrac{1}{2}\{1+(-1)^{l+m}\}P_l^m(x)\varepsilon(x)+\tfrac{1}{2}\{1-(-1)^{l+m}\}P_l^m(x), \qquad -1<x<1,$$

$$P_l'^m(0)\sum_{n=m}^{\infty}(-1)^{n+m}\gamma_{n,m}P_n^m(x)P_n^m(0)\{1/(l-n)-1/(l+n+1)\} \tag{47}$$
$$=\tfrac{1}{2}\{1-(-1)^{l+m}\}P_l^m(x)\varepsilon(x)+\tfrac{1}{2}\{1+(-1)^{l+m}\}P_l^m(x), \qquad -1<x<1.$$

Similar identities follow from (41).

Uniform convergence for equations (39)–(47) is the same as that of the respective parent equation from which each is derived.

5. Applications. The author has applied these identities to certain boundary value problems of mathematical physics.

Suppose we are working in an orthogonal curvilinear system of coordinates (u^1, u^2, u^3), in which Laplace's equation $\nabla^2\psi(u^1, u^2, u^3) = 0$ is separable [4]. The solution can be expressed in the form

$$\psi(u^1, u^2, u^3) = \psi_1(u^1)\psi_2(u^2)\psi_3(u^3).$$

If at least one of the functions $\psi_i(u^i)$ is a Legendre associated function $P_n^m(x_L)$, x_L being a function of u^i alone, then our identities are applicable to bodies bounded by two surfaces, one of which is given by $x_L = 0$, and the other is defined independently of x_L.

Consider the half space $z \geqq 0$. In spherical polar coordinates (where the solution to Laplace's equation takes the form $r^nP_n^m(\cos\theta)\cos\{m\phi\}$), the bounding surface $z = 0$ is given by $x_L = \cos\theta = \cos\{\frac{1}{2}\pi\} = 0$, the other surface being the hemisphere at infinity; the identities have been applied successfully, thereby providing a solution which could not otherwise be obtained formally.

Another example which has been dealt with is the interior of the prolate hemispheroid, where the solution to Laplace's equation is given in prolate spheroidal coordinates (ξ, η, ϕ) in the form

$$\sum_{n,m=0}^{\infty} B_{nm}P_n^m(\eta)Q_n^m(\xi)\cos\{m\phi\}, \qquad B_{nm} \text{ constant}.$$

Here the plane surface of the hemispheroid is given by $x_L = \eta = 0$; the curved surface is defined by $\xi = \alpha > 1$. The explicit verification of an identity obeyed by the Dirichlet Green's function of the interior of the prolate hemispheroid has been accomplished using these new identities [5].

These applications may also be regarded as alternative proofs (of special cases) of Theorem 3.

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NONLINEAR PERTURBATIONS OF THE ORR–SOMMERFELD EQUATION—ASYMPTOTIC EXPANSION OF THE LOGARITHMIC PHASE SHIFT ACROSS THE CRITICAL LAYER*

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Abstract. The equation governing the connection formulas across the critical layer is solved in the case in which the viscous effects asymptotically dominate the nonlinear effects in the critical layer. The dependence of the logarithmic phase shift across the critical layer on the amplitude of a wave disturbance to a parallel flow is calculated to the first order that includes nonlinear effects. The resulting asymptotic expansion of the phase shift agrees with a previous numerical calculation even when the viscous effects are only mildly more important than the nonlinear effects.

1. Introduction. Linearized disturbances to a parallel flow $\bar{u}(y)$ satisfy the Orr–Sommerfeld equation. For an inviscid fluid, the order of the differential equation is reduced. For large Reynolds number (corresponding to a nearly inviscid fluid), the resulting singular perturbation problem is more difficult than usual, since the inviscid equation (called the Rayleigh equation) has a singular point for neutrally stable waves at any critical point where the phase velocity of the wave c equals the mean parallel flow $\bar{u}(y_c) = c$. For an excellent account of the solution of the Rayleigh and Orr–Sommerfeld equations the reader is referred to Lin [11], [12]. More recently the method of matched asymptotic expansions has been utilized to obtain equivalent results (Eagles [6] and Reid [16]). One well-known result is that the inviscid solution, obtained by the method of Frobenius, contains a logarithmic singularity $\log(y - y_c)$ at the critical point. In order to calculate the neutral stability curve, connection formulas relating the solution above the critical point to the solution below must be determined. By reintroducing the viscous terms in a small region near the critical point (called the critical layer), Lin [11], [12] and others have shown that terms with $\log(y - y_c)$ should be analyzed as $\log|y - y_c| + i\phi$ below the critical layer, where $\phi = -\pi$. ϕ is called the logarithmic phase shift.

Benney and Bergeron [2] and Davis [5] independently observed that the Rayleigh equation is the result of two limiting processes not one. The Reynolds number R is large, but the fluid dynamical equations (Navier–Stokes) are linearized and hence the amplitude of the disturbance ε is small. They suggested that the inviscid singularity could be resolved by including the nonlinear terms near the critical point rather than including the viscous terms. One result of this nonlinear critical layer theory was that the logarithmic phase shift vanished in contrast to the viscous theory in which $\phi = -\pi$.

These two theories were connected by Haberman [9], who extended Benney and Bergeron's [2] analysis to allow for the dynamical balance in the critical layer between both the viscous and the nonlinear terms. It was shown that the

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parameter of importance was λ, the inverse of the local vertical Reynolds number,

$$\lambda = \frac{|u_c'|^{1/2}}{\alpha R(\varepsilon B)^{3/2}}, \tag{1.1}$$

where α is the wave number of the neutrally stable finite amplitude wave, $u_c' = du/dy$ evaluated at $y = y_c$, and εB is the amplitude of the stream function of the disturbance evaluated at $y = y_c$ using the Frobenius solution valid away from the critical layer. The logarithmic phase shift depends only on λ, its functional relationship being determined by a numerical solution of the asymptotic problem formulated in § 2. In order for the viscous theory to be a valid approximation, the amplitude of the perturbation cannot be too large; in particular, the condition $\lambda \gg 1$ must be satisfied, in which case $\phi \to -\pi$. As $\lambda \to 0$, the theory of the nonlinear critical layer applies and $\phi \to 0$. Analytic results were obtained for the logarithmic phase shift in the case in which λ is small.

In this paper, the effect of nonlinear perturbations are considered. For λ large, that is, for values of λ such that the viscous critical layer theory is nearly valid, the logarithmic phase shift across the critical layer is shown to be given by the following asymptotic formula:

$$\phi = -\pi + \frac{\pi}{2}\left(\frac{3}{2}\right)^{1/3}\Gamma\left(\frac{4}{3}\right)\lambda^{-4/3} + O(\lambda^{-8/3}), \tag{1.2}$$

where $\Gamma(x)$ is the Gamma function. It is shown that this formula agrees with numerically obtained values even when λ is not very large. By methods described in [9], formula (1.2) can be used to determine an analytic formula for the amplitude dependence of the asymptotic behavior of the upper branch of the neutral stability curve for nearly linear, long wave perturbations.

The connection formulas relating quantities above and below the critical layer are determined by the solution of a partial differential equation with subsidiary conditions provided by the method of matched asymptotic expansions. In § 2 a perturbation expansion in the case of λ being large is introduced to solve this problem. The first few ordered terms are most simply represented as quasi-Fourier integrals as derived in Appendix A. The logarithmic phase shift across the critical layer is then determined by using formulas derived in Appendix B for the asymptotic expansions of the necessary types of integrals.

2. Formulation and asymptotic solution for λ large. The logarithmic phase shift across the critical layer,

$$\phi \equiv C_+ - C_-, \tag{2.1}$$

need not always equal $-\pi$. Haberman [9] showed that it is determined by solving the following linear partial differential equation valid in the critical layer:

$$Y\Psi_{xYY} + \sin x\Psi_{YYY} = \lambda\Psi_{YYYY}, \tag{2.2}$$

with asymptotic conditions provided by the method of matched asymptotic expansions

$$\Psi = \frac{Y^3}{6} + Y \log|Y| \cos x + O\left(\frac{1}{Y}\right) \quad \text{as } Y \to +\infty, \tag{2.3a}$$

$$\begin{aligned} \Psi = \frac{Y^3}{6} &+ (H_- - H_+)Y^2 + Y \log|Y| \cos x + (A_- - A_+) Y \cos x \\ &+ (C_- - C_+) Y \sin x \\ &+ (B^*_{3_-/2} - B^*_{3_+/2}) \cos x + (D^*_{3_-/2} - D^*_{3_+/2}) \sin x + O\left(\frac{1}{Y}\right) \quad \text{as } Y \to -\infty. \end{aligned} \tag{2.3b}$$

Y is the stretched vertical coordinate in the critical layer based on the scaling developed for the nonlinear critical layer theory by Benney and Bergeron [2],

$$y - y_c = \operatorname{sgn}(u_c') \left(\frac{\varepsilon B}{|u_c'|}\right)^{1/2} Y. \tag{2.4}$$

The difference between + and − expressions represent unknown jump or connection formulas. Equations (2.2) and (2.3) are derived by considering weakly nonlinear perturbations to the Navier–Stokes equations. The diffusive term $\lambda \Psi_{YYYY}$ represents the linear viscous effect in the critical layer, where Ψ is a higher order stream function which incorporates part of the mean flow. The terms $Y\Psi_{xYY}$ and $\sin x \Psi_{YYY}$ are also present in the linear theory, but in that theory the term $\sin x \Psi_{YYY}$ is simplified as Ψ_{YYY} is only due to the parallel flow contribution to Ψ, i.e., $\Psi = Y^3/6$. These terms represent the interaction between the known parallel flow and the neutrally stable periodic wave perturbation. In the nonlinear theory, the term Ψ_{YYY} arises not only from $Y^3/6$, but also from the rest of the terms in (2.3). It is due to the additional weak nonlinear interaction between the linearized wave and itself, no longer ignored as in the linearized procedure. The notation is that used by Haberman [9], to which the reader is referred for details of this nonlinear stability theory.

For a solution to this problem, the following relationships among the jumps were shown to exist for all λ [9]:

$$\begin{aligned} &H_- - H_+ = (C_- - C_+)/(4\lambda), \\ &B^*_{3_-/2} - B^*_{3_+/2} = -2(H_- - H_+) = -(C_- - C_+)/(2\lambda), \\ &D^*_{3_-/2} - D^*_{3_+/2} = 0. \end{aligned} \tag{2.5}$$

Thus the only independent jump conditions remaining unknown are $A_+ - A_-$ and $C_+ - C_-$. They are functions of λ and must be determined such that (2.2) is satisfied along with matching conditions (2.3).

In this paper, solutions are sought to this problem for λ large. In the viscous critical layer theory, $\lambda \to \infty$, the viscous term $\lambda \Psi_{YYYY}$ is of the same order of magnitude as the inertial term $Y\Psi_{xYY}$. For large λ this suggests the scaling

$$Y = \lambda^{1/3} \eta, \tag{2.6}$$

in which case (2.2) becomes

$$\eta\Psi_{x\eta\eta}-\Psi_{\eta\eta\eta\eta}=-\lambda^{-2/3}\sin x\Psi_{\eta\eta\eta}. \tag{2.7}$$

Equation (2.7) expresses the fact that for large λ the term representing nonlinear wave interactions, $\sin x\Psi_{\eta\eta\eta}$, is small in an asymptotic sense. The thickness of this viscous critical layer is determined by letting $\eta=O(1)$ and yields the well-known result

$$y-y_c=O\{(\alpha u_c'R)^{-1/3}\}. \tag{2.8}$$

We thus will calculate the weakly nonlinear corrections to the viscous theory.

Equation (2.7) can also be derived from the Navier–Stokes equations by directly considering the critical layer dynamics based on (2.8) with $\lambda \gg 1$ and the method of matched asymptotic expansions. However, the derivation is lengthy and the result is mathematically equivalent to the viscous limit ($\lambda \gg 1$) of (2.2) and (2.3).

It is concluded that the expansions for large λ of the jump conditions across the critical layer are

$$\begin{aligned}-\phi=C_--C_+&=C_0+\lambda^{-2/3}C_1+\lambda^{-4/3}C_2+\lambda^{-6/3}C_3+\cdots,\\ A_--A_+&=A_0+\lambda^{-2/3}A_1+\lambda^{-4/3}A_2+\lambda^{-6/3}A_3+\cdots,\end{aligned} \tag{2.9}$$

since these quantities are functions of λ and since $\lambda^{-2/3}$ is the small parameter of (2.7) (furthermore the asymptotic conditions also suggest an expansion in powers of $\lambda^{-2/3}$). From the viscous theory, $\lambda\to\infty$, it is well known that $C_0=\pi$ and $A_0=0$. Expansions (2.9) are based on this fact, but in this paper these values of the leading order jump conditions will be rederived; in particular, the nonlinear corrections to these values are desired.

The scaling (2.6) when applied to (2.2) and (2.3) imply that Ψ has the asymptotic expansion

$$\begin{aligned}\Psi\sim\lambda\frac{\eta^3}{6}+\frac{1}{3}(\lambda\log\lambda)\eta\cos x+\lambda^{1/3}&\\ \cdot(\Psi_1+\lambda^{-2/3}\Psi_2+\lambda^{-4/3}\Psi_3+\lambda^{-6/3}\Psi_4+\cdots)&\quad\text{as }\lambda\to\infty.\end{aligned} \tag{2.10}$$

The asymptotic behavior as $\eta\to\pm\infty$ for each Ψ_i is as follows:

$$\Psi_1=\begin{cases}\eta\log|\eta|\cos x+O(1/\eta) & \text{as }\eta\to+\infty,\\ \eta\log|\eta|\cos x+C_0\eta\sin x+A_0\eta\cos x+O(1/\eta) & \text{as }\eta\to-\infty,\end{cases}$$

$$\Psi_2=\begin{cases}O(1/\eta) & \text{as }\eta\to+\infty,\\ C_0\eta^2/4+A_1\eta\cos x+C_1\eta\sin x+O(1/\eta) & \text{as }\eta\to-\infty,\end{cases} \tag{2.11}$$

$$(n\geqq 3)\quad \Psi_n=\begin{cases}O(1/\eta) & \text{as }\eta\to+\infty,\\ C_{n-2}\eta^2/4+A_{n-1}\eta\cos x+C_{n-1}\eta\sin x-C_{n-3}(\cos x)/2+O(1/\eta) & \text{as }\eta\to-\infty.\end{cases}$$

The equations for Ψ_n are

$$\eta\Psi_{1_{x\eta\eta}}-\Psi_{1_{\eta\eta\eta\eta}}=-\sin x, \tag{2.12a}$$

$$\eta\Psi_{n_{x\eta\eta}}-\Psi_{n_{\eta\eta\eta\eta}}=-\sin x\Psi_{n-1_{\eta\eta\eta}}\qquad(n\geqq 2). \tag{2.12b}$$

It is easily shown that all solutions which are periodic in x of the related homogeneous equation,

$$\eta \Psi_{x\eta\eta} - \Psi_{\eta\eta\eta\eta} = 0,$$

are exponentially growing either as $\eta \to +\infty$ or $\eta \to -\infty$ (except for certain cubic polynomial solutions in η). Therefore it follows that solutions to the nonhomogeneous equation (if they exist) are unique. Clearly from (2.12a) and (2.11),

$$\Psi_1 = \alpha_1(\eta) \cos x + \beta_1(\eta) \sin x.$$

Since the equation for Ψ_2 is forced by $-\sin x \Psi_{1_{\eta\eta\eta}}$, Ψ_2 will consist of only a mean term and a second harmonic,

$$\Psi_2 = \alpha_0(\eta) + \alpha_2(\eta) \cos 2x + \beta_2(\eta) \sin 2x,$$

in other words, only even harmonics. By continuing this argument it is seen that Ψ_n for n odd will contain only odd harmonics, while Ψ_n for n even will contain only even harmonics. Consequently, without an explicit solution, it is concluded from the asymptotic behavior of Ψ_n as $\eta \to \infty$ that $A_n = C_n = 0$ for n odd. Thus although the expansion of Ψ involves powers of $\lambda^{-2/3}$, the jump conditions across the critical layer are expanded in powers of $\lambda^{-4/3}$,

$$(2.13) \qquad \begin{aligned} -\phi = C_- - C_+ &= C_0 + \lambda^{-4/3} C_2 + \lambda^{-8/3} C_4 + \cdots, \\ A_- - A_+ &= A_0 + \lambda^{-4/3} A_2 + \lambda^{-8/3} A_4 + \cdots. \end{aligned}$$

To determine the jump conditions, Ψ_1 will be first calculated. Although Lommel functions can be used as the representation of the solution of (2.12a), they are not the most convenient. Instead, an integral representation of the solution exists:

$$(2.14) \qquad \Psi_{1\eta\eta} = -\mathrm{Im}\left(e^{ix} \int_0^\infty e^{-i\eta t} e^{-t^3/3}\, dt\right),$$

as derived in Appendix A. An integration of (2.14) yields

$$(2.15) \qquad \Psi_{1_\eta} = -\mathrm{Im}\left(e^{ix} \int_0^\infty \frac{e^{-i\eta t} - 1}{-it} e^{-t^3/3}\, dt\right) + f_1(x),$$

where $f_1(x)$ is the "constant" of integration equal to Ψ_{1_η} at $\eta = 0$. The asymptotic expansion for large η of this integral will be sufficient to determine A_0 and C_0 and will verify that $A_0 = 0$ and $C_0 = \pi$. Appendix B developes the asymptotic expansion of integrals of the kind in (2.15), which are related to Fourier type integrals. Applying these results yields

$$(2.16) \qquad \psi_{1_\eta} = \cos x(\log|\eta| + J) - \frac{\pi}{2}(\mathrm{sgn}\ \eta) \sin x + f_1(x) + O\left(\frac{1}{\eta}\right)$$

as $\eta \to \pm\infty$, where

$$\mathrm{sgn}\ \eta = \begin{cases} 1 & \eta > 0, \\ -1 & \eta < 0, \end{cases}$$

and where

$$J=\int_0^\infty \frac{e^{-t^3/3}-\cos t}{t}\,dt=\frac{1}{3}(\log 3+2\gamma)$$

(γ being Euler's constant).[1] In order to satisfy the asymptotic conditions given by (2.11), both as $\eta\to+\infty$ and as $\eta\to-\infty$, it follows that

$$f_1(x)+J\cos x-\frac{\pi}{2}\sin x=0,$$

$$f_1(x)+J\cos x+\frac{\pi}{2}\sin x=C_0\sin x+A_0\cos x.$$

Consequently, by eliminating $f_1(x)$, the well-known viscous critical layer results are rederived, $C_0=\pi$ and $A_0=0$.

The perturbation expansion (2.10) must be calculated at least through the Ψ_3 term in order to determine the nonlinear corrections to this essentially viscous result. Using (2.14), the equation for Ψ_2 becomes

$$\eta\Psi_{2_{x\eta\eta}}-\Psi_{2_{\eta\eta\eta\eta}}=\mathrm{Im}\left(\frac{1-e^{2ix}}{2}\int_0^\infty t\,e^{-i\eta t}e^{-t^3/3}\,dt\right). \tag{2.17}$$

An integral representation of the solution to (2.17) is obtained by again using the results of Appendix A, namely,

$$\Psi_{2_{\eta\eta}}=-\frac{1}{2}\int_0^\infty \frac{\sin\eta t}{t}e^{-t^3/3}\,dt-\mathrm{Im}\left(\frac{e^{2ix}}{4}\int_0^\infty e^{-i\eta t}e^{-t^3/6}\int_0^t \tau e^{-\tau^3/6}\,d\tau\,dt\right)+g_2, \tag{2.18}$$

where integration by parts has been used and where g_2 is an arbitrary constant corresponding to the cubic polynomial in η solution. It can be verified that g_2 can be determined in order to satisfy the asymptotic conditions on Ψ_2.

The solution of the linear partial differential equation for Ψ_3 (following from (2.12b)) has an integral representation obtained by again using the results of Appendix A:

$$\begin{aligned}\Psi_{3_{\eta\eta}}=\mathrm{Im}\Bigg[&e^{ix}\left(\frac{1}{4}\int_0^\infty e^{-i\eta t}t\,e^{-t^3/3}\,dt\right.\\&\left.+\frac{1}{8}\int_0^\infty e^{-i\eta t}e^{-t^3/3}\int_0^t T e^{T^3/6}\int_0^T \tau e^{-\tau^3/6}\,d\tau\,dT\,dt\right)\\&+e^{-ix}\left(\frac{1}{4}\int_{-\infty}^0 e^{-i\eta T}e^{T^3/3}\,dT\int_\infty^0 e^{-2t^3/3}\,dt\right.\\&\left.+\frac{1}{4}\int_0^\infty e^{-i\eta t}e^{t^3/3}\int_\infty^t e^{-2T^3/3}\,dT\,dt\right)\\&-\frac{e^{3ix}}{24}\int_0^\infty e^{-i\eta t}e^{-t^3/9}\int_0^t T e^{-T^3/18}\int_0^T \tau e^{-\tau^3/6}\,d\tau\,dT\,dt\Bigg].\end{aligned} \tag{2.19}$$

[1] Equation (2.16) can also be obtained directly from the asymptotics of integrals of Lommel functions (see Luke [13]).

The arbitrary constant from the quadratic mean term of Ψ_3 is ignored since $\Psi_{3_{\eta\eta}} \to 0$ as $\eta \to \pm\infty$. The third harmonic term, e^{3ix}, is not necessary in order to determine the jump relationships at the order of magnitude of interest. Integrating with respect to η introduces an arbitrary function of x, $f_3(x)$, and transforms terms of the form

$$e^{-i\eta w}$$

in (2.19) into

$$\frac{e^{-i\eta w}-1}{-iw}.$$

Using the asymptotic expansion developed in Appendix B (equation (B.7)) yields

$$\begin{aligned}\Psi_{3_\eta} = \operatorname{Im}\Bigg[\frac{e^{ix}}{8i}\int_0^\infty \frac{e^{-t^3/3}\int_0^t T e^{T^3/6}\int_0^T \tau e^{-\tau^3/6}\,d\tau\,dT}{t}\,dt \\ + e^{-ix}\bigg(\frac{\operatorname{sgn}\eta}{2}\int_0^\infty e^{-2T^3/3}\,dT\int_0^\infty \frac{\sin t}{t}\,dt \\ + \frac{1}{4i}\int_0^\infty \frac{e^{-t^3/3}\int_0^\infty e^{-2T^3/3}\,dT + e^{t^3/3}\int_\infty^t e^{-2T^3/3}\,dT}{t}\,dt\bigg)\Bigg] \\ + f_3(x) + O\left(\frac{1}{\eta}\right) \quad \text{as } \eta \to \pm\infty.\end{aligned} \tag{2.20}$$

The terms involving sgn η again clearly indicate differences in the asymptotic expansions as $\eta \to \pm\infty$. Comparing this asymptotic behavior of Ψ_{3_η} as $\eta \to \pm\infty$ with (2.11) implies that

$$A_2 \cos x + C_2 \sin x = \operatorname{Im}\left[e^{-ix}\int_0^\infty e^{-2T^3/3}\,dT\int_0^\infty \frac{\sin t}{t}\,dt\right], \tag{2.21}$$

and consequently,

$$\begin{aligned} A_2 &= 0, \\ C_2 &= -\int_0^\infty e^{-2T^3/3}\,dT\int_0^\infty \frac{\sin t}{t}\,dt. \end{aligned} \tag{2.22}$$

Explicitly evaluating both integrals yields

$$C_2 = -\frac{\pi}{2}\left(\frac{3}{2}\right)^{1/3}\Gamma(4/3). \tag{2.23}$$

Thus it is concluded that

$$-\phi = C_- - C_+ = \pi - \frac{\pi}{2}\left(\frac{3}{2}\right)^{1/3}\Gamma(4/3)\lambda^{-4/3} + O(\lambda^{-8/3}) \tag{2.24}$$

and

$$A_- - A_+ = O(\lambda^{-8/3}). \tag{2.25}$$

Based on numerical calculations, it has been hypothesized (Haberman [9]) that $A_- = A_+$ for all λ. This has not yet been proved. In this paper it has been shown that it is $O(\lambda^{-8/3})$ for large λ. Presumably further calculations could show that $A_- - A_+ = O(\lambda^{-n})$ for arbitrarily large n.

Values of the logarithmic phase shift were computed based on numerical solutions of (2.2) and (2.3) and extrapolated as a smooth curve (Haberman [8], [9]). The calculation of the phase shift by (2.24) compares quite well with the computer values even for λ not very large as illustrated in Table 1.

TABLE 1
Logarithmic phase shift ϕ

λ	ϕ	
	from (2.24)	computer [8]
10	−3.067	−3.07
5	−2.954	−2.96
2	−2.504	−2.63
1.5	−2.206	−2.45
1	−1.536	−2.12

Appendix A. An integral representation of a certain class of nonhomogeneous problems. Equations (2.12) have the form of the following nonhomogeneous linear partial differential equation:

$$\eta\Psi_{x\eta\eta} - \Psi_{\eta\eta\eta\eta} = h(x, \eta). \tag{A.1}$$

In this appendix an integral representation of the solution will be obtained which is convenient for determining its asymptotic behavior for large η. The periodicity requirement in x suggests Fourier series techniques in x. The boundedness as $\eta \to \pm\infty$ of the x-dependent part of $\Psi_{\eta\eta}$ suggests Fourier transform techniques in η for the x-dependent Fourier coefficients.

To indicate the simplicity of the solution, first assume

$$h(x, \eta) = e^{i\alpha x} e^{-i\eta t}, \tag{A.2}$$

where t is a parameter (the Fourier transform in η variable). Thus by letting

$$\Psi_{\eta\eta} = e^{i\alpha x} S(\eta; t), \tag{A.3}$$

it follows that

$$i\alpha\eta S - S_{\eta\eta} = e^{-it\eta}. \tag{A.4}$$

Equation (A.4) must be solved for all values of t. Using a linear operator notation

$$L(S) = e^{-it\eta}, \tag{A.5}$$

where

$$L \equiv -\frac{\partial^2}{\partial\eta^2} + i\alpha\eta, \tag{A.6}$$

it is seen that

$$L(t^2S - \alpha S_t - e^{-i\eta t}) = 0. \tag{A.7}$$

Since there are no nontrivial bounded solutions of the homogeneous equation $L(S) = 0$, then S must solve the first order differential equation in t:

$$\alpha S_t - t^2 S = -e^{-i\eta t}. \tag{A.8}$$

Consequently

$$S(\eta; t) = -\frac{e^{t^3/3\alpha}}{\alpha} \int_0^t e^{-i\eta\tau} e^{-\tau^3/3\alpha}\, d\tau + S(\eta; 0)\, e^{t^3/3\alpha}. \tag{A.9}$$

However, the solution corresponding to $t = 0$, $S(\eta; 0)$, can be obtained by solving

$$i\alpha\eta S - S_{\eta\eta} = 1. \tag{A.10}$$

Furthermore, the solution to this equation is directly needed in § 2. (A.10) can be related to a nonhomogeneous Airy equation which has been frequently studied. Solutions are called for example Lommel functions (Nayfeh [14]), related Airy functions (Abramowitz and Stegun [1]), or Scorer's function (Olver [15]). Nonhomogeneous solutions can be obtained by variation of parameters (Watson [18], Lin [11], [12], Holstein [10]), but an integral representation will be advantageous. An integral representation is known for a similar equation (Nayfeh [14], Tumarkin [17], Olver [15]) and can be applied to (A.10) yielding

$$S(\eta; 0) = \frac{1}{|\alpha|} \int_0^\infty e^{-i\eta\tau \operatorname{sgn} \alpha}\, e^{-\tau^3/3|\alpha|}\, d\tau, \tag{A.11}$$

as can be verified by substitution. Consequently, from (A.9),

$$S(\eta; t) = \begin{cases} \dfrac{e^{t^3/3\alpha}}{\alpha} \displaystyle\int_t^\infty e^{-i\eta\tau} e^{-\tau^3/3\alpha}\, d\tau, & \alpha > 0, \\ -\dfrac{e^{t^3/3\alpha}}{\alpha} \displaystyle\int_{-\infty}^t e^{-i\eta\tau} e^{-\tau^3/3\alpha}\, d\tau, & \alpha < 0. \end{cases} \tag{A.12}$$

Formula (A.12) is used frequently in § 2. It is convenient for obtaining asymptotic expansions as $\eta \to \pm\infty$. An alternate form of this result is

$$S\Big((\operatorname{sgn} \alpha)\eta; (\operatorname{sgn} \alpha)t\Big) = \frac{e^{t^3/3\alpha}}{|\alpha|} \int_t^\infty e^{-i\eta\tau}\, e^{-\tau^3/3|\alpha|}\, d\tau.$$

In summary, (A.1) has a simple integral representation since the $h(x, t)$ of interest in § 2 have a Fourier series in x which can be Fourier transformed in η. The solution to

$$\eta \Psi_{x\eta\eta} - \Psi_{\eta\eta\eta\eta} = \sum_{\alpha=-\infty}^{\infty} e^{i\alpha x} \int_{-\infty}^{\infty} f(\alpha, t)\, e^{-it\eta}\, dt$$

is

$$\Psi_{\eta\eta} = \sum_{\alpha=-\infty}^{\infty} e^{i\alpha x} \int_{-\infty}^{\infty} f(\alpha, t) S(\eta; t)\, dt,$$

where this formula must be modified in a straightforward manner for $\alpha = 0$. Asymptotic expansions for large η of these expressions are calculated by first integrating by parts. Solving for Ψ_η by direct integration yields quasi-Fourier integrals, such that the results of Appendix B are needed.

Appendix B. Asymptotic expansion of some quasi-Fourier integrals. Integrals of the form

$$I(\eta) = \int_0^\infty F(t)\frac{e^{-i\eta t} - 1}{t}\,dt \tag{B.1}$$

appear frequently in the analysis of § 2. In this appendix, asymptotic expansions of this type of integral will be calculated. It is assumed that (B.1) is a convergent integral, that is, $F(t)/t$ is integrable for large t. Furthermore, although it is more restrictive than necessary, $F(t)$ will be assumed regular for $0 \leqq t < \infty$ since the functions of interest in § 2 have this property. Integrals of a slightly different kind have been asymptotically evaluated by Benney and Saffman [3] and Benney and Newell [4].

Before evaluating (B.1), the well-known asymptotic result for ordinary Fourier integrals derived by repeated integration by parts is noted:

$$\int_0^\infty e^{-i\eta t}F(t)\,dt = -\sum_{n=0}^{N-1} i^{n-1}F^{(n)}(0)(-i\eta)^{-n-1} + o(\eta^{-N}) \qquad \text{as } \eta \to \pm\infty \tag{B.2}$$

(see, for example, Erdélyi [7] or Olver [15]). It is only assumed that $F(t)$ is N times continuously differentiable for $0 \leqq t < \infty$ and $F^{(n)}(t) \to 0$ as $t \to \infty$ for $n = 0, 1, \cdots, N-1$ and provided $F^{(n)}(t)$ is integrable over $0 < t < \infty$.

Integrating (indefinitely) this integral with respect to η yields

$$\int_0^\infty F(t)\frac{e^{-i\eta t} - 1}{-it}\,dt = -F(0)\log|\eta| + \sum_{n=1}^{N-1} i^n \frac{F^{(n)}(0)}{n}(-i\eta)^{-n} + d_\pm + o(\eta^{-N+1}) \qquad \text{as } \eta \to \pm\infty, \tag{B.3}$$

where $d_\pm$ are constants of integration (possibly different constants as $\eta \to +\infty$ and $\eta \to -\infty$). Thus the entire asymptotic expansion of (B.1) is determined except for the values of the two constants d_+ and d_-. In fact the matching problems discussed in § 2 depend in an important manner on the difference between d_+ and d_-.

Unfortunately the integration by parts technique used to derive (B.2) cannot be extended to (B.1). Instead $d_\pm$ are determined in the following way. First it is noted that

$$\begin{aligned} \int_0^\infty F(t)\frac{e^{-i\eta t} - 1}{t}\,dt &= \int_0^\infty e^{-i\eta t}\frac{F(t) - F(0)}{t}\,dt \\ &\quad + \int_0^\infty \frac{F(0)\cos \eta t - F(t)}{t}\,dt - iF(0)\int_0^\infty \frac{\sin \eta t}{t}\,dt. \end{aligned} \tag{B.4}$$

The Fourier integral asymptotic formula can be directly applied to the first integral on the right-hand side since

$$\frac{d^n}{dt^n}\frac{F(t) - F(0)}{t}\bigg|_{t=0} = \frac{F^{(n+1)}(0)}{n+1}.$$

It is this term which yields all the terms of (B.3) which asymptotically tend to zero as $\eta \to \pm\infty$. Thus applying the well-known integration formulas

$$\int_0^\infty \frac{\cos \eta t - \cos t}{t}\, dt = -\log |\eta|,$$

$$\int_0^\infty \frac{\sin \eta t}{t}\, dt = \frac{\pi}{2} \operatorname{sgn}(\eta),$$

yields

$$\text{(B.5)} \quad iF(0) \log |\eta| - id_\pm = iF(0) \log |\eta| - iF(0)\frac{\pi}{2} \operatorname{sgn}(\eta) + \int_0^\infty \frac{F(0)\cos t - F(t)}{t}\, dt.$$

d_+ and d_- are thus determined as

$$\text{(B.6)} \qquad -id_\pm = -i\frac{\pi}{2} F(0) \operatorname{sgn}(\eta) + \int_0^\infty \frac{F(0)\cos t - F(t)}{t}\, dt$$

or

$$-i(d_+ - d_-) = -i\pi F(0).$$

In summary, it has been shown that as $\eta \to \pm\infty$

$$\text{(B.7)} \qquad \begin{aligned} \int_0^\infty F(t) \frac{e^{-i\eta t} - 1}{-it}\, dt &= -iF(0) \log |\eta| + \frac{\pi}{2} F(0) \operatorname{sgn}(\eta) \\ &\quad + i \int_0^\infty \frac{F(0)\cos t - F(t)}{t}\, dt + O\left(\frac{1}{\eta}\right), \end{aligned}$$

where the complete asymptotic expansion (if needed), represented by the expression $O(1/\eta)$, is contained in (B.3).

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THE BESSEL POLYNOMIALS AND THE STUDENT t DISTRIBUTION*

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Abstract. The quotient

$$\frac{{}_2F_0(-n+1, n; -; -1/2\sqrt{x})}{\sqrt{x}\,{}_2F_0(-n, n+1; -; -1/2\sqrt{x})} \equiv \frac{P_{n-1}(\sqrt{x})}{P_n(\sqrt{x})}$$

arose in connection with the problem of the infinite divisibility of the Student t distribution. It is shown that $P_{n-1}(\sqrt{x})/P_n(\sqrt{x})$ is completely monotonic in $[0, \infty)$ for $n=4, 5$ and 6. This implies that the Student t distribution is infinitely divisible for 9, 11 and 13 degrees of freedom. We show that certain power sums of the zeros of the simple Bessel polynomials are zero. This is then used to show that for every $n=0, 1, 2, \cdots$, there exists a $\theta_n > 0$ such that the inverse Laplace transform of $P_{n-1}(\sqrt{x})/P_n(\sqrt{x})$ is nonnegative in $[\theta_n, \infty)$. This supports our conjecture that $P_{n-1}(\sqrt{x})/P_n(\sqrt{x})$ is completely monotonic in $(0, \infty)$ for all n, and that the Student t distribution is infinitely divisible for odd degrees of freedom.

1. Introduction. Theorems related to random variables and probability distributions are often proved by examining the Fourier transform of the distribution. The Fourier transform of the Student t distribution involves the simple Bessel polynomials. These polynomials, in Luke's [12, p. 194] notation, are defined by

$$Q_n(1, z) = z^n {}_2F_0(-n, n+1; -; -1/z), \qquad n=0, 1, \cdots. \tag{1.1}$$

Krall and Frink [9] define them as

$$y_n(z) = {}_2F_0(-n, n+1; -; -z/2), \qquad n=0, 1, \cdots. \tag{1.2}$$

We shall study the polynomials

$$P_n(z) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!} \frac{z^{n-k}}{2^k k!}, \qquad n=0, 1, \cdots, \tag{1.3}$$

which are related to $Q_n(1, z)$, $y_n(z)$, and the modified Bessel function $K_{n+(1/2)}(z)$ by

$$P_n(z) = z^n y_n\left(\frac{1}{z}\right) = 2^{-n} Q_n(1, 2z) = \sqrt{\frac{2}{\pi}} e^z z^{n+(1/2)} K_{n+(1/2)}(z).$$

Using the above notation, the problem of showing that the Student t distribution is infinitely divisible reduces to showing that the quotient $P_{n-1}(\sqrt{x})/P_n(\sqrt{x})$ is completely monotonic in $(0, \infty)$; that is, its inverse Laplace transform is nonnegative there. We conjecture that this is true for all n. Kelker [8] proved it for $n=1, 2, 3$. We show that for every $n=0, 1, 2, \cdots$, the abovementioned inverse Laplace transform is nonnegative in $[\theta_n, \infty)$ for some $\theta_n > 0$, and we give an estimate for θ_n. Therefore for a given n, the computer may be used to prove the result for $x \in [0, \theta_n]$. In particular, we prove the result for $n=4, 5, 6$.

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In the course of our investigation, we prove that certain power sums of the zeros of the simple Bessel polynomials $y_n(z)$ are zero.

We note that quotients of hypergeometric polynomials similar to $P_{n-1}(\sqrt{x})/P_n(\sqrt{x})$ have been used in Padé approximations (VanRossum [16]).

2. Notation. We shall use $(\sigma)_j$ to denote the ascending factorials

$$(\sigma)_j = \begin{cases} 0 & \text{if } j=0, \\ \sigma(\sigma+1)\cdots(\sigma+j-1) & \text{if } j=1,2,\cdots. \end{cases}$$

The hypergeometric series ${}_2F_1(a, b; c; x)$ is defined by

$${}_2F_1(a, b; c; x) = \sum_{j=0}^{\infty} \frac{(a)_j(b)_j}{(c)_j j!} x^j,$$

while ${}_2F_0(a, b; -; x)$ is defined by

$${}_2F_0(a, b; -; x) = \sum_{j=0}^{\infty} \frac{(a)_j(b)_j}{j!} x^j.$$

The only ${}_2F_1$ and ${}_2F_0$ that are used in this paper are terminating ones; that is, one of the parameters a or b is a negative integer and the defining series reduce to polynomials; hence are defined for all x.

For other definitions and notation, the reader is referred to Feller [6], Rainville [14] and Widder [18].

3. The Bessel polynomials. It is known (see Grosswald [7], Luke [12, p. 194] and Olver [13]) that the zeros of the simple Bessel polynomials, in either notation, are distinct, they all lie in the left half-plane, and there is only one real zero for odd-degree polynomials and none for even-degree polynomials. The smallest absolute value of any of the zeros of $Q_n(1, z)$ is asymptotically equal to $1.32548n$. The zeros of the first fifteen Bessel polynomials $Q_n(1, z)$ are given in Krylov and Skoblya [10, pp. 52–55].

From the orthogonality of the Bessel polynomials we get the difference equation

$$P_n(z) = (2n-1)P_{n-1}(z) + z^2 P_{n-2}(z), \qquad n = 2, 3, 4, \cdots. \tag{3.1}$$

This equation can be obtained from the recurrence relations for the Bessel functions as found in Watson [17], as can the relationship

$$\frac{d}{dz} e^{-z} P_n(z) = -z\, e^{-z} P_{n-1}(z). \tag{3.2}$$

For more properties of the Bessel polynomials, the reader is referred to Al-Salam [1], [2], Al-Salam and Carlitz [3] and Rainville [14, p. 293].

LEMMA. *Let $\alpha_{n,j}$, $j = 1, 2, \cdots, n$, be the roots of $P_n(z)$. Then the partial fraction decomposition of $P_{n-1}(z)/P_n(z)$ is*

$$\sum_{j=1}^{n} [\alpha_{n,j}(\alpha_{n,j} - z)]^{-1}.$$

Proof. Let $P_{n-1}(z)/P_n(z)=\sum_{j=1}^n A_{n,j}/(\alpha_{n,j}-z)$. Then

$$A_{n,j}=\lim_{z\to\alpha_{n,j}}(\alpha_{n,j}-z)\frac{P_{n-1}(z)}{P_n(z)}=\lim_{z\to\alpha_{n,j}}\frac{(\alpha_{n,j}-z)P_{n-1}(z)\,e^{-z}}{P_n(z)\,e^{-z}}=\frac{1}{\alpha_{n,j}}$$

by virtue of l'Hôpital's rule and (3.2).

THEOREM 1. *Let $\alpha_{n,j}$, $j=1,2,\cdots,n$, be the zeros of $P_n(z)$, (which of course are also the zeros of $K_{n+(1/2)}(z)$), with $n\geqq 2$. Then the sums*

$$S_{n,k}=\sum_{j=1}^{n}\alpha_{n,j}^{-k} \tag{3.3}$$

vanish for $k=3,5,\cdots,2n-1$.

Proof. Let

$$\frac{P_{n-1}(z)}{P_n(z)}=\sum_{j=1}^{n}A_{n,j}/(\alpha_{n,j}-z).$$

Since $P_n(0)\neq 0$, $P_{n-1}(z)/P_n(z)$ will have a power series expansion in a neighborhood of the origin. The coefficient of z^k in the Maclaurin series of $P_{n-1}(z)/P_n(z)$ will therefore be $\sum_{j=1}^n A_{n,j}\alpha_{n,j}^{-(k+1)}$. From the above lemma $A_{n,j}=\alpha_{n,j}^{-1}$, so that the coefficient of z^k will be $S_{n,k+2}$.

We now proceed, using induction on n. The relation $(1+x)^{-1}=\sum_{k=0}^\infty(-x)^k$, $|x|<1$, will be used repeatedly. For $n=2$ we have

$$\begin{aligned}\frac{P_1(z)}{P_2(z)}&=\frac{1+z}{3+3z+z^2}=\frac{1}{3}\left\{1+\frac{z^2}{3(1+z)}\right\}^{-1}\\&=\frac{1}{3}\left[1-\frac{z^2}{3}\sum_{k=0}^{\infty}(-z)^k+\frac{z^4}{9}\left\{\sum_{k=0}^{\infty}(-z)^k\right\}^2-\cdots\right].\end{aligned} \tag{3.4}$$

Since $S_{n,3}$ is the coefficient of z in the power series expansion of $P_{n-1}(z)/P_n(z)$, we conclude that $S_{2,3}=0$.

Now assume $S_{n,k}=0$ for $k=3,4,\cdots,2n-1$. Using (3.1), we get

$$\frac{P_{n-1}(z)}{P_n(z)}=\frac{1}{(2n-1)}\left\{1+\frac{z^2P_{n-2}(z)}{(2n-1)P_{n-1}(z)}\right\}^{-1}=\sum_{j=0}^{\infty}S_{n,j+2}z^j. \tag{3.5}$$

Clearly $S_{n,2}=1/(2n-1)$, and the induction hypothesis is $S_{n,j}=0$ for $j=3,5,\cdots,2n-1$ and $S_{n,2n+1}\neq 0$. Therefore

$$\frac{P_n(z)}{P_{n+1}(z)}=\frac{1}{2n+1}\left\{1+\frac{z^2P_{n-1}(z)}{(2n+1)P_n(z)}\right\}^{-1}; \tag{3.6}$$

that is,

$$\sum_{j=0}^{\infty}S_{n+1,j+2}z^j=\frac{1}{2n+1}\left\{1-\frac{z^2}{2n+1}\sum_{j=0}^{\infty}S_{n,j+2}z^j+\frac{z^4}{(2n+1)^2}\left(\sum_{j=0}^{\infty}S_{n,j+2}z^j\right)^2-\cdots\right\}. \tag{3.7}$$

Note that the first nonzero coefficient of an odd power of z in the right-hand side of (3.7) is the coefficient of z^{2n+1}. Thus $S_{n+1,j}=0$ for $j=3,5,\cdots,2n+1$. This completes the proof.

COROLLARY. *The simple Bessel polynomials $Q_n(1,z)$ with zeros $\beta_{n,j}$, $j=1,2,\cdots,n$, and the Krall and Frink Bessel polynomials $y_n(z)$ with zeros $\gamma_{n,j}$, $j=1,2,\cdots,n$, have the property that*

$$\sum_{j=1}^{n} \beta_{n,j}^{-k}=0 \quad \textit{and} \quad \sum_{j=1}^{n} \gamma_{n,j}^{k}=0 \quad \textit{for } k=3,5,\cdots,2n-1.$$

This follows since $\beta_{n,j}=2\alpha_{n,j}$ and $\gamma_{n,j}=1/\alpha_{n,j}$.

THEOREM 2. *Let $\alpha_{n,j}$ be as in Theorem 1. Then for $n\geqq 2$, we have*

$$S_{n,2n+1}=(-1)^n/\{3\cdot 5\cdots(2n-1)\}^2 \tag{3.8}$$

and

$$S_{n,2n+3}=(-1)^n/(2n-1)\{3\cdot 5\cdots(2n-1)\}^2. \tag{3.9}$$

Proof of (3.8). We shall use induction on n. For $n=2$, the coefficient of z^3 in the power series expansion of $P_1(z)/P_2(z)$ is $S_{2,5}$. Relation (3.4) implies that this coefficient is $1/9$. Thus $S_{2,5}=1/3^2$.

Now assume that (3.8) holds for some n. Using (3.6) and (3.7), we see that the coefficient of z^{2n+3} in the power series expansion of $P_n(z)/P_{n+1}(z)$ is $-S_{n,2n+1}/(2n+1)^2$; that is

$$S_{n+1,2n+3}=-S_{n,2n+1}/(2n+1)^2,$$

and (3.8) is proved.

Proof of (3.9). The proof is very similar to that of (3.8). The coefficient of z^5 in the power series expansion of $P_1(z)/P_2(z)$ is, by (3.4), $\frac{1}{3}(\frac{1}{3}-\frac{2}{9})$. Thus $S_{2,7}=1/3^3$.

Now assume the result for an $n>2$. Equating coefficients of z^{2n+3} in both sides of (3.7), we get

$$\begin{aligned} S_{n+1,2n+5}&=-\frac{1}{(2n+1)^2}S_{n,2n+3}+\frac{2}{(2n+1)^3}S_{n,2n+1}S_{n,2} \\ &=\frac{(-1)^{n+1}}{(2n+1)(2n-1)\{3\cdot 5\cdots(2n+1)\}^2}\{(2n+1)-2\} \\ &=\frac{(-1)^{n+1}}{(2n+1)\{3\cdot 5\cdots(2n+1)\}^2} \end{aligned}$$

by the induction hypothesis. This completes the proof.

As an immediate corollary we have the following.

COROLLARY. *The zeros $\beta_{n,j}$, $j=1,\cdots,n$, of the Bessel polynomials $Q_n(1,z)$ and the zeros $\gamma_{n,j}$, $j=1,\cdots,n$, of the Bessel polynomials $y_n(z)$ satisfy*

$$\sum_{j=1}^{n} \beta_{n,j}^{-(2n+1)}=\frac{(-1)^n}{2}\left(\frac{n!}{(2n)!}\right)^2,$$

$$\sum_{j=1}^{n} \beta_{n,j}^{-(2n+3)}=\frac{(-1)^n}{8(2n-1)}\left(\frac{n!}{(2n)!}\right)^2,$$

$$\sum_{j=1}^{n} \gamma_{n,j}^{2n+1}=(-1)^n 2^{2n}\left(\frac{n!}{(2n)!}\right)^2,$$

and

$$\sum_{j=1}^{n} \gamma_{n,j}^{2n+3} = \frac{(-1)^n 2^n}{(2n-1)}\left(\frac{n!}{(2n)!}\right)^2.$$

Remark. The properties of the zeros of the simple Bessel polynomials indicated in Theorem 1 do not hold for the generalized Bessel Polynomials of Al-Salam [1].

We shall postpone the proof of the next theorem until the end of § 4.

THEOREM 3. *The simple Bessel polynomials* $Q_n(1, x)$ *have the property that* $Q_k(1, \sqrt{x})/Q_n(1, \sqrt{x})$ *is completely monotonic in* $(0, \infty)$ *for* $0 \leqq k < n$, $1 \leqq n \leqq 6$.

4. The Student *t* distribution. The information on infinitely divisible distributions that we use can be found in Feller [6, pp. 425–428]. A variance mixture of the normal distribution has the form $\int_0^\infty (e^{-(x^2/2u)}/\sqrt{2\pi u})\, dG(u)$, where G is the mixing distribution. This mixture is infinitely divisible if G is infinitely divisible. G is infinitely divisible if and only if the Laplace transform of G is of the form $e^{-h(x)}$ with $h(0) = 0$ and the derivative $h'(x)$ completely monotonic on $(0, \infty)$.

The probability density function for the Student t distribution with k degrees of freedom can be written as

$$(4.1)\quad \frac{\Gamma((k+1)/2)}{\sqrt{k\pi}\Gamma(k/2)}\left(1+\frac{x^2}{k}\right)^{-(k+1)/2} = \frac{(k/2)^{k/2}}{\sqrt{2\pi}\Gamma(k/2)}\int_0^\infty \frac{e^{-x^2/2u}}{\sqrt{u}} u^{-(k+2)/2}\, e^{-k/2u}\, du.$$

For odd degrees of freedom, say $n = 2k+1$, the Laplace transform of the mixing distribution is given by, say,

$$\exp(-\sqrt{nt})\frac{k!}{(2k)!}\sum_{r=0}^{k}\frac{(k+r)!}{(k-r)!}\frac{(nt)^{(k-r)/2}}{r!}2^{k-r} = \exp(-\sqrt{nt})\frac{2^k k!}{(2k)!}P_k(\sqrt{nt}) \equiv e^{-h(t)}.$$

To show that $h(t)$ has a completely monotonic derivative it is sufficient to show that $-\log(e^{-\sqrt{t}}P_k(\sqrt{t}))$ has a completely monotonic derviative. We shall give a general procedure for showing this, with the details worked out for $k = 4, 5$ and 6. For the cases $k = 1, 2$ and 3, the complete monotonicity was established by Kelker [8] using direct differentiation.

Using relation (3.2), we get

$$\frac{d}{dt}(-\log(e^{-\sqrt{t}}P_n(\sqrt{t}))) = \frac{P_{n-1}(\sqrt{t})}{2P_n(\sqrt{t})}.$$

Recall that the partial fraction decomposition of $P_{n-1}(\sqrt{x})/P_n(\sqrt{x})$ is given by

$$\frac{P_{n-1}(\sqrt{x})}{P_n(\sqrt{x})} = \sum_{j=1}^{n}\frac{1}{\alpha_{n,j}(\alpha_{n,j}-\sqrt{x})}.$$

To show that a function is completely monotonic on $(0, \infty)$, it suffices to show that the inverse Laplace transform of the function is nonnegative on $(0, \infty)$. This follows from Bernstein's theorem (see Widder [18, p. 161]). The inverse Laplace transform of $P_{n-1}(\sqrt{x})/P_n(\sqrt{x})$ is (see [5, p. 233])

$$-\sum_{j=1}^{n}\left[\exp(\alpha_{n,j}^2 x)\operatorname{erfc}(-\alpha_{n,j}\sqrt{x})+\frac{1}{\sqrt{\pi x}\alpha_{n,j}}\right]. \tag{4.2}$$

The object is to show that the expression (4.2) is nonnegative for all nonnegative x. Applying the method of integration by parts (Copson [4, pp. 13–14]) to the error function integral, we obtain the following asymptotic expansion:

$$e^{a^2}\operatorname{erfc}(a)\sim\frac{1}{a\sqrt{\pi}}\left\{1+\sum_{m=1}^{\infty}\frac{(-1)^m 3\cdot 5\cdot 7\cdots(2m-1)}{2^m a^{2m}}\right\},$$

with the remainder after k terms, say R_k, being given by

$$R_k(a)=\frac{(-1)^{k-1}3\cdot 5\cdots(2k-1)}{\sqrt{\pi}\ 2^{k-1}}e^{a^2}\int_a^{\infty}u^{-2k}e^{-u^2}\,du.$$

With the asymptotic series of k terms with the remainder R_k, the expression (4.2) becomes

$$\frac{1}{\sqrt{\pi}}\sum_{m=1}^{k-1}\frac{(-1)^m 3\cdot 5\cdots(2m-1)}{2^m x^{m+1/2}}S_{n,2m+1}+\sum_{j=1}^{n}R_k(-\sqrt{x}\alpha_{n,j}). \tag{4.3}$$

Let Re (α) denote the real part of α and remember that Re $(\alpha_{n,j})<0$. It is easily shown that

$$\left|\sum_{j=1}^{n}\exp(x\alpha_{n,j}^2)\int_{\sqrt{x}\alpha_{n,j}}^{\infty}u^{-(2k+2)}e^{-u^2}\,du\right|$$

$$\leqq\frac{x^{-(k+3/2)}}{2}\sum_{j=1}^{n}|\alpha_{n,j}|^{-(2k+2)}/\operatorname{Re}(-\alpha_{n,j}).$$

Now let $k=n+1$ in expression (4.3). From the lemma, $S_{n,2m+1}=0$ for $m=1,2,\cdots,n-1$. Hence, the expression (4.3) is positive if

$$(-1)^n\frac{3\cdot 5\cdots(2n-1)}{2^n x^{n+1/2}}S_{n,2n+1}>\frac{3\cdot 5\cdots(2n+1)}{2^{n+1}x^{n+3/2}}\sum_{j=1}^{n}|\alpha_{n,j}|^{-(2n+2)}/\operatorname{Re}(-\alpha_{n,j}). \tag{4.4}$$

Applying Theorem 2 and simplifying, we get

$$x>-\frac{2n+1}{2}(3\cdot 5\cdots(2n-1))^2\sum_{j=1}^{n}|\alpha_{n,j}|^{-(2n+2)}/\operatorname{Re}(\alpha_{n,j}). \tag{4.5}$$

Inequality (4.5) gives a very reasonable lower bound on x for small values of n, but we shall show that as n increases, the lower bound for x approaches $+\infty$. Since little is known about the behavior of Re $(\alpha_{n,j})$ for large n, we will use another approximation for the remainder integral:

$$\left|\sum_{j=1}^{n}\exp(x\alpha_{n,j}^2)\int_{-\sqrt{x}\alpha_{n,j}}^{\infty}u^{-(2k+2)}e^{-u^2}\,du\right|$$

$$\leqq\frac{\sqrt{\pi}}{2}x^{-(k+1)}\sum_{j=1}^{n}|\alpha_{n,j}|^{-(2k+2)}.$$

Inequality (4.5) is now

$$\sqrt{x} > \sqrt{\pi}\frac{(2n+1)}{2}(3\cdot 5\cdots(2n-1))^2 \sum_{j=1}^{n} |\alpha_{n,j}|^{-(2n+2)}. \tag{4.6}$$

Since $P_n(x) = Q_n(1, 2x)$, the smallest of the norms of the roots of $P_n(x)$ is asymptotically equal to $(1.32548/2)n$. Replace $|\alpha_{n,j}|$ in (4.6) by the asymptotic value; then the term on the right becomes

$$\frac{\sqrt{\pi}}{2}(2n+1)(3\cdot 5\cdots(2n-1)^2)n\left(\frac{1.32548}{2}n\right)^{-(2n+2)}$$

$$= 2\sqrt{\pi}(2n+1)\left(\frac{(2n)!}{n!}\right)^2 \frac{n}{(1.32548n)^{2n+2}}.$$

On using Stirling's formula to approximate the factorials, we find that $(4\sqrt{\pi}(2n+1)/(1.32548)^2)(4/1.32548e)^{2n}$. *Thus the right side of* (4.6) *approaches* $-\infty$ *as* n approaches ∞. We were unable to find a better approximation for the remainder integral than the one used.

For $n = 4$, 5 and 6 we computed the roots of $P_n(x)$ to twelve significant digits. For $n = 4$, inequality (4.5) becomes

$$x > -\frac{9(3\cdot 5\cdot 7)^2}{2}\sum_{j=1}^{4} |\alpha_{4,j}|^{-10}/\mathrm{Re}\,(\alpha_{4,j}).$$

We get $-\sum_{j=1}^4 |\alpha_{4,j}|^{-10}/\mathrm{Re}\,(\alpha_{4,j}) = .155769\times 10^{-4}$. Solving, we see that the inequality is satisfied if $x > .78$. For $n = 5$, we get $-\sum_{j=1}^5 |\alpha_{5,j}|^{-12}/\mathrm{Re}\,(\alpha_{5,j}) = .144177\times 10^{-6}$. Inequality (4.5) with $n = 5$ is now satisfied if $x > .71$. For $n = 6$, we have $-\sum_{j=1}^6 |\alpha_{6,j}|^{-14}/\mathrm{Re}\,(\alpha_{6,j}) = .965467\times 10^{-9}$, and inequality (4.5) becomes $x > .68$.

For $n = 4$, 5 and 6 the computer was used to evaluate expression (4.2) over the interval (0, 1), and it was found to be positive and decreasing over the interval.

Since we can explicitly evaluate the first two nonzero terms of the asymptotic series, we can set up an inequality using the first two terms and the remainder. But the bounds for x given above are smaller than the bounds obtained using the first two terms and the remainder. However, for large n, let $k = n+2$ in expression (4.3); apply Theorem 2, the second approximation for the remainder integral, and Stirling's formula; then expression (4.3) becomes

$$x^{3/2} - \frac{2n+1}{2(2n-1)}x^{1/2} > \frac{4\sqrt{\pi}(2n+1)(2n+3)}{n^3(1.32548)^4}\left(\frac{4}{1.32548e}\right)^{2n}.$$

For large n this inequality will give a smaller lower bound for x than the bound obtained using the first term and remainder of the asymptotic series.

Therefore for $n = 4$, 5 and 6, we have that expression (4.2) is positive for all positive x, and this implies that $P_{n-1}(\sqrt{x})/P_n(\sqrt{x})$ is completely monotonic in $(0, \infty)$, which implies that the mixing distribution in (4.1) is infinitely divisible, and this implies that the Student t distribution with $2n+1$ degrees of freedom is infinitely divisible. Adding the known cases, we now have the following result.

THEOREM 4. *The Student t distribution is infinitely divisible for* 1, 3, 5, 7, 9, 11 *and* 13 *degrees of freedom.*

For any odd number of degrees of freedom, the method of attack used above should work for showing whether or not the Student t distribution is infinitely divisible; but a different approach appears to be necessary for the general case.

THEOREM 5. *Let χ_k^2 be a chi-square variable with k degrees of freedom. Let $Y_k = (\chi_k^2)^{-1}$. Then Y_k is infinitely divisible for $k = 1, 3, 5, 7, 9, 11$ and* 13.

Proof. For any positive integer k, the probability density function of $((1/k)\chi_k^2)^{-1}$ is $(k/2)^{k/2}\Gamma(k/2)^{-1}x^{-(k/2)-1}\, e^{-k/2x}$, and this is the infinitely divisible mixing distribution of the Student t distribution for $k = 1, 3, 5, 7, 9, 11$ and 13.

We are now ready to prove Theorem 3. $Q_0(1, \sqrt{x}) \equiv 1$. $Q_1(1, \sqrt{x}) = (1 + (\sqrt{x}/2))^{-1}$, which is completely monotonic in $(0, \infty)$. To prove that the Student t distribution is infinitely divisible, in the cases considered here and in the cases considered in Kelker [8], it was shown that $P_{n-1}(\sqrt{x})/P_n(\sqrt{x})$ is completely monotonic. Thus

$$\frac{Q_{n-1}(1, \sqrt{x})}{Q_n(1, \sqrt{x})} = \frac{P_{n-1}(\sqrt{x}/2)}{P_n(\sqrt{x}/2)}$$

is completely monotonic for $n = 2, 3, 4, 5$ and 6. Now for $0 \leqq k < n$, $1 \leqq n \leqq 6$, we have that

$$\frac{Q_k(1, \sqrt{x})}{Q_n(1, \sqrt{x})} = \frac{Q_{n-1}(1, \sqrt{x})}{Q_n(1, \sqrt{x})} \cdot \frac{Q_{n-2}(1, \sqrt{x})}{Q_{n-1}(1, \sqrt{x})} \cdots \frac{Q_k(1, \sqrt{x})}{Q_{k+1}(1, \sqrt{x})}.$$

Each factor on the right is completely monotonic, so the product is completely monotonic. The conclusions of Theorem 3 are obviously still true if $\sqrt{x}$ is replaced by x^s with $0 < s \leqq \frac{1}{2}$.

5. Concluding remarks and a conjecture. As we have seen in the previous sections, we have very strong reasons to believe that

$$\frac{P_{n-1}(\sqrt{x})}{P_n(\sqrt{x})} = \frac{K_{n-(1/2)}(\sqrt{x})}{\sqrt{x}K_{n+(1/2)}(\sqrt{x})}$$

is completely monotonic on $(0, \infty)$ for $n = 1, 2, \cdots$. As a matter of fact, we believe the following conjecture is true.

Conjecture. The quotient $K_\nu(\sqrt{x})/\sqrt{x}K_{\nu+1}(\sqrt{x})$ is completely monotonic on $(0, \infty)$ for $\nu \geqq 0$.

Note that the conjecture will be false if $\sqrt{x}$ is replaced by x. Indeed, the second derivative of

$$\frac{K_{3/2}(x)}{xK_{5/2}(x)}$$

changes sign on $(0, \infty)$.

One might be tempted to prove the conjecture by trying to show $K_\nu(\sqrt{x})/K_{\nu+1}(\sqrt{x})$ is completely monotonic on $(0, \infty)$, as is $x^{-1/2}$; hence the result.

This is the wrong approach. It is known that $K_\nu(x)/K_u(x)$ is continuous, increasing and positive on $(0, \infty)$ for $0 \leqq \nu < u$. This is Theorem 3 of Lorch [11].

Recently Trlifaj [15] computed the asymptotic expansion of $K_\nu(x)/xK_{\nu+1}(x)$ for $\nu > 0$ and $x \geqq 0$ as $x \to \infty$. This particular quotient of Bessel functions appeared in solving Schrödinger's equation with a rectangular potential well. The same quotient also occurred in the nuclear model of K-harmonics. For references see Trlifaj [15].

We will conclude with an alternative proof of Theorem 1. From (1.3) it is clear that

$$P_n(x) = \prod_{j=1}^{n} (x - \alpha_{n,j}) = \frac{(2n)!}{2^n(n!)} \prod_{j=1}^{n} (1 - x\alpha_{n,j}^{-1}).$$

Taking the logarithm of the right side we get

$$\log P_n(x) = \log \frac{(2n)!}{2^n(n!)} - \sum_{k=1}^{\infty} S_{n,k} \frac{x^k}{k} \tag{5.1}$$

and

$$\log P_n(-x) = \log \frac{(2n)!}{2^n(n!)} - \sum_{k=1}^{\infty} (-1)^k S_{n,k} \frac{x^k}{k}. \tag{5.2}$$

Subtract (5.1) from (5.2) to obtain

$$\log \frac{P_n(-x)}{P_n(x)} = 2 \sum_{k=0}^{\infty} S_{n,2k+1} \frac{x^{2k+1}}{2k+1}$$

which is equivalent to

$$P_n(-x) = P_n(x) \exp\left\{2 \sum_{k=0}^{\infty} S_{n,2k+1} \frac{x^{2k+1}}{2k+1}\right\}.$$

Since $S_{n,1} = -1$, the above formula becomes

$$e^{2x} P_n(-x) = P_n(x) \exp\left\{2 \sum_{k=1}^{\infty} S_{n,2k+1} \frac{x^{2k+1}}{2k+1}\right\}. \tag{5.3}$$

Let $\prod_{j,1}$ and $\prod_{j,2}$ be the coefficients of x^j in $e^{2x}P_n(-x)$ and $P_n(x)$, respectively. The coefficients depend on n also, but we suppress this dependence for ease in printing. The crux of the proof is to show that $\prod_{j,1} = \prod_{j,2}$ for $0 \leqq j \leqq 2n$.

Observe that for $0 \leqq j \leqq 2n$, $\prod_{j,1}$ is the same as the coefficient of x^j in $\{\sum_{l=0}^{2n} (2x)^l/l!\} P_n(-x)$:

$$\prod_{j,1} = \sum_{l=0}^{j} \frac{2^{j-n}(2n-l)!(-1)^l}{(j-l)!\,l!\,(n-l)!} = \frac{2^{j-n}(2n)!}{(n!)(j!)} {}_2F_1(-j, -n; -2n; 1),$$

since $p!/(p-l)! = (-1)^l(-p)_l$ for $p, l = 0, 1, 2, \cdots$. Using Gauss' theorem in the form ${}_2F_1(-j, b; c; 1) = (c-b)_j/(c)_j$, $j = 0, 1, \cdots$, (see Rainville [14, p. 69]) we get

$$\prod_{j,1} = \frac{2^{j-n}(2n)!(-n)_j}{n!j!(-2n)_j} = \begin{cases} \dfrac{2^{j-n}(2n-j)!}{j!(n-j)!} & \text{for } 0 \leqq j \leqq n, \\ 0 & \text{for } n < j \leqq 2n. \end{cases}$$

Therefore $\prod_{j,1} = \prod_{j,2}$, $j = 0, 1, \cdots, 2n$.

On the other hand, (5.3) implies that $\exp\{2\sum_{k=1}^{\infty} S_{n,2k+1}(x^{2k+1}/(2k+1))\}$ does not contribute to the coefficient of x^j, $j = 1, 2, \cdots, 2n$ in $P_n(x)\exp\{2\sum_{k=1}^{\infty} S_{n,2k+1}(x^{2k+1}/(2k+1))\}$. This can only happen if the coefficients of $x^3, x^5, \cdots, x^{2n-1}$ in the power series expansion of $\exp\{2\sum_{k=1}^{\infty} S_{n,2k+1}(x^{2k+1}/(2k+1))\}$ vanish. The coefficient of x^3 in this series is $(1/3)S_{n,3}$. Therefore $S_{n,3} = 0$. By very easy induction we get $S_{n,j} = 0$, $j = 3, 5, \cdots, 2n-1$. This completes the proof.

Remark. The power sums $S_{n,2n+1}$, $S_{n,2n+3}$ can also be evaluated from (5.3). This follows from equating coefficients of x^{2n+1} and x^{2n-1} in both sides of (5.3).

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CONSTRUCTION OF A FAMILY OF POSITIVE KERNELS FROM JACOBI POLYNOMIALS*

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Abstract. Starting with the Jacobi polynomials ${}_2F_1(-n, n+\alpha+\beta+\gamma+\delta-1; \alpha+\beta; x)$ over $[0, 1]$ with $\alpha, \delta>0$ and $\beta, \gamma>\frac{1}{2}$, a symmetric, positive definite square-integrable kernel is constructed. For $1\leqq\beta<\gamma$, this kernel is also found to be continuous. Special limiting kernels are obtained by considering the limits, $\alpha\to0^+$, $\delta\to0^+$, $\gamma\to\infty$ and $\delta\to\infty$. All these kernels are shown to have stochastic properties. As a by-product, some bilinear formulas are obtained with the Jacobi and Laguerre polynomials.

1. Introduction. Recently we adopted a method of differential and integral ladder operators (also known as shift operators) to obtain some bilinear formulas involving associated Laguerre and Jacobi polynomials and their discrete counterparts, namely, the Meixner and Hahn polynomials (Cooper, Hoare and Rahman [10]]). However, the class of operators considered in that paper was allowed to shift one parameter at a time, for example, the Laguerre polynomial $L_n^\alpha(x)$ was allowed to shift to $L_n^{\alpha\pm1}(x)$. Likewise, for the Jacobi polynomials

$$P_n^{(c-1,a-c)}(x)=\frac{(c)_n}{n!}\,{}_2F_1\left(-n, n+a; c; \frac{1-x}{2}\right),$$

only the parameter c was seen to jump by $+1$ or -1 in one operation without changing n or a. The idea of one-parameter ladder operators for second order Sturm–Liouville problems is quite an old one (see, for example, Morse and Feshbach [29]), and the method is intimately connected with the factorizability of Sturm–Liouville operators (for references in this area see [10]). It would seem natural to extend this method to multiparameter ladder operators, but, to our knowledge, no serious attempts seem to have been made in this direction. For one thing, the relative simplicity of the one-parameter case changes abruptly when one tries a two-parameter ladder operator; secondly, it seems rather difficult to treat a family of multiparameter ladder operators in any general way. Needless to say, we did have a serious look at this problem, but failing to obtain any concrete result generally, we felt it is worth reporting, nevertheless, a rather interesting set of special results involving the Jacobi and Laguerre polynomials.

Starting with the Jacobi polynomials $J_n(\alpha+\beta+\gamma+\delta-1,\ \alpha+\beta;\ u/E) = {}_2F_1(-n, n+\alpha+\beta+\gamma+\delta-1;\ \alpha+\beta;\ u/E)$, where α, β, γ, δ, E are strictly positive parameters and $0\leqq u\leqq E$, we first construct a fairly complicated positive kernel, study its properties, obtain a whole set of positive-valued special kernels by considering special limiting values of the parameters, and finally exploit well-known theorems in Jacobi series to obtain some bilinear formulas. The

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reason for using a rather nonstandard definition of the Jacobi polynomials is that we make heavy use of the hypergeometric functions, and it seems more convenient to work on the interval [0, 1] rather than the usual interval [−1, 1] for $P_n^{(a,b)}(x)$. Needless to say, our results can be translated in terms of standard notations and definitions through obvious transformations.

It should be stressed that our search for an extension of the ladder-operator method did not "accidentally" lead us to the discovery of the general kernel $K_E(u, v; \alpha, \beta, \gamma, \delta)$. This kernel and most of its special forms that we have discussed here have, in fact, been known through the works of Hoare [18], Hoare and Thiele [19], Cooper and Hoare [11] and Cooper [12] on a class of stochastic models. We wish to make this acknowledgment more specific in a special note at the end of this paper. Our work simply gives an alternative mathematical approach of reproducing this family of kernels starting from their eigenfunctions.

It appears that after about 30 years of relative quiet, there has been a sudden burst of active interest in Jacobi polynomials and Jacobi series, thanks largely to the works of R. Askey, [2]–[5], G. Gasper [4], [14]–[17], T. H. Koornwinder [22]–[27] and others. We would like to thank Professor Askey for drawing our attention to this rather substantial volume of recent literature. A survey of this work is available in *Orthogonal Polynomials and Special Functions* by R. Askey. This monograph will be volume 21 in the SIAM series of Regional Conference Lectures.

However, the problem and approach of this paper are somewhat different from those of Askey–Gasper–Koornwinder, although there is an underlying common interest in the positivity of the kernels and the corresponding bilinear sums. Our approach seems to be more akin to that of Popov [30], who also derived some bilinear sums for Jacobi polynomials by first showing that these polynomials are the eigenfunctions of a certain kernel. Our method is quite elementary, based on a few well-known properties of the Gaussian and generalized hypergeometric functions (see, for example, Bateman Manuscript Project [9], Slater [32], Bailey [6].

2. The general kernel. Let us consider the Jacobi polynomials

$$J_n(\alpha+\beta+\gamma+\delta-1, \alpha+\beta; x) = {}_2F_1(-n, n+\alpha+\beta+\gamma+\delta-1; \alpha+\beta; x), \quad n = 0, 1, 2, \cdots. \tag{2.1}$$

We have deliberately introduced four parameters α, β, γ, δ, so that special results can be obtained for their special values. For the moment, the only assumption that we are making is that they are all positive. The polynomials $J_n(\alpha+\beta+\gamma+\delta-1, \alpha+\beta; x)$ are known to be orthogonal with respect to the weight function $x^{\alpha+\beta-1}(1-x)^{\gamma+\delta-1}$, and they form a complete orthogonal system on $L_2(0, 1)$. (See, for example, Morse and Feshbach [29].)

Let us perform the following operations: multiply ${}_2F_1(-n, n+\alpha+\beta+\gamma+\delta-1; \alpha+\beta; y)$ by $(y-z_1)^{\beta-1}(z_2-y)^{\gamma-1}$ and integrate over y from z_1 to z_2, where

$0 \leqq z_1 < z_2 \leqq 1$. We get

$$\begin{aligned}
&\int_{z_1}^{z_2} dy\,(y-z_1)^{\beta-1}(z_2-y)^{\gamma-1}{}_2F_1(-n, n+\alpha+\beta+\gamma+\delta-1; \alpha+\beta; y)\\
&\quad = (z_2-z_1)^{\beta+\gamma-1}\int_0^1 dy\, y^{\beta-1}(1-y)^{\gamma-1}{}_2F_1(-n, n+\alpha+\beta+\gamma+\delta-1; \alpha\\
&\qquad\qquad +\beta; z_1+(z_2-z_1)y)\\
&\quad = (z_2-z_1)^{\beta+\gamma-1}\sum_{s=0}^{n}\frac{(-n)_s(n+\alpha+\beta+\gamma+\delta-1)_s}{(\alpha+\beta)_s s!}\sum_{m=0}^{s}\binom{s}{m}z_2^m z_1^{s-m}\\
&\qquad\qquad \cdot\int_0^1 dy\, y^{m+\beta-1}(1-y)^{s-m+\gamma-1}\\
&\quad = (z_2-z_1)^{\beta+\gamma-1}\sum_{s=0}^{n}\frac{(-n)_s(n+\alpha+\beta+\gamma+\delta-1)_s}{(\alpha+\beta)_s s!}\\
&\qquad \cdot\sum_{m=0}^{s}\binom{s}{m}B(m+\beta, s-m+\gamma)z_1^{s-m}z_2^m,
\end{aligned}\tag{2.2}$$

where

$$B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$$

is the beta function.

Note that the integral in (2.2) diverges as $z_1 \to z_2$, unless $\beta+\gamma-1 \geqq 0$. As we shall see later, this fact has important bearing on the square-integrability of the kernel that we are about to derive.

Our aim is to perform a series of operations on $J_n(\alpha+\beta+\gamma+\delta-1, \alpha+\beta; y)$, so that the end result is $J_n(\alpha+\beta+\gamma+\delta-1, \alpha+\beta; x)$. Since the weight function associated with $J_n(\alpha+\beta+\gamma+\delta-1, \alpha+\beta; x)$ is $x^{\alpha+\beta-1}(1-x)^{\gamma+\delta-1}$, the next obvious step is to divide (2.2) by $(z_2-z_1)^{\beta+\gamma-1}$, multiply by $(1-z_2)^{\delta-1}(z_2-x)^{\gamma-1}$, and integrate over z_2 from x to 1. Thus

$$\begin{aligned}
&\int_x^1 dz_2\frac{(1-z_2)^{\delta-1}(z_2-x)^{\gamma-1}}{(z_2-z_1)^{\beta+\gamma-1}}\int_{z_1}^{z_2} dy\,(y-z_1)^{\beta-1}(z_2-y)^{\gamma-1}\\
&\qquad\qquad \cdot{}_2F_1(-n, n+\alpha+\beta+\gamma+\delta-1; \alpha+\beta; y)\\
&\quad = \sum_{s=0}^{n}\frac{(-n)_s(n+\alpha+\beta+\gamma+\delta-1)_s}{(\alpha+\beta)_s s!}\sum_{m=0}^{s}\binom{s}{m}B(m+\beta, s-m+\gamma)z_1^{s-m}\\
&\qquad\qquad \cdot\int_x^1 dz_2 z_2^m(1-z_2)^{\delta-1}(z_2-x)^{\gamma-1}\\
&\quad = (1-x)^{\gamma+\delta-1}\sum_{s=0}^{n}\frac{(-n)_s(n+\alpha+\beta+\gamma+\delta-1)_s}{(\alpha+\beta)_s s!}\\
&\qquad \cdot\sum_{m=0}^{s}\binom{s}{m}B(m+\beta, s-m+\gamma)z_1^{s-m}\\
&\qquad \cdot\sum_{k=0}^{m}\binom{m}{k}B(k+\gamma, m-k+\delta)x^{m-k}.
\end{aligned}\tag{2.3}$$

The final step is to multiply this by $z_1^{\alpha-1}(x-z_1)^{\beta-1}$, and integrate over z_1 from 0 to x. We have

$$\int_0^x dz_1 z_1^{\alpha-1}(x-z_1)^{\beta-1}\int_x^1 dz_2 \frac{(1-z_2)^{\delta-1}(z_2-x)^{\gamma-1}}{(z_2-z_1)^{\beta+\gamma-1}}$$

(2.4)
$$\cdot\int_{z_1}^{z_2} dy(y-z_1)^{\beta-1}(z_2-y)^{\gamma-1}{}_2F_1(-n, n+\alpha+\beta+\gamma+\delta-1; \alpha+\beta; y)$$

$$= x^{\alpha+\beta-1}(1-x)^{\gamma+\delta-1}M_n(x),$$

say, where

$$M_n(x)=\sum_{s=0}^{n}\frac{(-n)_s(n+\alpha+\beta+\gamma+\delta-1)_s}{(\alpha+\beta)_s s!}$$

(2.5)
$$\cdot\sum_{m=0}^{s}\binom{s}{m}B(m+\beta, s-m+\gamma)B(s-m+\alpha, \beta)$$

$$\cdot\sum_{k=0}^{m}\binom{m}{k}B(k+\gamma, m-k+\delta)x^{s-k}.$$

The operations on the left-hand side of (2.4) can be seen as

$$\int_0^1 dy\, {}_2F_1(-n, n+\alpha+\beta+\gamma+\delta-1; \alpha+\beta; y)$$

$$\cdot\int_0^1 dz_1\int_0^1 dz_2\, \frac{z_1^{\alpha-1}(x-z_1)^{\beta-1}(y-z_1)^{\beta-1}(z_2-x)^{\gamma-1}(z_2-y)^{\gamma-1}(1-z_2)^{\delta-1}}{(z_2-z_1)^{\beta+\gamma-1}}$$

$$\cdot H(x-z_1)H(y-z_1)H(z_2-x)H(z_2-y)$$

(2.6)
$$=\int_0^1 dy\, {}_2F_1(-n, n+\alpha+\beta+\gamma+\delta-1; \alpha+\beta; y)$$

$$\cdot\int_0^{\min(x,y)} dt\, t^{\alpha-1}(x-t)^{\beta-1}(y-t)^{\beta-1}$$

$$\cdot\int_{\max(x,y)}^1 dz\frac{(1-z)^{\delta-1}(z-x)^{\gamma-1}(z-y)^{\gamma-1}}{(z-t)^{\beta+\gamma-1}},$$

where

$$H(t)=\begin{cases}0 & \text{if } t<0,\\ 1 & \text{if } t>0.\end{cases}$$

Hence if we denote

$$K(x, y; \alpha, \beta, \gamma, \delta)=\frac{x^{-\alpha-\beta+1}(1-x)^{-\gamma-\delta+1}}{B(\alpha, \beta)B(\beta, \gamma)B(\gamma, \delta)}$$

(2.7)
$$\cdot\int_0^{\min(x,y)} dt\, t^{\alpha-1}(x-t)^{\beta-1}(y-t)^{\beta-1}$$

$$\cdot\int_{\max(x,y)}^1 dz\frac{(1-z)^{\delta-1}(z-x)^{\gamma-1}(z-y)^{\gamma-1}}{(z-t)^{\beta+\gamma-1}},$$

then it follows from (2.4) that

$$\int_0^1 dy\ K(x, y; \alpha, \beta, \gamma, \delta) J_n(\alpha+\beta+\gamma+\delta-1, \alpha+\beta; y)$$

(2.8)
$$= \frac{M_n(x)}{B(\alpha, \beta)B(\beta, \gamma)B(\gamma, \delta)}.$$

It is obvious from (2.5) that $M_n(x)$ is a polynomial of degree n. What we shall now prove is that $M_n(x)$ is a multiple of $J_n(\alpha+\beta+\gamma+\delta-1, \alpha+\beta; x)$ itself.

For a nonnegative integer p such that $0 \leqq p \leqq n$, the coefficient of x^p in $M_n(x)$ is, clearly,

(2.9)
$$\begin{aligned} &\sum_{s=p}^{n} \frac{(-n)_s(n+\alpha+\beta+\gamma+\delta-1)_s}{(\alpha+\beta)_s s!} \sum_{m=s-p}^{s} \binom{s}{m}\binom{m}{s-p} B(m+\beta, s-m+\gamma) \\ &\qquad \cdot B(s-m+\alpha, \beta)B(s-p+\gamma, m-s+p+\delta) \\ &= \frac{(-n)_p(n+\alpha+\beta+\gamma+\delta-1)_p}{(\alpha+\beta)_p p!} B(\alpha, \beta)B(\beta, \gamma)B(\gamma, \delta)S(n; p), \end{aligned}$$

where

(2.10)
$$\begin{aligned} S(n; p) = &\frac{1}{B(\alpha, \beta)B(\beta, \gamma)B(\gamma, \delta)} \sum_{l=0}^{n-p} \frac{(-n+p)_l(n+\alpha+\beta+\gamma+\delta+p-1)_l}{(\alpha+\beta+p)_l(p+1)_l} \\ &\cdot \sum_{k=0}^{p} \binom{p+l}{k+l}\binom{k+l}{l} B(k+l+\beta, p-k+\gamma)B(p-k+\alpha, \beta) \\ &\cdot B(l+\gamma, k+\delta). \end{aligned}$$

After some simplifications, this reduces to

(2.11)
$$\begin{aligned} S(n; p) = &\frac{p!}{(\beta+\gamma)_p} \sum_{k=0}^{p} \frac{(\alpha)_{p-k}(\beta)_k(\gamma)_{p-k}(\delta)_k}{(\alpha+\beta)_{p-k}(\gamma+\delta)_k(p-k)!k!} \\ &\cdot {}_4F_3\left[\begin{matrix} -n+p, & n+\alpha+\beta+\gamma+\delta+p-1, \beta+k, \gamma \\ \alpha+\beta+p, & \beta+\gamma+p, \quad \gamma+\delta+k \end{matrix} ; 1\right], \end{aligned}$$

where

(2.12)
$${}_4F_3\left[\begin{matrix} a, b, c, d \\ e, f, g \end{matrix} ; t\right] = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c)_r(d)_r}{(e)_r(f)_r(g)_r} \frac{t^r}{r!}$$

is a generalized hypergeometric function.

If we set $p = 0$ in (2.11), we obtain

(2.13)
$$S(n; 0) = {}_4F_3\left[\begin{matrix} -n, n+\alpha+\beta+\gamma+\delta-1, \beta, \gamma \\ \alpha+\beta, \beta+\gamma, \gamma+\delta \end{matrix} ; 1\right].$$

We show in the Appendix that $S(n; p)$ is, indeed, independent of p, and therefore equal to $S(n; 0)$.

Thus from (2.8) and (2.9) it follows that

$$\int_0^1 dy\, K(x, y; \alpha, \beta, \gamma, \delta) J_n(\alpha+\beta+\gamma+\delta-1, \alpha+\beta; y) = S(n; 0) J_n(\alpha+\beta+\gamma+\delta-1, \alpha+\beta; x). \tag{2.14}$$

This proves that for each nonnegative integer n, $J_n(\alpha+\beta+\gamma+\delta-1, \alpha+\beta; x)$ is an eigenfunction of the integral equation

$$\lambda\phi(x) = \int_0^1 K(x, y; \alpha, \beta, \gamma, \delta)\phi(y)\, dy, \tag{2.15}$$

and the corresponding eigenvalue is given by

$$\lambda = \lambda_n = S(n; 0). \tag{2.16}$$

If we write

$$w(x) = x^{\alpha+\beta-1}(1-x)^{\gamma+\delta-1}, \tag{2.17}$$

then the kernel (2.7) can be "symmetrized" as

$$G(x, y; \alpha, \beta, \gamma, \delta) = \sqrt{\frac{w(x)}{w(y)}}\, K(x, y; \alpha, \beta, \gamma, \delta). \tag{2.18}$$

The corresponding integral equation,

$$\lambda f(x) = \int_0^1 G(x, y; \alpha, \beta, \gamma, \delta) f(y)\, dy, \tag{2.19}$$

then has the same eigenvalue (2.17) for each n with the eigenfunction

$$f_n(x) = N_n\sqrt{w(x)}\, {}_2F_1(-n, n+\alpha+\beta+\gamma+\delta-1; \alpha+\beta; x), \tag{2.20}$$

where N_n is a normalizing constant given by

$$N_n^2 = \{(2n+\alpha+\beta+\gamma+\delta-1)\Gamma(\alpha+\beta+n)\Gamma(n+\alpha+\beta+\gamma+\delta-1)\}/\{n!\Gamma^2(\alpha+\beta)\Gamma(n+\gamma+\delta)\}. \tag{2.21}$$

It is well known that (see [29])

$$\int_0^1 f_m(x) f_n(x)\, dx = \delta_{mn}. \tag{2.22}$$

Before passing to the next section, we note that a transformation

$$x \to u/E, \quad y \to v/E \tag{2.23}$$

transforms the integral equation (2.15) to

$$\lambda\phi(u) = \int_0^E K_E(u, v; \alpha, \beta, \gamma, \delta)\phi(v)\, dv, \tag{2.24}$$

where

$$K_E(u, v; \alpha, \beta, \gamma, \delta) = [B(\alpha, \beta)B(\beta, \gamma)B(\gamma, \delta)]^{-1} u^{-\alpha-\beta+1}(E-u)^{-\gamma-\delta+1}$$

(2.25)
$$\cdot \int_0^{\min(u,v)} dt\, t^{\alpha-1}(u-t)^{\beta-1}(v-t)^{\beta-1}$$

$$\cdot \int_{\max(u,v)}^{E} dz \frac{(E-z)^{\delta-1}(z-u)^{\gamma-1}(z-v)^{\gamma-1}}{(z-t)^{\beta+\gamma-1}}$$

and

(2.26)
$$\phi(u) = {}_2F_1(-n, n+\alpha+\beta+\gamma+\delta-1; \alpha+\beta; u/E).$$

Note that the eigenvalues of K_E remain the same as in (2.16). Also note that if $\phi(u)$ is an eigenfunction of (2.24) with eigenvalue λ, then $\phi(E-u)$ is an eigenfunction of (2.24) with $u, v, \alpha, \beta, \gamma, \delta$ replaced by $E-u, E-v, \delta, \gamma, \beta, \alpha$, respectively.

3. Special cases.

Case I. $\alpha \to 0^+$, $\beta, \gamma, \delta > 0$. The double integral in $K_E(u, v)$ diverges as $\alpha \to 0^+$, but when it is divided by $B(\alpha, \beta)$, the limit exists. Using the integral representation of the hypergeometric function (see, e.g., [7])

(3.1)
$$B(b, c-b)\, {}_2F_1(a, b; c; z) = \int_0^1 dt\, t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}, \qquad \operatorname{Re} c > \operatorname{Re} b > 0,$$

it can be shown that

$$[B(\alpha, \beta)]^{-1} \int_0^{\min(u,v)} dt\, t^{\alpha-1}(u-t)^{\beta-1}(v-t)^{\beta-1}$$

$$\cdot \int_{\max(u,v)}^{E} dz \frac{(E-z)^{\delta-1}(z-u)^{\gamma-1}(z-v)^{\gamma-1}}{(z-t)^{\beta+\gamma-1}}$$

(3.2)
$$= [\min(u, v)]^{\alpha+\beta-1}[\max(u, v)]^{\beta-1} \sum_{k=0}^{\infty} \binom{\beta-1}{k}$$

$$\cdot \left[-\frac{\min(u, v)}{\max(u, v)}\right]^k \frac{\Gamma(k+\alpha)\Gamma(k+\beta)}{\Gamma(\alpha)\Gamma(k+\alpha+\beta)}$$

$$\cdot \int_{\max(u,v)}^{E} dz \frac{(E-z)^{\delta-1}(z-u)^{\gamma-1}(z-v)^{\gamma-1}}{z^{\beta+\gamma-1}}$$

$$\cdot\, {}_2F_1\left(\beta+\gamma-1, k+\alpha; k+\alpha+\beta; \frac{\min(u, v)}{z}\right).$$

Now

$$\lim_{\alpha \to 0^+} \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)} = \begin{cases} 1, & k = 0, \\ 0, & k = 1, 2, \cdots. \end{cases}$$

Also

$$\lim_{\alpha\to 0^+} {}_2F_1\left(\beta+\gamma-1, \alpha; \beta+\alpha; \frac{\min(u, v)}{z}\right) = 1.$$

Hence

$$K_E(u, v; 0, \beta, \gamma, \delta) = \lim_{\alpha\to 0^+} K_E(u, v; \alpha, \beta, \gamma, \delta)$$

$$(3.3) \qquad = [B(\beta, \gamma)B(\gamma, \delta)]^{-1} \frac{v^{\beta-1}}{(E-u)^{\gamma+\delta-1}}$$

$$\cdot \int_{\max(u,v)}^{E} dz \frac{(E-z)^{\delta-1}(z-u)^{\gamma-1}(z-v)^{\gamma-1}}{z^{\beta+\gamma-1}}$$

The eigenfunctions of this limiting kernel are the Jacobi polynomials

$${}_2F_1(-n, n+\beta+\gamma+\delta-1; \beta; u/E),$$

and the corresponding eigenvalues are

$$(3.4) \qquad \begin{aligned} \lambda_n &= {}_4F_3\left[\begin{matrix} -n, n+\beta+\gamma+\delta-1; \beta; \gamma \\ \beta, \beta+\gamma, \gamma+\delta \end{matrix}; 1\right] \\ &= {}_3F_2\left[\begin{matrix} -n, n+\beta+\gamma+\delta-1, \gamma \\ \beta+\gamma, \gamma+\delta \end{matrix}; 1\right] \end{aligned}$$

By using the Saalschutzian theorem (Slater [32]), this can be evaluated.

We may evaluate it directly by considering the integral representation of ${}_3F_2(1)$ (Slater [32]). Thus

$$(3.5) \qquad \begin{aligned} \lambda_n &= \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma)\Gamma(\delta)} \cdot \int_0^1 t^{\gamma-1}(1-t)^{\delta-1} {}_2F_1(-n, n+\beta+\gamma+\delta-1; \beta+\gamma; t)\, dt \\ &= \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma)\Gamma(\delta)} \cdot \frac{\Gamma(\beta+\gamma)\Gamma(\gamma)\Gamma(\delta+n)\Gamma(\beta+n)}{\Gamma(\beta+\gamma+n)\Gamma(\beta)\Gamma(\gamma+\delta+n)} \\ &= \frac{(\beta)_n(\delta)_n}{(\beta+\gamma)_n(\gamma+\delta)_n} \quad \text{(Bateman [9, p. 398, (2)]).} \end{aligned}$$

Case II. $\delta\to 0^+$, $\alpha, \beta, \gamma > 0$. Let us first determine the limit of

$$[B(\gamma, \delta)]^{-1} \int_{\max(u,v)}^{E} dz \frac{(E-z)^{\delta-1}(z-u)^{\gamma-1}(z-v)^{\gamma-1}}{(z-t)^{\beta+\gamma-1}}$$

as $\delta\to 0^+$.

After making use of two obvious transformations and a binomial expansion, this integral reduces to

$$\frac{[E-\max(u,v)]^{\gamma+\delta-1}[E-\min(u,v)]^{\gamma-1}[E-t]^{-\beta-\gamma+1}}{B(\gamma,\delta)}\sum_{k=0}^{\infty}\binom{\gamma-1}{k}\cdot\left[-\frac{E-\max(u,v)}{E-\min(u,v)}\right]^k$$

$$\cdot\int_0^1 dz\, z^{k+\delta-1}(1-z)^{\gamma-1}\left[1-\frac{E-\max(u,v)}{E-t}z\right]^{-\beta-\gamma+1}.$$

As in Case I, only the $k=0$ term survives as $\delta\to 0^+$ and the integral divided by $B(\gamma,\delta)$ approaches 1. Hence

$$\begin{aligned}K_E(u,v;\alpha,\beta,\gamma,0)&=\lim_{\delta\to 0^+}K_E(u,v;\alpha,\beta,\gamma,\delta)\\ &=[B(\alpha,\beta)B(\beta,\gamma)]^{-1}\frac{(E-v)^{\gamma-1}}{u^{\alpha+\beta-1}}\\ &\quad\cdot\int_0^{\min(u,v)}dt\frac{t^{\alpha-1}(u-t)^{\beta-1}(v-t)^{\beta-1}}{(E-t)^{\beta+\gamma-1}}\end{aligned}\tag{3.6}$$

The eigenfunctions are again basically the same, namely, ${}_2F_1(-n,n+\alpha+\beta+\gamma-1;\alpha+\beta;u/E)$, while the eigenvalues are given by

$$\begin{aligned}\lambda_n&={}_4F_3\left[\begin{matrix}-n,n+\alpha+\beta+\gamma-1,\beta,\gamma\\ \alpha+\beta,\beta+\gamma,\gamma\end{matrix};1\right]\\ &={}_3F_2\left[\begin{matrix}-n,n+\alpha+\beta+\gamma-1,\beta\\ \alpha+\beta,\beta+\gamma\end{matrix};1\right]\\ &=\frac{(\alpha)_n(\gamma)_n}{(\alpha+\beta)_n(\beta+\gamma)_n}.\end{aligned}\tag{3.7}$$

Case III. $\alpha,\beta,\gamma>0$, $E,\delta\to\infty$ such that $E/\delta=\text{const}$. There is obviously no loss of generality in assuming that this constant ratio is equal to 1. In this case,

$$\frac{(E-z)^{\delta-1}}{B(\gamma,\delta)(E-u)^{\gamma+\delta-1}}=\frac{\Gamma(\gamma+\delta)(\delta-z)^{\delta-1}}{\Gamma(\gamma)\Gamma(\delta)(\delta-u)^{\gamma+\delta-1}}=\frac{1}{\Gamma(\gamma)}\frac{\Gamma(\gamma+\delta)}{\Gamma(\delta)\delta^{\gamma}}\cdot\frac{(1-z/\delta)^{\delta-1}}{(1-u/\delta)^{\gamma+\delta-1}}.$$

As $\delta\to\infty$, this approaches the limit $e^{-z+u}/\Gamma(\gamma)$.

Therefore

$$\begin{aligned}K_\infty(u,v;\alpha,\beta,\gamma,\infty)&=\lim_{\substack{E\to\infty\\ \delta\to\infty\\ E/\delta=1}}K_E(u,v;\alpha,\beta,\gamma,\delta)\\ &=[B(\alpha,\beta)B(\beta,\gamma)\Gamma(\gamma)]^{-1}e^{u}u^{-\alpha-\beta+1}\\ &\quad\cdot\int_0^{\min(u,v)}dt\,t^{\alpha-1}(u-t)^{\beta-1}(v-t)^{\beta-1}\\ &\quad\cdot\int_{\max(u,v)}^{\infty}dz\frac{e^{-z}(z-u)^{\gamma-1}(z-v)^{\gamma-1}}{(z-t)^{\beta+\gamma-1}}.\end{aligned}\tag{3.8}$$

The eigenfunctions of this kernel are

$$\lim_{\delta\to\infty} {}_2F_1(-n, n+\alpha+\beta+\gamma+\delta-1; \alpha+\beta; u/\delta) = {}_1F_1(-n; \alpha+\beta; u)$$

(3.9)

$$= \frac{n!}{(\alpha+\beta)_n} L_n^{(\alpha+\beta-1)}(u),$$

where $L_n^{(\alpha+\beta-1)}(u)$ is the associated Laguerre polynomial (see, for example, Bateman [7] and [8]). The eigenvalue corresponding to a given integer n is

(3.10)

$$\lambda_n = \lim_{\delta\to\infty} {}_4F_3\left[\begin{matrix}-n, \alpha+\beta+\gamma+\delta-1, \beta, \gamma \\ \alpha+\beta, \beta+\gamma, \gamma+\delta\end{matrix} ; 1\right]$$

$$= {}_3F_2\left[\begin{matrix}-n, \beta, \gamma \\ \alpha+\beta, \beta+\gamma\end{matrix} ; 1\right].$$

The ${}_3F_2[1]$ in (3.10) is neither Saalschutzian nor well-poised (Slater [32]). It therefore does not seem possible to express it in finite terms as in (3.5) or (3.7).

Case IV. $\alpha\to 0^+$, $E=\delta\to\infty$, β, $\gamma>0$. Combining Cases I and III, we now obtain

(3.11)

$$K_\infty(u, v; 0, \beta, \gamma, \infty) = \lim_{\alpha\to 0^+} \lim_{\substack{\delta\to\infty \\ E\to\infty \\ E/\delta=1}} K_E(u, v; \alpha, \beta, \gamma, \delta)$$

$$= [B(\beta, \gamma)\Gamma(\gamma)]^{-1} e^u v^{\beta-1} \int_{\max(u,v)}^{\infty} dz \frac{e^{-z}(z-u)^{\gamma-1}(z-v)^{\gamma-1}}{z^{\beta+\gamma-1}}.$$

The eigenfunctions are simply $(n!/(\beta)_n(L_n^{(\beta-1)}(u)$, while the eigenvalues take a simpler form,

(3.12)

$$\lambda_n = \lim_{\alpha\to 0^+} {}_3F_2\left[\begin{matrix}-n, \beta, \gamma \\ \alpha+\beta, \beta+\gamma\end{matrix} ; 1\right]$$

$$= {}_2F_1(-n, \gamma; \beta+\gamma; 1)$$

$$= \frac{(\beta)_n}{(\beta+\gamma)_n}$$

Case V. $\delta\to 0^+$, $\gamma=E\to\infty$, α, $\beta>0$. Setting $\gamma=E$ in (3.6) and passing to the limit $\gamma\to\infty$, we obtain, in a manner similar to Case III, the kernel

(3.13)

$$K_\infty(u, v; \alpha, \beta, \infty, 0) = \lim_{\substack{\delta\to 0^+ \\ E\to\infty \\ \gamma\to\infty \\ E/\gamma=1}} K_E(u, v; \alpha, \beta, \gamma, \delta)$$

$$= [B(\alpha, \beta)\Gamma(\beta)]^{-1} \frac{e^{-v}}{u^{\alpha+\beta-1}} \int_0^{\min(u,v)} dt\, e^t t^{\alpha-1}(u-t)^{\beta-1}(v-t)^{\beta-1}.$$

The eigenfunctions are the same as in (3.9), but the eigenvalues reduce to

$$\lambda_n = \lim_{\gamma\to\infty} \frac{(\alpha)_n(\gamma)_n}{(\alpha+\beta)_n(\beta+\gamma)_n} = \frac{(\alpha)_n}{(\alpha+\beta)_n}. \tag{3.14}$$

Case VI. $\alpha \to 0^+$, $\delta \to 0^+$, β, $\gamma > 0$. Finally, combining Cases I and II, we obtain the degenerate kernel

$$K_E(u, v; 0, \beta, \gamma, 0) = \lim_{\alpha\to o^+} \lim_{\delta\to 0^+} K_E(u, v; \alpha, \beta, \gamma, \delta) = \frac{v^{\beta-1}(E-v)^{\gamma-1}}{B(\beta, \gamma)E^{\beta+\gamma-1}}. \tag{3.15}$$

There is only one eigenvalue and one eigenfunction, and both are equal to 1.

4. Properties of the kernel $K(x, y; \alpha, \beta, \gamma, \delta)$. In this section, we shall first list a number of interesting properties of the kernel $K(x, y; \alpha, \beta, \gamma, \delta)$ which are, more or less, evident from the manner in which the kernel has been constructed.

Property 1. For $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, $\alpha, \beta, \gamma, \delta > 0$,

$$K(x, y; \alpha, \beta, \gamma, \delta) \geqq 0. \tag{4.1}$$

This is obvious from (2.7).

Property 2. For $n = 0$, $J_n(\alpha+\beta+\gamma+\delta-1, \alpha+\beta; z) = 1 = \lambda_0$. Hence

$$\int_0^1 K(x, y; \alpha, \beta, \gamma, \delta)\, dy = 1. \tag{4.2}$$

Property 3. We shall prove in Theorem 1 that $K(x, y; \alpha, \beta, \gamma, \delta)$ is continuous and therefore bounded on the unit square in the parameter-range $\alpha, \delta > 0$, $\beta, \gamma \geqq 1$.

Property 4. The kernel has the "detailed-balance" property

$$w(x)K(x, y; \alpha, \beta, \gamma, \delta) = w(y)K(y, x; \alpha, \beta, \gamma, \delta). \tag{4.3}$$

Property 5. The symmetric kernel $G(x, y; \alpha, \beta, \gamma, \delta)$ is square-integrable for all $\alpha, \delta > 0$, $\beta, \gamma > \frac{1}{2}$. This is proved in Theorem 1 below. However, if we make use of the property that the set of functions $\{f_n(x)\}_{n=0}^{\infty}$ (see (2.20)) constitutes a complete orthonormal basis for $L_2(0, 1)$, then the square-integrability of the kernel G also follows from the fact that $\sum_{n=0}^{\infty} \lambda_n^2 < \infty$, which we have proved in the Appendix, and the known relation (Tricomi [33])

$$\int_0^1 \int_0^1 dx\, dy\, G^2(x, y; \alpha, \beta, \gamma, \delta) = \sum_{n=0}^{\infty} \lambda_n^2. \tag{4.4}$$

Property 6. The kernel $G(x, y; \alpha, \beta, \gamma, \delta)$ is positive definite over $L_2(0, 1)$. This follows from the fact that the eigenvalues of G are all positive, which has been shown in the Appendix.

Property 7. Properties 1–4 enable us to interpret $K(x, y; \alpha, \beta, \gamma, \delta)$ as a stochastic kernel. For $\alpha, \delta > 0$, β, $\gamma \geqq 1$, it can be regarded as the transition probability for a Markovian stochastic process [12]. Now we shall prove the statements of Properties 3 and 5 in the following theorem.

THEOREM 1.

Part A. *Let* $\alpha, \beta, \gamma, \delta$ *be four real parameters such that*

$$\alpha > 0, \quad \delta > 0, \quad \beta > \tfrac{1}{2}, \quad \gamma > \tfrac{1}{2}. \tag{4.5}$$

Then the kernel

$$\begin{aligned} G(x, y; \alpha, \beta, \gamma, \delta) = {} & [B(\alpha, \beta)B(\beta, \gamma)B(\gamma, \delta)]^{-1}(xy)^{-(\alpha+\beta-1)/2}[(1-x) \\ & \cdot (1-y)]^{-(\gamma+\delta-1)/2} \\ & \cdot \int_0^{\min(x,y)} dt\, t^{\alpha-1}(x-t)^{\beta-1}(y-t)^{\beta-1} \\ & \cdot \int_{\max(x,y)}^1 dz \frac{(1-z)^{\delta-1}(z-x)^{\gamma-1}(z-y)^{\gamma-1}}{(z-t)^{\beta+\gamma-1}} \end{aligned} \tag{4.6}$$

has singularities on the line $y = x$, *but is square-integrable over* $(0, 1)$ *if*

$$\begin{aligned} & \text{(i) } \tfrac{1}{2} < \beta \leqq \gamma < 1 \quad \textit{or} \quad \text{(ii) } \tfrac{1}{2} < \gamma < \beta < \gamma + \tfrac{1}{2}, \\ & \textit{or} \quad \text{(iii) } \tfrac{1}{2} < \beta < 1, \quad \gamma \geqq 1 \quad \textit{or} \quad \text{(iv) } 1 \leqq \beta = \gamma. \end{aligned} \tag{4.7}$$

Part B. *If* $\alpha, \delta > 0$ *and* $1 \leqq \beta < \gamma$, *then* $G(x, y; \alpha, \beta, \gamma, \delta)$ *is continuous and bounded in the closed square* $0 \leqq x \leqq 1, 0 \leqq y \leqq 1$.

Proof of Part A. To fix ideas, let us suppose that $0 \leqq x \leqq y \leqq 1$. By obvious transformations, the kernel G can be expressed as

$$\begin{aligned} G(x, y; \alpha, \beta, \gamma, \delta) = {} & [B(\alpha, \beta)B(\beta, \gamma)B(\gamma, \delta)]^{-1}\left(\frac{x}{y}\right)^{(\alpha+\beta-1)/2} \\ & \cdot \left(\frac{1-y}{1-x}\right)^{(\gamma+\delta-1)/2} R(x, y; \alpha, \beta, \gamma, \delta), \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} R(x, y; \alpha, \beta, \gamma, \delta) = {} & \int_0^1 dt\, t^{\alpha-1}(1-t)^{\beta-1}(y-xt)^{\beta-1} \\ & \cdot \int_0^1 dz \frac{z^{\gamma-1}(1-z)^{\delta-1}[y-x+(1-y)z]^{\gamma-1}}{[y-x+(1-t)x+(1-y)z]^{\beta+\gamma-1}}. \end{aligned} \tag{4.9}$$

It is clear that with the parameters $\alpha, \beta, \gamma, \delta$ restricted by (4.5), $R(x, y; \alpha, \beta, \gamma, \delta)$ is well-behaved in the open region

$$0 < x < y < 1.$$

Hence, for the purpose of square-integrability of G, it is sufficient to investigate the behavior of R as $x \to 0$, $y \to 1$ and $y - x \to 0$. First of all,

$$\begin{aligned} R(0, y; \alpha, \beta, \gamma, \delta) & = y^{\beta-1} \int_0^1 dt\, t^{\alpha-1}(1-t)^{\beta-1} \int_0^1 dz \frac{z^{\gamma-1}(1-z)^{\delta-1}}{[y+(1-y)z]^{\beta}} \\ & = B(\alpha, \beta)B(\gamma, \delta)y^{\beta-1}\, {}_2F_1(\beta, \delta; \gamma+\delta; 1-y). \end{aligned} \tag{4.10}$$

Then

$$R(x, 1; \alpha, \beta, \gamma, \delta) = B(\alpha, \beta)B(\gamma, \delta)(1-x)^{\gamma-1}\,{}_2F_1(\gamma, \alpha; \alpha+\beta; x). \tag{4.11}$$

As long as $y \neq 0$ and $x \neq 1$, both $R(0, y)$ and $R(x, 1)$ are bounded and, indeed,

$$R(0, 1; \alpha, \beta, \gamma, \delta) = B(\alpha, \beta)B(\gamma, \delta). \tag{4.12}$$

However, if $\gamma > \beta$, then

$$R(0, y) \sim y^{-(1-\beta)} \qquad \text{as } y \to 0,$$

and

$$R(x, 1) \sim (1-x)^{-(1-\beta)} \quad \text{as } x \to 1.$$

In deriving these order relations, we have used the following well-known properties of the hypergeometric function:

$$\begin{aligned} {}_2F_1(a, b; c; z) &= (1-z)^{c-a-b}\,{}_2F_1(c-a, c-b; c; z), \\ {}_2F_1(a, b; c; 1) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \operatorname{Re}(c-a-b) > 0, \quad \operatorname{Re} c > 0. \end{aligned} \tag{4.13}$$

(See, for example, [7]).

On the other hand, if $\beta > \gamma$, then

$$R(0, y) \sim y^{-(1-\gamma)} \qquad \text{as } y \to 0,$$

and

$$R(x, 1) \sim (1-x)^{-(1-\gamma)} \quad \text{as } x \to 1.$$

However, if $\frac{1}{2} < \beta < 1$ and $\frac{1}{2} < \gamma < 1$, $\beta \neq \gamma$, there exist singularities at the endpoints $(0, 0)$ and $(1, 1)$, but $R(x, y)$, and therefore $G(x, y)$, remain square-integrable over $(0, 1)$. If $\frac{1}{2} < \beta = \gamma < 1$, then there is an additional logarithmic singularity at both ends, but this also does not affect the square-integrability of $G(x, y)$.

To investigate the behavior of $R(x, y)$ at other points of the diagonal, we note that in the region of integration and for $\beta > \frac{1}{2}$, $\gamma > \frac{1}{2}$,

$$[y - x + (1-t)x + (1-y)z]^{-(\beta+\gamma-1)} \leqq [y - x + (1-y)z]^{-(\beta+\gamma-1)}.$$

Hence

$$\begin{aligned} R(x, y; \alpha, \beta, \gamma, \delta) &\leqq \int_0^1 dt\, t^{\alpha-1}(1-t)^{\beta-1} \\ &\quad \cdot \int_0^1 dz\, z^{\gamma-1}(1-z)^{\delta-1}[y - x\ (1-y)z]^{-\beta} \\ &= B(\alpha, \beta)B(\gamma, \delta)R'(x, y), \end{aligned} \tag{4.14}$$

where

$$R'(x, y) = y^{\beta-1}(1-x)^{-\beta}\,{}_2F_1(1-\beta, \alpha; \alpha+\beta; x/y)\,{}_2F_1\left(\beta, \delta; \gamma+\delta; \frac{1-y}{1-x}\right). \tag{4.15}$$

If $x \neq 0, 1$, $y \neq 0, 1$ and $\gamma > \beta > \frac{1}{2}$, then $R'(x, y)$ has no singularity on the diagonal or anywhere else. However, if $\gamma < \beta$, then we can write, by using the first of the relations (4.13),

$$R'(x, y) = y^{\beta-1}(1-x)^{-\gamma}(y-x)^{\gamma-\beta}\,{}_2F_1(1-\beta, \alpha; \alpha+\beta; x/y) \cdot {}_2F_1\left(\gamma+\delta-\beta, \gamma; \gamma+\delta; \frac{1-y}{1-x}\right).$$

There appear singularities all along the line $y = x$, but $R'(x, y)$ and hence $G(x, y)$, nevertheless, remains $L_2(0, 1)$ if

$$2(\beta-\gamma) < 1, \quad \text{i.e.,} \quad \beta < \gamma + \tfrac{1}{2}.$$

In the event $\beta = \gamma$, the singularity on the diagonal is logarithmic and hence square-integrable on $(0, 1)$.

Now, if $\frac{1}{2} < \beta < 1$, $\gamma \geqq 1$, then

$$R(0, y) \sim y^{-(1-\beta)} \qquad \text{as } y \to 0,$$

and

$$R(x, 1) \sim (1-x)^{-(1-\beta)} \quad \text{as } x \to 1,$$

but there are no singularities elsewhere.

Finally, if $1 \leqq \beta = \gamma$, there is a singularity of the type $\log(y-x)$. However, if $\alpha, \delta \geqq 1$ and $1 \leqq \beta = \gamma$, then this logarithmic singularity remains only at the corners $(0, 0)$ and $(1, 1)$. For, in this case,

$$R(x, y; \alpha, \beta, \gamma, \delta) \leqq \int_0^1 dt\,(y-xt)^{\beta-1} \int_0^1 \frac{dz}{[y-xt+(1-y)z]^{\beta}}$$

$$(4.16) \qquad = \begin{cases} \dfrac{1}{x(1-y)}[(y-x)\log(y-x) - y\log y \\ \qquad\qquad -(1-x)\log(1-x)] & \text{if } \beta = 1, \\ \left\{x - (1-y)\displaystyle\int_{(y-x)/(1-x)}^{y} \frac{u^{\beta-1}}{(1-u)^2}\,du\right\} \Big/ \{(\beta-1)x(1-y)\} & \\ & \text{if } \beta > 1. \end{cases}$$

Use of L'Hôpital's rule will confirm the above statement.

Proof of Part B. When $\beta \geqq 1$ and $\gamma \geqq 1$, equations (4.10) and (4.11) show that there are no singularities at $(0, 0)$ and $(1, 1)$. Further, if $\gamma > \beta$, equation (4.15) shows that there are no singularities anywhere else. Hence $G(x, y; \alpha, \beta, \gamma, \delta)$ is continuous on the closed unit square.

It may be remarked that $K(x, y; \alpha, \beta, \gamma, \delta)$ is bounded whenever $G(x, y; \alpha, \beta, \gamma, \delta)$ is. Even when G has logarithmic singularities at the points $(0, 0)$ and $(1, 1)$, $K(x, y; \alpha, \beta, \gamma, \delta)$ is bounded provided $\alpha + \beta > 1$ and $\gamma + \delta > 1$ for

$$(4.17) \qquad K(x, x; \alpha, \beta, \gamma, \delta) = x^{(\alpha+\beta-1)/2}(1-x)^{(\gamma+\delta-1)/2} G(x, x; \alpha, \beta, \gamma, \delta).$$

5. Properties of the limiting kernels. The limiting kernels that we derived in § 3 all share Properties 1, 2, 3 and 4, and the positive definiteness of the general kernel $K(x, y; \alpha, \beta, \gamma, \delta)$.

According to Theorem 1, the square-integrability and boundedness of the symmetric kernel $G(x, y; \alpha, \beta, \gamma, \delta)$ depend mostly on the relative values of β and γ. Since the limiting kernels are obtained by taking different limiting values of α and δ, it is expected that the conclusions of Theorem 1 will apply, roughly speaking, also to the limiting kernels. However, it is possible to find better bounds for the limiting kernels, being simpler in form. Besides, in the limit $E=\delta\to\infty$ or $\gamma=E\to\infty$, we are no longer in the $L_2(0, 1)$ *space*, rather in $L_2(0,\infty)$. For these reasons we wish to take up each of the limiting kernels and briefly discuss their special properties in various parameter-ranges.

Kernel $K(x, y; 0, \beta, \gamma, \delta)$. By setting $E=1$ in (3.3) and multiplying by an obvious symmetrizing factor, we obtain the limiting ($\alpha\to 0^+$) symmetric kernel

$$G(x, y; 0, \beta, \gamma, \delta)=[B(\beta, \gamma)B(\gamma, \delta)]^{-1}\frac{(xy)^{(\beta-1)/2}}{[(1-x)(1-y)]^{(\gamma+\delta-1)/2}} \cdot\int_{\max(x,y)}^{1} dz\frac{(1-z)^{\delta-1}(z-x)^{\gamma-1}(z-y)^{\gamma-1}}{z^{\beta+\gamma-1}}. \tag{5.1}$$

For $0\leqq x\leqq y\leqq 1$, we get

$$G(x, y; 0, \beta, \gamma, \delta)=[B(\beta, \gamma)B(\gamma, \delta)]^{-1}(xy)^{(\beta-1)/2}\left(\frac{1-y}{1-x}\right)^{(\gamma+\delta-1)/2} \cdot\int_0^1 dz\frac{z^{\gamma-1}(1-z)^{\delta-1}[y-x+(1-y)z]^{\gamma-1}}{[y+(1-y)z]^{\beta+\gamma-1}}. \tag{5.2}$$

On the diagonal $y=x$,

$$\begin{aligned} G(x, x; 0, \beta, \gamma, \delta)&=[B(\beta, \gamma)B(\gamma, \delta)]^{-1}x^{\beta-1}(1-x)^{\gamma-1} \\ &\quad\cdot\int_0^1 dz\, z^{\delta-1}(1-z)^{2\gamma-1}[1-(1-x)z]^{-(\beta+\gamma-1)} \\ &=[B(\beta, \gamma)B(\gamma, \delta)]^{-1}B(\delta, 2\gamma-1)x^{\beta-1}(1-x)^{\gamma-1} \\ &\quad\cdot{}_2F_1(\beta+\gamma-1, \delta; 2\gamma+\delta-1; 1-x). \end{aligned} \tag{5.3}$$

For $\delta>0$ and $\frac{1}{2}<\beta<\gamma<1$, this is continuous except at $(0, 0)$ and $(1, 1)$, where the singularities are of the type $x^{-(1-\beta)}$ and $(1-x)^{-(1-\gamma)}$ respectively, and hence $G(x, y; 0, \beta, \gamma, \delta)$ is $L_2(0, 1)$. If $\beta=\gamma$, there appears, in addition, a logarithmic singularity at $(0, 0)$. If $\frac{1}{2}<\beta<1$ and $\gamma\geqq 1$, the only singularity is at $(0, 0)$ and is of type $x^{-(1-\beta)}$. On the other hand, if $\frac{1}{2}<\gamma<1$, then singularities of type $x^{-(1-\gamma)}$ and $(1-x)^{-(1-\gamma)}$ occur at both $(0, 0)$ and $(1, 1)$. Finally, if $\beta\geqq 1$, $\gamma\geqq 1$, the only possible singularity is at (0, 0) which is (i) logarithmic if $\beta=\gamma$; (ii) nonexistent if $\beta\neq\gamma$.

Hence we conclude that $G(x, y; 0, \gamma, \delta)$ is

(i) $L_2(0, 1)$ if $\delta>0$ and $\beta, \gamma>\frac{1}{2}$;

(ii) continuous in $0\leqq x\leqq 1$, $0\leqq y\leqq 1$ if $\beta, \gamma\geqq 1$ and $\beta\neq\gamma$.

Kernel $K(x, y; \alpha, \beta, \gamma, 0)$. The properties of the symmetrized kernel $G(x, y; \alpha, \beta, \gamma, 0)$ are, indeed, identical to those of $G(x, y; 0, \beta, \gamma, \delta)$ with the interchange of β and γ and the condition $\delta>0$ replaced by $\alpha>0$.

Kernel $K_\infty(x, y; \alpha, \beta, \gamma, \infty)$. The symmetrized kernel in this case reduces, for $0\leqq x\leqq y<\infty$, *to*

$$G(x, y; \alpha, \beta, \gamma, \infty)=[B(\alpha, \beta)B(\beta, \gamma)\Gamma(\gamma)]^{-1}\, e^{(x-y)/2}\left(\frac{x}{y}\right)^{(\alpha+\beta-1)/2} \cdot Q(x, y; \alpha, \beta, \gamma, \infty), \tag{5.4}$$

where

$$Q(x, y; \alpha, \beta, \gamma, \infty)=\int_0^1 dt\, t^{\alpha-1}(1-t)^{\beta-1}(y-xt)^{\beta-1} \cdot \int_0^\infty dz\, \frac{e^{-z}z^{\gamma-1}(y-x+z)^{\gamma-1}}{(y-xt+z)^{\beta+\gamma-1}}. \tag{5.5}$$

Note that

$$\begin{aligned} Q(x, x; \alpha, \beta, \gamma, \infty) &= x^{\gamma-1}\int_0^1 dt\, t^{\alpha-1}(1-t)^{\beta+\gamma-2}\int_0^\infty dz\, \frac{z^{2\gamma-2}\, e^{-x(1-t)z}}{(1+z)^{\beta+\gamma-1}} \\ &= \Gamma(2\gamma-1)x^{\gamma-1}\int_0^1 dt\, t^{\alpha-1}(1-t)^{\beta+\gamma-2} \\ &\quad \cdot U(2\gamma-1, \gamma-\beta+1; x(1-t)), \end{aligned} \tag{5.6}$$

where

$$U(a, b; z)=\frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}, \qquad \operatorname{Re} a>0, \tag{5.7}$$

is a confluent hypergeometric function having a singularity at the origin (see, for example, [1]).

If there are any singularities of $G(x, y; \alpha, \beta, \gamma, \infty)$, one should be able to spot them in $Q(x, x; \alpha, \beta, \gamma, \infty)$ alone. Since for large y, $Q(x, y; \alpha, \beta, \gamma, \infty)$ decreases at least like $1/y$, the behavior of G at ∞ is controlled by the exponential term $e^{-y/2}$.

Now the necessary conditions for the convergence of the integral in $Q(x, x; \alpha, \beta, \gamma, \infty)$ are

$$\alpha>0, \quad \beta+\gamma>1, \quad \gamma>\tfrac{1}{2}. \tag{5.8}$$

When these inequalities are satisfied, the only possible singularity remains at $x=0$. For small x and $0<t<1$,

$$U(2\gamma-1,\gamma-\beta+1;x(1-t))$$

$$(5.9)\qquad =\begin{cases} \dfrac{\Gamma(\gamma-\beta)}{\Gamma(2\gamma-1)}[x(1-t)]^{\beta-\gamma}+O([x(1-t)]^{\gamma-\beta-1}), & \gamma-\beta>1,\\ \dfrac{\Gamma(\gamma-\beta)}{\Gamma(2\gamma-1)}[x(1-t)]^{\beta-\gamma}+O(\log x(1-t)), & \gamma-\beta=1,\\ \dfrac{\Gamma(\gamma-\beta)}{\Gamma(2\gamma-1)}[x(1-t)]^{\beta-\gamma}+O(1), & 0<\gamma-\beta<1,\\ -\dfrac{\log x(1-t)}{\Gamma(2\gamma-1)}+O([x(1-t)\log x(1-t)], & \gamma=\beta,\\ \dfrac{\Gamma(\beta-\gamma)}{\Gamma(\beta+\gamma-1)}+O([x(1-t)]^{\beta-\gamma}), & 0<\beta-\gamma<1,\\ \dfrac{1}{\Gamma(2\gamma)}+O([x(1-t)\log x(1-t)], & \beta-\gamma=1,\\ \dfrac{\Gamma(\beta-\gamma)}{\Gamma(\beta+\gamma-1)}+O([x(1-t)), & \beta-\gamma>1. \end{cases}$$

(See [1, p. 508].)

Hence $Q(x,x;\alpha,\beta,\gamma,\infty)$ has a singularity at $x=0$ of the type $x^{-(1-\beta)}$ or $x^{-(1-\gamma)}$ if $\frac{1}{2}<\beta\neq\gamma\leqq 1$, and a logarithmic singularity at $x=0$ if $\beta=\gamma$ whether or not they are less than, equal to or greater than 1. The singularity at the origin disappears if $\beta\neq\gamma$ *and* $\beta\geqq 1$, $\gamma\geqq 1$.

Therefore the kernel $G(x,y;\alpha,\beta,\gamma,\infty)$ is bounded in any closed interval $0\leqq x\leqq\xi$, $0\leqq y\leqq\xi$, $\xi<\infty$ if $\beta\geqq 1$, $\gamma\geqq 1$ *and* $\beta\neq\gamma$.

Kernel $K_\infty(x,y;0,\beta,\gamma,\infty)$. For this case we have

$$(5.10)\qquad \begin{aligned} G(x,y;0,\beta,\gamma,\infty)&=[B(\beta,\gamma)\Gamma(\gamma)]^{-1}e^{(x+y)/2}(xy)^{(\beta-1)/2}\\ &\quad\cdot\int_{\max(x,y)}^{\infty}dz\,\frac{e^{-z}(z-x)^{\gamma-1}(z-y)^{\gamma-1}}{z^{\beta+\gamma-1}}. \end{aligned}$$

For $0\leqq x\leqq y\leqq\infty$,

$$(5.11)\qquad G(x,y;0,\beta,\gamma,\infty)=[B(\beta,\gamma)\Gamma(\gamma)]^{-1}e^{(x-y)/2}\left(\frac{x}{y}\right)^{(\beta-1)/2}Q(x,y;0,\beta,\gamma,\infty),$$

where

$$(5.12)\qquad Q(x,y;0,\beta,\gamma,\infty)=\int_0^\infty dz\,z^{\gamma-1}e^{-yz}(y-x+yz)^{\gamma-1}(1+z)^{-(\beta+\gamma-1)}.$$

In particular,

$$(5.13)\qquad Q(x,x;0,\beta,\gamma,\infty)=\Gamma(2\gamma-1)x^{\gamma-1}U(2\gamma-1,\gamma-\beta+1;x).$$

For large x, $Q\sim x^{-\gamma}$, and for small x, it is $O(x^{\beta-1})$ if $\gamma\geqq\beta+1$ or $0<\gamma<\beta+1$; of the order $O(x^{\gamma-1})$ if $\beta\geqq\gamma+1$ or $0<\beta\leqq\gamma+1$, and $O(\log x)$ if $\beta=\gamma$. Hence our conclusions remain the same as in the previous case.

Kernel $K_\infty(x, y; \alpha, \beta, \infty, 0)$. Here we have

$$G(x, y; \alpha, \beta, \infty, 0) \tag{5.14}$$
$$= [B(\alpha, \beta)\Gamma(\beta)]^{-1} e^{-(x+y)/2}(xy)^{-(\alpha+\beta-1)/2} \int_0^{\min(x,y)} dt\, e^t t^{\alpha-1}(x-t)^{\beta-1}(y-t)^{\beta-1}$$

and for $0 \leqq x \leqq y < \infty$,

$$G(x, y; \alpha, \beta, \infty, 0) \tag{5.15}$$
$$= [B(\alpha, \beta)\Gamma(\beta)]^{-1} e^{-(x+y)/2}\left(\frac{x}{y}\right)^{-(\alpha+\beta-1)/2} \int_0^1 dt\, e^{xt} t^{\alpha-1}(1-t)^{\beta-1}(y-xt)^{\beta-1}.$$

It can be easily seen, by arguments similar to the previous cases, that $G(x, y; \alpha, \beta, \infty, 0)$ is square-integrable in $L_2(0, \infty)$ if $\alpha > 0$, $\beta > \frac{1}{2}$, and is bounded everywhere except $x = 0$ if $\beta < 1$. For $\alpha > 0$, $\beta \geqq 1$, G is in $L_2(0, \infty)$ as well as bounded everywhere.

6. Bilinear formulas. Now that we have completed the discussion of the square-integrability and continuity of the symmetric kernel $G(x, y; \alpha, \beta, \gamma, \delta)$ and its various limiting forms, we may write down a number of bilinear formulas involving the Jacobi and Laguerre polynomials.

Formula I.

$$\begin{aligned}
&[B(\alpha, \beta)B(\beta, \gamma)B(\gamma, \delta)]^{-1}(xy)^{-(\alpha+\beta-1)/2}[(1-x)(1-y)]^{-(\gamma+\delta-1)/2}\\
&\quad \cdot \int_0^{\min(x,y)} dt\, t^{\alpha-1}(x-t)^{\beta-1}(y-t)^{\beta-1}\\
&\quad \cdot \int_{\max(x,y)}^1 dz \frac{(1-z)^{\delta-1}(z-x)^{\gamma-1}(z-y)^{\gamma-1}}{(z-t)^{\beta+\gamma-1}}\\
&= \sum_{n=0}^{\infty} \lambda_n f_n(x) f_n(y),
\end{aligned} \tag{6.1}$$

where λ_n is given by (2.13) and (2.16) and $f_n(x)$ is defined by (2.20) and (2.21).

Since the kernel on the left-hand side is continuous on the closed square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ for $\alpha, \delta > 0$, $1 \leqq \beta < \gamma$, by Theorem 1, the infinite series on the right is uniformly and absolutely convergent for all x, y in this parameter-domain, according to Mercer's theorem (Tricomi [33]). For other parameter-values for which the kernel is square-integrable, the convergence of (6.1) has to be understood as convergence in the mean.

However, even if the kernel is continuous in a restricted region $\varepsilon_1 \leqq x \leqq 1 - \varepsilon_1$, $\varepsilon_1 > 0$ and $\varepsilon_2 \leqq y \leqq 1 - \varepsilon_2$, $\varepsilon_2 > 0$ (see Theorem 1), and has a piecewise continuous derivative in each variable, expansion formula (6.1) remains valid in this restricted region, following a result of Rau [31]. It can be shown that in the cases $\alpha, \delta > 0$, $\frac{1}{2} < \beta < \gamma < 1$ or $\frac{1}{2} < \beta < 1$, $\gamma \geqq 1$ or $1 \leqq \beta = \gamma$, conditions of Rau's theorem are satisfied.

Formula II.

$$(6.2)\quad \begin{aligned}&[B(\beta,\gamma)B(\gamma,\delta)]^{-1}[(1-x)(1-y)]^{-(\gamma+\delta-1)}\\ &\cdot\int_{\max(x,y)}^{1} dz\,\frac{(1-z)^{\delta-1}(z-x)^{\gamma-1}(z-y)^{\gamma-1}}{z^{\beta+\gamma-1}}\\ &=\sum_{n=0}^{\infty}\frac{(\beta)_n(\delta)_n}{(\beta+\gamma)_n(\gamma+\delta)_n}\,\frac{(2n+\beta+\gamma+\delta-1)\Gamma(n+\beta)\Gamma(n+\beta+\gamma+\delta-1)}{\Gamma^2(\beta)\Gamma(n+\gamma+\delta)n!}\\ &\cdot {}_2F_1(-n,n+\beta+\gamma+\delta-1;\beta;x)\,{}_2F_1(-n,n+\beta+\gamma+\delta-1;\beta;y).\end{aligned}$$

If $\delta>0$, $\beta\neq\gamma$ and $\beta,\gamma\geqq 1$, this formula is valid for all x, y in $[0,1]$. If $\beta=\gamma$ or $\beta,\gamma>\frac{1}{2}$ but one of them is less than 1, then by Rau's theorem, (6.2) applies in the restricted region $0<x<1$ and $0<y<1$.

Formula III.

$$(6.3)\quad \begin{aligned}&[B(\alpha,\beta)B(\beta,\gamma)\Gamma(\gamma)]^{-1}e^{x+y}(xy)^{-(\alpha+\beta-1)}\\ &\cdot\int_0^{\min(x,y)} dt\,t^{\alpha-1}(x-t)^{\beta-1}(y-t)^{\beta-1}\\ &\cdot\int_{\max(x,y)}^{\infty} dz\,\frac{e^{-z}(z-x)^{\gamma-1}(z-y)^{\gamma-1}}{(z-t)^{\beta+\gamma-1}}\\ &=\sum_{n=0}^{\infty}{}_3F_2\left[\begin{matrix}-n,\beta,\gamma\\ \alpha+\beta,\beta+\gamma\end{matrix};1\right]\frac{n!}{\Gamma(\alpha+\beta+n)}L_n^{(\alpha+\beta-1)}(x)L_n^{(\alpha+\beta-1)}(y).\end{aligned}$$

If $\alpha>0$, $\beta\geqq 1$, $\gamma\geqq 1$ and $\beta\neq\gamma$, then (6.3) is valid for every x, y such that $0\leqq x\leqq\xi$, $0\leqq y\leqq\xi$, $\xi<\infty$. If $\beta=\gamma$ and/or $\frac{1}{2}<\beta,\gamma<1$, then the convergence is in the mean, but if we exclude the point $x=0$, then the infinite series can be shown to be uniformly and absolutely convergent in any region $\varepsilon_1\leqq x<\infty$, $\varepsilon_2\leqq y<\infty$, $\varepsilon_1,\varepsilon_2>0$.

Formula IV.

$$(6.4)\quad \begin{aligned}&[\Gamma(\gamma)]^{-2}e^{x+y}\int_{\max(x,y)}^{\infty} dz\,\frac{e^{-z}(z-x)^{\gamma-1}(z-y)^{\gamma-1}}{z^{\beta+\gamma-1}}\\ &=\sum_{n=0}^{\infty}\frac{n!}{\Gamma(\beta+\gamma+n)}L_n^{(\beta-1)}(x)L_n^{(\beta-1)}(y).\end{aligned}$$

The regions of validity of this formula are the same as for Formula III. When $\beta=1$, $\gamma=1$, this formula reduces to Koschmeider's formula [28], [34].

Formula V.

$$(6.5)\quad \begin{aligned}&[\Gamma(\beta)]^{-2}\int_0^{\min(x,y)} dt\,e^t t^{\alpha-1}(x-t)^{\beta-1}(y-t)^{\beta-1}\\ &=(xy)^{\alpha+\beta-1}\sum_{n=0}^{\infty}\frac{n!\Gamma(\alpha+n)}{\Gamma^2(\alpha+\beta+n)}L_n^{(\alpha+\beta-1)}(x)L_n^{(\alpha+\beta-1)}(y).\end{aligned}$$

For $\alpha>0$, $\beta>\frac{1}{2}$, this formula is valid for all x, y such that $\varepsilon_1\leqq x<\infty$, $\varepsilon_2\leqq y<\infty$, $\varepsilon_1,\varepsilon_2>0$. For $\alpha>0$ and $\beta\geqq 1$, this is valid even at the origin $(0,0)$. This formula is essentially the same as obtained by Erdélyi [13].

7. Note on previous work. As was indicated in the Introduction, the kernel (2.7), and some of the limiting kernels, have been known for some years in the theory of certain processes in statistical mechanics. In this context, they arise naturally from a class of "urn-models" for model stochastic processes, whose transition probabilities depend on particularly simple combinations of random variables. In most cases, their eigenvalue problems have been solved, and the consequent bilinear formulas are thus known as spectral resolutions of the appropriate transition kernels. Specifically, the kernel $K_\infty(x, y; 0, \beta, \gamma, \infty)$ was derived from statistical considerations by Hoare [18], who solved the eigenvalue problem for $K_\infty(x, y; 0, 1, 1, \infty)$. Later Hoare and Thiele [19] derived the kernel $K_E(u, v; 0, 1, 1, \delta)$ and showed its eigenfunctions to be Jacobi polynomials. Still later Hoare and Cooper [11], [12] solved the eigenvalue problem for $K_\infty(x, y; 0, \beta, \gamma, \infty)$, obtaining the bilinear formula (6.4), and have since extended their results to the kernel $K(x, y; 0, \beta, \gamma, \delta)$, obtaining the eigenfunctions (2.1) and the spectral resolution (6.2) [12]. These authors have derived the full kernel $K_E(u, v; \alpha, \beta, \gamma, \delta)$ and obtained its eigenvalues but, at the time of writing, do not appear to have obtained the eigenfunctions (2.1) [20]. In all this work, the parameters $\alpha, \beta, \gamma, \delta$ arise as positive integers representing stochastic "degrees of freedom", and the starting point has invariably been the integral operator and its eigenvalue problem, rather than the reverse construction considered in this paper. For a full account of the probabilistic implications of this class of kernels, see [21].

Appendix A.

THEOREM A.1. *For a positive integer n and an integer p such that* $0 \leqq p \leqq n$,

$$S(n; p) = \frac{p!}{(\beta+\gamma)_p} \sum_{k=0}^{p} \frac{(\alpha)_{p-k}(\beta)_k(\gamma)_{p-k}(\delta)_k}{(\alpha+\beta)_{p-k}(\gamma+\delta)_k(p-k)!k!} \cdot {}_4F_3\left[\begin{matrix} -n+p, n+\alpha+\beta+\gamma+\delta+p-1, \beta+k, \gamma \\ \alpha+\beta+p, \beta+\gamma+p, \gamma+\delta+k \end{matrix}; 1\right]$$

is independent of p and is equal to

$$\lambda_n = S(n; 0) = {}_4F_3\left[\begin{matrix} -n, n+\alpha+\beta+\gamma+\delta-1, \beta, \gamma \\ \alpha+\beta, \beta+\gamma, \gamma+\delta \end{matrix}; 1\right].$$

Proof. The theorem is proved very easily if we make use of the following two known results for Saalschutzian series:

$$\text{(A.1)} \qquad {}_3F_2\left[\begin{matrix} a, b, -m \\ c, 1+a+b-c-m \end{matrix}; 1\right] = \frac{(c-a)_m(c-b)_m}{(c)_m(c-a-b)_m},$$

where m is a positive integer, and

(A.2)

$${}_4F_3\left[\begin{matrix} x, y, z, -p \\ u, v, w \end{matrix}; 1\right] = \frac{(v-z)_p(w-z)_p}{(v)_p(w)_p} {}_4F_3\left[\begin{matrix} u-x, u-y, z, -p \\ 1-v+z-p, 1-w+z-p, u \end{matrix}; 1\right],$$

where p is, again, a positive integer and

$$u+v+w=x+y+z-p+1 \tag{A.3}$$

(see Bailey [6]).

By using the identity

$$(a)_{N-n}=(-1)^n(a)_N/(1-a-N)_n, \tag{A.4}$$

where $N \geqq n$ and both are positive integers, we may write $S(n;p)$ in the form

$$S(n;p)=\sum_{l=0}^{n-p}\frac{(-n+p)_l(n+\alpha+\beta+\gamma+\delta+p-1)_l(\beta)_l(\gamma)_l}{(\alpha+\beta+p)_l(\beta+\gamma+p)_l(\gamma+\delta)_l l!}A_{p,l}, \tag{A.5}$$

with

$$\begin{aligned} A_{p,l}&=\frac{(\alpha)_p(\gamma)_p}{(\alpha+\beta)_p(\beta+\gamma)_p}\sum_{k=0}^{p}\frac{(-p)_k(1-\alpha-\beta-p)_k(\beta+l)_k(\delta)_k}{(1-\alpha-p)_k(1-\gamma-p)_k(\gamma+\delta+l)_k k!}\\ &=\frac{(\alpha)_p(\gamma)_p}{(\alpha+\beta)_p(\beta+\gamma)_p}\,{}_4F_3\left[\begin{matrix}1-\alpha-\beta-p,\ \delta,\ \beta+l,\ -p\\ l+\gamma+\delta,\ 1-\alpha-p,\ 1-\gamma-p\end{matrix};\ 1\right].\end{aligned} \tag{A.6}$$

Note that the parameters of this ${}_4F_3$ satisfy the Saalschutzian condition (A.3).

By (A.2), we have the transformation

$$\begin{aligned} A_{p,l}&=\frac{(\alpha)_p(\gamma)_p(1-\alpha-\beta-p-l)_p(1-\beta-\gamma-p-l)_p}{(\alpha+\beta)_p(\beta+\gamma)_p(1-\alpha-p)_p(1-\gamma-p)_p}\\ &\quad\cdot {}_4F_3\left[\begin{matrix}p+l+\alpha+\beta+\gamma+\delta-1,\ \gamma+l,\ \beta+l,\ -p\\ \alpha+\beta+l,\ \beta+\gamma+l,\ \gamma+\delta+l\end{matrix};\ 1\right]\\ &=\frac{(\alpha+\beta+l)_p(\beta+\gamma+l)_p}{(\alpha+\beta)_p(\beta+\gamma)_p}\,{}_4F_3\left[\begin{matrix}p+l+\alpha+\beta+\gamma+\delta-1,\ \beta+l,\ \gamma+l,\ -p\\ \alpha+\beta+l,\ \beta+\gamma+l,\ \gamma+\delta+l\end{matrix};1\right].\end{aligned} \tag{A.7}$$

In deriving the last expression for $A_{p,l}$, we have also made use of (A.4).

Now

$$\frac{(\alpha+\beta+l)_p}{(\alpha+\beta)_p(\alpha+\beta+p)_l}=\frac{(\alpha+\beta)_{p+l}}{(\alpha+\beta)_l(\alpha+\beta)_{p+l}}=\frac{1}{(\alpha+\beta)_l},\quad \text{etc.}$$

Hence

$$\begin{aligned} &S(n;p)\\ &=\sum_{l=0}^{n-p}\sum_{k=0}^{p}\frac{(-n+p)_l(n+\alpha+\beta+\gamma+\delta+p-1)_l(\beta)_l(\gamma)_l(l+p+\alpha+\beta+\gamma+\delta-1)_k(\beta+l)_k(\gamma+l)_k(-p)_k}{(\alpha+\beta)_l(\beta+l)_l(\gamma+\delta)_l(\alpha+\beta+l)_k(\beta+\gamma+l)_k(\gamma+\delta+l)_k l!k!}\\ &=\sum_{l=0}^{n-p}\sum_{k=0}^{p}\frac{(-n+p)_l(n+\alpha+\beta+\gamma+\delta+p-1)_l(-p)_k(\beta)_{k+l}(\gamma)_{k+l}(p+\alpha+\beta+\gamma+\delta-1)_{k+l}}{(p+\alpha+\beta+\gamma+\delta-1)_l(\alpha+\beta)_{k+l}(\beta+\gamma)_{k+l}(\gamma+\delta)_{k+l}l!k!}.\end{aligned} \tag{A.8}$$

Let us make the transformation

$$k+l=m, \quad l=m-k. \tag{A.9}$$

Then m runs from 0 to n, while k goes from 0 to m.

Hence

$$S(n;p)=\sum_{m=0}^{n}\frac{(p+\alpha+\beta+\gamma+\delta-1)_m(\beta)_m(\gamma)_m}{(\alpha+\beta)_m(\beta+\gamma)_m(\gamma+\delta)_m}B_{n,m}, \tag{A.10}$$

where

$$\begin{aligned}B_{n,m}&=\sum_{k=0}^{m}\frac{(-n+p)_{m-k}(n+\alpha+\beta+\gamma+\delta+p-1)_{m-k}(-p)_k}{(p+\alpha+\beta+\gamma+\delta-1)_{m-k}(m-k)!\,k!}\\&=\frac{(n+\alpha+\beta+\gamma+\delta+p-1)_m(-n+p)_m}{(p+\alpha+\beta+\gamma+\delta-1)_m m!}\\&\quad\cdot\sum_{k=0}^{m}\frac{(-m)_k(-p)_k(2-p-\alpha-\beta-\gamma-\delta-m)_k}{(1+n-p-m)_k(2-n-\alpha-\beta-\gamma-\delta-p-m)_k k!}\\&=\frac{(-n+p)_m(n+\alpha+\beta+\gamma+\delta+p-1)_m}{(p+\alpha+\beta+\gamma+\delta-1)_m m!}\\&\quad\cdot{}_3F_2\left[\begin{matrix}-p,\,2-p-\alpha-\beta-\gamma-\delta-m,\,-m\\1+n-p-m,\,2-n-\alpha-\beta-\gamma-\delta-p-m\end{matrix};\;1\right].\end{aligned} \tag{A.11}$$

The ${}_3F_2$ series in (A.11) is of the form (A.1) and, therefore, we obtain

$$\begin{aligned}B_{n,m}&=\frac{(-n+p)_m(n+\alpha+\beta+\gamma+\delta+p-1)_m}{(p+\alpha+\beta+\gamma+\delta-1)_m m!}\\&\quad\cdot\frac{(1+n-m)_m(n+\alpha+\beta+\gamma+\delta-1)_m}{(1+n-p-m)_m(n+\alpha+\beta+\gamma+\delta+p-1)_m}\\&=\frac{(-n)_m(n+\alpha+\beta+\gamma+\delta-1)_m}{(p+\alpha+\beta+\gamma+\delta-1)_m m!}.\end{aligned} \tag{A.12}$$

Finally, then,

$$\begin{aligned}S(n;p)&=\sum_{m=0}^{n}\frac{(-n)_m(n+\alpha+\beta+\gamma+\delta-1)_m(\beta)_m(\gamma)_m}{(\alpha+\beta)_m(\beta+\gamma)_m(\gamma+\delta)_m m!}\\&={}_4F_3\left[\begin{matrix}-n,\,n+\alpha+\beta+\gamma+\delta-1,\,\beta,\,\gamma\\\alpha+\beta,\,\beta+\gamma,\,\gamma+\delta\end{matrix};\;1\right].\end{aligned}$$

THEOREM A.2. *If* $\alpha, \beta, \gamma, \delta>0$, *then*

$$0<\lambda_n\leqq 1. \tag{A.13}$$

If, in addition, $\beta>\frac{1}{2}$, $\gamma>\frac{1}{2}$, *then*

$$\sum_{n=0}^{\infty}\lambda_n^p<\infty, \qquad p\geqq 1. \tag{A.14}$$

Proof. From Theorem A.1 it follows that

$$\lambda_n = {}_4F_3\left[\begin{matrix}-n,\ n+\alpha+\beta+\gamma+\delta-1,\ \beta,\ \gamma \\ \alpha+\beta,\ \beta+\gamma,\ \gamma+\delta\end{matrix};\ 1\right] \tag{A.15}$$

$$\frac{n!}{(\beta+\gamma)_n}\sum_{k=0}^{n}\frac{(\alpha)_{n-k}(\beta)_k(\gamma)_{n-k}(\delta)_k}{(\alpha+\beta)_{n-k}(\gamma+\delta)_k(n-k)!k!}.$$

For α, β, γ, $\delta>0$, all the terms in the finite sum are positive, and hence $\lambda_n>0$. Also,

$$\frac{(\alpha)_{n-k}}{(\alpha+\beta)_{n-k}}\leqq 1,\quad \frac{(\delta)_k}{(\gamma+\delta)_k}\leqq 1,\qquad 0\leqq k\leqq n.$$

Hence

$$\lambda_n \leqq \frac{n!}{(\beta+\gamma)_n}\sum_{k=0}^{n}\frac{(\gamma)_{n-k}(\beta)_k}{(n-k)!k!}=\frac{(\gamma)_n}{(\beta+\gamma)_n}\sum_{k=0}^{n}\frac{(-n)_k(\beta)_k}{(1-\gamma-n)_k k!}=1.$$

To prove (A.14), let us consider the sum $\sum_{n=0}^{\infty} c_n x^n$, $0\leqq x<1$, where

$$c_n=\sum_{k=0}^{n}\frac{(\alpha)_{n-k}(\gamma)_{n-k}}{(\alpha+\beta)_{n-k}(n-k)!}\cdot\frac{(\beta)_k(\delta)_k}{(\gamma+\delta)_k k!}. \tag{A.16}$$

From the convolution nature of this sum it is obvious that

$$\sum_{n=0}^{\infty} c_n x^n=\left[\sum_{n=0}^{\infty}\frac{(\alpha)_n(\gamma)_n}{(\alpha+\beta)_n n!}x^n\right]\left[\sum_{n=0}^{\infty}\frac{(\beta)_n(\delta)_n}{(\gamma+\delta)_n n!}x^n\right] \tag{A.17}$$

$$={}_2F_1(\alpha,\gamma;\alpha+\beta;x)\,{}_2F_1(\beta,\delta;\gamma+\delta;x),$$

whenever the infinite sums converge. For $0\leqq x<1$, the infinite sum on the left obviously does converge, but we are here interested in the limit $x\to 1-$.

Suppose $\gamma>\beta>0$. Then

$$\lim_{x\to 1-}{}_2F_1(\beta,\delta;\gamma+\delta;x)=\frac{\Gamma(\gamma+\delta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)\Gamma(\gamma+\delta-\beta)}.$$

But ${}_2F_1(\alpha,\gamma;\alpha+\beta;x)$ diverges like

$$(1-x)^{-(\gamma-\beta)}=\sum_{n=0}^{\infty}\frac{(\gamma-\beta)_n}{n!}x^n.$$

Hence for large n we can say

$$c_n\sim(\gamma-\beta)_n/n!.$$

Therefore

$$\lambda_n=O\left[\frac{n!}{(\gamma+\beta)_n}\cdot\frac{(\gamma-\beta)_n}{n!}\right]=O[n^{-2\beta}].$$

We conclude, then, $\sum_{n=0}^{\infty}\lambda_n<\infty$ if $\beta>\frac{1}{2}$. Similarly, if $\beta>\gamma$, $\sum_{n=0}^{\infty}\lambda_n<\infty$ if $\gamma>\frac{1}{2}$. Finally, let $\beta=\gamma$. Then both the hypergeometric functions diverge as $x\to 1-$, but they diverge like $\log(1-x)$. Since

$$[\log(1-x)]^2=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n-1}\frac{1}{k(n-k)}\right)x^n,$$

it follows that

$$c_n \sim \sum_{k=1}^{n}\frac{1}{k(n-k)}=\frac{2}{n}\sum_{1}^{n}\frac{1}{k}\sim\frac{2\log n}{n}.$$

Hence

$$\lambda_n=O\left[\frac{n!}{(2\beta)_n}\frac{\log n}{n}\right]=O\left[\frac{\log n}{n^{2\beta}}\right].$$

For $\beta>\frac{1}{2}$, $\sum_{n=0}^{\infty}\lambda_n$ is again convergent. It follows trivially that $\sum_{n=0}^{\infty}\lambda_n^p<\infty$ for $p>1$, hence the theorem.

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MODULI OF MONOTONICITY WITH APPLICATIONS TO MONOTONE POLYNOMIAL APPROXIMATION*

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Abstract. This article introduces new concepts called the moduli of monotonicity of a real function defined on an interval. They are a one-sided analogue of the well-known modulus of continuity, and are a measure of the extent by which a given function fails to be monotone. It is shown that they naturally arise in the process of approximating a real function by nondecreasing polynomials. Upper and lower bounds on the "degree of approximation" by monotone polynomials are derived in terms of these moduli.

1. Introduction. The main purpose of this article is to introduce certain new concepts called the moduli of monotonicity of a function and indicate their applications to approximation theory. Roughly speaking, the decreasing and increasing moduli of monotonicity are a one-sided analogue of the well-known modulus of continuity and are a measure of the extent by which a given real function defined on an interval fails to be monotone. It is shown that they arise naturally in the process of approximating a continuous function by monotone polynomials on an interval. Bounds on the "degree of approximation" by monotone polynomials are derived by making use of the Friedrichs mollifier functions (Morrey [9]) and the moduli mentioned above. It is a well-known fact (Lorentz [5], Meinardus [8]) that the modulus of continuity plays an important role in the theory of approximation of a continuous function by polynomials (not necessarily monotone); however, it will be seen from the results of this article that moduli of monotonicity, and not the modulus of continuity, appear predominantly in the analysis of the problem of approximation by monotone polynomials.

To introduce the relevant concepts, let B denote the set of all bounded real-valued functions on a closed real interval $I = [a, b]$ of length $l = b - a$. For any f in B, define

$$\omega(\delta) = \omega(f, \delta) = \sup_{x,y\in I,\, |x-y|\leqq\delta} |f(y) - f(x)|, \qquad \delta \in [0, l]. \tag{1.1}$$

The nonnegative bounded function $\omega(f, \cdot)$, defined on $[0, l]$, for f fixed, is known as the modulus of continuity of f. Analogously, for any f in B, we define on $[0, l]$ two nonnegative bounded functions $\underline{\mu}(f, \cdot)$ and $\bar{\mu}(f, \cdot)$ by

$$\underline{\mu}(\delta) = \underline{\mu}(f, \delta) = \sup_{x,y\in I,\, 0\leqq y-x\leqq\delta} (f(y) - f(x)), \qquad \delta \in [0, l], \tag{1.2}$$

$$\bar{\mu}(\delta) = \bar{\mu}(f, \delta) = \sup_{x,y\in I,\, 0\leqq y-x\leqq\delta} (f(x) - f(y)), \qquad \delta \in [0, l]. \tag{1.3}$$

The functions $\underline{\mu}(f, \cdot)$ and $\bar{\mu}(f, \cdot)$ are called the moduli of monotonicity, decreasing and increasing respectively, of the function f. As was observed before, it is easily

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seen that $\underline{\mu}(f, \delta)$ ($\bar{\mu}(f, \delta)$) is a measure of the extent by which the function f in B fails to be nonincreasing (nondecreasing) on an interval of length δ. It is also obvious that $\omega = \max(\underline{\mu}, \bar{\mu})$. Thus $\underline{\mu}$ and $\bar{\mu}$ give a decomposition of ω in this sense.

Let P_n denote the class of all nondecreasing polynomials on I of degree at most n. Given a continuous function f defined on I, not necessarily nondecreasing, the problem of monotone polynomial approximation is to find a q_n in P_n such that $\|f - q_n\|$ minimizes $\|f - p_n\|$ for all p_n in P_n, where $\|\cdot\|$ is the uniform or supremum norm. The number $E_n(f)$ defined by

$$(1.4) \qquad E_n(f) = \|f - q_n\| = \min_{p_n \in P_n} \|f - p_n\|$$

is known as the "degree of approximation" of f by the polynomials of the class P_n. The existence of such a minimizing q_n can be easily demonstrated by using standard compactness arguments applied to finite-dimensional spaces.

In § 2, we examine briefly the properties of $\underline{\mu}$ and $\bar{\mu}$ which are similar to those of ω. In § 3, we investigate the existence of a continuous function f defined on I such that its moduli of monotonicity $\underline{\mu}(f, \cdot)$, $\bar{\mu}(f, \cdot)$ equal respectively two given functions on $[0, l]$ having properties of a modulus of continuity. This investigation parallels a similar well-known question concerning the existence of a continuous function f on I, whose modulus of continuity $\omega(f, \cdot)$ equals a given modulus of continuity τ defined on $[0, l]$. The existence of such a function f is trivially established by setting $f(x) = \tau(x - a)$ for all $x \in I$, however, the issues raised in § 3 are more difficult to answer. The applications part, § 4, is devoted to the analysis necessary to establish upper and lower bounds on $E_n(f)$, the degree of approximation by monotone polynomials. It will be shown that the moduli of monotonicity play a prime role in these bounds. This situation again corresponds to the one encountered in determining bounds on the degree of approximation by polynomials, not necessarily nondecreasing, wherein the modulus of continuity plays an important part. (See Jackson [3], Lorentz [5], Meinardus [8].) Lorentz and Zeller [7] have obtained bounds on $E_n(f)$ when f itself is continuous and nondecreasing. Shisha [11] and Roulier [10] consider the problem of approximating a continuous function by polynomials p_n of degree at most n satisfying $p_n^{(k)}(x) \geqq 0$ for all $x \in I$, for a fixed $k \geqq 1$, and obtain bounds on the degree of approximation from this class of polynomials under various differentiability and other conditions on f. We examine the case when f is continuous but not nondecreasing, and obtain both upper and lower bounds on $E_n(f)$, without imposing any additional restrictions on f. Thus our results complement those of Lorentz and Zeller. Roughly speaking, we show that for a fixed continuous f which is not nondecreasing and any fixed $k \geqq 1$, $E_n(f)$, which is bounded below by $(1/2)\bar{\mu}(f, l)$, converges to $(1/2)\bar{\mu}(f, l)$ as $n \to \infty$ at least as fast as

$$(1.5) \qquad \left(\prod_{j=1}^{k-1} (n + 1 - j)\right)^{-1} \min\{c_1(n + 1 - k)^{-1}, c_2\underline{\mu}(f, l(n + 1 - k)^{-1})\},$$

where c_1 and c_2 are independent of n. Here the empty product means unity. The values of the constants c_1 and c_2 are given in Theorems 2 and 3 of § 4. There we compare several known bounds with (1.5).

2. Properties of the moduli of monotonicity. The properties of $\underline{\mu}$ and $\bar{\mu}$ are similar to those of ω. They are stated in this section with brief proofs. Some of these proofs are similar to those used to establish the properties of ω. See, e.g., Lorentz [5].

PROPOSITION 1. *Let $f, f_1, f_2 \in B$ and μ denote $\underline{\mu}$ or $\bar{\mu}$.*

(i) *μ is nonnegative, bounded and $\mu(0) = 0$.*

(ii) *μ is nondecreasing.*

(iii) *μ is subadditive, that is, if $0 \leqq \delta_1,\ \delta_2 \leqq \delta_1 + \delta_2 \leqq l$, then $\mu(\delta_1 + \delta_2) \leqq \mu(\delta_1) + \mu(\delta_2)$.*

(iv) *$\underline{\mu}(f, \cdot) \equiv 0 \Leftrightarrow f$ is nonincreasing on $[a, b]$.*
$\bar{\mu}(f, \cdot) \equiv 0 \Leftrightarrow f$ is nondecreasing on $[a, b]$.

(v) *$\omega(f, \delta) = \max\{\underline{\mu}(f, \delta), \bar{\mu}(f, \delta)\}$.*

(vi) *$\underline{\mu}(\alpha f, \delta) = \alpha\underline{\mu}(f, \delta)$ if $\alpha \geqq 0$.*
$\underline{\mu}(\alpha f, \delta) = -\alpha\bar{\mu}(f, \delta)$ if $\alpha \leqq 0$.

(vii) *$\mu(f_1 + f_2, \delta) \leqq \mu(f_1, \delta) + \mu(f_2, \delta)$.*

Proof. We establish (iii). Others follow directly from the definitions of $\underline{\mu}$ or $\bar{\mu}$. To prove (iii), suppose $\mu = \bar{\mu}$. If $0 \leqq y - x \leqq \delta_1 + \delta_2$, then there exists $z \in [a, b]$ such that $0 \leqq z - x \leqq \delta_1$ and $0 \leqq y - z \leqq \delta_2$. Since

$$f(y) - f(x) = f(y) - f(z) + f(z) - f(x) \leqq \bar{\mu}(f, \delta_2) + \bar{\mu}(f, \delta_1),$$

the result follows for $\bar{\mu}$. The proof for $\underline{\mu}$ is similar.

PROPOSITION 2. *Let μ denote $\underline{\mu}$ or $\bar{\mu}$.*

(i) *If f is continuous on $[a, b]$, then*

$$\lim_{\delta \downarrow 0} \mu(\delta) = 0. \tag{2.1}$$

(ii) *Properties* (ii) *and* (iii) *of Proposition* 1 *and* (2.1) *imply that μ is continuous on $[0, l]$.*

(iii) *f is continuous on $[a, b] \Leftrightarrow$ both $\underline{\mu}$ and $\bar{\mu}$ are continuous on $[0, l]$.*

(iv) *Properties* (ii) *and* (iii) *of Proposition* 1 $\Rightarrow$ *if $\alpha > 0$, then $\mu(\alpha\delta) \leqq ((\alpha) + 1)\mu(\delta)$, where (α) is the largest integer* less *than α.*

(v) *Let $f \in B$ and $0 < \delta_1 < \delta_2 \leqq l$. Then $\mu(\delta_1) > 0 \Leftrightarrow \mu(\delta_2) > 0$.*

Proof. (i) This follows at once from the continuity of f.

(ii) Let $0 < \delta_1 \leqq \delta < \delta_1 + \delta \leqq l$. Using (ii) and (iii) of Proposition 1, we may easily show that $|\mu(\delta \pm \delta_1) - \mu(\delta)| \leqq \mu(\delta_1)$. The continuity of μ follows now from (2.1).

(iii) If both $\underline{\mu}$ and $\bar{\mu}$ are continuous, then from (v) of Proposition 1, ω is continuous, and it follows that f is continuous. The converse follows from (i) and (ii).

(iv) Let n be a nonnegative integer such that $n < \alpha \leqq n + 1$. Then by (ii) of Proposition 1, we have $\mu(\alpha\delta) \leqq \mu((n + 1)\delta)$. Again, by a successive application of (iii) of Proposition 1, we have $\mu((n + 1)\delta) \leqq (n + 1)\mu(\delta)$.

(v) Let $\alpha = \delta_2/\delta_1 > 0$. Then using (iv), we have $0 < \mu(\delta_2) \leqq ((\alpha) + 1)\mu(\delta_1)$, which gives the required result.

3. The μ-function. It was shown in § 2 that the moduli of monotonicity of continuous functions had, among others, the properties (i), (ii), (iii) of Proposition 1 and (2.1). A function having these properties is called a μ-function. Thus a μ-function is a real function μ defined on $[0, l]$ which is nonnegative, nondecreasing, subadditive and satisfies $\lim_{\delta \downarrow 0} \mu(\delta) = \mu(0) = 0$. Note that a μ-function is continuous by Proposition 2 (ii). The modulus of continuity ω of a continuous function is also a μ-function.

Given a μ-function, it is easy to determine continuous functions f_1, f_2, f_3 defined on $[a, b]$ such that $\omega(f_1, \cdot) = \underline{\mu}(f_2, \cdot) = \bar{\mu}(f_3, \cdot) = \mu(\cdot)$. One simply lets $f_1(x) = f_2(x) = \mu(x - a)$ and $f_3(x) = -\mu(x - a)$ for all $x \in [a, b]$. In this case, $\bar{\mu}(f_2, \cdot) = \underline{\mu}(f_3, \cdot) \equiv 0$. Now one may ask the following question: Given two μ-functions μ_1 and μ_2, does there exist a continuous function f on $[a, b]$ such that $\bar{\mu}(f, \cdot) = \mu_1(\cdot)$ and $\underline{\mu}(f, \cdot) = \mu_2(\cdot)$ hold simultaneously? In this section we seek an answer to this question.

Let μ be a μ-function, and let

$$\sup(\mu) = \sup_{0 \leqq \delta \leqq l} \mu(\delta).$$

Also let

$$\delta^*(\mu) = \inf\{\delta : 0 \leqq \delta \leqq l, \mu(\delta) = \sup(\mu)\}.$$

Since μ is continuous, we have $\sup(\mu) = \mu(\delta^*(\mu))$. We now state and prove the following.

THEOREM 1. *Given two μ-functions μ_1 and μ_2, in order that there exist a continuous function f on $[a, b]$ such that $\bar{\mu}(f, \cdot) = \mu_1(\cdot)$, $\underline{\mu}(f, \cdot) = \mu_2(\cdot)$, it is necessary that at least one of the following conditions* (a), (b), (c) *is satisfied*:

(a) $\delta^*(\mu_1) + \delta^*(\mu_2) \leqq l$,

(b) $\delta^*(\mu_1) < \delta^*(\mu_2)$ *and* $\sup(\mu_1) < \sup(\mu_2)$,

(c) $\delta^*(\mu_2) < \delta^*(\mu_1)$ *and* $\sup(\mu_2) < \sup(\mu_1)$.

Further, condition (a) *is sufficient for such an f to exist.*

Proof. Necessity. Suppose that f is continuous, $\underline{\mu}(f, \cdot) = \mu_2(\cdot)$ and $\bar{\mu}(f, \cdot) = \mu_1(\cdot)$. Assume first that both $\delta^*(\mu_1)$ and $\delta^*(\mu_2) > 0$. Then clearly both $\sup(\mu_1)$ and $\sup(\mu_2) > 0$. Let

$$X_i = \{(x, y) : a \leqq x \leqq y \leqq b, y - x = \delta^*(\mu_i)\}, \qquad i = 1, 2,$$

where (x, y) denotes an ordered pair. Obviously, X_i are not empty. Suppose $(x_1, y_1) \in X_1$, $(x_2, y_2) \in X_2$, with $f(x_1) - f(y_1) = \sup(\mu_1)$ and $f(y_2) - f(x_2) = \sup(\mu_2)$. Then $a \leqq x_1 < y_1 \leqq b$ and $a \leqq x_2 < y_2 \leqq b$. Several cases arise. These are listed below:

(i) $a \leqq x_1 \leqq x_2 < y_2 \leqq y_1 \leqq b$,

(i′) $a \leqq x_2 \leqq x_1 < y_1 \leqq y_2 \leqq b$,

(ii) $a \leqq x_1 < y_1 \leqq x_2 < y_2 \leqq b$,

(ii′) $a \leqq x_2 < y_2 \leqq x_1 < y_1 \leqq b$,

(iii) $a \leqq x_1 < x_2 < y_1 < y_2 \leqq b$,

(iii′) $a \leqq x_2 < x_1 < y_2 < y_1 \leqq b$.

The cases (i′), (ii′), (iii′) are obtained by interchanging subscripts 1 and 2 in the cases (i), (ii), (iii) respectively. We first treat the cases (i), (ii) and (iii).

(i) Note that $f(x_1) - f(y_1) = \sup(\mu_1)$ and $f(y_2) - f(x_2) = \sup(\mu_2)$. We assert that $x_1 < x_2 < y_2 < y_1$. Suppose, on the contrary, that $x_1 = x_2$. Then we have $f(y_2) - f(y_1) = f(y_2) - f(x_2) + f(x_1) - f(y_1) = \sup(\mu_2) + \sup(\mu_1) > \sup(\mu_1)$, and $0 \leqq y_1 - y_2 < y_1 - x_1 = \delta^*(\mu_1)$. These contradictions to the definitions of $\sup(\mu_1)$ and $\delta^*(\mu_1)$ show that $x_1 < x_2$. The case when $y_2 = y_1$ may be treated similarly. This establishes the validity of our assertion, and it follows that $\delta^*(\mu_2) < \delta^*(\mu_1)$. We further assert that

$$f(y_1) < f(x) < f(x_1) \quad \text{for all } x,\ x_1 < x < y_1. \tag{3.1}$$

Suppose, on the contrary, that $f(x) \leqq f(y_1)$ for some x such that $x_1 < x < y_1$. Then we must have

$$f(x_1) - f(x) \geqq f(x_1) - f(y_1) = \sup(\mu_1).$$

But since $x - x_1 < \delta^*(\mu_1)$, a contradiction is reached. The other case, when $f(x) \geqq f(x_1)$ for some x such that $x_1 < x < y_1$, may be similarly treated. This establishes (3.1), and we conclude that $f(y_1) < f(x_2) < f(y_2) < f(x_1)$. Hence $\sup(\mu_2) = f(y_2) - f(x_2) < f(x_1) - f(y_1) = \sup(\mu_1)$, and condition (c) holds.

(ii) In this case, we have

$$l = b - a \geqq (y_1 - x_1) + (y_2 - x_2) = \delta^*(\mu_1) + \delta^*(\mu_2),$$

and thus condition (a) holds.

(iii) We show that this case cannot occur. By arguments similar to those used in case (i) to establish (3.1), it may be shown that $f(x_2) < f(x) < f(y_2)$ holds for all x such that $x_2 < x < y_2$. Hence $f(x_2) < f(y_1) < f(y_2)$. Then we must have

$$f(x_1) - f(x_2) > f(x_1) - f(y_1) = \sup(\mu_1),$$

and also $0 < x_2 - x_1 < y_1 - x_1 = \delta^*(\mu_1)$, which are contradictions.

The proofs for cases (i′), (ii′) and (iii′) are similar to those for (i), (ii) and (iii). Specifically, (i′) ⇒ (b), (ii′) ⇒ (a) and (iii′) cannot occur.

Since $\delta^*(\mu_i) = 0$ if and only if $\mu_i \equiv 0$, it follows that in the case $\delta^*(\mu_i) = 0$ for some i, the necessary conditions are satisfied trivially. Thus the necessity is established in all the cases.

Sufficiency of condition (a). In this case, $\delta^*(\mu_1) + \delta^*(\mu_2) \leqq l = (b - a)$. Define a continuous function f on $[a, b]$ by

$$f(x) = \begin{cases} -\mu_1(x - a), & a \leqq x \leqq a + \delta^*(\mu_1), \\ -\sup(\mu_1), & a + \delta^*(\mu_1) \leqq x \leqq b - \delta^*(\mu_2), \\ -\sup(\mu_1) + \mu_2(x - b + \delta^*(\mu_2)), & b - \delta^*(\mu_2) \leqq x \leqq b. \end{cases}$$

It is easy to verify that $\bar{\mu}(f, \cdot) = \mu_1(\cdot)$ and $\underline{\mu}(f, \cdot) = \mu_2(\cdot)$. The proof of the theorem is now complete.

4. Applications of moduli of monotonicity to approximation by monotone polynomials. In this section, we consider the monotone polynomial approximation problem described in § 1, and, making use of the moduli of monotonicity, obtain bounds on $E_n(f)$, the degree of approximation, defined by (1.4).

We have already introduced the notation $l = b - a$, B, P_n, ω, $\underline{\mu}$ and $\bar{\mu}$ in § 1. In addition, we let $K \subset B$ denote the set of real nondecreasing functions on $[a, b]$ and C, C^∞ denote, respectively, the set of continuous functions and the set of infinitely differentiable functions defined on $[a, b]$. The kth derivative of a function f in B at a point x in $[a, b]$, if it exists, is designated by $f^{(k)}(x)$. The norm notation $\|\cdot\|$ is used throughout to indicate the uniform norm defined by $\|f\| = \sup_{x\in[a,b]} |f(x)|$, where $f \in B$.

Immediately below, we give references to the relevant literature concerning the degree of approximation by monotone polynomials. Let $f(x)$ be continuous on $[0, 1]$, and $b_n(f, x)$ be its Bernstein polynomial defined by

$$b_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1 - x)^{n-k}, \qquad 0 \leqq x \leqq 1.$$

It is known that (Meinardus [8])

$$\max_{0\leqq x\leqq 1} |f(x) - b_n(f, x)| \leqq c\omega(f, n^{-1/2}), \tag{4.1}$$

where c is an absolute constant. The greatest lower bound of all numbers c such that (4.1) holds for all continuous functions f on $[0, 1]$ and all integers n is shown by Sikkema [12] (see also Meinardus [8]) to be equal to

$$\kappa = (4306 + 837\sqrt{6})/5832 = 1.0898873 \cdots. \tag{4.2}$$

It is further known that if f is nondecreasing on $[0, 1]$, then $b_n(f, x)$ is also nondecreasing (Lorentz [4]). By Proposition 1 (iv), (v), if f is nondecreasing, then $\omega(f, \cdot) = \underline{\mu}(f, \cdot)$. These observations and simple arguments then allow us to transform (4.1) to a relation giving a bound on $E_n(f)$ for a function f defined on an arbitrary interval $[a, b]$ as follows. If $f \in K \cap C$, then

$$E_n(f) \leqq \kappa\underline{\mu}(f, ln^{-1/2}), \tag{4.3}$$

where κ is given by (4.2) and $l = b - a$. A substantial improvement of this bound is due to Lorentz and Zeller [7]. Using simple arguments, it may be deduced from their Theorem 2 that if $f \in K \cap C$, then

$$E_n(f) \leqq c_0\underline{\mu}(f, l(2n)^{-1}), \tag{4.4}$$

where c_0 is an absolute constant. (For an expository article on monotone approximation, see Lorentz [6]). Since bounds on $E_n(f)$ are available when $f \in K \cap C$ (expression (4.4)), we restrict our attention to the case where $f \in C - K$, and establish upper and lower bounds on $E_n(f)$. But before we state our results, we need to introduce a class of functions called the mollifier functions, which are used extensively in the analysis presented in the sequel.

A real-valued function ϕ, defined on the real line, is called a Friedrichs mollifier function if (i) $\phi(x) \geqq 0$ for all $x \in (-\infty, \infty)$, (ii) ϕ is infinitely differentiable on $(-\infty, \infty)$, (iii) it vanishes outside $[0, 1]$, that is, has support in $[0, 1]$, and (iv) $\int_0^1 \phi(x)\,dx = 1$. As an example of a mollifier function, we may take

$$\phi(x) = \begin{cases} A^{-1} \exp((x(x-1))^{-1}), & 0 < x < 1, \\ 0 & \text{otherwise}, \end{cases}$$

where $A = \int_0^1 \exp((x(x-1))^{-1})\,dx$. Mollifier functions find considerable applications in the calculus of variations (see Morrey [9, p. 20]). The definition of a mollifier function, which we have given, is slightly different from the one usually employed. Generally it is assumed that the support of ϕ is contained in $[-1, 1]$. We find that this modification of the definition is convenient for the purpose of application to our problem. This fact will be verified by the reader in the proofs of Theorems 2, 3 and the subsequent remarks. In the proofs, the mollifier functions will be used to generate infinitely differentiable functions from continuous functions as follows. Let u be a continuous real function defined on the real line. Let $0 < \rho < \infty$, and define

$$u_\rho(x) = \int_x^{x+\rho} u(\xi)\phi_\rho^*(\xi - x)\,d\xi, \qquad x \in (-\infty, \infty),$$

where

$$\phi_\rho^*(\xi) = \rho^{-1}\phi(\rho^{-1}\xi), \qquad \xi \in (-\infty, \infty). \tag{4.5}$$

It is easily seen that $u_\rho(x)$ is an infinitely differentiable function with its kth derivative $u_\rho^{(k)}(x)$ given by

$$u_\rho^{(k)}(x) = (-1)^k \int_x^{x+\rho} u(\xi)\phi_\rho^{*(k)}(\xi - x)\,d\xi, \qquad x \in (-\infty, \infty),$$

for all $k = 1, 2, \cdots$. The function $u_\rho(x)$ is called the ϕ-mollified function of u.

We now state our results.

THEOREM 2. *Let $f \in C - K$. Then for every positive integer k and for all $n \geqq k$,*

$$\begin{aligned} 0 < (1/2)\bar{\mu}(f, l) \leqq E_n(f) \leqq (1/2)\bar{\mu}(f, l) \\ + \gamma(f, l, k)\left(\prod_{j=1}^{k-1} (n + 1 - j)\right)^{-1} \underline{\mu}\left(f, \frac{l}{n + 1 - k}\right), \end{aligned} \tag{4.6}$$

where

$$\gamma(f, l, k) = 2\left(1 + \frac{4}{\pi}\right)\left(\frac{\pi l}{4\lambda}\right)^k \inf_{\phi \in \Phi} \int_0^1 |\phi^{(k)}(\xi)|\,d\xi, \tag{4.7}$$

Φ is the class of all the Friedrichs mollifier functions ϕ and

$$\lambda = \sup\{\delta \in [0, l] : \omega(f, \delta) = \bar{\mu}(f, l)\} > 0. \tag{4.8}$$

(Empty product in (4.6) means 1.)

THEOREM 3. *Let $f \in C - K$. Then for every positive integer k and for all $n \geqq k$,*

$$0 < (1/2)\bar{\mu}(f, l) \leqq E_n(f) \leqq (1/2)\bar{\mu}(f, l) + \theta(f, l, k)\left(\prod_{j=1}^{k} (n + 1 - j)\right)^{-1}, \tag{4.9}$$

where

$$\theta(f, l, k) = \frac{8}{\pi}(\|f\| + (1/2)\bar{\mu}(f, l))\left(\frac{\pi l}{4\lambda}\right)^{k+1} \inf_{\phi \in \Phi} \int_0^1 |\phi^{(k+1)}(\xi)|\, d\xi, \tag{4.10}$$

Φ and λ are as in the statement of Theorem 2.

The prime role of moduli of monotonicity is clearly demonstrated by (4.4). (4.6), (4.8) and (4.9). The validity of (1.5) can now be easily verified by combining the results of both the above theorems. Clearly, $c_1 = \theta(f, l, k)$ and $c_2 = \gamma(f, l, k)$. Compare (1.5) with (4.4). The values of $\inf_{\phi \in \Phi} \int_0^1 |\phi^{(k)}(\xi)|\, d\xi$ for all k and several other related results are obtained in Ubhaya [15]. They enable one to determine the values of constants γ and θ given by (4.7) and (4.10) respectively. We state here two simple results only.

$$\inf_{\phi \in \Phi} \int_0^1 |\phi^{(k)}(\xi)|\, d\xi \begin{cases} = k!2^{2k-1}, & k = 1, 2, 3, \\ \leqq (2k)^k & \text{for all } k = 1, 2, \cdots. \end{cases}$$

Before we proceed to the proofs of Theorems 2 and 3, we establish some preliminary results.

LEMMA 1. *Let $f \in C - K$. Then the set*

$$\Delta(f) = \{\delta \in [0, l] : \omega(f, \delta) = \bar{\mu}(f, l)\} \tag{4.11}$$

is not empty. Moreover, if $\lambda = \sup \Delta(f)$, then $\lambda > 0$.

Proof. Since $f \in C - K$, by Proposition 1, (iv), we conclude that $\bar{\mu}(f, l) > 0$. Also, $\omega(f, l) \geqq \bar{\mu}(f, l)$ by Proposition 1, (v). If $\omega(f, l) = \bar{\mu}(f, l)$, then $l \in \Delta(f)$, and $\lambda = l > 0$. If $\omega(f, l) > \bar{\mu}(f, l)$, then since $\omega(f, \cdot)$ is continuous and $\lim_{\delta \downarrow 0} \omega(f, \delta) = 0$, it follows again that $\Delta(f) \neq \emptyset$ and $\lambda > 0$.

The following lemma also follows from the results in Ubhaya [14, part I].

LEMMA 2. *Let $f \in C$ and define*

$$h(x) = \max_{z \in [a, x]} f(z) - (1/2)\bar{\mu}(f, l), \qquad x \in [a, b], \tag{4.12}$$

$$k(x) = \min_{z \in [x, b]} f(z) + (1/2)\bar{\mu}(f, l), \qquad x \in [a, b]. \tag{4.13}$$

Then $h, k \in K \cap C$, $h(x) \leqq k(x)$ for all $x \in [a, b]$, and

$$(1/2)\bar{\mu}(f, l) = \min_{g \in K} \|f - g\| = \min_{g \in K \cap C} \|f - g\| = \|f - h\| = \|f - k\|. \tag{4.14}$$

Proof. Let $g \in K$, x, $y \in [a, b]$ and $x \leqq y$. Since $g(x) \leqq g(y)$, we have

$$f(x) - f(y) \leqq (f(x) - g(x)) - (f(y) - g(y)) \leqq 2\|f - g\|.$$

Hence by (1.3), the inequality $(1/2)\bar{\mu}(f, l) \leqq \|f - g\|$ holds for all $g \in K$. Since $f \in C$, it may be easily shown that h, $k \in C$. Clearly h, $k \in K$. We now show that

$(1/2)\bar{\mu}(f, l) = \|f - h\|$. The proof for k is similar. By (4.12), we have $h(x) \geqq f(x) - (1/2)\bar{\mu}(f, l)$. Again by the continuity of f, $h(x) = f(y) - (1/2)\bar{\mu}(f, l)$ holds for some $y \in [a, x]$. Since $y \leqq x$, using (1.3), we conclude that $f(y) - f(x) \leqq \bar{\mu}(f, l)$. Hence

$$h(x) = f(y) - (1/2)\bar{\mu}(f, l) \leqq f(x) + (1/2)\bar{\mu}(f, l),$$

and thus $(1/2)\bar{\mu}(f, l) \geqq \|f - h\|$. But since $h \in K$, the reverse inequality holds, and therefore $(1/2)\bar{\mu}(f, l) = \|f - h\|$. If $u, v \in [a, b]$ and $u \leqq v$, then again by (1.3),

$$f(u) - (1/2)\bar{\mu}(f, l) \leqq f(v) + (1/2)\bar{\mu}(f, l),$$

and it follows that $h \leqq k$.

PROPOSITION 3. *Let* $f \in C - K$, *and* h *be as given by* (4.12). *Define a function* h_1 *on the real line by*

$$h_1(x) = \begin{cases} h(x) & \text{if } x \in [a, b], \\ h(a) & \text{if } x \in (-\infty, a), \\ h(b) & \text{if } x \in (b, \infty). \end{cases} \tag{4.15}$$

Let $\lambda = \sup(\Delta f)$, *where* $\Delta(f)$ *is given by* (4.11) ($\lambda > 0$ *by Lemma* 1). *For each* ρ, $0 < \rho < \lambda$, *define a function* $h_\rho(x)$ *on* $[a, b]$ *by*

$$h_\rho(x) = \int_x^{x+\rho} h_1(\xi)\phi_\rho^*(\xi - x)\, d\xi, \qquad x \in [a, b], \tag{4.16}$$

where ϕ_ρ^* *is given by* (4.5) *and* ϕ *is any mollifier function. Then* $h_\rho \in K \cap C^\infty$ *and*

$$(1/2)\bar{\mu}(f, l) = \min_{g \in K} \|f - g\| = \min_{g \in K \cap C^\infty} \|f - g\| = \|f - h_\rho\|. \tag{4.17}$$

Proof. It is easy to verify from (4.16) and (4.5) that

$$h_\rho(x) = \int_0^1 h_1(x + \rho\xi)\phi(\xi)\, d\xi, \qquad x \in [a, b]. \tag{4.18}$$

We first show that $h_\rho \in K$. Suppose $x, y \in [a, b]$, $x \leqq y$. Then

$$h_\rho(y) - h_\rho(x) = \int_0^1 (h_1(y + \rho\xi) - h_1(x + \rho\xi))\phi(\xi)\, d\xi.$$

Since h is in K, (4.15) shows that h_1 is nondecreasing, and consequently, $h_1(y + \rho\xi) \geqq h_1(x + \rho\xi)$ for all ξ, $0 \leqq \xi \leqq 1$. Using nonnegativity of ϕ, we conclude that $h_\rho(y) \geqq h_\rho(x)$. Thus $h_\rho \in K$. From the discussion on the mollifier functions preceding the statement of Theorem 2, it follows that $h_\rho \in C^\infty$. We now show that

$$h(x) \leqq h_1(x + \lambda) \leqq k(x) \quad \text{for all } x \in [a, b]. \tag{4.19}$$

The first inequality is obviously true. Now by (4.12), (4.13), (4.15) and the continuity of f, there exist $y \in [a, x + \lambda] \cap [a, b]$ and $t \in [x, b]$ such that

$$h_1(x + \lambda) = f(y) - (1/2)\bar{\mu}(f, l)$$

and

$$k(x) = f(t) + (1/2)\bar{\mu}(f, l).$$

Suppose $y \leqq t$. Then by (1.3), we have $f(y) - f(t) \leqq \bar{\mu}(f, l)$, and thus $h_1(x + \lambda) \leqq k(x)$. On the other hand, if $y > t$, then $x \leqq t < y \leqq x + \lambda$, and it follows from the definition of λ that

$$|f(y) - f(t)| \leqq \omega(f, \lambda) = \bar{\mu}(f, l),$$

which gives

$$h_1(x + \lambda) = f(y) - (1/2)\bar{\mu}(f, l) \leqq f(t) + (1/2)\bar{\mu}(f, l) = k(x).$$

Thus (4.19) is established, and since h_1 is nondecreasing, we conclude that

$$h(x) = h_1(x) \leqq h_1(x + \alpha) \leqq h_1(x + \lambda) \leqq k(x) \tag{4.20}$$

for all $x \in [a, b]$ for all α, $0 \leqq \alpha \leqq \lambda$. Hence if $0 < \rho \leqq \lambda$, then

$$\int_0^1 h(x)\phi(\xi)\, d\xi \leqq \int_0^1 h_1(x + \rho\xi)\phi(\xi)\, d\xi \leqq \int_0^1 k(x)\phi(\xi)\, d\xi \tag{4.21}$$

holds for all $x \in [a, b]$. By (4.18) and the fact that $\int_0^1 \phi(\xi)\, d\xi = 1$, we have $h(x) \leqq h_\rho(x) \leqq k(x)$ for all $x \in [a, b]$. Thus

$$\|f - h_\rho\| \leqq \max\{\|f - h\|, \|f - k\|\},$$

and from (4.14), we conclude that $(1/2)\bar{\mu}(f, l) = \|f - h_\rho\|$. The proof of the proposition is now complete.

PROPOSITION 4. *Let* h_ρ, $0 < \rho \leqq \lambda$, *be as defined by* (4.16). *Then*

(i) $\|h_\rho\| \leqq \|f\| + (1/2)\bar{\mu}(f, l)$,

(ii) $\|h_\rho^{(k)}\| \leqq \rho^{-k}(\|f\| + (1/2)\bar{\mu}(f, l))(\int_0^1 |\phi^{(k)}(\xi)|\, d\xi)$,

(iii) $\underline{\mu}(h_\rho, \delta) \leqq \underline{\mu}(f, \delta)$, $0 \leqq \delta \leqq l$,

(iv) $\omega(h_\rho^{(k)}, \delta) \leqq \rho^{-k}\underline{\mu}(f, \delta)(\int_0^1 |\phi^{(k)}(\xi)|\, d\xi)$, $0 \leqq \delta \leqq l$.

Proof. By (4.12) we verify that

$$\|h\| \leqq \|f\| + (1/2)\bar{\mu}(f, l). \tag{4.22}$$

By (4.15) and (4.18) we have, for all $x \in [a, b]$,

$$|h_\rho(x)| \leqq (\sup_{x \in [a,b]} |h_1(x)|) \int_0^1 \phi(\xi)\, d\xi = \|h\|.$$

Hence $\|h_\rho\| \leqq \|h\|$, and this together with (4.22) establishes (i). Using (4.5) and differentiating (4.16), we obtain

$$\begin{aligned} h_\rho^{(k)}(x) &= (-1)^k \int_x^{x+\rho} h_1(\xi)\phi_\rho^{*(k)}(\xi - x)\, d\xi \\ &= (-\rho)^{-k} \int_0^1 h_1(x + \rho\xi)\phi^{(k)}(\xi)\, d\xi. \end{aligned} \tag{4.23}$$

It follows, for all $x \in [a, b]$, that

$$|h_\rho^{(k)}(x)| \leqq \rho^{-k}(\sup_{x\in[a,b]} |h_1(x)|) \int_0^1 |\phi^{(k)}(\xi)|\, d\xi.$$

Thus

$$\|h_\rho^{(k)}\| \leqq \rho^{-k}\|h\| \int_0^1 |\phi^{(k)}(\xi)|\, d\xi.$$

This inequality together with (4.22) gives (ii).

To prove (iii) and (iv), we first show that

$$\underline{\mu}(h, \delta) \leqq \underline{\mu}(f, \delta). \tag{4.24}$$

Let $x, y \in [a, b]$ and $0 \leqq y - x \leqq \delta$. Since $h \in K$, we have $h(y) \geqq h(x)$. If $h(y) = h(x)$, then clearly $h(y) - h(x) \leqq \underline{\mu}(f, \delta)$. Now suppose that $h(y) > h(x)$. Then, using (4.12), we may write

$$\begin{aligned} h(y) &= \max\,(h(x), \max_{z\in[x,y]} f(z) - (1/2)\bar{\mu}(f, l)) \\ &= \max_{z\in[x,y]} f(z) - (1/2)\bar{\mu}(f, l). \end{aligned}$$

By continuity of f, there exists $t \in (x, y]$ such that $h(y) = f(t) - (1/2)\bar{\mu}(f, l)$. Also by (4.12), $h(x) \geqq f(x) - (1/2)\bar{\mu}(f, l)$. Since $0 < t - x \leqq \delta$, we conclude, using (1.2), that

$$0 < h(y) - h(x) \leqq f(t) - f(x) \leqq \underline{\mu}(f, \delta),$$

and (4.24) follows.

To show (iii), again let $x, y \in [a, b]$ and $0 \leqq y - x \leqq \delta$. Then by (4.15), (4.18), (1.2) and the fact that h_1 is nondecreasing, we have

$$\begin{aligned} h_\rho(y) - h_\rho(x) &= \int_0^1 (h_1(y + \rho\xi) - h_1(x + \rho\xi))\phi(\xi)\, d\xi \\ &\leqq \max\,\{(h_1(v) - h_1(u)) : a \leqq u \leqq v \leqq b + \rho, 0 \leqq v - u \leqq \delta\} \int_0^1 \phi(\xi)\, d\xi \\ &\leqq \underline{\mu}(h, \delta). \end{aligned}$$

It follows that $\underline{\mu}(h_\rho, \delta) \leqq \underline{\mu}(h, \delta)$. This together with (4.24) establishes (iii).

To verify (iv), let $x, y \in [a, b]$ be chosen as before. Then since h_1 is nondecreasing, we have, using (4.23),

$$\begin{aligned} |h_\rho^{(k)}(y) - h_\rho^{(k)}(x)| &= \rho^{-k} \left| \int_0^1 (h_1(y + \rho\xi) - h_1(x + \rho\xi))\phi^{(k)}(\xi)\, d\xi \right| \\ &\leqq \rho^{-k} \max\,\{(h_1(v) - h_1(u)) : a \leqq u \leqq v \leqq b + \rho, 0 \leqq v - u \leqq \delta\} \\ &\quad \cdot \int_0^1 |\phi^{(k)}(\xi)|\, d\xi \\ &\leqq \rho^{-k} \underline{\mu}(h, \delta) \int_0^1 |\phi^{(k)}(\xi)|\, d\xi. \end{aligned}$$

Hence we conclude that

$$\omega(h_\rho^{(k)}, \delta) \leqq \rho^{-k}\underline{\mu}(h, \delta) \int_0^1 |\phi^{(k)}(\xi)|\, d\xi.$$

The above inequality together with (4.24) proves (iv).

The proof of the proposition is now complete.

We now make some remarks. It was pointed out in the beginning of this section that, as far as our problem was concerned, there was a definite advantage in letting the support of a mollifier function be contained in $[0, 1]$ rather than in $[-1, 1]$, as is conventionally done. It will be easily verified by the reader that this modification of the definition enables us to establish (4.21) from (4.20), and subsequently various properties of h_ρ. Obviously h_ρ is defined using h. Symmetrically can we, using k given by (4.13), define a function k_ρ in $K \cap C^\infty$ having properties similar to h_ρ? This indeed can be done. But for this purpose we need to alter the definition of the mollifier function as shown below. Let ψ be a real-valued, nonnegative, infinitely differentiable function defined on the real line, having support in $[-1, 0]$ satisfying $\int_{-1}^{0} \psi(\xi)\, d\xi = 1$. Then, analogous to (4.15) and (4.16), we may define

$$k_1(x) = \begin{cases} k(x) & \text{if } x \in [a, b], \\ k(a) & \text{if } x \in (-\infty, a), \\ k(b) & \text{if } x \in (b, \infty), \end{cases}$$

and

$$k_\rho(x) = \int_{x-\rho}^{x} k_1(\xi)\psi_\rho^*(\xi - x)\, d\xi, \qquad x \in [a, b],$$

where

$$\psi_\rho^*(\xi) = \rho^{-1}\psi(\rho^{-1}\xi), \qquad \xi \in (-\infty, \infty).$$

It is easy to establish the results for k which are similar in nature to those given by Propositions 3 and 4 for h_ρ. However, it will be seen in the sequel that our bounds on the degree of approximation are independent of the choice of h_ρ or k_ρ made to establish the intermediate results. For approximation and optimization problems on partially or totally ordered sets and their duality implications, the reader is referred to Ubhaya [13], [14] and other references stated there.

To prove Theorems 2 and 3, we make use of two results by Shisha [11], which we quote below for the convenience of the reader. Shisha obtained these results by following methods of Jackson [3] and making use of investigations by Farvad [2] and Ahiezer and Krein [1].

(i) Let r and k be integers, $1 \leqq r \leqq k$, and let a real function f satisfy throughout $[a, b]$, $f^{(r)}(x) \geqq 0$, $|f^{(k)}(x)| \leqq M$, M being a constant. Then for every integer $n \geqq k$, there exists a real polynomial $q_n(x)$ of degree at most n such that $q_n^{(r)}(x) \geqq 0$ for all

$x \in [a, b]$, and

$$\|f - q_n\| \leqq 2(1 + \pi/4)(\pi/4)^{k-r} l^k \left(r! \prod_{j=r}^{k-1} (n + 1 - j) \right)^{-1} \omega(f^{(k)}, l/(n + 1 - k)). \tag{4.25}$$

(Empty product means 1).

(ii) Let r and k be integers $1 \leqq r \leqq k$, and let a real function f satisfy throughout $[a, b]$, $f^{(r)}(x) \geqq 0$ and

$$|f^{(k)}(x) - f^{(k)}(y)| \leqq \Lambda |x - y|, \tag{4.26}$$

Λ being a constant. Then for every integer $n \geqq k$, there exists a real polynomial $q_n(x)$ of degree at most n such that $q_n^{(r)}(x) \geqq 0$ for all $x \in [a, b]$, and

$$\|f - q_n\| \leqq 2\Lambda(\pi/4)^{k-r+1} l^{k+1} \left(r! \prod_{j=r}^{k} (n + 1 - j) \right)^{-1}. \tag{4.27}$$

Proof of Theorem 2. Since $f \in C - K$, by Proposition 1 (iv), $\bar{\mu}(f, l) > 0$. Consider h_λ as defined by (4.16), with $\rho = \lambda > 0$, and let ϕ be any mollifier function. Since $h_\lambda \in K \cap C^\infty$, we have $h_\lambda^{(1)}(x) \geqq 0$ for all $x \in [a, b]$. Letting $r = 1$ and $f = h_\lambda$ in (4.25), we have for every positive integer k and all $n \geqq k$,

$$E_n(h_\lambda) \leqq 2(1 + \pi/4)(\pi/4)^{k-1} l^k \left(\prod_{j=1}^{k-1} (n + 1 - j) \right)^{-1} \omega(h_\lambda^{(k)}, l/(n + 1 - k)).$$

Substituting from Proposition 4; (iv) in the above relation, we have

$$E_n(h_\lambda) \leqq 2\left(1 + \frac{4}{\pi}\right)\left(\frac{\pi l}{4\lambda}\right)^k \left(\int_0^1 |\phi^{(k)}(\xi)|\, d\xi \right) \left(\prod_{j=1}^{k-1} (n + 1 - j) \right)^{-1} \underline{\mu}\left(f, \frac{l}{n + 1 - k}\right). \tag{4.28}$$

If $p_n \in P_n$ and satisfies $E_n(h_\lambda) = \|h_\lambda - p_n\|$, then by (4.17),

$$\|f - h_\lambda\| = \min_{g \in K \cap C} \|f - g\| \leqq E_n(f) \leqq \|f - p_n\| \leqq \|f - h_\lambda\| + \|h_\lambda - p_n\|.$$

Again by (4.17), $\|f - h_\lambda\| = (1/2)\bar{\mu}(f, l) > 0$, and we obtain from the above expression

$$(1/2)\bar{\mu}(f, l) \leqq E_n(f) \leqq (1/2)\bar{\mu}(f, l) + E_n(h_\lambda).$$

Substituting from (4.28) for $E_n(h_\lambda)$ and taking the infimum of $\int_0^1 |\phi^{(k)}(\xi)|\, d\xi$ over Φ, we get (4.6).

Proof of Theorem 3. Since $f \in C - K$, $\bar{\mu}(f, l) > 0$. Let h_λ be as defined by (4.16) with $\rho = \lambda > 0$ and $x, y \in [a, b]$. Then

$$h_\lambda^{(k)}(x) - h_\lambda^{(k)}(y) = h_\lambda^{(k+1)}(z)(x - y) \quad \text{for some } z \in [x, y].$$

Hence

$$|h_\lambda^{(k)}(x) - h_\lambda^{(k)}(y)| \leqq \|h_\lambda^{(k+1)}\|\, |x - y|.$$

Substituting from Proposition 4 (ii) in the above relation, we get

$$|h_\lambda^{(k)}(x) - h_\lambda^{(k)}(y)| \leqq \lambda^{-(k+1)}(\|f\| + (1/2)\bar{\mu}(f, l))\left(\int_0^1 |\phi^{(k+1)}(\xi)|\, d\xi\right)|x - y|. \tag{4.29}$$

Since $h_\lambda^{(1)}(x) \geqq 0$ for all $x \in [a, b]$, comparing (4.26) and (4.29) and letting $r = 1$, $f = h_\lambda$ in (4.27), we get

$$E_n(h_\lambda) \leqq \frac{8}{\pi}(\|f\| + (1/2)\bar{\mu}(f, l))\left(\frac{\pi l}{4\lambda}\right)^{k+1}\left(\int_0^1 |\phi^{(k+1)}(\xi)|\, d\xi\right)\left(\prod_{j=1}^{k}(n + 1 - j)\right)^{-1}.$$

The proof may now be completed as in Theorem 2.

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SINGULARITIES OF SOLUTIONS TO EXTERIOR ANALYTIC BOUNDARY VALUE PROBLEMS FOR THE HELMHOLTZ EQUATION IN THREE INDEPENDENT VARIABLES. I: THE PLANE BOUNDARY

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Abstract. A method is developed for locating singularities of solutions to boundary value problems for the Helmholtz (or Laplace) equation in three independent variables. It relates singularities in the analytically continued boundary data to real singularities in the solution. On the plane boundary $z = 0$, an analytic Neumann, Dirichlet, or linear boundary condition is prescribed. For the first two, the unknown boundary data are determined by integrals over the boundary, whereas in the third case the unknown satisfies a two-dimensional, linear integral equation. The kernels of the integrals are singular, but a method of E. E. Levi is used to extend them analytically into the complex domain of x and y on $z = 0$ as far as their singularities. For the third boundary condition, the integral equation is solved iteratively in the large in the complex domain, and the singularities of the boundary data are located. Under certain conditions, it is found that the singularities in the unknown data coincide with those in the prescribed data. They may be carried through the complex x,y,z-domain on characteristic surfaces, and possible real singularities are found where the characteristics pierce the real domain. For purposes of illustration, the method is applied to an elementary problem for the Laplace equation. However, a second example shows that this naive application of characteristic theory may not yield all the real singularities of the solution, and indicates that further examination of this aspect of the problem is warranted.

1. Introduction. When solving an analytic boundary value problem for an elliptic partial differential equation, one often finds that it would be useful to know the location of singularities in the analytic continuation of the solution across the boundary. Their importance stems from the fact that they are the fundamental sources of the solution, in terms of which it may be represented in a more or less elementary fashion. For example, Handelsman and Keller [13] have obtained solutions to axially symmetric potential problems in the exterior of slender bodies by relating the solution to an axial source distribution interior to the body, and Geer and Keller [8] have studied analogous two-dimensional problems. More recently, Miloh [27] has stressed the importance of knowing the system of singularities of an exterior potential field, and has located them within a triaxial ellipsoid with a view to application in problems of ship hydrodynamics. The singularities also play an important role in the so-called inverse problems of geophysics. This is discussed briefly in [23], where additional references are given; see also [28]. Knowledge of the location of singularities is useful as well in a somewhat different, if related, context. When a formal solution to the Laplace or Helmholtz equation is obtained by separation of variables in polar coordinates, the domain in which the series converges is determined by the geometry of the singularities of the solution [23], [24].

It is clear that procedures for locating singularities are of considerable practical importance; moreover, knowledge of their position and character is of some theoretical interest, since they determine the extent to which analytic continuation of the solution is possible. Of course, if the results of such procedures

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are to be useful in determining representations for the solution, it should not be necessary to determine the unknown boundary values first.

The problem of locating singularities of a solution to a linear analytic elliptic equation of the second order in two independent variables, satisfying an analytic boundary condition, has been recently discussed [25], [26]. It was shown how a representation for the solution u in terms of its boundary data—values of u and of its normal derivative $\partial u/\partial \nu$—and a fundamental solution could be used to continue the data analytically into the complex domain of arc length on the boundary. In particular, without first solving the problem for the unknown data, we located their singularities. Then, by considering the continuation problem for u as a Cauchy problem in the complex domain [7, Chap. 16], we were able to use the theory of characteristics to relate these singularities to real singularities of u that lay outside the original domain of definition.

The present paper is a first attempt to extend these ideas to second order equations in three independent variables. Here the literature is much less extensive than for problems in the plane. Most relevant, perhaps, are the papers of Filippenko [6] and Lewy [19]; they considered the possibility of continuing a solution to the Laplace equation, satisfying a linear analytic boundary condition, across a bounded portion of a plane boundary and throughout the reflection of the initial bounded domain. By using function-theoretic methods, Colton [2] has delimited a domain containing the singularities of axisymmetric solutions to the Helmholtz equation when the far-field pattern, rather than a boundary condition, is given. Results of a related nature had been obtained for vector (electromagnetic) problems by Weston, Bowman and Ar [33]. Sleeman [29] has succeeded in removing the limitation of axial symmetry from Colton's analysis. However, in none of [2], [29] and [33] have the singularities been located precisely. Singularities of classes of harmonic functions of three independent variables have been discussed by Gilbert [9], who uses integral operators and representations. More recently, Gilbert [34, Chap. VII] has examined the singularities of such functions that satisfy Cauchy data of a certain kind on a plane. The possibility of a relationship between Gilbert's work and that of the present paper has not yet been explored.

The extension of the earlier analysis to three independent variables presents certain difficulties, and a corresponding degree of generality has yet to be attained. There are, however, two particular classes of exterior three-dimensional problems that are less complex than that in which the boundary is a general closed analytic surface. One of these is the class of axisymmetric problems, where the axial symmetry reduces the number of independent variables by one; it is likely that these can be treated as completely as were the strictly two-dimensional problems.

The other class consists of problems for which the boundary is a plane, and it is these that we consider here. Specifically, we shall confine our present attention to solutions of the Helmholtz equation

$$u_{xx}+u_{yy}+u_{zz}+k^2u=0, \tag{1.1}$$

that are nonsingular in the half-space $z \geqq 0$, and we shall search for singularities in $z<0$. Usually, we shall assume that $k>0$, although in one instance we take $k=0$.

On the boundary $z=0$, $-\infty<x<\infty$, $-\infty<y<\infty$, an analytic Neumann, Dirichlet, or linear boundary condition is prescribed. It is assumed that these data are holomorphic for all real values of x and y; real singularities in the data would lead to real singularities in the solution at the same points, and would complicate the analytic continuation process. (A related problem in two dimensions has been discussed in [26].)

It is worthwhile to look at this problem for several reasons. The analysis provides guidance for the treatment of more general equations and boundaries. In contrast to axisymmetric problems, it is strictly three-dimensional, as is reflected in the form of singularity of the fundamental solution. It is complicated slightly by the fact that the boundary is infinite, but its planar character simplifies other aspects of the analysis. The problem differs from those considered by Filippenko [6] and Lewy [19], in that the initial domain of definition here is unbounded.

Because of the simple geometry, there is a variety of ways in which we could formulate the problem. However, we shall only introduce ideas and methods that can be used in more general situations.

As our point of departure, we again observe that the initially unknown boundary data u and/or $\partial u/\partial \nu$ are analytic and can be continued into the complex domain of their arguments. This is effected by using the integral equations for the unknown data that follow from the Helmholtz representation for the solution when the field point (x, y, z) approaches the boundary. Here the fundamental solution becomes infinite as $1/r$, r denoting distance between $(x, y, 0)$ and an integration point $(\xi, \eta, 0)$ on the boundary, and to discuss the analyticity of the integrals we must employ a method devised by Levi [18].

On account of the plane boundary, for Neumann and Dirichlet problems we shall see that the integral equations reduce to integral representations for the unknown boundary data. Nevertheless, the singularities in the analytic continuation of the data, and of the solution into $z<0$, are not located immediately. For the more general linear boundary condition, such simplification is not found, and we are obliged to determine analytic properties of the solution to a two-dimensional integral equation in the complex domain. The results are summarized in Theorems 3.1, 4.1, and 5.1, and conditions are given to guarantee that all possible singularities of the unknown data in the finite domain coincide with the prescribed singularities of the data. Then, having located all singularities in the data, we use the theory of characteristics to determine real singularities of the solution.

In § 2, we formulate the problem and derive the integral equations. For the Neumann problem, the analytic continuation of the boundary data is described in § 3, and a simple example is discussed. The Dirichlet problem is treated in § 4, and in § 5 the linear boundary condition is considered. The emphasis throughout is on locating singularities of the data, but the example of § 3 is used again in § 6, where we employ the theory of characteristics to relate singularities of the boundary data to real singularities of the solution in $z<0$. A few concluding remarks are made in § 7, and necessary properties of the distance function are derived in the Appendix.

2. Formulation. Let u be a complex-valued analytic solution to the equation

$$u_{xx}+u_{yy}+u_{zz}+k^2u=0, \qquad k\geqq 0, \tag{2.1}$$

holomorphic in the unbounded region $D=\{(x, y, z)|-\infty<x<\infty, -\infty<y<\infty, z>0\}$ of R^3 and on its boundary S: $z=0$. At a point $P=(x, y, z)\in D$, we assume that $u(x, y, z)$ may be represented as an integral over S in terms of the fundamental solution e^{ikr}/r (the Helmholtz representation):

$$4\pi u(P)=\int_S\left[u(Q')\frac{\partial}{\partial\nu'}-\frac{\partial u}{\partial\nu'}(Q')\right]\frac{e^{ikr}}{r}dS, \qquad P\in D. \tag{2.2}$$

Here $\partial/\partial\nu'$ denotes differentiation along the unit normal to S directed into D at the integration point Q' and r is the distance between P and Q'. Henceforth we shall denote $(\partial u/\partial\nu')(Q')$ by $v(Q')$, and integration is over S unless otherwise indicated.

For the validity of (2.2), it suffices that u satisfies an outgoing radiation condition at infinity if $k>0$; more specifically, we shall suppose that

$$u(x, y, z)=f(\theta, \phi)\, e^{ikR}/R+O(R^{-2}) \tag{2.3}$$

as $R\to\infty$, where (R, θ, ϕ) are spherical polar coordinates of P with respect to the origin 0. When $k=0$, we shall assume regularity in the sense of Kellogg [16, p. 217]; that is,

$$\begin{aligned} &u(x, y, z)=O(R^{-1}),\\ &u_\xi(x, y, z)=O(R^{-2}) \quad \text{for } \xi=x, y, \text{ or } z, \end{aligned} \tag{2.4}$$

as $R\to\infty$.

We have made the assumptions (2.3) and (2.4) to ensure the validity of (2.2) as well as the uniform convergence of some subsequently occurring integrals. It is known that (2.2) is valid under milder restrictions than these which, roughly speaking, mean that the singularities of u are confined to a bounded region of $z<0$.

If so-called surface waves are present, (2.3) is violated, and such solutions are not included directly in our considerations. Nevertheless, in any given circumstances, our results will remain valid, even if (2.3) or (2.4) are not satisfied, provided that the representation (2.2) (or a modified version thereof, in which a known function is added to the right-hand side) still holds, and that uniform convergence of certain integrals can be established.

If f_1 denotes the derivative of f with respect to its first argument, then for $k>0$,

$$v(Q')=-f_1(\tfrac{1}{2}\pi, \phi)\, e^{ik\rho}/\rho^2+O(\rho^{-3}) \tag{2.5}$$

as $\rho\to\infty$; here (ρ, ϕ) are polar coordinates of Q' in S. Thus, when $k\geqq 0$, the integrand in (2.2) is $O(\rho^{-3})$ as $\rho\to\infty$, and the integral converges uniformly with respect to x, y, and z in closed subsets of D.

If in (2.2) we let $P\to Q\in S$, the first term in the integrand vanishes because S is a plane, and we obtain the integral equation

$$2\pi u(Q)=-\int v(Q')\frac{e^{ikr}}{r}\,dS. \tag{2.6}$$

If we differentiate (2.2) along the normal ν at Q, we find

$$2\pi v(Q)=\frac{\partial}{\partial\nu}\int u(Q')\frac{\partial}{\partial\nu'}\left(\frac{e^{ikr}}{r}\right)dS. \tag{2.7}$$

To facilitate the following discussion, we shall bring the normal derivative in (2.7) under the sign of integration. This requires some care on two accounts: the integrand is singular at $r=0$, and the domain of integration is of infinite extent. The problem of the singularity is obviated by subtracting from, and adding to, the integrand suitable terms, and then proceeding in the manner described by Kellogg [16, Chap. 6]. Thus we find

$$\begin{aligned}&\frac{\partial}{\partial\nu}\int u(Q')\frac{\partial}{\partial\nu'}\left(\frac{e^{ikr}}{r}\right)dS\\ &\quad=\frac{\partial}{\partial\nu}\int [u(Q')-u(Q)]\frac{\partial}{\partial\nu'}\left(\frac{e^{ikr}}{r}\right)dS+u(Q)\int\frac{\partial^2}{\partial\nu\,\partial\nu'}\left(\frac{e^{ikr}}{r}-\frac{1}{r}\right)dS.\end{aligned} \tag{2.8}$$

Here we have used the result that at all points of D, the potential of a uniform double layer is constant, so

$$\frac{\partial}{\partial\nu}\int\frac{\partial}{\partial\nu'}\left(\frac{1}{r}\right)dS=0, \qquad P\in D.$$

Justification for differentiating under the sign of integration in the last integral of (2.8) rests on the uniform convergence of the integral. For this reason, too, we may take the normal derivative of the first integral on the right-hand side of (2.8) under the sign of integration, provided that we interpret the result as a singular integral [22]. Then

$$\begin{aligned}2\pi v(Q)=&\int^{*}[u(Q')-u(Q)]\frac{\partial^2}{\partial\nu\,\partial\nu'}\left(\frac{e^{ikr}}{r}\right)dS\\ &+u(Q)\int\frac{\partial^2}{\partial\nu\,\partial\nu'}\left(\frac{e^{ikr}}{r}-\frac{1}{r}\right)dS.\end{aligned} \tag{2.9}$$

Here

$$\int^{*}\equiv\lim_{\varepsilon\to 0}\int_{S-S_\varepsilon},$$

where S_ε denotes a circular disc in S, with center at Q and radius ε.

It is (2.6) and (2.9) that we shall use to continue the boundary data analytically into the complex domain, and to locate their possible singularities. Of course, it could be argued that an equation simpler than (2.9) can be found by using the well-known Green's function for the half-space. But the Green's function for most other regions is not known explicitly, whereas integrals similar to (2.9) arise in all cases. Thus we shall consider (2.9), in keeping with our stated plan to avoid introducing methods that cannot be carried over to the study of more general boundaries.

3. Neumann problem. We examine first the Neumann problem. Then $v(Q')$ is a prescribed analytic function of ξ and η (where (ξ, η) are the coordinates of Q') satisfying (2.5) and is holomorphic for real ξ and η. Thus (2.6) becomes an integral representation for the unknown $u(Q)$. These values may be inserted into (2.2) to give an explicit representation for $u(P)$ in terms of known functions. Nevertheless, the singularities of its analytic continuation into $z<0$ are not given directly.

We shall determine the possible singularities of an integral like that in (2.6), which we rewrite as

$$2\pi u(x, y) = -[I_1(x, y) + iI_2(x, y)], \tag{3.1}$$

where

$$I_1(x, y) \equiv \int v(\xi, \eta)(K_1(r^2)/r)\, d\xi\, d\eta \tag{3.2}$$

and

$$I_2(x, y) \equiv \int v(\xi, \eta) K_2(r^2)\, d\xi\, d\eta. \tag{3.3}$$

Here K_1 and K_2 are entire functions of r^2 ($\equiv (x-\xi)^2 + (y-\eta)^2$) and, hence, of x, y, ξ, and η:

$$K_1(r^2) \equiv \cos kr, \tag{3.4}$$

$$K_2(r^2) \equiv (\sin kr)/r. \tag{3.5}$$

The function I_2 vanishes identically when $k=0$.

Since $v(\xi, \eta)$ is holomorphic when ξ and η are real, and is $O(\rho^{-2})$ as $\rho \equiv (\xi^2+\eta^2)^{1/2} \to \infty$, the continuation of I_2 is almost immediate: we need only replace x and y formally by complex variables x_1+ix_2, y_1+iy_2, and check that the resultant integral still converges uniformly, to conclude that I_2 is an entire function of x and y.

In order to verify convergence, we must examine the behavior of $K_2(r^2)$ in the complex domain. To preserve continuity, this is done in the Appendix, where properties of r will be found. There it is shown that for $k>0$,

$$\begin{aligned} K_2(r^2) = \frac{1}{2i\rho}\{&\exp[ik\{\rho - \rho_0 \cos(\phi_0 - \phi)\} + k\gamma \cos(\phi - \delta)] \\ &- \exp[-ik\{\rho - \rho_0 \cos(\phi_0 - \phi)\} - k\gamma \cos(\phi - \delta)]\}\{1 + O(\rho^{-1})\}, \end{aligned} \tag{3.6}$$

where

$$x_2 = \gamma \cos \delta, \quad y_2 = \gamma \sin \delta, \quad \gamma \geqq 0,$$

and the remaining quantities may be determined from Fig. 4. Thus, from (2.5) and (3.6), it is seen that the integral in (3.3) converges uniformly for x and y in closed subsets of $\mathbb{C}^2$, as required.

On the other hand, $K_1(r^2)/r$ is singular when $r=0$ and a similar straightforward approach is not possible. To prove analyticity of I_1, and to continue it into the complex domain of x and y, we employ a procedure devised by Levi [18]; it has been further exploited by Hopf [14] and others, and is described briefly in [1, Chap. II, § 6]. Levi developed his method to extend an integral like I_1 to complex values of x and y, and to demonstrate its analyticity in a sufficiently small complex neighborhood of a real point. Our intent is to use Levi's procedure and the Cauchy–Poincaré theorem [30, Chap. IV, § 22] to discuss the continuation of I_1 in the large, and to locate its possible singularities.

For real x and y, $I_1(x, y)$ is defined as an integral over the real manifold S. This surface contains the point $\xi = x$, $\eta = y$ at which $1/r$ is singular. Following Levi, we define $I_1(x, y)$ for complex x and y as a functional of the integration manifold, by continuously deforming part of S through the complex ξ, η-domain in such a manner that the point $\xi = x$, $\eta = y$ always remains on it. The area element $dS \equiv d\xi\, d\eta$ may then be interpreted as a complex differential form [30, Chap. IV]. For the deformed portion of S, Levi chose the cone-like manifold with the complex point (x, y) at its vertex, generated by lines passing through (x, y) and a suitable circle C on S that contains in its interior T the real initial point from which the continuation has taken place.

It is then possible to show that I_1 is analytic in x and y. Proofs for a bounded integration domain may be found, for example, in [18], [14], or [1, Chap. II, § 6]. Then for the unbounded manifold S, we need only verify that the integral over $S - T$ converges uniformly at infinity. In the Appendix, it is shown that

$$
\begin{aligned}
K_1(r^2)/r = (1/2\rho)\{&\exp[ik\{\rho - \rho_0 \cos(\phi_0 - \phi)\} + k\gamma \cos(\phi - \delta)] \\
&+ \exp[-ik\{\rho - \rho_0 \cos(\phi_0 - \phi)\} - k\gamma \cos(\phi - \delta)]\}\{1 + O(\rho^{-1})\},
\end{aligned}
\tag{3.7}
$$

which, with (2.5), suffices to guarantee uniform convergence. Thus I_1 is analytic in x and y in a complex neighborhood of S.

Levi's deformation of S is not the only possible choice, for the Cauchy–Poincaré theorem [30, Chap. IV, § 22] permits us to deform any bounded portion of a piecewise-smooth integration manifold through the complex domain without changing the value of the integral, if in so doing we avoid all singularities of the integrand. Now it is not difficult to show that the singularity manifold, defined in the complex ξ, η-space by $r = 0$ for x and y fixed, meets the integration manifold (deformed in accordance with Levi's procedure) only in the complex point $\xi = x$, $\eta = y$, provided that the imaginary parts of x and y are sufficiently small relative to the radius of C. Consequently we may obtain the analytic continuation of I_1 from real to complex x and y by deforming S arbitrarily through the complex ξ, η-space while ensuring that the deformed manifold contains the singularity $\xi = x$, $\eta = y$ and always avoids other singularities of the integrand.

In this manner, we can prove analyticity of I_1, first in a neighborhood of S, and thereafter step by step through the complex domain. A singularity may occur when it is no longer possible to prevent the integration manifold from sweeping across other singularities of the integrand as x and y are varied. For example, it may happen that $\xi = x$, $\eta = y$ approaches another singularity of the integrand; but these circumstances also arise when the integration manifold becomes trapped or pinched between two or more singularity manifolds of the integrand that tend to touch, one from each side of the integration manifold. (See, for example, [5, Chap. 2], [15], or [9, Chap. 1, § 3]. Certain finer points of analysis that were overlooked in earlier work are discussed in [35].) Points (x, y) for which the integration manifold is pinched in this way are possible singularities of the integral.

Necessary conditions for a pinch to occur are well known [9], [5], [15]. We need only consider the simplest possibility in which the integration manifold is pinched by $r^2=0$ and a (fixed) singularity manifold of $v(\xi, \eta)$. For it is easy to see that $r^2=0$ consists of two, two-dimensional manifolds M_1 and M_2 in the $\xi=\xi_1+i\xi_2$, $\eta=\eta_1+i\eta_2$ space:

$$(3.8)\qquad \begin{aligned} &\xi_1-x_1+\eta_2-y_2=0, \qquad \xi_2-x_2-\eta_1+y_1=0, \qquad (M_1)\\ &\xi_1-x_1-\eta_2+y_2=0. \qquad \xi_2-x_2+\eta_1-y_1=0. \qquad (M_2)\end{aligned}$$

Due to their linearity, it is not possible for either M_1 or M_2 to form a pinch with itself, nor can M_1 and M_2 form a pinch: their only point of intersection is $\xi=x$, $\eta=y$. Thus this simplest possibility is the only possibility in the present case.

Let us suppose that the jth singularity manifold of v is determined by the analytic relation $F^j(\xi, \eta)=0$. We assume also that F^j_ξ and F^j_η do not vanish simultaneously on $F^j=0$, so that in a neighborhood of any point of $F^j=0$ we have either $\xi=f_j(\eta)$ or $\eta=g_j(\xi)$, with f_j and g_j analytic. Then according to [9], [5], or [15], the location of a possible pinch of the integration manifold by this singularity manifold and $r^2(\xi, \eta; x, y)\equiv(x-\xi)^2+(y-\eta)^2=0$ is determined by ξ and η that simultaneously satisfy the equations

$$(3.9)\qquad F^j(\xi, \eta)=r^2(\xi, \eta; x, y)=0,$$

$$(3.10)\qquad F^j_\xi/(r^2)_\xi=F^j_\eta/(r^2)_\eta.$$

This latter equation imposes the condition that $F^j=0$ and $r^2=0$ be tangent at the points of intersection. (To decide whether or not such solutions do indeed correspond to a pinch of the integration manifold requires further examination.) Then the corresponding (x, y) determines a possible singularity—more precisely, a point on a singularity manifold—of the integral. If, however, (3.9) and (3.10) have no solution except, possibly, $\xi=x$, $\eta=y$, then I_1 can be continued through the complex x, y-domain up to singularities of v. From (3.1) we conclude that u will be holomorphic in this same domain.

It is possible to describe qualitatively the effect of a pinch in the context of the present geometry, and this is suggested in Fig. 1. Suppose we have continued the integral as far as a singularity manifold $F^j=0$ of v, and assume that the integration manifold has been shrunk to the greatest possible extent onto the two faces of a triangle determined by M_1 and M_2. (This process is described in detail after (5.9).) We may think of this triangle as a probe into the complex domain. As (x, y) moves along $F^j=0$, it may happen that it reaches a point (x_0, y_0) at which M_1 (say) is tangent to $F^j=0$: this will be a pinch. Denote this tangential manifold by M_1^0. Then unless M_1^0 actually intersects $F^j=0$ at (x_0, y_0), the point (x, y) will not be able to continue moving along $F^j=0$: a portion of M_1^0 is a barrier to further analytic continuation. Thus the manifold of possible singularity of the integral consists of a part of $F^j=0$ and a part of the tangential manifold M_1^0.

We may summarize our results in the following theorem.

THEOREM 3.1. *Let $u(x, y)$ be determined for real x and y by* (2.6), *in which $v(Q')\equiv v(\xi, \eta)$ is holomorphic for real ξ and η and satisfies* (2.5) *if $k>0$, or the second of* (2.4) *if $k=0$. Suppose that the singularity manifolds of v, p in number, may be represented in the form $F^j(\xi, \eta)=0$ ($j=1, 2, \ldots, p$), where the F^j are*

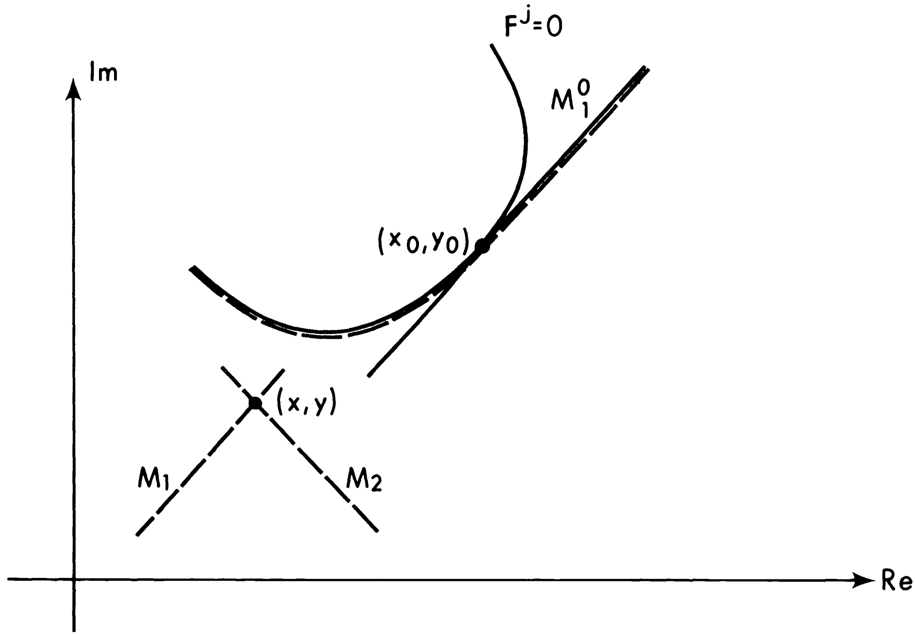

FIG. 1. *Effect of a pinch. Bròken line represents barrier to further continuation of the integral by present method.*

analytic and where F^j_ξ, F^j_η do not vanish simultaneously on $F^j=0$ ($j=1,2,\dots,p$). If none of the p sets of simultaneous equations (3.9) *and* (3.10) *has a solution $(\xi,\eta)\neq(x,y)$, then u can be continued analytically into the complex x, y-domain as far as a singularity of v. If one (or more) of these sets of equations has a solution $(\xi,\eta)\neq(x,y)$, then the corresponding point (x,y) may lie on a singularity manifold of u.*

Remark. Since $F^j=0$ defines a two-dimensional manifold in the four-dimensional space $\mathbb{C}^2$, the singularity manifolds of v do not form all or part of the boundary of a domain in $\mathbb{C}^2$. Thus all points on every singularity manifold of v can be reached by analytic continuation from the real domain, provided (3.9) and (3.10) have no solution $(\xi,\eta)\neq(x,y)$.

3.1. *An example.* We shall illustrate the above conclusion by a simple example, with a known solution, for the Laplace equation ($k=0$). Consider the potential of a point singularity at $(0,0,-h)$ ($h>0$) in the ξ,η,ζ-coordinate system. The potential at (x,y,z) is $u(x,y,z)=1/R$, where $R^2=x^2+y^2+(z+h)^2$; $u(x,y,z)$ is regular at infinity in the sense of Kellogg. [This problem has the following geophysical significance. Suppose that the Earth is represented by the half-space $\zeta\leqq 0$. Its density is assumed to decrease as ζ decreases, so that the resultant gravitational field is finite and uniform in $\zeta\geqq 0$. Let a homogeneous (or even radially stratified) sphere of radius $\leqq h$ be imbedded in this medium with its center at $(0,0,-h)$. Then the uniform field in $\zeta\geqq 0$ is perturbed by that of the sphere. The singularity in the potential of the perturbing field will be at the center of the sphere, so the following discussion, coupled with that in § 6, helps to locate the perturbing body.]

Let us suppose that $z>0$. As in § 2, we express $u(x, y, z)$ in terms of the (known) data on $\zeta=0$. On letting z tend to zero, we obtain the integral representation (see (2.6))

$$2\pi u(x, y)=-\int (v(\xi, \eta)/r)\, d\xi\, d\eta,$$

in which

$$v(\xi, \eta)=-h(\xi^2+\eta^2+h^2)^{-3/2}.$$

Here $p=1$,

$$F^1(\xi, \eta)\equiv \xi^2+\eta^2+h^2, \tag{3.11}$$

and F^1_ξ, F^1_η do not vanish simultaneously on $F^1=0$.

It is not difficult to verify that (3.9), (3.10) have no solution in any finite region of ξ, η-space. Therefore no pinch can develop, and u can be continued, and is analytic, except on the singularity manifold of v. This agrees with the known form of $u(x, y)$ ($\equiv u(x, y, 0)$):

$$u(x, y)=(x^2+y^2+h^2)^{-1/2}.$$

We shall return to this example later when we wish to locate the real singularities from knowledge of the singularities in the boundary data.

4. Dirichlet problem. When u is prescribed, and is holomorphic for real ξ and η, we use (2.9). On evaluation of the derivatives, this becomes

$$\begin{aligned} 2\pi v(x, y) &= \int^* [u(\xi, \eta)-u(x, y)]\frac{e^{ikr}}{r^3}(1-ikr)\, d\xi\, d\eta \\ &\quad + u(x, y)\int \left[\frac{e^{ikr}(1-ikr)-1}{r^3}\right] d\xi\, d\eta. \end{aligned} \tag{4.1}$$

Let $r_0>0$ and write (4.1) as

$$\begin{aligned} 2\pi v(x, y) &= \int^*_{r'<r_0} [u(\xi, \eta)-u(x, y)]\frac{e^{ikr}}{r^3}(1-ikr)\, d\xi\, d\eta \\ &\quad + u(x, y)\int_{r'<r_0} \left[\frac{e^{ikr}(1-ikr)-1}{r^3}\right] d\xi\, d\eta \\ &\quad + \int_{r'>r_0} \left[\frac{u(\xi, \eta)\, e^{ikr}\,(1-ikr)-u(x, y)}{r^3}\right] d\xi\, d\eta, \end{aligned} \tag{4.2}$$

where $r'\equiv[(x_1-\xi)^2+(y_1-\eta)^2]^{1/2}$. It is easy to verify that the third integral in (4.2) converges uniformly as $\xi^2+\eta^2\to\infty$ when x and y are complex and $k\geqq 0$. Furthermore, if we choose r_0 so that $r_0^2>x_2^2+y_2^2$, then r will not vanish (see (A.3)) and this integral will be a holomorphic function of x and y in a neighborhood of the real x, y-domain.

Consider next the first two integrals in (4.2). If we write $e^{ikr}=1+(\cos kr-1)+i\sin kr$, we see that they may be written as

$$\int^*_{r'<r_0} \frac{u(\xi, \eta)-u(x, y)}{r^3}\, d\xi\, d\eta \tag{4.3}$$

plus integrals with less singular integrands at $r=0$, of the type discussed previously. Then it is easy to apply Levi's method to show that these integrals, excepting (4.3), are holomorphic in x and y in a complex neighborhood of the real domain; their analytic continuation may be effected in the above manner.

We show that (4.3), too, is analytic in x and y, by expressing it as an ordinary convergent integral to which Levi's method applies. For real x and y, we have

$$\begin{aligned}(4.4)\qquad &\int_{r'<r_0}^{*} \frac{u(\xi,\eta)-u(x,y)}{r^3}\,d\xi\,d\eta \\ &= \int_{r'<r_0} \frac{u(\xi,\eta)-u(x,y)-(\xi-x)u_\xi(x,y)-(\eta-y)u_\eta(x,y)}{r^3}\,d\xi\,d\eta \\ &\quad + \int_{r'<r_0}^{*} \frac{(\xi-x)u_\xi(x,y)+(\eta-y)u_\eta(x,y)}{r^3}\,d\xi\,d\eta.\end{aligned}$$

The last integral in (4.4) vanishes, as is seen by introducing polar coordinates on S, with the pole at (x, y). Again, an application of Levi's method demonstrates the analyticity of the other integral on the right-hand side. We find

$$\begin{aligned}(4.5)\qquad 2\pi v(x,y) &= \int_{r'<r_0} \frac{U(\xi,\eta;x,y)-(\xi-x)u_\xi(x,y)-(\eta-y)u_\eta(x,y)}{r^3}\,d\xi\,d\eta \\ &\quad + \int_{r'<r_0} U(\xi,\eta;x,y)\frac{(\cos kr-1)}{r^3}\,d\xi\,d\eta \\ &\quad + i\int_{r'<r_0} \frac{U(\xi,\eta;x,y)}{r^2}\frac{\sin kr}{r}\,d\xi\,d\eta \\ &\quad - ik\int_{r'<r_0} \frac{U(\xi,\eta;x,y)}{r^2}e^{ikr}\,d\xi\,d\eta \\ &\quad + u(x,y)\int_{r'<r_0}\left[\frac{e^{ikr}(1-ikr)-1}{r^3}\right]d\xi\,d\eta \\ &\quad + \int_{r'>r_0}\left[\frac{u(\xi,\eta)\,e^{ikr}(1-ikr)-u(x,y)}{r^3}\right]d\xi\,d\eta,\end{aligned}$$

wherein

$$(4.6)\qquad U(\xi,\eta;x,y) \equiv u(\xi,\eta)-u(x,y).$$

Having established the analyticity of these integrals (and hence of v) by Levi's method, we may continue them into the complex domain by deforming that part of S determined by $r'<r_0$ in the manner described in § 3. The process is repeated step by step until a singularity is encountered. As in § 3, we have the following.

THEOREM 4.1. *Let $v(x, y)$ be determined for real x and y by* (2.9), *in which $u(Q')\equiv u(\xi,\eta)$ is holomorphic for real ξ and η and satisfies* (2.3) *if $k>0$, or is regular at infinity if $k=0$. Suppose that the singularity manifolds of u may be represented in the form $F^j(\xi,\eta)=0$ $(j=1,2,\cdots,q)$, where the F^j are analytic and where F^j_ξ, F^j_η do not vanish simultaneously on $F^j=0$ $(j=1,2,\cdots,q)$. If none of the q sets of simultaneous equations* (3.9) *and* (3.10) *has a solution $(\xi,\eta)\neq(x,y)$,* *then v can be continued analytically into the complex x, y-domain as far as a singularity of u. If one (or more) of these sets of equations has a solution* $(\xi, \eta) \neq (x, y)$, *then the corresponding point* (x, y) *may lie on a singularity manifold of v.*

(The remark following Theorem 3.1 is pertinent here too. We could also illustrate our conclusions by the example of § 3.1.)

5. Linear boundary condition. We consider now the boundary condition

$$(2\pi)^{-1}v(\xi, \eta) = a(\xi, \eta)u(\xi, \eta) + b(\xi, \eta), \tag{5.1}$$

a and b being prescribed and holomorphic for real values of their arguments. Since (2.3) and (2.5), or (2.4), must still be valid, we assume further that a and b have appropriate behavior as $\rho \equiv (\xi^2 + \eta^2)^{1/2} \to \infty$:

$$\begin{aligned} a(\xi, \eta) &= A(\phi)/\rho + O(\rho^{-2}), \\ b(\xi, \eta) &= B(\phi)\, e^{ik\rho}/\rho^2 + O(\rho^{-3}) \end{aligned} \tag{5.2}$$

for $k \geqq 0$. Then (2.6) gives the integral equation

$$\begin{aligned} u(x, y) = &-\int [a(\xi, \eta)u(\xi, \eta) + b(\xi, \eta)]\frac{K_1(r^2)}{r}\, d\xi\, d\eta \\ &- i\int [a(\xi, \eta)u(\xi, \eta) + b(\xi, \eta)]K_2(r^2)\, d\xi\, d\eta. \end{aligned} \tag{5.3}$$

We shall suppose that this equation admits of a solution that is consistent with (2.3) or (2.4) as the case may be. We assume that $u(x, y)$ has been determined for real x and y and, by standard arguments, we see that u is a continuous function of x and y. Then we may show, as in § 3, that

$$\psi(x, y) \equiv -i\int [a(\xi, \eta)u(\xi, \eta) + b(\xi, \eta)]K_2(r^2)\, d\xi\, d\eta \tag{5.4}$$

is an entire function of x and y. Similarly,

$$\chi(x, y) \equiv -\int b(\xi, \eta)\frac{K_1(r^2)}{r}\, d\xi\, d\eta \tag{5.5}$$

is, in principle, a known function; it may be continued into the complex x, y-domain in the manner discussed previously. Consequently we may write

$$u(x, y) = -\int_S a(\xi, \eta)\frac{K_1(r^2)}{r}u(\xi, \eta)\, d\xi\, d\eta + \Phi(x, y), \tag{5.6}$$

in which

$$\Phi(x, y) \equiv \psi(x, y) + \chi(x, y) \tag{5.7}$$

is known in principle, and is holomorphic for real x and y.

We shall extend (5.6) into the complex domain and show that the resulting integral equation has a unique, analytic solution that reduces to the solution to (5.6) when x and y are real. Further, we shall show that $u(x, y)$ can be continued analytically up to singularities of a or of b, if no pinches develop.

Our equation is a special case of the system considered by Hopf [14, (6.1)]. (Although S is unbounded, the integral converges uniformly; we could write $\int = \int_\Lambda + \int_{S-\Lambda}$, where Λ is bounded and $(x, y) \in \Lambda$, and absorb the integral over $S - \Lambda$ into the function Φ.) Consequently, we shall extend (5.6) into the complex domain in the manner of Hopf, and we may make use of his results when convenient. In particular, we note that Hopf's analysis shows that $u(x, y)$ reduces appropriately when x and y become real, and also demonstrates the analyticity of u in a complex neighborhood of the real domain. Hopf obtained an iterative solution and, to infer analyticity in the large, we must refine his bounds for the iterates somewhat in the present linear case. We obtain bounds that do not depend on values of u by shrinking part of his two-dimensional integration manifold in the four-dimensional complex domain $\mathbb{C}^2$ onto the two faces of a triangle. The base of this triangle lies in S, and $r^2 = 0$ determines its remaining sides. Since all functions involved are holomorphic in the domain through which this deformation takes place, we may use the Cauchy–Poincaré theorem to conclude that the value of the integral is unchanged, and that the unique solution that we then obtain by iteration must be identical to the analytic solution found by Hopf. Thus we need not explicitly demonstrate the analyticity of our solution. Since (5.6) is linear, we can continue $u(x, y)$ analytically in the above fashion step by step into the complex domain until further progress is prevented.

Let $\xi = \xi_1 + i\xi_2$, $\eta = \eta_1 + i\eta_2$. We shall deform S through the complex ξ, η-domain in the manner described in § 3; now, however, we must analyze the procedure in detail. Suppose S deforms into $S - T + W$, where W is a two-dimensional cone-like surface in $\mathbb{C}^2$. Its vertex is at (x, y) $(x = x_1 + ix_2,\ y = y_1 + iy_2)$, and its trace in the real ξ, η-domain is the circular boundary of the disc T, as illustrated suggestively in Fig. 2. (This figure is accurate if $x_2 = 0$ or $y_2 = 0$. In these cases, we may take $\xi_2 = 0$ or $\eta_2 = 0$, respectively, and the axis marked Im then corresponds to η_2 or ξ_2.) The manifold W is specified by

$$(5.8)\qquad \begin{aligned} \xi_1 &= x_1 + \tau(p_1 - x_1), & \xi_2 &= x_2(1-\tau), \\ \eta_1 &= y_1 + \tau(p_2 - y_1), & \eta_2 &= y_2(1-\tau); \end{aligned}$$

see [14, (7.1)]. Here τ is a parameter $(0 \leqq \tau \leqq 1)$, and (p_1, p_2) is a real point on the circumference of T, depending on a single parameter. The disc T is sufficiently large (or x_2 and y_2 sufficiently small) that the manifolds M_1, M_2 of $r^2 = 0$ (see (3.8)) meet W only in $\xi = x$, $\eta = y$, and that the point $(x_1, y_1) \in T$.

Thus, in a complex neighborhood of the real domain, we define $u(x, y)$ implicitly by

$$(5.9)\qquad u(x, y) = -\left\{\iint_W + \int_{S-T}\right\} a(\xi, \eta)\frac{K_1(r^2)}{r}u(\xi, \eta)\, d\xi\, d\eta + \Phi(x, y).$$

Hopf has shown that this equation possesses an analytic solution if $\gamma \equiv (x_2^2 + y_2^2)^{1/2}$ is sufficiently small; we assume that γ is thus restricted.

The manifolds M_1, M_2, on which $r^2 = 0$, are defined by (3.8). They meet the real domain in the points $(x_1 + y_2, -x_2 + y_1)$ and $(x_1 - y_2, x_2 + y_1)$ respectively; these points lie in T. Let us now collapse the disc T onto the line segment l joining these points. Then W will deform into the two faces of a triangle. Its base will be l. When

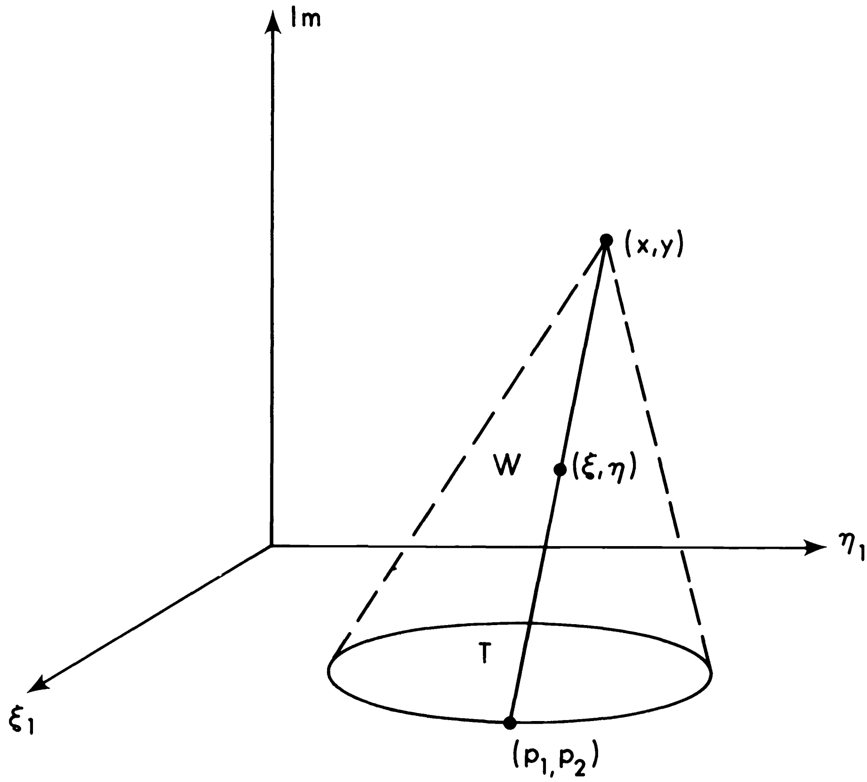

FIG. 2. *Integration manifold W in the complex domain*

we take $p_1 = x_1 + y_2$, $p_2 = -x_2 + y_1$, we see that W will now touch M_1 along the line segment l_1 given by

$$\xi_1 = x_1 + \tau y_2, \qquad \xi_2 = x_2(1-\tau), \tag{5.10}$$
$$\eta_1 = y_1 - \tau x_2, \qquad \eta_2 = y_2(1-\tau),$$

for $0 \leqq \tau \leqq 1$. Similarly, W will touch M_2 along the line segment l_2:

$$\xi_1 = x_1 - \tau y_2, \qquad \xi_2 = x_2(1-\tau), \tag{5.11}$$
$$\eta_1 = y_1 + \tau x_2, \qquad \eta_2 = y_2(1-\tau)$$

for $0 \leqq \tau \leqq 1$. The situation is suggested by Fig. 3 for a typical case in which all of x_1, x_2, y_1, y_2 are positive.

Since the integrand is singular along l_1 and l_2, the two faces of the triangle into which W deforms must be joined by small cylindrical or cone-like surfaces enclosing l_1 and l_2. We shall now show that, in the limit as these surfaces shrink onto l_1 and l_2, the integrals over them contribute nothing. To be specific, let the surface enclosing l_1 be generated by straight lines that pass through $\xi = x$, $\eta = y$ and through a circle of radius ε in the ξ_1, η_1-plane, centered on $(x_1 + y_2, -x_2 + y_1)$:

$$\xi_1 = x_1 + s(y_2 + \varepsilon \cos \theta), \qquad \xi_2 = x_2(1-s), \tag{5.12}$$
$$\eta_1 = y_1 - s(x_2 - \varepsilon \sin \theta), \qquad \eta_2 = y_2(1-s),$$

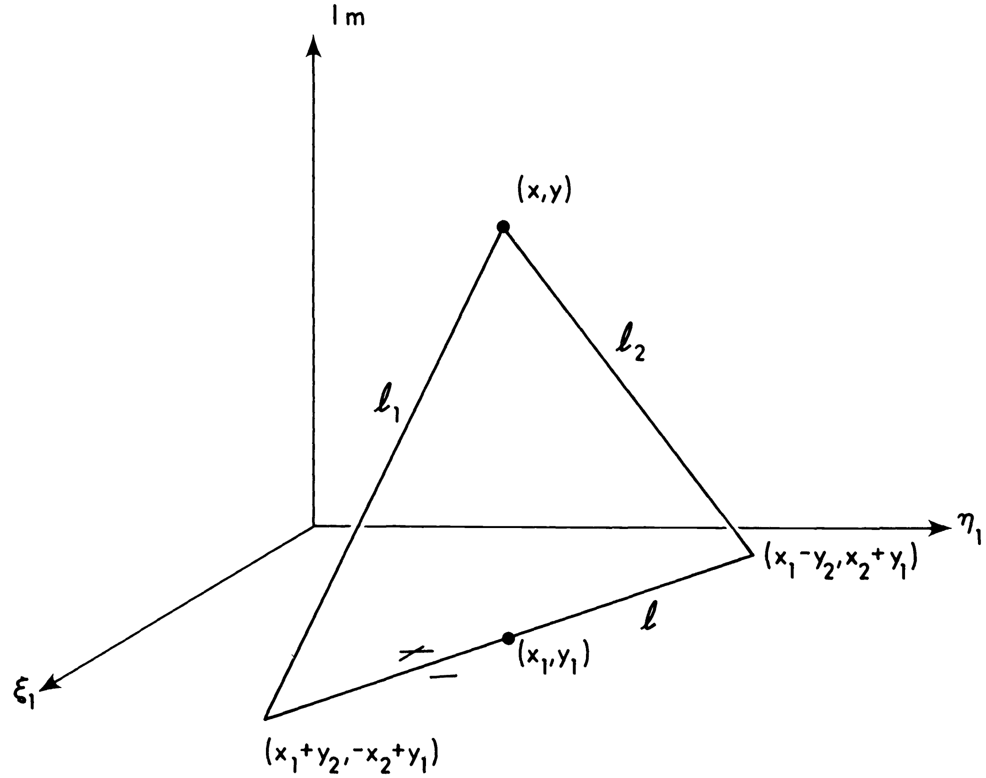

FIG. 3. *Integration manifold W is shrunk onto the triangle defined by line segments l, l_1, and l_2.*

for $0 \leqq s \leqq 1$, $0 \leqq \theta \leqq 2\pi$. Then on this surface,

$$r^2 = s^2 \varepsilon\, e^{-i\theta}[\varepsilon \cos\theta + 2y_2 + i(\varepsilon \sin\theta - 2x_2)]. \tag{5.13}$$

Consequently,

$$|r|^2 \geqq s^2 \varepsilon |\varepsilon - 2\gamma|, \tag{5.14}$$

where we have used $x_2 = \gamma \cos\delta$, $y_2 = \gamma \sin\delta$, $\gamma > 0$. Choose $\varepsilon < 2\gamma$. Then

$$|r|^{-1} \leqq s^{-1} \varepsilon^{-1/2} (2\gamma - \varepsilon)^{-1/2}. \tag{5.15}$$

On this surface, the theory of complex differential forms [30, Chap. 4] gives

$$d\xi\, d\eta = s\varepsilon[\varepsilon - i(x_2 + iy_2)\, e^{-i\theta}]\, ds\, d\theta. \tag{5.16}$$

Thus $|d\xi\, d\eta|/|r|$ is of order $e^{1/2}$ as $\varepsilon \to 0$, uniformly in s $(0 \leqq s \leqq 1)$, and the contribution to the integral from this surface vanishes as $\varepsilon \to 0$. Similarly, the conical surface enclosing l_2 contributes nothing in the limit.

Consider next the integral over the two faces of the triangle. To be definite, we assume that (x_1, y_1) lies in the first quadrant and that x_2 and y_2 are positive.

(This assumption is for convenience only, and does not restrict the validity of Theorem 5.1, that follows.) Thus the base (l) of the triangle will be situated in the ξ_1, η_1-plane in the general manner illustrated in Fig. 3. We label the side of l that is toward the origin as positive (l_+), and the side away from the origin as negative (l_-). Then we can label the two faces of the triangle in the corresponding manner : W_+ will denote the face whose trace is l_+, and similarly for W_-.

We are interested in

$$\int_{W_+ + W_-} a(\xi, \eta)\frac{K_1(r^2)}{r}u(\xi, \eta)\, d\xi\, d\eta.$$

Now the integrals over W_+ and W_- are equal. For we know that $a(\xi, \eta)$, $K_1(r^2)$, and $u(\xi, \eta)$ are holomorphic in a neighborhood of W. (Recall that we know that u is holomorphic, on the basis of Hopf's work.) However, in passing from a point on W_+ to the corresponding point on W_-, r changes sign; this is discussed in the Appendix. But W_+ and W_- are oriented in opposite senses, and the assertion follows.

The integral equation (5.9) now becomes

$$u(x, y) = -2\int_{W_-} a(\xi, \eta)\frac{K_1(r^2)}{r}u(\xi, \eta)\, d\xi\, d\eta + \Psi(x, y), \tag{5.17}$$

with

$$\Psi(x, y) \equiv -\int_S a(\xi, \eta)\frac{K_1(r^2)}{r}u(\xi, \eta)\, d\xi\, d\eta + \Phi(x, y). \tag{5.18}$$

Note that neither term on the right-hand side of (5.17) is analytic.

As a consequence of [14], we know that (5.17) has an analytic solution, provided that γ is sufficiently small. Next we shall show that it has only one solution, and we shall obtain bounds on the infinite series of iterates that in principle enable us to continue $u(x, y)$ analytically in the large.

A point on W_- is determined by (5.8), for appropriate choice of p_1, p_2, and τ. (p_1, p_2) is now a point on the line segment l_-; it is determined by a single parameter μ, where

$$p_1 = x_1 - \mu y_2, \qquad p_2 = y_1 + \mu x_2, \qquad -1 \leqq \mu \leqq 1. \tag{5.19}$$

Then on W_-,

$$\begin{aligned} \xi_1 &= x_1 - \tau\mu y_2, \qquad & \xi_2 &= x_2(1-\tau), \\ \eta_1 &= y_1 + \tau\mu x_2, \qquad & \eta_2 &= y_2(1-\tau), \end{aligned} \tag{5.20}$$

and the differential form $d\xi\, d\eta$ becomes

$$d\xi\, d\eta = -i\gamma^2\tau\, d\tau\, d\mu. \tag{5.21}$$

It is not difficult to see that W_- and the region $G = \{(\tau, \mu) | 0 \leqq \tau \leqq 1, -1 \leqq \mu \leqq 1\}$ have the same orientation: we may refer to Fig. 3, and take $y_2 = 0$ (so that the Im axis corresponds to ξ_2) for verification in this special case. For the general case in which $y_2 \neq 0$, the result follows by continuity. Moreover, on W_-,

$$r^2 = -\gamma^2\tau^2(1-\mu^2) \tag{5.22}$$

and, since $\arg r = -\frac{1}{2}\pi$ on W_- (see Appendix),

$$r = -i\gamma\tau(1-\mu^2)^{1/2}, \tag{5.23}$$

where $(1-\mu^2)^{1/2}$ is nonnegative on $-1 \leqq \mu \leqq 1$. Thus

$$\int_{W_-} a(\xi, \eta)\frac{K_1(r^2)}{r}u(\xi, \eta)\, d\xi\, d\eta = \gamma\int_0^1 d\tau \int_{-1}^1 a(\xi, \eta)\frac{K_1(r^2)}{(1-\mu^2)^{1/2}}u(\xi, \eta)\, d\mu. \tag{5.24}$$

Let us now solve (5.17) by successive approximations:

$$\begin{aligned} u_0(x, y) &= \Psi(x, y), \\ u_{n+1}(x, y) &= -2\int_{W_-} a(\xi, \eta)\frac{K_1(r^2)}{r}u_n(\xi, \eta)\, d\xi\, d\eta + \Psi(x, y), \qquad n = 0, 1, 2, \cdots. \end{aligned} \tag{5.25}$$

Then

$$u_{m+1}(x, y) = \sum_{n=0}^{m} [u_{n+1}(x, y) - u_n(x, y)] + \Psi(x, y), \tag{5.26}$$

and we must demonstrate the convergence of this series as $m \to \infty$. Our proof will be modeled on Garabedian [7, Chap. 4, § 2]. Set

$$\begin{aligned} v_{n+1}(x, y) &\equiv u_{n+1}(x, y) - u_n(x, y), \qquad n = 0, 1, 2, \cdots, \\ v_0(x, y) &\equiv u_0(x, y), \\ u_{-1}(x, y) &\equiv 0, \end{aligned} \tag{5.27}$$

and make the change of variable $1-\tau = \sigma/\gamma$ in the integral. Then

$$v_{n+1}(x, y) = 2\int_0^\gamma d\sigma \int_{-1}^1 a(\xi, \eta)\frac{K_1(r^2)}{(1-\mu^2)^{1/2}}v_n(\xi, \eta)\, d\mu, \tag{5.28}$$

where

$$\begin{aligned} \xi &= x_1 - \mu(1-\sigma/\gamma)y_2 + i\sigma x_2/\gamma, \\ \eta &= y_1 + \mu(1-\sigma/\gamma)x_2 + i\sigma y_2/\gamma. \end{aligned} \tag{5.29}$$

Suppose that $(x, y) \in H$, where H is a closed convex, complex neighborhood of the real disc $(\xi_1 - x_0)^2 + (\eta_1 - y_0)^2 < r_0^2$, for some $r_0 > 0$; here (x_0, y_0) is any point on S. We assume that a and b are holomorphic in H. Hopf has shown that H may be chosen so that the line segments l_1, l_2 (and, hence, W_-) are contained in H for all $(x, y) \in H$. (For H, we may take the closure of the domain $(R)_\gamma$ defined by Hopf [14, § 6].) We shall show that (5.26) converges uniformly and absolutely for $(x, y) \in H$; since (x_0, y_0) is an arbitrary point, the result will hold in a complex neighborhood of S.

From (5.29), we note first that, for $(\xi, \eta) \in W_-$,

$$\xi_2^2 + \eta_2^2 = \sigma^2. \tag{5.30}$$

Then we define a function V_n by

$$V_n(\sigma) = \max_{(\xi,\eta)\in H^\sigma} |v_n(\xi, \eta)|, \tag{5.31}$$

where $H^\sigma = \{(\xi, \eta) | (\xi, \eta) \in H, \xi_2^2 + \eta_2^2 = \sigma^2\}$. Thus, since $W_- \subset H$,

$$\max_{(\xi,\eta)\in W_-^\sigma} |v_n(\xi, \eta)| \leqq V_n(\sigma), \tag{5.32}$$

where $W_-^\sigma = \{(\xi, \eta) | (\xi, \eta) \in W_-, \xi_2^2 + \eta_2^2 = \sigma^2\}$.

Suppose that

$$|a(\xi, \eta) K_1(r^2)| \leqq A, \qquad (\xi, \eta) \in H. \tag{5.33}$$

Then (5.28) gives

$$\begin{aligned} |v_{n+1}(x, y)| &\leqq 2A \int_0^\gamma V_n(\sigma)\, d\sigma \int_{-1}^1 \frac{d\mu}{(1-\mu^2)^{1/2}} \\ &= 2\pi A \int_0^\gamma V_n(\sigma)\, d\sigma. \end{aligned} \tag{5.34}$$

This inequality is true for all $(x, y) \in H$, with $x_2^2 + y_2^2 = \gamma^2$. Thus we may replace the left-hand side by its maximum, for $(x, y) \in H$ and $x_2^2 + y_2^2 = \gamma^2$, to give

$$V_{n+1}(\gamma) \leqq 2\pi A \int_0^\gamma V_n(\sigma)\, d\sigma. \tag{5.35}$$

Then if

$$V_0(\sigma) \leqq M, \tag{5.36}$$

for some constant M, and for all $(\xi, \eta) \in H$ subject to (5.30), we find

$$V_n(\gamma) \leqq M(2\pi A)^n \gamma^n / n!, \qquad n = 0, 1, 2, \cdots. \tag{5.37}$$

Thus, provided that (5.36) is satisfied, $v_n \to 0$ and the right-hand side of (5.26) converges as $m \to \infty$, uniformly and absolutely throughout H. Hence the sequence defined by (5.25) converges uniformly and absolutely in H to a solution $u(x, y)$ to (5.17).

This solution is unique, for let $U(x, y)$ be another solution that fulfills all the above requirements, and consider $w_n \equiv U - u_n$. Then w_n and w_{n+1} are related by (5.28), in which v is replaced by w. It follows that $w_n \to 0$ as $n \to \infty$, and $U(x, y) \equiv u(x, y)$. Therefore this solution must coincide with Hopf's analytic solution, in a complex neighborhood of S.

It only remains to verify the inequality (5.36). We have $v_0(x, y) = \Psi(x, y)$, where Ψ is defined by (5.18) in which $\Phi = \psi + \chi$ by (5.7). The function ψ, given by (5.4), is an entire function of x and y. χ, as defined by (5.5), is holomorphic in a complex neighborhood of the real domain; its singularities can be found by the methods of § 3, and we choose H sufficiently small that it is holomorphic in H. As a consequence, we have

$$|\Phi(x, y)| \leqq M_1, \qquad (x, y) \in H, \tag{5.38}$$

for some constant M_1. Thus we need only verify the boundedness in H of

$$\int_S a(\xi, \eta) \frac{K_1(r^2)}{r} u(\xi, \eta)\, d\xi\, d\eta. \tag{5.39}$$

This integral is similar in form to $I_1(x, y)$, which we considered in § 3. When we take account of (5.2), (2.3), (2.5), we see that the integrand of (5.39) behaves at infinity like that of (3.1). The singularities at $r = 0$ are integrable (see, for example, (5.15)). Hence the above integral converges uniformly, and is bounded, for $(x, y) \in H$. Thus

$$|\Psi(x, y)| \leqq M, \qquad (x, y) \in H,$$

and consequently (5.36) holds, as required.

We have seen that (5.9) (or (5.17)) has a solution that is holomorphic in a complex neighborhood of the real x, y-domain. Hence if we let x and y take on complex values, with $x_2^2 + y_2^2$ sufficiently small, we may use the Cauchy–Poincaré theorem to deform S in (5.6) by a small amount through the complex domain, in such a manner that $\xi = x$, $\eta = y$ remains on the integration manifold. Then we may repeat all our former arguments to prove that $u(x, y)$ is analytic in a neighborhood of the new integration manifold. Since our bounds on the iterates in the series solution are independent of u, we may continue in this fashion, step by step, until further progress is prevented. This will occur if (x, y) encounters a singularity of Φ or of the integral in (5.6). All singularities of Φ arise through the function χ; these will be singularities of b, but others may arise if a pinch develops when we continue χ into the complex domain. If (x, y) approaches a singularity of the function a, or if a pinch develops between $r = 0$ and a singularity manifold of either a or Φ, then it will be no longer possible to prove analyticity of the integral in (5.6) and, hence, of $u(x, y)$. Thus we have the following.

THEOREM 5.1. *Let $u(x, y)$ be a solution to* (5.3) *for real values of x and y. Suppose that a and b are holomorphic for real values of their arguments, and satisfy* (5.2). *Let χ be defined by* (5.5). *Then $u(x, y)$ can be defined in a consistent manner for complex x and y and, if no pinches develop, can be continued analytically up to singularities of a and b.*

The corresponding singularities of v may be found from (5.1).

6. Location of real singularities. In [25], we employed two methods to locate real singularities of the solution when the position of singularities in the boundary data was known. Here we shall consider only the second procedure, in which the problem is regarded as a Cauchy problem for the Helmholtz equation in the complex domain. The initial data on $\zeta = 0$ are $u(\xi, \eta)$ and the values $v(\xi, \eta)$ of the ζ-derivative, continued analytically throughout the complex ξ, η-domain.

It is well known that discontinuities in the solution to a linear initial value problem can occur only across the characteristics of the differential operator; see, for example, [3, Chap. III, V, VI], [21]. For linear, analytic problems in the complex domain, Leray [17] has stated that singularities in the data are borne by characteristic manifolds. This was verified in an explicit analysis for initial data that contain poles or essential singularities by Hamada [10] when the characteristics are simple. Hamada's results have been generalized to systems of equations by Wagschal [31]. The case of multiple characteristics has been examined by Hamada [11], [12] and de Paris [4]. Subsequently, Wagschal [32] has shown that under suitable conditions the conclusions are valid with no hypothesis about the type of singularity in the initial data.

These results are all local, and their validity has been demonstrated only in a neighborhood of the hyperplane that bears the initial data. Moreover, the singularity submanifold in the initial hyperplane is itself assumed to be a hyperplane. The results also obtain for solutions to the Helmholtz equation near a smooth, analytic singularity manifold, since we may introduce this manifold locally as a coordinate hypersurface, and transform the problem into one of the above kind.

For solutions to the Helmholtz equation, one might be tempted to suppose that all real singularities lie where the characteristics that emanate from an analytic singularity manifold of the data intersect the real domain. As we shall see, this global result is not true in general. Nevertheless, as an illustrative example, let us return to the problem for the Laplace equation that we discussed earlier (§ 3.1). This concerned the potential of a point singularity at $(0, 0, -h)$, and we found that the analytic singularity manifold for the boundary data on $\zeta = 0$ was

$$\xi^2 + \eta^2 + h^2 = 0; \tag{6.1}$$

see (3.11). Evidently the initial manifold $\zeta = 0$ is nowhere characteristic.

We wish to obtain the two complex characteristics that issue from (6.1) in $\zeta = 0$. That is, we want to determine functions $\phi_i(\xi, \eta, \zeta)$ for $i = 1, 2$, such that $\phi_i(\xi, \eta, \zeta) = 0$ is a characteristic manifold, and

$$\phi_i(\xi, \eta, 0) \equiv \xi^2 + \eta^2 + h^2. \tag{6.2}$$

In the present circumstances, the problem may be simplified by exploiting its axial symmetry to reduce the number of independent variables by one. Thus we set $\rho = (\xi^2 + \eta^2)^{1/2}$, with $\rho > 0$ if $\xi^2 + \eta^2 > 0$. The singularity submanifold (6.1) then transforms into the two points $\rho = \pm ih$ in the complex ρ-plane, and the equation for a characteristic $\psi(\rho, \zeta) = 0$ is

$$\psi_\rho^2 + \psi_\zeta^2 = 0; \tag{6.3}$$

see [3, Chap. II]. The appropriate solutions then are

$$\psi(\rho, \zeta) = \rho \pm i\zeta \pm ih, \tag{6.4}$$

in which any combination of the signs is permissible. Thus the characteristics through $\rho = ih$ are $\psi_1 = \psi_2 = 0$ and those through $\rho = -ih$ are $\psi_3 = \psi_4 = 0$, where

$$\begin{aligned} \psi_1(\rho, \zeta) &\equiv \rho + i(\zeta + h), \qquad \psi_2(\rho, \zeta) \equiv \rho - i(\zeta - h), \\ \psi_3(\rho, \zeta) &\equiv \rho + i(\zeta - h), \qquad \psi_4(\rho, \zeta) \equiv \rho - i(\zeta + h). \end{aligned} \tag{6.5}$$

Had we not utilized the axial symmetry of this problem, we would have obtained the two solutions

$$
\begin{aligned}
\phi_1(\xi, \eta, \zeta) &= h^2+\xi^2+\eta^2-\zeta^2+2i\zeta(\xi^2+\eta^2)^{1/2}, \\
\phi_2(\xi, \eta, \zeta) &= h^2+\xi^2+\eta^2-\zeta^2-2i\zeta(\xi^2+\eta^2)^{1/2}.
\end{aligned}
\tag{6.6}
$$

It is evident that $\phi_1 \equiv \psi_1\psi_3$ and $\phi_2 \equiv \psi_2\psi_4$.

We see that $\psi_1 = \psi_4 = 0$ meet the real ρ, ζ-domain in $\rho = 0$, $\zeta = -h$, whereas $\psi_2 = \psi_3 = 0$ meet it in $\rho = 0$, $\zeta = h$. Consequently, in the real ξ, η, ζ-domain there are two possible singularities of the solution u, at $\xi = \eta = 0$, $z = \pm h$. But only that at $(0, 0, -h)$ exists; it coincides with the given singularity of the solution. There is no singularity at $(0, 0, h)$ since for a boundary value problem the data $u(\xi, \eta)$ and $v(\xi, \eta)$ must be related in such a manner that u is holomorphic in $\zeta \geqq 0$. However, had we arbitrarily prescribed analytic data on $\zeta = 0$ with singularity manifold (6.1), real singularities would have appeared at both points.

The fact that the above analysis leads directly to the singularity of the solution is fortuitous. For we can give another relatively simple example in which this straightforward but naive application of the theory of characteristics does not locate all singularities in the solution. Suppose we consider the exterior potential problem for a prolate spheroid situated in $z < 0$, for which the foci lie on the negative z-axis and on which the potential is constant. This has a closed-form solution [16, p. 56], and one may verify that its analytic continuation into the interior of the spheroid is logarithmically singular on the line segment joining the foci. As in § 3.1, we may represent the solution in $z > 0$ in terms of the known values of u and $\partial u/\partial z$ on $z = 0$. If we continue these data into the complex x, y-domain, we find that they are singular only on the manifolds that would correspond to point singularities at the foci: there are no singularities in the data that correspond to those in the solution between the foci, and our naive application of the theory of characteristics will not determine all the singularities of the solution. Evidently a much more careful analysis of this aspect of the problem, one that involves the geometry of the singularity manifolds and the characteristics emanating from them, is necessary. In particular, this axially-symmetric example shows that methods that are useful for locating singularities in two-dimensional problems for equations with holomorphic coefficients must in general be modified when applied to equations with singular coefficients.

7. Concluding remarks. In this paper, we have been interested in continuing a solution to the Helmholtz equation across a plane surface on which one of the usual boundary conditions is imposed. The boundary here is infinite so, in a sense, our work complements that of Filippenko [6], who considered the continuation of a harmonic function across a portion of a plane, and whose boundary condition corresponds to (5.1) with $b \equiv 0$ and a a polynomial. However, Filippenko was concerned with constructing the continuation, whereas we have confined our attention to locating the singularities and so determining the extent to which continuation is possible. It seems likely that the present methods could be modified to study continuation of a solution u across a portion S_0 of an infinite plane. Then conditions such as (2.3), (2.4) and (2.5) would be irrelevant; however, our integral equations would involve not only integrals on S_0, but integrals on S_1, where S_1 lies in the region where u is initially defined and is such that S_0+S_1 is a closed surface.

We have already mentioned that we have imposed rather strong conditions at infinity in order to ensure uniform convergence of certain integrals and their consequent analyticity in x and y. These conditions may be weakened, provided that the integrals remain analytic.

The analysis of the problem when the bounding surface is closed and analytic seems possible. Certainly axisymmetric problems can be handled, for they reduce to problems in two independent variables. When axial symmetry is lacking, the three boundary conditions studied here all give rise to integral equations analogous to (5.9). However, integration is with respect to the parameters in terms of which the bounding surface is defined, and complications arise on this account.

We conclude with two remarks that refer to this and earlier work. First, the use of an integral equation to study singularities of its solution is not novel; see, for example, [20] (Math. Reviews, 25 (1963), # 1413), in which a one-dimensional equation is examined. This work predates [23] and [24] by several years. However, the application to problems of the present type and the use of two-dimensional equations seem to be new.

Secondly, we note that it is because of the linearity of (5.1) that we are able to continue $u(x, y)$ globally. This is also true for problems in two independent variables, and we take this opportunity to correct a statement in [25, § 2]. There it was asserted that global continuation from the real axis was possible in the case of the boundary condition $v=f(u)$, provided that f was an entire function. The error in that statement is apparent when we note that the nonlinear integral equation

$$\phi(t)+\int_0^t [\phi(s)]^2\,ds=-1/c, \qquad c=\text{const.}\neq 0, \tag{7.1}$$

has the solution $\phi(t)=1/(t-c)$. This is singular at $t=c$, although the right-hand side of (7.1) is everywhere regular. In particular, if we take $c=c_1+ic_2$ $(c_2\neq 0)$, then $\phi(t)$ is holomorphic for real t; but we may choose c_2 to make $\phi(t)$ singular at a point as close as we please to the real axis.

Appendix. Behavior of *r* in the complex domain. The distance r is defined by

$$r^2=(x-\xi)^2+(\eta-y)^2, \tag{A.1}$$

and $r\geqq 0$ for all real x, y, ξ, and η. Let $x=x_1+ix_2$, $y=y_1+iy_2$, $\xi=\xi_1+i\xi_2$, $\eta=\eta_1+i\eta_2$, where x_j, y_j, ξ_j, η_j $(j=1,2)$ are real. Then

$$\begin{aligned} r^2=&(x_1-\xi_1)^2+(y_1-\eta_1)^2-(x_2-\xi_2)^2-(y_2-\eta_2)^2\\ &+2i[(x_1-\xi_1)(x_2-\xi_2)+(y_1-\eta_1)(y_2-\eta_2)]. \end{aligned} \tag{A.2}$$

In the real ξ, η-domain, this becomes

$$r^2=\lambda^2-\gamma^2-2i\lambda\gamma\cos(\omega-\delta), \tag{A.3}$$

where (see Fig. 4)

$$\begin{aligned} &\xi_1=x_1+\lambda\cos\omega, \qquad \eta_1=y_1+\lambda\sin\omega,\\ &x_2=\gamma\cos\delta, \qquad y_2=\gamma\sin\delta, \end{aligned} \tag{A.4}$$

with $\lambda\geqq 0$, $\gamma\geqq 0$.

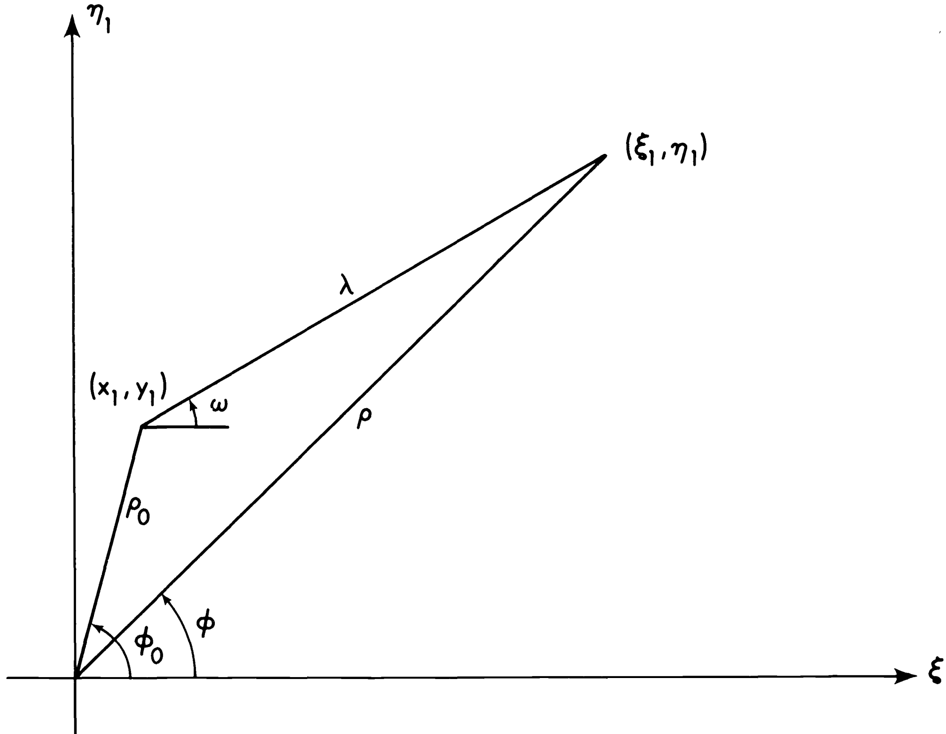

FIG. 4. *Geometry associated with discussion of r in the complex domain*

In the ξ_1, η_1-plane, let C_0 denote the circle defined by $\lambda = \gamma$ when x_1 and y_1 are fixed, and C_1 the line $(x_1 - \xi_1)x_2 + (y_1 - \eta_1)y_2 = 0$. Then Re $r^2 > 0$ for all points in this plane that lie outside C_0. Define $\Delta \equiv \arg r$ by

$$\Delta = \tfrac{1}{2} \arg r^2$$
(A.5)
$$= -\tfrac{1}{2} \tan^{-1} \frac{2\lambda\gamma \cos(\omega - \delta)}{\lambda^2 - \gamma^2}.$$

The behavior of Δ depends on the signs of x_2 and y_2. As x_2 and y_2 vary, the radius of C_0 increases or decreases, and C_1 rotates about the center (x_1, y_1). For purposes of illustration, we assume that $x_2 > 0$, $y_2 > 0$. Values of Δ are shown in Fig. 5 for $x_1 > 0$, $y_1 > 0$, $x_2 > 0$, $y_2 > 0$.

For large λ,

(A.6) $$\Delta = -(\gamma/\lambda) \cos(\omega - \delta) + O(\lambda^{-3})$$

and

(A.7) $$|r| = \lambda + O(\lambda^{-3}).$$

Consider the function $K_2(r^2) \equiv (\sin kr)/r$. On writing the sine in exponential form and using (A.6) and (A.7), we find

(A.8) $$K_2(r^2) = (1/2i\lambda)\{\exp[k\gamma \cos(\omega - \delta) + ik\lambda] - \exp[-k\gamma \cos(\omega - \delta) - ik\lambda]\}\{1 + O(\lambda^{-2})\}$$

as $\lambda \to \infty$.

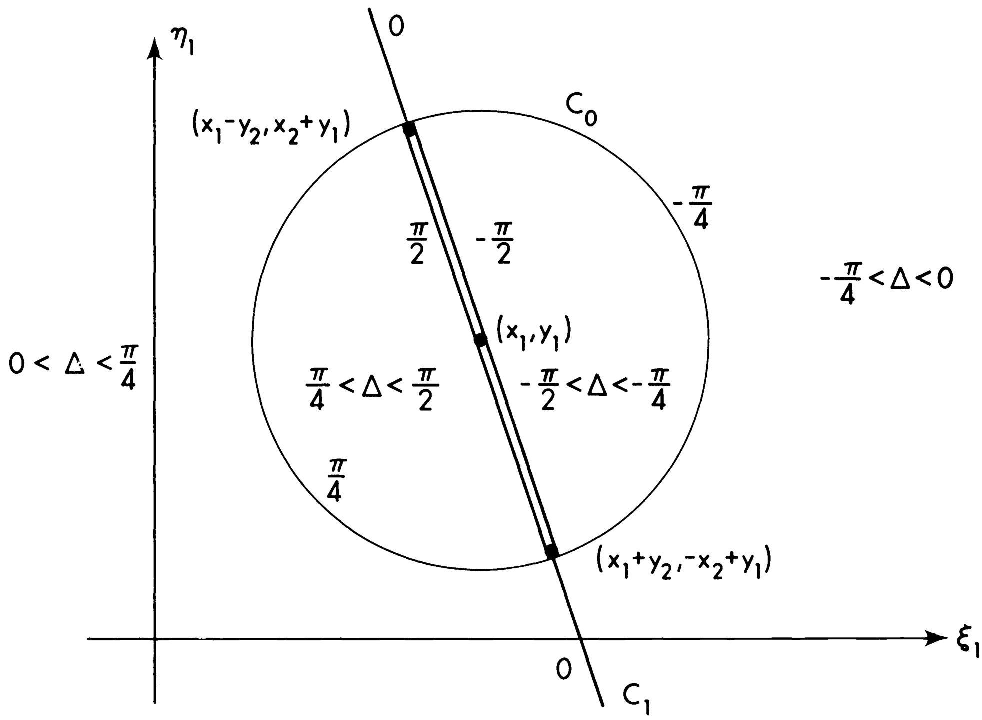

FIG. 5. *Variation of* arg *r in the* ξ_1, η_1*-plane*

If

$$\begin{aligned} &\xi_1 = \rho \cos \phi, &&\eta_1 = \rho \sin \phi, \\ &x_1 = \rho_0 \cos \phi_0, &&y_1 = \rho_0 \sin \phi_0, \end{aligned} \tag{A.9}$$

then

$$\begin{aligned} \lambda &= \rho - \rho_0 \cos(\phi_0 - \phi) + O(\rho^{-1}), \\ \cos \omega &= \cos \phi + O(\rho^{-1}), \\ \sin \omega &= \sin \phi + O(\rho^{-1}) \end{aligned} \tag{A.10}$$

as $\rho \to \infty$.

It is easy to see that the O-relations in the above equations are uniform for x and y in closed subsets of $\mathbb{C}^2$. Consequently, on substituting (A.10) into (A.8), we find

$$\begin{aligned} K_2(r^2) = \frac{1}{2i\rho} &\{\exp[ik\{\rho - \rho_0 \cos(\phi_0 - \phi)\} + k\gamma \cos(\phi - \delta)] \\ &- \exp[-ik\{\rho - \rho_0 \cos(\phi_0 - \phi)\} - k\gamma \cos(\phi - \delta)]\}\{1 + O(\rho^{-1})\}, \end{aligned} \tag{A.11}$$

uniformly in x and y, as $\rho \to \infty$.

In a similar manner, we see that

$$\begin{aligned}K_1(r^2)/r = \frac{1}{2\rho}\{&\exp[ik\{\rho-\rho_0\cos(\phi_0-\phi)\}+k\gamma\cos(\phi-\delta)] \\ &+\exp[-ik\{\rho-\rho_0\cos(\phi_0-\phi)\}-k\gamma\cos(\phi-\delta)]\}\{1+O(\rho^{-1})\},\end{aligned} \tag{A.12}$$

uniformly in x and y, as $\rho\to\infty$.

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A CONNECTION PROBLEM FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH TWO IRREGULAR SINGULAR POINTS*

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Abstract. Using Heaviside's exponential series, a power series solution of the differential equation is split into formal solutions with known asymptotic expansions.

1. Introduction. We consider second order linear differential equations with rational coefficients and two irregular singular points of integer ranks:

$$t^2\frac{d^2x}{dt^2}+\left\{\sum_{i=-r}^{R} a_i t^i\right\}t\frac{dx}{dt}+\left\{\sum_{i=-2r}^{2R} b_i t^i\right\}x=0. \tag{1.1}$$

The ranks are

$$r \quad \text{at } t=0 \quad \text{and} \quad R \quad \text{at } t=\infty,$$

where $0<r$, $R<\infty$.

The coefficients c_n of a power series solution

$$x(t)=\sum_{n=-\infty}^{\infty} c_n t^{n+\rho}, \qquad 0<|t|<\infty, \tag{1.2}$$

and the characteristic exponent ρ can be found numerically, if one interprets the recursive relations

$$(n+\rho)(n+\rho-1)c_n+\sum_{i=-r}^{R} a_i(n+\rho-i)c_{n-i}+\sum_{i=-2r}^{2R} b_i c_{n-i}=0 \tag{1.3}$$

as a nonlinear eigenvalue problem; the eigenvalue ρ is to be determined so that there exists $\{c_n\}$ with

$$0<\sum_{-\infty}^{\infty}|c_n|^2<\infty; \tag{1.4}$$

ρ and c_n generally will be complex numbers.

For numerical methods that will be published elsewhere (Naundorf [16]), we give a brief sketch: The nonlinear eigenvalue problem (1.3), (1.4) is treated by a Newton iteration process (Ruhe [22]). At every iteration step a linear system of infinite equations is to be computed, but this does not make trouble, because of the band structure and the property of "quasi-regularity" (Kantorowitsch and Krylow [10]) of the system.

For the Newton-iteration process, starting values for ρ and c_n are needed. This can be done according to a technique described in Bieberbach [3, pp. 138–140]. We suggest the following modification: The circuit matrix (Wasow [23, p. 10] is computed by numerical integration of the differential equation (1.1) on the unit circle from $t=1$ to $t=\exp(2\pi i)$. The eigenvalues of the circuit matrix are

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exp $(2\pi i\rho_1)$ and exp $(2\pi i\rho_2)$. If $\rho_1 \not\equiv \rho_2 \pmod 1$, then there exist two independent solutions of the form (1.2); in the other case, the second solution may have logarithmic terms. A control for the numerical integration process is

$$\rho_1+\rho_2\equiv -a_0 \pmod 1 \tag{1.5}$$

which follows, if one expresses the determinant of the circuit matrix by Wronskians.

For the differential equation

$$\frac{d^2x}{dt^2}+\frac{1}{t}\frac{dx}{dt}+\left(\kappa^2-\frac{\nu^2}{t^2}-\frac{\omega^2}{t^4}\right)x=0 \tag{1.6}$$

(κ, ν, ω are complex numbers), there is a well-known iteration process, based on continued fractions (Morse–Feshbach [15, pp. 557–559]) that gives ρ and c_n.

Analytic methods. If the rank r at $t=0$ is zero, then the characteristic exponent ρ is a solution of a polynomial equation of degree 2. If $r>0$, then ρ is a solution of a transcendental equation, given by infinite determinants. The aim is a transformation into an equation of only elementary transcendental functions. There are examples in Morse–Feshbach [15, pp. 560–562] and Whittaker–Watson [24, pp. 36–37 and pp. 415–416].

Formal solutions. The differential equation (1.1) has two formal solutions of the form ($k=1,2$):

$$x^k_{asy}(t) := \exp\left(\sum_{i=1}^{R}\frac{\alpha^k_i}{i}t^i\right)t^{\mu_k}\sum_{s=0}^{\infty}h^k_s t^{-s}, \tag{1.7}$$

$$\lambda_k := \alpha^k_R \quad \text{is determined from} \quad (\lambda)^2+a_R\lambda+b_{2R}=0.$$

We assume that

$$0<|\lambda_k|<|\lambda_1-\lambda_2|, \qquad k=1,2. \tag{1.8}$$

The condition is fulfilled, for example, if $a_R=0$ and $b_{2R}\neq 0$; for then $\lambda_1=-\lambda_2\neq 0$. In any case, the differential equation may be transformed by elementary substitutions so that assumption (1.8) becomes valid. For every λ_k, $\alpha^k_{R-1}, \ldots, \alpha^k_1$, μ_k then follow by comparison of coefficients.

From formula (2.07) of Olver and Stenger [19], we see that

$$\mu_1+\mu_2=-a_0-(R-1).$$

Equation (1.5) implies

$$\rho_1+\rho_2\equiv\mu_1+\mu_2 \pmod 1. \tag{1.9}$$

From the formulas (2.12) and (2.13) of Olver and Stenger [19], it follows that there are recursive relations for h^k_s of the following form:

$$h_s=\frac{s}{\Delta\lambda}h_{s-R}+\sum_{i=1}^{I}(A_i+s^{-1}B_i)h_{s-i} \tag{1.10}$$

($h_s=0$ if $s<0$), with

$$\Delta\lambda := \begin{cases}\lambda_1-\lambda_2 & \text{if } h^1_s \text{ is to be determined,}\\ \lambda_2-\lambda_1 & \text{if } h^2_s \text{ is to be determined.}\end{cases}$$

Here I is less than ∞ because $r<\infty$, and $\{A_i\}$ and $\{B_i\}$ depend on $\alpha_R, \ldots, \alpha_1, \mu$. We normalize (1.7) by $h_0=1$.

DEFINITION 1.1. With

$$x(t) \sim \sum_{k=1}^{2} T_k x_{asy}^k(t), \qquad \alpha < ph(t) < \beta$$

($x(t)$ is the solution (1, 2), T_k are complex numbers), it is meant that for $t\to\infty$ in this sector, and all integers N and $\sigma(\sigma \geqq 1)$,

$$\sum_{n=N}^{\infty} c_n t^{n+\rho} = \sum_{k=1}^{2} T_k \left\{ \exp\left(\sum_{i=1}^{R} \frac{\alpha_i^k}{i} t^i \right) t^{\mu_k} \sum_{s=0}^{\sigma} h_s^k t^{-s} + O(t^{-\sigma-1}) \right\} + O(t^{N-1+\rho})$$

(O is the Landau symbol for $t\to\infty$).

Remark 1.2. As we do not restrict the real part of the characteristic exponent ρ to be in some interval of length $=1$, it is sufficient to take $N=0$ in the above definition.

In this paper we give a solution for the following connection problem: How does a power series solution (1.2) of the differential equation (1.1) behave near the singular point at $t=0$ and $t=\infty$? Without loss of generality, we consider only $t=\infty$, because the singularity at $t=0$ may be transformed to $t=\infty$ by the substitution $t\to t^{-1}$.

Using Heavisides's exponential series (Hardy [8, pp. 36–41]) we split the power series solution (1.2) into formal power series solutions with known asymptotic behavior. We extend the work of Kohno [12], [13], [14], who considered nth order differential equations with one regular and one irregular singular point, to second order differential equations with two irregular singular points.

The proposed technique is easier to apply than "Hopf's principle" (see Hopf [9], Knobloch, [11]), because one process of integration has then already been done. Originally Hopf used a method to find convergent solutions of a differential equation with specified asymptotic behavior. (Spoken in terms of Bessel's differential equation, he constructed Hankel functions.) This implied serious restrictions, which are not present if one determines the asymptotic expansion of a given power series solution (1.2).

As an example, we prove a connection formula given by Fubini and Stroffolini [6] for the differential equation (1.6). Another approach for (1.6) is made by Erdélyi [5] and Bühring [4] by using Laplace-type integrals.

General assumptions. Let (1.8) be valid.

Lemma 3.8 below is proved by a result of Kohno [14]. For this lemma we further assume that

$$(1.11) \qquad \rho \not\equiv \mu_k \pmod 1, \qquad k=1,2.$$

Notation. With integer p (and λ given by (1.7)) we define the sector

$$(1.12) \qquad S(\lambda, p) := \{t; |ph(\lambda t^R) - 2\pi p| < \pi\}.$$

$S(\lambda, p)$ has central angle $2\pi/R$. $S(\lambda, p)$ and $S(\lambda, p+1)$ are separated by one ray with phase

$$(1.13) \qquad ((2p+1)\pi - ph(\lambda))/R.$$

Remark 1.3. Because of assumption (1.8), there is at least one set

$$\bigcup_{p_1=-\infty}^{\infty} S(\lambda_1, p_1) \quad \text{or} \quad \bigcup_{p_2=-\infty}^{\infty} S(\lambda_2, p_2)$$

that contains a given $t \neq 0$.

Remark 1.4. If $|ph(\lambda_k t^R) - 2\pi(p_k + \frac{1}{2})| < \pi/2$ for any integer p_k, then the exponential term of $x^k_{asy}(t)$ vanishes for $|t| \to \infty$.

2. Heaviside's exponential series.

DEFINITION 2.1. By

$$F(t) \sim \sum_{n=-\infty}^{\infty} a_n \cdot t^{n+\delta}, \qquad \alpha < ph(t) < \beta,$$

with complex number δ, and $F(t)$ holomorphic for sufficiently large $|t|$ in the sector $\alpha < ph(t) < \beta$, we mean (a) $\sum_{n=0}^{\infty} a_n t^n$ is an entire function, (b) $\sum_{n=-\infty}^{-1} a_n t^{n+\delta}$ is an asymptotic series for $F(t) - \sum_{n=0}^{\infty} a_n t^{n+\delta}$ in the sector $\alpha < ph(t) < \beta$ for $t \to \infty$.

Heaviside's exponential series

$$\sum_{n=-\infty}^{\infty} \frac{t^{n+\delta}}{(n+\delta)!}$$

is equal to exp (t) for every integer-valued δ; otherwise this series is divergent everywhere in the complex t-plane, but (Barnes [2, pp. 268–269])

$$\exp(t) \sim \sum_{n=-\infty}^{\infty} \frac{t^{n+\delta}}{(n+\delta)!}, \qquad |ph(t)| < \pi. \tag{2.1}$$

From this series we obtain in $|ph(\lambda t^2)| < \pi$,

$$\begin{aligned}\exp\left(\frac{\lambda}{2}t^2 + \alpha t\right) &\sim \sum_{n=-\infty}^{\infty} \frac{((\lambda/2)t^2)^{n+\delta/2}}{(n+(\delta/2))!} \cdot \sum_{m=0}^{\infty} \frac{(\alpha t)^m}{m!} \\ &= \sum_{n=-\infty}^{\infty} t^{n+\delta} \cdot \sum_{k=-\infty}^{[n/2]} \frac{(\lambda/2)^{k+\delta/2} \cdot \alpha^{n-2k}}{(k+(\delta/2))!(n-2k)!}.\end{aligned}$$

Replacing δ by $\delta+1$, we get a second series, linearly independent of the first:

$$\exp\left(\frac{\lambda}{2}t^2 + \alpha t\right) \sim \sum_{n=-\infty}^{\infty} t^{n+\delta} \cdot \sum_{k=-\infty}^{[(n-1)/2]} \frac{(\lambda/2)^{k+((\delta+1)/2)} \cdot \alpha^{n-1-2k}}{(k+((\delta+1)/2))!(n-1-2k)!}.$$

More generally, we get for every integer $R \geqq 1$, a set of R linearly independent series indexed by $L = 0, 1, \ldots, R-1$:

$$\exp\left(\sum_{i=1}^{R} \frac{\alpha_i}{i} t^i\right) \sim \sum_{n=-\infty}^{\infty} g_n^L \cdot t^{n+\delta}, \qquad |ph(\lambda t^R)| < \pi, \tag{2.2}$$

with $\lambda := \alpha_R \neq 0$. These series are obtained by multiplying

$$\exp\left(\frac{\lambda}{R} t^R\right) \sim \sum_{n=-\infty}^{\infty} \frac{(\lambda/R)^{n+((\delta+L)/R)}}{(n+((\delta+L)/R))!} \cdot t^{nR+L+\delta} \tag{2.3}$$

by the Taylor series of exp $(\sum_{i=1}^{R-1} ((\alpha_i/i)t^i)$.

We proceed to other sectors, which we index by the integer p: in $S(\lambda, p)$,

$$\exp\left(\sum_{i=1}^{R} \frac{\alpha_i}{i} t^i\right) \sim \exp\left(-2\pi i p \frac{L+\delta}{R}\right) \cdot \sum_{n=-\infty}^{\infty} g_n^L t^{n+\delta}. \tag{2.4}$$

This formula is obtained as follows. If t is in $S(\lambda, p)$, then $t \exp(-i(2\pi p/R))$ is in $S(\lambda, 0)$. Hence formula (2.3) transforms into

$$\exp\left(\frac{\lambda}{R} t^R\right) \sim \exp\left(-2\pi i p \frac{L+\delta}{R}\right) \sum_{n=-\infty}^{\infty} \frac{(\lambda/R)^{n+((L+\delta)/R)} t^{nR+L+\delta}}{(n+((L+\delta)/R))!}.$$

By differentiating (2.2), we get a recurrence relation for the g_n^L:

$$(n+\delta) g_n = \sum_{i=1}^{R} \alpha_i g_{n-i}. \tag{2.5}$$

The g_n^L $(L=0, \cdots, R-1)$ form a fundamental system of solutions for this relation (2.5).

LEMMA 2.2. *For every not identical vanishing solution of* (2.5),

$$\limsup_{n\to\infty} \{|g_n| \cdot (n!)^{1/R}\}^{1/n} = |\lambda|^{1/R}.$$

Proof. See Appendix A.

Remark 2.3. Kohno [14] defines another fundamental system of solutions for (2.5). There is just one linear transformation that relates his system to ours.

3. The connection problem.

Problem 3.1. Compute complex numbers $T_k (k=1, 2)$, so that for a given ray $ph(t)$ and $|t| \to \infty$,

$$x(t) \sim \sum_{k=1}^{2} T_k x_{asy}^k(t) \tag{3.1}$$

is valid in the sense of Definition 1.1.

THEOREM 3.2. *If* (1.8) *and* (1.11) *are fulfilled, then Problem* 3.1 *is solved by the following Method* 3.3.

Method 3.3.

Step 1. For each index k, there are R series of the form (2.2):

$$\exp\left(\sum_{i=1}^{R} \frac{\alpha_i^k}{i} t^i\right) \sim \sum_{n=-\infty}^{\infty} g_n^{k,L} t^{n+\rho-\mu_k}, \qquad t \in S(\lambda_k, 0) \quad L=0, \cdots, R-1.$$

Step 2. Formal multiplication:

$$\sum_{n=-\infty}^{\infty} f_n^{k,L} t^{n+\rho} := \left\{\sum_{n=-\infty}^{\infty} g_n^{k,L} t^{n+\rho-\mu_k}\right\} t^{\mu_k} \sum_{s=0}^{\infty} h_s^k t^{-s}, \tag{3.2}$$

$$f_n^{k,L} := \sum_{s=0}^{\infty} g_{n+s}^{k,L} h_s^k. \tag{3.3}$$

Step 3. The coefficients c_n of the convergent series solution (1.2) depend linearly on the $f_n^{k,L}$:

$$c_n = \sum_{k=1}^{2} \sum_{L=0}^{R-1} \beta_{k,L} f_n^{k,L} \quad \text{for all } n. \tag{3.4}$$

Compute the $2R$ complex numbers $\beta_{k,L}$ by a system of $2R$ linear equations (3.4) with $N \leqq n < N+2R$ for sufficiently large N. Take N so that for all $n \geqq N$,

$$\sum_{i=-r}^{R} |a_i(n+2R-i+\rho)| + \sum_{i=-2r}^{2R} |b_i| < |(n+2R+\rho)(n+2R-1+\rho)|. \tag{3.5}$$

Step 4. For integers p_k we define

$$T_k(p_k) := \sum_{L=0}^{R-1} \exp\left(2\pi i p_k \frac{L+\rho-\mu_k}{R}\right)\beta_{k,L}. \tag{3.6}$$

Then set

$$T_k := T_k(p_k) \quad \text{if } t \in S(\lambda_k, p_k), \tag{3.7$_1$}$$

$$T_k := \tfrac{1}{2}(T_k(p_k) + T_k(p_k+1)) \tag{3.7$_2$}$$

if t is on the boundary ray that separates $S(\lambda_k, p_k)$ and $S(\lambda_k, p_k+1)$.

Remark 3.4. We will prove (3.7$_1$), while the choice of T_k in formula (3.7$_2$) has been arbitrary according to Remark 1.4.

Proof of Theorem 3.2.

Step 2 *of Method* 3.3.

LEMMA 3.5. *For each solution of the recurrence relation* (1.10), ($|\Delta\lambda| := |\lambda_1 - \lambda_2|$):

$$\limsup_{s\to\infty} (|h_s|(s!)^{-1/R})^{1/s} \leqq |\Delta\lambda|^{-1/R}.$$

Proof. See Appendix A.

Remark 3.6. This result agrees with Kohnos [14, Thm. 3.3] in the special case of a second order differential equation.

LEMMA 3.7. *If the condition* (1.8) *is fulfilled, then the series* (3.3) *converges absolutely as fast as the geometric series* $\sum_{s=0}^{\infty} |\lambda/(\Delta\lambda)|^{s/R}$, *and* $\sum_{n=0}^{\infty} f_n t^n$ *is an entire function.*

Proof. See Appendix A.

Justification of Step 3 *of Method* 3.3. See Appendix B.

Step 4 *of Method* 3.3.

LEMMA 3.8. *If* (1.11) *is fulfilled, then*

$$\sum_{n=0}^{\infty} \left(f_n - \sum_{s=0}^{\sigma} g_{n+s} h_s\right) t^{n+\rho} = O\left(\exp\left(\sum_{i=1}^{R} \frac{\alpha_i}{i} t^i\right) t^{\mu-\sigma-1}\right) + O(t^{\rho+R-1})$$

for $|ph(t)| \leqq \pi$.

Proof. By the behavior of h_s as given in Lemma 3.5 and assumption (1.11), Kohno [14] derives this result (see [14, pp. 335–336, (5.9) and (5.11)]). He chooses another fundamental system for the recurrence relation (2.5) as we did. They are, however, linearly dependent.

LEMMA 3.9. *For every integer* $\sigma \geqq 1$ *and* $t \in S(\lambda, 0)$ *and the assumptions* (1.8) *and* (1.11), *one has*

$$\sum_{n=0}^{\infty} f_n t^{n+\rho} = \exp\left(\sum_{i=1}^{R} \frac{\alpha_i}{i} t^i\right) t^{\mu} \left\{ \sum_{s=0}^{\sigma} h_s t^{-s} + O(t^{-\sigma-1}) \right\} + O(t^{\rho-1}).$$

Proof. With $\delta := \rho - \mu$ we have

$$\begin{aligned}\sum_{n=0}^{\infty} \left(\sum_{s=0}^{\sigma} g_{n+s} h_s \right) t^{n+\rho} &= \sum_{s=0}^{\sigma} h_s t^{\mu-s} \sum_{n=0}^{\infty} g_{n+s} t^{n+s+\delta} \\ &= \sum_{s=0}^{\sigma} h_s t^{\mu-s} \sum_{n=s}^{\infty} g_n t^{n+\delta} \\ &= \sum_{s=0}^{\sigma} h_s t^{\mu-s} \left(\exp\left(\sum_{i=1}^{R} \frac{\alpha_i}{i} t^i \right) + O(t^{s-1+\delta}) \right) \\ &= \exp\left(\sum_{i=1}^{R} \frac{\alpha_i}{i} t^i \right) t^{\mu} \sum_{s=0}^{\sigma} h_s t^{-s} + O(t^{\rho-1}).\end{aligned}$$

By Lemma 3.8 we get

$$\sum_{n=0}^{\infty} f_n t^{n+\rho} = \left[\exp\left(\sum_{i=1}^{R} \frac{\alpha_i}{i} t^i \right) t^{\mu} \left\{ \sum_{s=0}^{\sigma} h_s t^{-s} + O(t^{-\sigma-1}) \right\} \right] + O(t^{\rho+R-1}).$$

Similarly we can derive

$$\sum_{n=-R}^{\infty} f_n t^{n+\rho} = [\,\cdot\,] + O(t^{\rho-1}) \quad \text{or} \quad \sum_{n=0}^{\infty} f_n t^{n+\rho} = [\,\cdot\,] + O(t^{\rho-1}),$$

and this is the assertion.

LEMMA 3.10. *If* $t \in S(\lambda_k, p_k)$, *then*

$$\sum_{n=0}^{\infty} f_n^{k,L} t^{n+\rho} = \exp\left(2\pi i p_k \frac{L+\rho-\mu_k}{R} \right) \cdot E^k + O(t^{\rho-1}),$$

where

$$E^k := \exp\left(\sum_{i=1}^{R} \frac{\alpha_i^k}{i} t^i \right) t^{\mu_k} \left\{ \sum_{s=0}^{\sigma} h_s^k t^{-s} + O(t^{-\sigma-1}) \right\}$$

is an abbreviation

Proof. This results from formula (2.4) and Lemma 3.9.

COROLLARY 3.11. *For each* $k = 1, 2$ *and* $t \in S(\lambda_k, p_k)$,

$$\sum_{n=0}^{\infty} \left(\sum_{L=0}^{R-1} \beta_{k,L} f_n^{k,L} \right) t^{n+\rho} = T_k(p_k) \cdot E^k + O(t^{\rho-1}).$$

Here E^k is the abbreviation used in Lemma 3.10 and $T_k(p_k)$ is defined in formula (3.6).

Proof. The left-hand side is written as

$$\sum_{n=0}^{\infty}\sum_{L=0}^{R-1}\left\{\exp\left(2\pi i p_k\frac{L+\rho-\mu_k}{R}\right)\beta_{k,L}\right\}\left\{\exp\left(-2\pi i p_k\frac{L+\rho-\mu_k}{R}\right)f_n^{k,L}t^{n+\rho}\right\}$$

$$=\sum_{L=0}^{R-1}\left\{\exp\left(2\pi i p_k\frac{L+\rho-\mu_k}{R}\right)\beta_{k,L}\right\}\sum_{n=0}^{\infty}\exp\left(-2\pi i p_k\frac{L+\rho-\mu_k}{R}\right)f_n^{k,L}t^{n+\rho}.$$

By Lemma 3.10 the assertion then follows.

With Remark 3.4, Step 4 of Method 3.3 is justified.

4. An example. The differential equation (1.6):

$$\frac{d^2x}{dt^2}+\frac{1}{t}\cdot\frac{dx}{dt}+\left(\kappa^2-\frac{\nu^2}{t^2}-\frac{\omega^2}{t^4}\right)\cdot x=0,$$

has power series solutions

$$x(t)=\sum_{n=-\infty}^{\infty}c_{2n}t^{2n+\rho},\qquad 0<|t|<\infty, \tag{4.1}$$

with

$$\kappa^2c_{2n-2}+\{(2n+\rho)^2-\nu^2\}\cdot c_{2n}-\omega^2\cdot c_{2n+2}=0. \tag{4.2}$$

There is a well-known equation for ρ involving only elementary transcendental functions (see Morse and Feshbach [15, p. 562]). If the characteristic exponent ρ is not an integer, then the two solutions (4.1) and $x((\omega/(i\kappa))\cdot t^{-1})$ (if $\kappa\neq 0$ and $\omega\neq 0$) are linearly independent.

The formal solutions (1.7) are ($k=1,2$)

$$\begin{aligned} x_{asy}^k(t)&=\exp(\lambda_k t)\cdot t^{\mu_k}\cdot\sum_{s=0}^{\infty}h_s^k\cdot t^{-s},\\ \lambda_{1,2}&=\pm i\kappa\quad\text{and}\quad \mu_1=\mu_2=-\tfrac{1}{2}, \end{aligned} \tag{4.3}$$

The h_s^k are determined by $h_s^1=i^{-s}A_s$ and $h_s^2=(-i)^{-s}A_s$, where A_s is defined by $A_s=0$ if $s<0$, $A_0=1$ and

$$2\kappa sA_s=\{(s-\tfrac{1}{2})^2-\nu^2\}A_{s-1}+\omega^2A_{s-3}.$$

Application of Method 3.3.

Step 1. With the abbreviation $\delta:=\rho+\frac{1}{2}$, one has

$$\exp(\lambda_k t)\sim\sum_{n=-\infty}^{\infty}g_n^k t^{n+\delta},\qquad t\in S(\lambda_k,0),$$

where $g_n^k:=(\lambda_k)^{n+\delta}/(n+\delta)!$

Step 2.

$$f_n^k:=\sum_{s=0}^{\infty}g_{n+s}^k h_s^k.$$

As $\lambda_k=\exp(\pm i(\pi/2))\kappa$, we get $f_n^k=\exp(\pm i(\pi/2)(\delta+n))\kappa^{\delta+n}S_n$ with $S_n:=\sum_{s=0}^{\infty}(A_s\kappa^s)/(n+s+\delta)!$. In these formulas $+$ is to be taken if $k=1$, and $-$ is to be taken if $k=2$.

Lemma 3.5 implies that $\limsup_{s\to\infty}\sqrt[s]{|h^k_s|/s!}\leqq 1/|\lambda_1-\lambda_2|$ or $\limsup_{s\to\infty}\sqrt[s]{|A_s\kappa^s/s!|}\leqq\frac{1}{2}$. Hence we get

$$\begin{aligned} S_n &= \sum_{s=0}^{\infty}\frac{A_s\kappa^s}{(n+s+\delta)!} \\ &= \frac{1}{(n+\delta)!}\left\{1+\sum_{s=1}^{\infty}\frac{1}{\binom{n+s+\delta}{s}}\left(\frac{A_s\kappa^s}{s!}\right)\right\} \qquad (4.4)\\ &= \frac{1}{(n+\delta)!}\{1+O(n^{-1})\}. \end{aligned}$$

Step 3. There are complex numbers β_k so that

$$c_n=\sum_{k=1}^{2}\beta_k f^k_n \quad \text{for all } n.$$

Especially we have, for fixed n,

$$\sum_{k=1}^{2}\beta_k f^k_{2n-1}=c_{2n-1}=0,$$
$$\sum_{k=1}^{2}\beta_k f^k_{2n}=c_{2n}.$$

From

$$\sum_{k=1}^{2}\beta_k f^k_n=(\kappa^{\delta+n}S_n)\left\{\beta_1\exp\left(i\frac{\pi}{2}(\delta+n)\right)+\beta_2\exp\left(-i\frac{\pi}{2}(\delta+n)\right)\right\}$$

we obtain

$$\beta_1\exp\left(i\frac{\pi}{2}\delta\right)-\beta_2\exp\left(-i\frac{\pi}{2}\delta\right)=0,$$

$$\beta_1\exp\left(i\frac{\pi}{2}\delta\right)+\beta_2\exp\left(-i\frac{\pi}{2}\delta\right)=D,$$

with the abbreviation

$$D:=c_{2n}\{(i\kappa)^{2n}\kappa^{\delta}S_{2n}\}^{-1}.$$

D is independent of n, because β_1 and β_2 are independent of n.

$$\beta_1=\frac{D}{2}\exp\left(-i\frac{\pi}{2}\delta\right),\qquad \beta_2=\frac{D}{2}\exp\left(i\frac{\pi}{2}\delta\right). \tag{4.5}$$

There are two particularly important ways for computing D. First we may take $n=0$ and get

$$D=c_0\{\kappa^{\delta}S_0\}^{-1}. \tag{4.6a}$$

Secondly we may compute the limit for $n\to\infty$. By (4.4) we obtain

$$D=\lim_{n\to\infty}\frac{c_{2n}(2n+\delta)!}{(i\kappa)^{2n}\kappa^{\delta}}. \tag{4.6b}$$

Step 4. We determine the connection coefficients T_k for the sectors $S(\lambda_k, p_k)=\{t; |ph(\lambda_k t)-2\pi p_k|<\pi\}$. Let us assume for simplicity of notation that κ is real >0. Then $ph(\lambda_1)=\pi/2$ and $ph(\lambda_2)=-\pi/2$.

$$S(\lambda_1, 0): -\frac{3}{2}\pi<ph(t)<\frac{\pi}{2},$$

$$S(\lambda_1, 1): \frac{\pi}{2}<ph(t)<\frac{5}{2}\pi,$$

$$S(\lambda_2, -1): -\frac{5}{2}\pi<ph(t)<-\frac{\pi}{2},$$

$$S(\lambda_2, 0): -\frac{\pi}{2}<ph(t)<\frac{3}{2}\pi.$$

We set $T_k(p_k) \coloneqq \exp(2\pi i p_k\delta)\beta_k$.

Restricting $ph(t)$ to $-\pi<ph(t)\leqq\pi$, we have computed the coefficients T_k in the expansion $x(t)\sim\sum_{k=1}^{2} T_k x_{asy}^k(t)$ as

$$T_1=\begin{cases} T_1(0) & \text{in } -\pi<ph(t)<\pi/2,\\ T_1(1) & \text{in } \pi/2<ph(t)\leqq\pi,\\ \frac{1}{2}(T_1(0)+T_1(1)) & \text{if } ph(t)=\pi/2,\end{cases}$$

(4.7)

$$T_2=\begin{cases} T_2(0) & \text{in} -\pi/2<ph(t)\leqq\pi,\\ T_2(-1) & \text{in} -\pi<ph(t)<-\pi/2,\\ \frac{1}{2}(T_2(0)+T_2(-1)) & \text{if } ph(t)=-\pi/2.\end{cases}$$

Especially in the sector $-\pi/2<ph(t)<\pi/2$, we obtained

$$\frac{2}{D}x(t)\sim\exp\left(-i\frac{\pi}{2}\delta\right)x_{asy}^1(t)+\exp\left(i\frac{\pi}{2}\delta\right)x_{asy}^2(t), \tag{4.8}$$

while in $\pi/2<|ph(t)|<\pi$, we have obtained the formulas (4.9a) and (4.9b) below. As

$$\exp(\lambda_1 t)\to 0 \quad \text{in } 0<ph(t)<\pi$$

and

$$\exp(\lambda_2 t)\to 0 \quad \text{in } -\pi<ph(t)<0,$$

we may write formula (4.8) for $-\pi<ph(t)<\pi$ too.

$$\frac{2}{D}x(t)\sim\exp(2\pi i\delta)\exp\left(-i\frac{\pi}{2}\delta\right)x_{asy}^1(t)+\exp\left(i\frac{\pi}{2}\delta\right)x_{asy}^2(t) \quad \text{in } \frac{\pi}{2}<ph(t)<\pi, \tag{4.9a}$$

$$\frac{2}{D}x(t)\sim\exp\left(-i\frac{\pi}{2}\delta\right)x_{asy}^1(t)+\exp(-2\pi i\delta)\exp\left(i\frac{\pi}{2}\delta\right)x_{asy}^2(t) \quad \text{in } -\pi<ph(t)<-\frac{\pi}{2}. \tag{4.9b}$$

However, it is better to take the expansions (4.9a, b) in $\pi/2<|ph(t)|<\pi$ than to take (4.8) as is indicated in § 5.

Application to the work of Bühring [4]. Bühring chooses $\kappa=1$ (without loss of generality) and defines two types of convergent solutions of the differential equation (1.6). The solutions $g_{\pm\rho}(t)$ are the power series solutions (4.1). The solutions $g^{(1)}(t)$ and $g^{(2)}(t)$ are defined such that they have the asymptotic expansions ($t\to\infty$):

$$g^{(1)}(t)\sim\exp\left(-i\frac{\pi}{4}\right)\cdot x^1_{asy}(t)\quad\text{in } -\pi<ph(t)<2\pi,$$

$$g^{(2)}(t)\sim\exp\left(i\frac{\pi}{4}\right)\cdot x^2_{asy}(t)\quad\text{in } -2\pi<ph(t)<\pi.$$

Thus Bühring generalizes the theory of Bessel functions ($g_{\pm\rho}(t)$) and Hankel functions ($g^{(1)}(t)$ and $g^{(2)}(t)$) to the more difficult case of the differential equation (1.6).

Using Laplace-type integrals he shows (equation (85)):

$$\nabla_\sigma\cdot g_\sigma(t)=\exp\left(-i\frac{\pi}{2}\sigma\right)\cdot g^{(1)}(t)+\exp\left(i\frac{\pi}{2}\sigma\right)\cdot g^{(2)}(t).$$

$\sigma=\pm\rho$ and ∇_σ being a factor of proportionality. In the sector $-\pi<ph(t)<\pi$, we get the asymptotic expansion

$$\nabla_\rho g_\rho(t)\sim\exp\left(-i\frac{\pi}{2}\delta\right)x^1_{asy}(t)+\exp\left(i\frac{\pi}{2}\delta\right)x^2_{asy}(t).$$

But $x(t)$ in (4.1) is equal to $g_\rho(t)$, if we normalize $c_0=1$; hence comparing this last asymptotic relation with (4.8), we see that

$$\nabla_\rho=2/D.$$

Using (4.6a), we get in the case $c_0=1$ and $\kappa=1$,

$$\nabla_\rho=2\sum_{s=0}^{\infty}\frac{A_s}{(s+\rho+\frac{1}{2})!}$$

and

$$\nabla_{-\rho}=2\sum_{s=0}^{\infty}\frac{A_s}{(s-\rho+\frac{1}{2})!}$$

These formulas were given by Bühring [4, p. 1459].

A connection formula of Fubini and Stroffolini [6]. In order to apply (4.6b), we have to determine the asymptotic behavior of c_{2n} for large n. This can be done using infinite determinants.

Introducing $d_n:=c_{2n}$, we get from (4.2) the equations (if $\rho\not\equiv\pm\nu(\text{mod } 2)$):

$$\text{(4.10)}\qquad \frac{\kappa^2}{(2n+\rho)^2-\nu^2}\cdot d_{n-1}+d_n-\frac{\omega^2}{(2n+\rho)^2-\nu^2}\cdot d_{n+1}=0.$$

Fubini and Stroffolini then define $\Delta_\rho(M,N)$ to be the finite determinants of (4.10) with the diagonal elements all equal to 1 and containing the rows and columns of (4.10) with indexes $\geqq M$ and $\leqq N$. As is seen from a well-known criterion

(Whittaker and Watson [24, pp. 36–37]), there exist $\Delta_\rho(m,\infty)$, $\Delta_\rho(-\infty, n)$ and $\Delta_\rho(-\infty,\infty)$, which are defined by the corresponding limits. Also we have

$$\lim_{m\to\infty} \Delta_\rho(m,\infty)=1, \tag{4.11a}$$

$$\lim_{m\to\infty} \Delta_\rho(-\infty,-m)=1. \tag{4.11b}$$

The characteristic exponent ρ obeys

$$\Delta_\rho(-\infty,\infty)=0. \tag{4.12}$$

$\Delta_\rho(n,\infty)$ can be computed, using (4.11a) and

$$\Delta_\rho(n,\infty)=\Delta_\rho(n+1,\infty)+\frac{\kappa^2\cdot\omega^2\cdot\Delta_\rho(n+2,\infty)}{\{(2n+\rho)^2-\nu^2\}\{(2n+2+\rho)^2-\nu^2\}}. \tag{4.13a}$$

For computation of $\Delta_\rho(-\infty,-n)$ one can use (4.11b) and

$$\Delta_\rho(-\infty,-n)=\Delta_\rho(-\infty,-n-1)+\frac{\kappa^2\cdot\omega^2\cdot\Delta_\rho(-\infty,-n-2)}{\{(2n-\rho)^2-\nu^2\}\{(2n+2-\rho)^2-\nu^2\}}. \tag{4.13b}$$

From (4.13a) we get

$$d_n^+=\left(\frac{i\kappa}{2}\right)^{2n}\cdot\frac{((\rho+\nu)/2)!\cdot((\rho-\nu)/2)!}{(n+((\rho+\nu)/2))!(n+((\rho-\nu)/2))!}\cdot\Delta_\rho(-\infty,-1)\cdot\Delta_\rho(n+1,\infty). \tag{4.14a}$$

From (4.13b) we get

$$d_{-n}^-=\left(\frac{\omega}{2}\right)^{2n}\cdot\frac{(-(\rho+\nu)/2)!(-(\rho-\nu)/2)!}{(n-((\rho+\nu)/2))!(n-((\rho-\nu)/2))!}\cdot\Delta_\rho(1,\infty)\cdot\Delta_\rho(-\infty,-n,-1). \tag{4.14b}$$

In these formulas we assume $n\geqq 0$. We remark:

$$d_0=\Delta_\rho(-\infty,-1)\cdot\Delta_\rho(1,\infty). \tag{4.15}$$

If (4.12) is fulfilled, then $d_n^+=d_n^-$. By means of Stirling's formula we find

$$\lim_{n\to\infty}\frac{(2n+\rho+\frac{1}{2})!}{2^{2n+\rho}\cdot(n+((\rho+\nu)/2))!(n+((\rho-\nu)/2))!}=\sqrt{\frac{2}{\pi}},$$

and conclude that

$$d_n^+\sim\left(\frac{\rho+\nu}{2}\right)!\left(\frac{\rho-\nu}{2}\right)!\sqrt{\frac{2}{\pi}}\cdot 2^\rho\cdot\Delta_\rho(-\infty,-1)\cdot\frac{(i\kappa)^{2n}}{(2n+\rho+\frac{1}{2})!}. \tag{4.16}$$

Now we apply (4.6b) (with $c_{2n} = d_n^+$):

$$D = \left(\frac{\rho+\nu}{2}\right)!\left(\frac{\rho-\nu}{2}\right)!\frac{\Delta_\rho(-\infty,\,-1)}{\sqrt{\pi}}\cdot\left(\frac{2}{\kappa}\right)^{\rho+(1/2)}.$$

From equations (4.5) we obtain

$$(4.17)\quad \beta_k = \frac{\Delta_\rho(-\infty,-1)}{2\cdot\sqrt{\pi}}\cdot \exp\left\{\mp i\frac{\pi}{2}\left(\rho+\frac{1}{2}\right)\right\}\cdot\left(\frac{\rho+\nu}{2}\right)!\left(\frac{\rho-\nu}{2}\right)!\left(\frac{2}{\kappa}\right)^{\rho+(1/2)},$$

where the $-$ sign is to be taken if $k=1$, and $+$ if $k=2$. The connection coefficients are finally

$$T_1 = \beta_1 \quad \text{valid in } |ph(\kappa t) + (\pi/2| < \pi,$$

(4.18) and

$$T_2 = \beta_2 \quad \text{valid in } |ph(\kappa t) - (\pi/2)| < \pi.$$

The results here have been obtained with the assumption that $\rho \not\equiv \pm\nu \pmod 2$, but are also valid in the exceptional case (Naundorf [16]).

A special case, where this exception holds, is given for Bessel's equation

$$\frac{d^2x}{dt^2} + \frac{1}{t}\frac{dx}{dt} + \left(1-\frac{\nu^2}{t^2}\right)x = 0.$$

Then $\rho = \pm\nu$ and $\Delta(-\infty,\,-1) = 1$, so we get from (4.17),

$$2\beta_k = \sqrt{\frac{2}{\pi}}\exp\left(\mp i\frac{\pi}{2}\left(\rho+\frac{1}{2}\right)\right)(\rho!)2^\rho.$$

Here $c_0 = 1$. If we normalize $c_0 = 1/(2^\rho\rho!)$, we have to replace $2\beta_k$ by $\sqrt{2/\pi}\exp(\mp i(\pi/2)(\rho+\frac{1}{2}))$ and we obtain the asymptotic expansion of the Bessel's function $J_\rho(t)$:

$$(4.19)\qquad 2J_\rho(t) \sim H^1_{\rho,asy}(t) + H^2_{\rho,asy}(t)$$

in the sector $|ph(t)| < \pi/2$.

Here we have set

$$H^1_{\rho,asy}(t) := \sqrt{\frac{2}{\pi}}\exp\left(-i\frac{\pi}{2}\left(\rho+\frac{1}{2}\right)\right)x^1_{asy}(t)$$

$$= \sqrt{\frac{2}{\pi t}}\exp\left\{i\left(t-\frac{\pi}{2}\left(\rho+\frac{1}{2}\right)\right)\right\}\left(1+i\frac{4\rho^2-1}{8t}-\cdots\right),$$

$$H^2_{\rho,asy}(t) := \sqrt{\frac{2}{\pi}}\exp\left(i\frac{\pi}{2}\left(\rho+\frac{1}{2}\right)\right)x^2_{asy}(t)$$

$$= \sqrt{\frac{2}{\pi t}}\exp\left\{-i\left(t-\frac{\pi}{2}\left(\rho+\frac{1}{2}\right)\right)\right\}\left(1-i\frac{4\rho^2-1}{8t}-\cdots\right).$$

We remember that with Hankel functions $H_\rho^k(t)$ there is

$$2J_\rho(t) = H_\rho^1(t) + H_\rho^2(t)$$

and

$$H_\rho^1(t) \sim H_{\rho,asy}^1(t) \quad \text{in } -\pi < ph(t) < 2\pi,$$

$$H^2(t) \sim H_{\rho,asy}^2(t) \quad \text{in } -2\pi < ph(t) < \pi.$$

According to (4.9a), we have in $\pi/2 < ph(t) < \pi$ the expansion

$$2J_\rho(t) \sim \exp(2\pi i(\rho + \tfrac{1}{2}))H_{\rho,asy}^1(t) + H_{\rho,asy}^2(t). \tag{4.20}$$

5. Comparisons of equivalent asymptotic expansions in the case of Bessel functions. From the formulas 9.1.37 and 9.1.38 of Abramowitz and Stegun [1], one obtains

$$H_\rho^2(t) = -H_\rho^1(t)(e^{2\pi i\rho} + 1) - H_\rho^2(t\, e^{-2\pi i}),$$

hence by an argument found in Olver [18, p. 240] one has

$$H_\rho^2(t) \sim -(e^{2\pi i\rho} + 1)H_{\rho,asy}^1(t) + H_{\rho,asy}^2(t) \tag{5.1}$$

in the sector $0 < ph(t) < 2\pi$. In $0 < ph(t) < \pi$, this expansion and

$$H_\rho^2(t) \sim H_{\rho,asy}^2(t) \tag{5.2}$$

are asymptotically equivalent. But the error bound (Olver [18]) of (5.1) is less than the one of (5.2) in $\pi/2 < ph(t) < \pi$, while in $0 < ph(t) < \pi/2$, the opposite is true. A complete discussion is given by Olver [18, p. 268].

Hence it is suggested that one expand $J_\rho(t)$ in $0 < ph(t) < \pi$ as follows (see Table 1):

TABLE 1

$\lvert t\rvert$	$J_0(\lvert t\rvert \exp(i\frac{3}{4}\pi))$	$(H_{0,asy}^1(t) + H_{0,asy}^2(t))/2$	$(-H_{0,asy}^1(t) + H_{0,asy}^2(t))/2$
0.0	1.00 + i 0.00	—	—
0.5	1.00 + i 0.06	0.77 − i 0.47	1.11 + i 0.11
1.0	0.98 + i 0.25	0.72 + i 0.07	1.04 + i 0.23
1.5	0.92 + i 0.56	0.73 + i 0.51	0.94 + i 0.55
2.0	0.75 + i 0.97	0.68 + i 0.99	0.76 + i 0.97
2.5	0.40 + i 1.46	0.33 + i 1.50	0.40 + i 1.45
3.0	−0.22 + i 1.94	−0.25 + i 1.98	−0.22 + i 1.94
3.5	−1.19 + i 2.28	−1.20 + i 2.32	−1.19 + i 2.28
4.0	−2.56 + i 2.29	−2.56 + i 2.32	−2.56 + i 2.29
4.5	−4.30 + i 1.69	−4.29 + i 1.70	−4.30 + i 1.69
5.0	−6.23 + i 0.12	−6.22 + i 0.12	−6.23 + i 0.12

The numerical values of

$$J_0(\lvert t\rvert \exp(i\tfrac{3}{4}\pi)) = ber_0(\lvert t\rvert) + i\,bei_0(\lvert t\rvert)$$

were taken from Abramowitz and Stegun [1, p. 430].
The asymptotic series appearing in $H_{0,asy}^k(t)$ were computed until the smallest absolute value appeared.

$$2J_\rho(t) \sim H_{\rho,asy}^1(t) + H_{\rho,asy}^2(t), \qquad 0 < ph(t) < \pi/2 \tag{5.3}$$

$$2J_\rho(t) \sim \exp(2\pi i(\rho + \tfrac{1}{2}))H_{\rho,asy}^1(t) + H_{\rho,asy}^2(t) \quad \text{in } \pi/2 < ph(t) < \pi. \tag{5.4}$$

These formulas we obtained in (4.19), (4.20).

Appendix A. The recurrence relations (1.10) and (2.5) are Perron–Kreuser difference equations of the following form (A.1). Gautschi [7] reviewed this theory on page 35 in the special case of three-term recurrence relations.

$$y_{n+m}+A_1 n^{k_1}(1+o(1))y_{n+m-1}+\cdots+A_m n^{k_m}(1+o(1))y_n=0 \quad \text{for } n=0,1,2,\cdots. \tag{A.1}$$

By the symbol $o(1)$ is meant an expression that vanishes as $n\to\infty$. A_i are complex, k_i are real numbers. We set $k_0:=0$ and $k_i:=-\infty$ if $A_i=0$. $P_i:=(i,k_i)$ are $m+1$ points in a plane.

DEFINITION A.1. The Newton–Puiseux diagram for (A.1) is an upward convex polygon connecting P_0 and P_m such that no P_i lies above the polygon and the only corners are contained in the set of the points P_i, say,

$$P_{e_0}=P_0, \qquad P_{e_1},\cdots,P_{e_g}=P_m.$$

Remark A.2. Let q_j be the directional derivative of the line $\overline{P_{e_j}P_{e_{j+1}}}$. Then by construction of the polygon,

$$q_o>q_1>\cdots>q_{g-1}.$$

The following Lemma is part of Perron's [21] Theorem A.

LEMMA A.3. *There is a fundamental system of solutions of* (A.1) *such that for each solution of this system there is just one line derivative* q_j, $0\leqq j<g$, *with*

$$\limsup_{n\to\infty}\sqrt[n]{|y_n|(n!)^{-q_j}}=|\sigma|\neq 0,$$

and $\sigma\neq 0$ *is a root of the equation*

$$\textstyle\sum^{(j)}A_i z^{-i}=0, \tag{A.2}$$

where $A_i z^{-i}$ *appears in the sum, if* P_i *lies on the line* $\overline{P_{e_j}P_{e_{j+1}}}$.

Proof of Lemma 2.2. The Newton–Puiseux diagram for the recurrence relation

$$g_n=\sum_{i=1}^{R}\frac{\alpha_i}{n+\delta}g_{n-i}, \qquad \alpha_R=\lambda\neq 0,$$

consists of one line connecting the points $(0,0)$ and $(R,-1)$. The directional derivative of the line is $q_0=-1/R$, and equation (A.2) is $1-\lambda z^{-R}=0$. Hence all solutions of the recurrence relation that do not vanish identically, obey

$$\limsup_{n\to\infty}\sqrt[n]{|g_n|(n!)^{1/R}}=|\lambda|^{1/R}.$$

Proof of Lemma 3.5. The Newton–Puiseux diagram for the recurrence relation

$$h_s=\frac{s}{\Delta\lambda}h_{s-R}+\sum_{i=1}^{I}(A_i+s^{-1}B_i)h_{s-i}$$

consists of at least 2 lines, if $I > R$. Line 1 connects the points (0, 0) and $(R, 1)$, $q_0 = 1/R$ is the line derivative, and equation (A.2) is

$$1 - \frac{1}{\Delta\lambda} z^{-R} = 0.$$

Then a second line begins in $(R, 1)$ with line derivative $q_1 < 0$.

Hence there exists a fundamental system of solutions which may be separated into two classes. In the first class, the solutions obey

$$\limsup_{s\to\infty} \sqrt[s]{|h_s|(s!)^{-1/R}} = \left|\frac{1}{\Delta\lambda}\right|^{1/R}.$$

These solutions all dominate those of the second class, which proves Lemma 3.5.

Proof of Lemma 3.7. The Lemmas 2.2 and 3.5 imply that for every positive ε, there exist constants M_1 and M_2 such that for all $n \geqq 0$ and $s \geqq 0$,

$$|g_n| < M_1(|\lambda|^{1/R} + \varepsilon)^n (n!)^{-1/R},$$

and

$$|h_s| < M_2(|\Delta\lambda|^{-1/R} + \varepsilon)^s (s!)^{1/R}.$$

Hence there is an $\tilde{\varepsilon}$ that will be arbitrarily small for the appropriate choice of ε, so that

$$|g_{n+s}h_s| \leqq M_1 M_2 \frac{(|\lambda|^{1/R} + \varepsilon)^n}{(n!)^{1/R}} \cdot \frac{(|\lambda/\Delta\lambda|^{1/R} + \tilde{\varepsilon})^s}{\binom{n+s}{s}^{1/R}}.$$

So there is a constant M_3 with

$$|f_n| \leqq M_3 \frac{(|\lambda|^{1/R} + \varepsilon)^n}{(n!)^{1/R}}$$

if $|\lambda| < |\Delta\lambda|$. From this estimate it is seen that $\sum_{n=0}^{\infty} f_n t^n$ is an entire function.

Appendix B.

LEMMA B.1 (Perron [20]). *The coefficients of the system of linear equations*

$$\text{(B.1)} \qquad \sum_{\nu=0}^{\infty} (a_\nu + b_{\mu\nu}) y_{\mu+\nu} = w_\nu, \qquad \mu = 0, 1, 2, \cdots,$$

may fulfill

$$a_0 + b_{\mu 0} \neq 0, \qquad \mu = 0, 1, 2, \cdots,$$

$$|b_{\mu\nu}| \leqq k_\mu \partial^\nu, \qquad 0 < \partial < 1,$$

with

$$\lim_{\mu\to\infty} k_\mu = 0, \qquad \limsup_{\mu\to\infty} \sqrt[\mu]{|w_\mu|} \leqq 1,$$

and the function $F(z) := \sum_{\nu=0}^{\infty} a_\nu z^\nu$ *regular in* $|z| \leqq 1$.

If $n(\geqq 0)$ is the number of zeros of $F(z)$ in $|z|\leqq 1$ (counted according to their multiplicity), then the general solution of (B.1) *with the constraint*

$$\limsup_{\mu\to\infty} \sqrt[\mu]{|y_\mu|}\leqq 1$$

has exactly n arbitrary constants B_λ and has the form

$$y_\nu = y_{\nu 0} + \sum_{\lambda=1}^{n} B_\lambda x_{\nu\lambda}.$$

If M is a sufficiently large index, then there is just one such solution, for which the n unknowns $y_M, y_{M+1}, \cdots, y_{M+n-1}$ have given values.

Remark B.2. How to choose M is described in Perron [20, p. 8].

We now justify Step 3 of Method 3.3.

LEMMA B.3. *$f_n^{k,L}$ are solutions of* (1.3) *with*

$$\limsup_{n\to\infty} \sqrt[n]{|f_n^{k,L}|}=0.$$

The coefficients c_n of the multiplicative solution (1.2) *also have the behavior*

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|}=0.$$

Proof. The formal series $\sum_{n=-\infty}^{\infty} f_n t^{n+\rho}$ is a formal solution of (1.1), hence the f_n are solutions of (1.3). As $\sum_{n=0}^{\infty} f_n t^n$ and $\sum_{n=0}^{\infty} c_n t^n$ are entire functions, the behavior of f_n, c_n for $n\to\infty$ follows.

LEMMA B.4. *There are constants $\beta_{k,L}$ (uniquely) with*

$$c_n = \sum_{k=1}^{2}\sum_{L=0}^{R-1} \beta_{k,L} f_n^{k,L} \quad \textit{for all } n.$$

Proof. We write the recurrence relations (1.3) in the form

$$c_n + \sum_{i=-r}^{R} \frac{a_i(n+\rho-i)}{(n+\rho)(n+\rho-1)} c_{n-i} + \sum_{i=-2r}^{2R} \frac{b_i}{(n+\rho)(n+\rho-1)} c_{n-i} = 0.$$

With the abbreviation $m := n-2R$ we get

$$\begin{aligned}
&\frac{b_{2R}}{(m+2R+\rho)(m+2R+\rho-1)} c_m \\
&\quad + \cdots + \left\{1 + \frac{a_0}{(m+2R-1+\rho)} \right. \\
&\qquad\qquad \left. + \frac{b_0}{(m+2R+\rho)(m+2R-1+\rho)}\right\} c_{m+2R} \\
&\quad + \cdots + \frac{b_{-2r}}{(m+2R+\rho)(m+2R-1+\rho)} c_{m+2(R+r)} = 0.
\end{aligned}$$

Now we apply Lemma B.1 with $\partial=\frac{1}{2}$,

$$k_m=(1/m)2^{2(R+r)}\max\{|a_i|,|b_j|\}$$

and $F(z)=z^{2R}$. $F(z)$ has $2R$ zeros in $|z|\leqq 1$. As the $2R$ solutions $f_n^{k,L}$ of the recurrence relation are independent, we get the assertion, using Lemma B.3.

LEMMA B.5. *For sufficiently large N, the system of linear equations for the unknowns* $\beta_{k,L}$,

$$\sum_{k=1}^{2}\sum_{L=0}^{R-1}\beta_{k,L}f_n^{k,L}=c_n, \qquad N\leqq n<N+2R,$$

has exactly one solution.

Proof. The proof results from the last part of Lemma B.1 and from Lemma B.4.

LEMMA B.6. *N can be chosen as given in* (3.5).

Proof. See Naundorf [16].

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ON THE BOUNDEDNESS AND STABILITY OF SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS OF THE FIFTH ORDER*

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Abstract. The paper studies the equation (1.1) in three cases: (i) $p \equiv 0$, (ii) $p\ (\neq 0)$ satisfies $|p(t, x, y, z, w, u)| \leqq (A_0 + |y| + |z| + |w| + |u|)\psi(t)$, where ψ is a nonnegative function of t; *(iii)* $p\ (\neq 0)$ satisfies $|p(t, x, y, z, w, u)| \leqq A < \infty$, where A is a positive constant. In case (i) the asymptotic stability (in the large) of the solution $x = 0$ is studied; in case (ii) a general estimate and a boundedness result are deduced for solutions of (1.1); in case (iii) the ultimate boundedness of all solutions of (1.1) is proved in such a way that the ultimate bounding constant is independent of any solutions chosen. The major contribution of the paper is the successful generalization of the Routh–Hurwitz criteria for linear constant coefficient fifth order equations to the nonlinear case, a generalization that involves the use of Lyapunov functions. The results obtained seem to be as general as recent analogous treatments of fourth order equations.

1. Introduction. Although not of such universal occurrence as second, third and fourth order equations, fifth order systems do arise in a number of applications, for example, in some three loop electric circuit problems and in control theory. (See Rosenvasser [13].) It is, therefore, of some importance that the quantitative behavior of solutions of such equations be investigated. Inspired by researches on fourth order equations by Ezeilo [4], [5], Harrow [9], [10], [11], Lalli and Skrapek [12] and Sinha and Hoft [14], we initiated such a study in [2]. The results obtained are comparable in generality with the more recent results of Harrow [10]. The present paper considers a different fifth order equation, namely,

$$x^{(5)} + a\ddddot{x} + f_2(\dddot{x}) + c\ddot{x} + f_4(x) + f_5(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}), \tag{1.1}$$

or its equivalent system

$$\begin{aligned} &\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = w, \quad \dot{w} = u, \\ &\dot{u} = -au - f_2(w) - cz - f_4(y) - f_5(x) + p(t, x, y, t, w, u), \end{aligned} \tag{1.2}$$

obtained from (1.1) on setting

$$\frac{dx}{dt} = \dot{x} = y, \quad \frac{d^2x}{dt^2} = \ddot{x} = z, \quad \frac{d^3x}{dt^3} = x = w,$$

$$\frac{d^4x}{dt^4} = x = \ddddot{u}, \qquad x^{(5)} = \frac{d^5x}{dt^5} = \dot{u}.$$

It is assumed as basic that a and c are constants, and f_2, f_4, f_5 and p are continuous functions which depend only on the arguments displayed and are such that the existence and uniqueness of solutions, as well as their continuous dependence on the initial conditions, are guaranteed.

The paper investigates (1.2) in three cases: (i) $p \equiv 0$; (ii) $p\ (\neq 0)$ satisfies $|p(t, x, y, z, w, u)| \leqq (A_0 + |y| + |z| + |w| + |u|)\psi(t)$, where ψ is a nonnegative function of t and A_0 constant; (iii) $p\ (\neq 0)$ satisfies $|p(t, x, y, z, w, u)| \leqq A < \infty$, where A is a constant. In the first case, the asymptotic stability (in the large) of the trivial

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solution $x=0$ is proved, and in the second case, an interesting bound is deduced for the solutions of (1.2). In the third case, sufficient conditions for ultimate boundedness are stated and are such that the ultimate bounding constant is independent of any solutions chosen. The methods of investigation are those of Lyapunov and its refinement by Yoshizawa [17]. While Lyapunov's second method is used to study the first two cases, the well-known Yoshizawa type argument adapted to a fifth order system is used to prove the ultimate boundedness of solutions of (1.2).

When the first two cases are compared with analogous treatment in [10], the results here are more general; they are comparable to the recent work of Ezeilo and Tejumola [8]. Ultimate boundedness results of nonlinear differential equations (treated in case (iii)) are both important and interesting. For the status of such research see Tejumola [16] for the second order, Ezeilo [7] and Chukwu [1] for the third, Ezeilo and Tejumola [8] and Chukwu [3] for the fourth. The present work is the first attempt to obtain sufficient conditions for ultimate boundedness of solutions of fifth order differential equations. The result obtained is comparable in generality to the work of Ezeilo and Tejumola [8] on fourth order equations.

To appreciate what the author considers to be the main contribution of the paper, consider the linear constant coefficient differential equations,

$$x^{(5)}+a\ddddot{x}+b\dddot{x}+c\ddot{x}+d\dot{x}+ex=0. \tag{1.3}$$

A necessary and sufficient condition that all solutions of (1.3) tend to the trivial solution $x=0$ as $t\to\infty$ is the Routh–Hurwitz criterion:

$$\begin{aligned} &a>0, \quad ab-c>0, \quad (ab-c)c-(ad-e)a>0,\\ &\Delta=(dc-be)(ab-c)-(ad-e)^2>0, \qquad e>0. \end{aligned} \tag{1.4}$$

It follows immediately that

$$a>0, \quad b>0, \quad c>0, \quad d>0, \quad e>0, \quad ad-e>0, \quad dc-be>0. \tag{1.5}$$

The asymptotic stability result to be proved below has hypotheses generalizations of the Routh–Hurwitz criterion (1.4): the nonlinearities are required to satisfy these generalizations. The major contribution of the paper is the successful generalization of (1.4) for linear constant coefficient equation (1.3) to the nonlinear case (1.2). As will be seen below the generalization rests on an explicit Lyapunov function.

2. Statement of results.

THEOREM 1. *Suppose in* (1.2) $p(t,x,y,z,w,u)\equiv 0$; *and*

(i) *the constants* a, b, c, d, e *are such that* (1.4) *and* (2.1) *below holds*:

$$\begin{aligned} &\Delta_1=\frac{(dc-be)(ab-c)}{ad-e}-(af_4'(y)-e)>2\varepsilon b \quad \textit{for all } y,\\ &\Delta_2=\frac{(dc-be)}{ad-e}-\frac{\gamma(ad-e)}{d(ab-c)}-\frac{\varepsilon}{a}>0, \end{aligned} \tag{2.1}$$

where

$$\gamma = \begin{cases} \dfrac{f_4(y)}{y}, & y \neq 0, \\ f_4'(0), & y = 0, \end{cases} \tag{2.2}$$

and ε *is a sufficiently small positive constant*;

(ii) $$f_i(0) = 0, \qquad i = 2, 4, 5;$$

(iii) $$\frac{f_2(w)}{w} \geqq b, \quad w \neq 0, \quad \frac{f_4(y)}{y} \geqq d, \quad y \neq 0, \quad f_5'(x) \leqq e$$

for all x;

(iv) $f_5(x) \operatorname{sgn} x > 0, \quad x \neq 0, \quad F_5(x) \equiv \int_0^x f_5(s)\, ds \to \infty \quad as\ |x| \to \infty;$

(v) $$0 \leqq \frac{f_2(w)}{w} - b \leqq \varepsilon_1, \quad |f_4'(y) - d| \leqq \varepsilon_2, \quad e - f_5'(x) \leqq \varepsilon_3;$$

for all x, y, w $(\neq 0)$ *and for sufficiently small* ε_1, ε_2 *and* ε_3,

$$f_4'(y) - \frac{f_4(y)}{y} \leqq \beta < \frac{e\Delta}{d^2(ab - c)}, \qquad y \neq 0. \tag{2.3}$$

Then every solution $(x(t), y(t), z(t), w(t), u(t))$ *of* (1.2) *satisfies*

$$x^2(t) + y^2(t) + z^2(t) + w^2(t) + u^2(t) \to 0 \quad as\ t \to \infty \tag{2.4}$$

Remark. The special case

$$f_2(w) = bw, \quad f_4(y) = dy, \quad f_5(x) = ex,$$

where b, d, e are constants should be noted.

In this case the hypotheses (ii), (iii), (iv) and (v) are trivially satisfied, while condition (i) reduces to the Routh–Hurwitz conditions for the asymptotic stability (in the large) of the trivial solution of (1.3).

THEOREM 2. *Suppose in* (1.2) *that*

$$f_2(0) = 0 = f_4(0),$$

and the conditions (i), (iii) *and* (v) *of Theorem* 1 *hold. Furthermore*

(i) $$f_5(x) \operatorname{sgn} x > 0 \quad for\ |x| \geqq 1,$$

(ii) *the function* $p(t, x, y, z, w, u)$ *satisfies*

$$|p(t, x, y, z, w, y)| \leqq \{A_0 + |y| + |z| + |w| + |u|\}\psi(t) \tag{2.5}$$

for all t, x, y, z, w, u, *where* A_0 *is a constant and* $\psi(t) \geqq 0$ *is a continuous function of* t. *Then for any given finite* x_0, y_0, z_0, w_0, u_0, *there exist constants*

$$K_i = K_i(x_0, y_0, z_0, w_0, u_0), \qquad i = 0, 1,$$

and constant $\lambda > 0$ *whose magnitude is independent of* x_0, y_0, z_0, w_0, u_0, *such that any solution* $x(t), y(t), z(t), w(t), u(t)$, *of* (1.2) *determined by*

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0, \quad w(0) = w_0, \quad u(0) = u_0 \tag{2.6}$$

satisfies for all $t \geqq 0$,

$$y^2(t) + z^2(t) + w^2(t) + u^2(t) \leqq K_0\left\{1 + \chi^{-1}(t)\left[1 + \int_0^t \psi(\tau)\chi(\tau)\, d\tau\right]\right\}, \tag{2.7}$$

$$F_5(x(t)) = \int_0^{x(t)} f_5(s)\, ds \leqq K_1\left\{1 + \chi^{-1}(t)\left[1 + \int_0^t \psi(\tau)\chi(\tau)\, d\tau\right]\right\}, \tag{2.8}$$

where

$$\chi(\tau) = \exp(-\lambda \int_0^t \psi(\tau)\, d\tau).$$

COROLLARY. *Suppose in addition to the conditions of Theorem* 2 *that*

$$F_5(x) \to +\infty \quad \text{as } |x| \to \infty$$

and

$$\int_0^\infty \psi(t)\, dt < \infty.$$

Then there exists some constant $K_2 = K_2(x_0, y_0, z_0, w_0, u_0)$ *such that the unique solution* $(x(t), y(t), z(t), w(t), u(t))$ *of* (1.2) *determined by* (2.6) *satisfies*

$$|x(t)| \leqq K_2, \quad |y(t)| \leqq K_2, \quad |z(t)| \leqq K_2, \quad |w(t)| \leqq K_2, \quad |u(t)| \leqq K_2, \tag{2.9}$$

for all $t \geqq 0$.

The corollary follows immediately from Theorem 2. Indeed if

$$\int_0^\infty \psi(t)\, dt < \infty,$$

then

$$\left\{1 + \chi^{-1}(t)\left[1 + \int_0^t \psi(\tau)\chi(\tau)\, d\tau\right]\right\} < \infty.$$

Because of (2.7) and (2.8) there exists a constant $D < \infty$ such that

$$y^2(t) + z^2(t) + w^2(t) + u^2(t) \leqq DK_0;$$

and

$$F_5(x(t)) \leqq DK_1.$$

Since $F_5(x) \to \infty$ as $|x| \to \infty$, these two estimates imply (2.9).

THEOREM 3. *In* (1.2) *suppose* $f_2(0) = 0 = f_4(0)$ *and that*

(i) $\qquad f_5(x) \operatorname{sgn} x \to +\infty \quad$ *as* $|x| \to \infty$;

(ii) *the constants* a, b, c, d, e *are such that* (1.4) *and* (2.1) *hold*;

(iii) $\dfrac{f_2(w)}{w} \geqq b, \quad w \neq 0, \quad \dfrac{f_4(y)}{y} \geqq d, \quad y \neq 0, \quad f_5'(x) \leqq e$ *for all* x;

(iv) $$0 \leqq \frac{f_2(w)}{w} - b \leqq \varepsilon_1, \quad |f_4'(y) - d| \leqq \varepsilon_2, \quad e - f_5'(x) \leqq \varepsilon_3$$

for all x, y, w $(\neq 0)$ *and for sufficiently small positive* ε_1, ε_2 *and* ε_3;

$$f_4'(y) - \frac{f_4(y)}{y} \leqq \beta < \frac{e\Delta}{d^2(ab-c)}, \qquad |y| \geqq 1; \tag{2.10}$$

(v) *for all values of* t, x, y, z, w, u, p (t, x, y, z, w, u) *satisfies*

$$|p(t, x, y, z, w, u)| \leqq A < \infty,$$

where A *is a positive constant. Then there exists a constant* K *whose magnitude depends only on* a, b, c, d, e, Δ, Δ_1, Δ_2, ε, A *as well as on the functions* f_2, f_4 *and* f_5 *such that every solution* $(x(t), y(t), z(t), w(t), u(t))$ *of* (1.2) *ultimately satisfies*

$$|x(t)| \leqq K, \quad |y(t)| \leqq K, \quad |z(t)| \leqq K, \quad |w(t)| \leqq K, \quad |u(t)| \leqq K. \tag{2.11}$$

Note that the conditions on a, b, c, d and e in (1.4) of (ii) correspond to the Routh–Hurwitz criteria for the asymptotic stability of the trivial solution of (1.3). For the equation (1.3), conditions (2.1) of (ii) are implied by (1.4), while the other conditions of Theorem 3 are trivially verified when (1.2) is specialized to (1.3).

3. Notation. In what follows we shall use the letter D for positive constants whose magnitudes depend on a, b, c, d, e, A, A_0 and f_i $(i = 2, 4, 5)$. No two D's are ever the same unless they are numbered, but all the D's: D_1, D_2, D_3, $\cdots$, with suffixes attached retain their identities throughout the sequel.

4. Some lemmas. The main tool, beside the system (1.2) itself, in the proof of the theorems is the function $V_1 = V_1(x, y, z, w, u)$ defined by

$$\begin{aligned} 2V_1 = {} & u^2 + 2auw + \frac{2d(ab-c)uz}{ad-e} + 2\delta yu + 2\int_0^w f_2(s)\,ds \\ & + \left[a^2 - \frac{d(ab-c)}{ad-e}\right]w^2 + 2\left[c + \frac{ad(ab-c)}{ad-e} - \delta\right]wz + 2a\delta wy \\ & + 2wf_4(y) + 2wf_5(x) + \left[ac + \frac{bd(ab-c)}{ad-e} - d - a\delta\right]z^2 + 2\delta byz \\ & + azf_4(y) - 2ezy + 2azf_5(x) + \frac{2d(ab-c)}{ad-e}\int_0^y f_4(s)\,ds + (\delta c - ea)y^2 \\ & + \frac{2d(ab-c)}{ad-e} f_5(x) + 2\delta\int_0^x f_5(s)\,ds, \end{aligned} \tag{4.1}$$

where δ is defined by $\delta = e(ab-c)/(ad-e) + \varepsilon$.

LEMMA 1. *Subject to the conditions of Theorem* 1, $V_1(0,0,0,0,0)=0$ *and there are constants* D_i $(i=1,2,3,4,5)$ *such that*

$$V_1 \geqq D_1F_5(x)+D_2y^2+D_3z^2+D_4w^2+D_5u^2, \tag{4.2}$$

for all x, y, z, w and u, provided ε *is chosen sufficiently small.*

Proof. Trivially $V_1(0,0,0,0,0)=0$. To verify (4.2) we recall that

$$\gamma=\begin{cases}\dfrac{f_4(y)}{y}, & y\neq 0,\\ f_4'(0), & y=0;\end{cases}$$

and rewrite (4.1) as follows:

$$\begin{aligned}2V_1=&\left[u+aw+\frac{d(ab-c)z}{ad-e}+\delta y\right]^2+\frac{d}{\gamma}\left(\frac{ad-e}{ab-c}\right)\\ &\cdot\left[\frac{(ab-c)}{ad-e}f_5(x)+\frac{(ab-c)}{ad-e}\gamma y+\frac{a}{d}\gamma z+\frac{\gamma}{d}w\right]^2\\ &+\frac{d\Delta}{(ad-e)^2}\left[z+\frac{ey}{d}\right]^2+\Delta_2[w+az]^2+\sum_{i=1}^{4}Wi,\end{aligned}\tag{4.3}$$

where

$$\begin{aligned}W_1&=2\delta\int_0^x f_5(s)\,ds-\frac{d}{\gamma}\left(\frac{ab-c}{ad-e}\right)f_5^2(x),\\ W_2&=\frac{2(ab-c)}{ad-e}\left[2\int_0^y f_4(s)\,ds-yf_4(y)\right]\\ &\quad+\left[\delta c-ea-\frac{e^2\Delta}{d(ad-e)^2}-\delta^2\right]y^2,\\ W_3&=\frac{\varepsilon}{a}w^2+2\int_0^w f_2(s)\,ds-bw^2,\\ W_4&=2\frac{\varepsilon(dc-be)}{ad-e}zy.\end{aligned}\tag{4.4}$$

Evidently,

$$\begin{aligned}W_1&=2\varepsilon\int_0^x f_5(s)\,ds+\frac{2(ab-c)}{ad-e}\\ &\quad\cdot\int_0^x f_5(s)\left[e-\frac{d}{\gamma}f_5'(s)\right]ds-\frac{d}{\gamma}\frac{(ab-c)}{ad-e}f_5^2(0)\\ &\geqq 2\epsilon\int_0^x f_5(s)\,ds-\frac{d}{\gamma}\frac{(ab-c)}{ad-e}f_5^2(0),\end{aligned}\tag{4.5}$$

since $f_5(x)\operatorname{sgn} x>0$, $x\neq 0$ by (iv);

$$f_5'(x)\leqq e \quad\text{and}\quad \frac{d}{\gamma}\leqq 1 \qquad \text{by (iii).}$$

Because $f_5(0)=0$ by (ii), we obtain

$$W_1 \geqq 2\varepsilon \int_0^x f_5(s)\,ds. \tag{4.6}$$

Now note that

$$yf_4(y)=\int_0^y f_4(s)\,ds+\int_0^y sf_4'(s)\,ds.$$

Therefore

$$\begin{aligned} W_2 &= \int_0^y \left[\frac{2e\Delta}{d(ad-e)}-\frac{d(ab-c)}{ad-e}\left\{f_4'(s)-\frac{f_4(s)}{s}\right\}\right.\\ &\qquad \left. -2\varepsilon\left\{\varepsilon+\frac{2(ab-c)e}{ad-e}-c\right\}\right] s\,ds \\ &\geqq \int_0^y \left[\frac{2e\Delta}{d(ad-e)}-\frac{d(ab-c)}{ad-e}\beta-2\varepsilon\left\{\varepsilon+\frac{2(ab-c)e}{ad-e}-c\right\}\right] s\,ds \\ &\geqq \int_0^y \left[\frac{e\Delta}{d(ad-e)}-2\varepsilon\left\{\varepsilon+\frac{2(ab-c)e}{ad-e}-c\right\}\right] s\,ds, \end{aligned}$$

by (2.3). Hence

$$W_2 \geqq \frac{e\Delta y^2}{2d(ad-e)} \geqq 0, \tag{4.7}$$

provided

$$\frac{e\Delta}{4d(ad-e)} > \varepsilon\left(\varepsilon+\frac{2(ab-c)e}{ed-e}-c\right), \tag{4.8}$$

which we now assume. Also

$$W_3 \geqq \frac{\varepsilon}{a} w^2 \qquad \text{by (iii)}. \tag{4.9}$$

On gathering the estimates (4.6), (4.7) and (4.9) into (4.3) we obtain, since $d/\gamma \geqq 1$, $(ad-e)/(ab-c)>0$, that

$$\begin{aligned} 2V_1 &\geqq \left[u+aw+\frac{d(ab-c)}{ad-e}z+\delta y\right]^2 \\ &\quad +\frac{d\Delta}{(ad-e)^2}\left[z+\frac{e}{d}y\right]^2+\Delta_2[w+az]^2 \\ &\quad +\frac{\varepsilon w^2}{a}+\frac{e\Delta y^2}{2d(ad-e)}+2\varepsilon\int_0^x f_5(s)\,ds+2\varepsilon\frac{(dc-be)}{ad-e}zy. \end{aligned}$$

It follows from the first six terms of this inequality that there exist constants D_i $(i=1, 2, 3, 4, 5)$ such that

$$2V_1 \geqq D_1\int_0^x f_5(s)\,ds+2D_2y^2+2D_3z^2+D_4w^2+D_5u^2+2\varepsilon\left(\frac{dc-be}{ad-e}\right)zy. \tag{4.10}$$

Now consider the terms

$$W_5 = D_2 y^2 + \frac{2\varepsilon(dc-be)}{ad-e} zy + D_3 z^2 \tag{4.11}$$

which are contained in (4.10). In view of the inequality

$$|yz| \leqq \tfrac{1}{2}(y^2+z^2),$$

it is evident that W_5 (defined in (4.11)) satisfies

$$W_5 \geqq D_2 y^2 + D_3 z^2 - \left(\frac{(dc-be)}{\varepsilon d - e}\right)\varepsilon(y^2+z^2) \geqq D(y^2+z^2)$$

for some D if

$$\varepsilon \leqq \frac{1}{2}\left(\frac{ad-e}{dc-be}\right) \min[D_2, D_3]. \tag{4.12}$$

Hence

$$2V_1 \geqq D_1 F_5(x) + D_2 y^2 + D_3 z^2 + D_4 w^2 + D_5 u^2, \tag{4.13}$$

provided ε is so small that (4.8) and (4.12) hold, which proves the lemma.

LEMMA 2. *Under conditions* (i)–(v) *of Theorem* 1 *there exist constants* D_6, D_7 *and* D_8 *such that whenever* (x, y, z, w, u) *is any solution of* (1.2) *with* $p(t, x, y, z, w, u) \equiv 0$, *then*

$$\dot{V}_1 \equiv \frac{d}{dt} V_1(x, y, z, w, u) \leqq -(D_6 y^2 + D_7 z^2 + D_8 w^2), \tag{4.14}$$

provided $\varepsilon_0 = \max[\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3]$ *is sufficiently small.*

Proof. Assume that $w \neq 0$. Then a straightforward calculation using the identity

$$\dot{V}_1 = \frac{\partial V_1}{\partial u}\dot{u} + \frac{\partial V_1}{\partial w}u + \frac{\partial V_1}{\partial z}w + \frac{\partial V_1}{\partial y}z + \frac{\partial V_1}{\partial x}y$$

yields

$$\begin{aligned} \dot{V}_1 = {} & -w^2\left[\frac{af_2(w)}{w} - \left\{c + \frac{ad(ab-c)}{ad-e} - \delta\right\}\right] \\ & - z^2\left[\frac{dc(ab-c)}{ad-e} - \delta b - (af_4'(y) - e)\right] \\ & - y^2\left[\delta\frac{f_4(y)}{y} - \frac{d(ab-c)}{ad-e}f_5'(x)\right] \\ & - wz\left[\frac{d(ab-c)}{ad-e}\right]\left[\frac{f_2(w)}{w} - b\right] - wz[d - f_4'(x)] \\ & - yw[e - f_5'(x)] - \delta yw\left[\frac{f_2(w)}{w} - b\right]. \end{aligned} \tag{4.15}$$

It is clear that the coefficient of $-w^2$ in the first term is the same as

$$a\left[\frac{f_2(w)}{w}-b\right]+\varepsilon;$$

while the coefficient of $-z^2$ in the second term is the same as

$$\frac{(ab-c)(dc-be)}{ad-e}-(af_4'(y)-e)-b\varepsilon>\frac{1}{2}\left[\frac{(dc-be)(ab-c)}{ad-e}-(af_4'(y)-e)\right],$$

by (2.1). It is also immediate on using the definition of δ in (4.1) and hypothesis (iii) that the coefficient of $-y^2$ in the third term satisfies

$$\frac{\delta f_4(y)}{y}-\frac{d(ab-c)}{ad-e}f_5'(x)\geqq\varepsilon d+\frac{(ab-c)d}{ad-e}[e-f_5'(x)].$$

Thus the first three terms, involving w^2, z^2 and y^2, are majorizable by

$$-\left(\frac{\varepsilon}{4}w^2+\frac{\Delta_1}{8}z^2+\frac{\varepsilon d}{4}y^2\right).$$

Now let $R(x, y, z, w)$ denote the sum of the remaining four terms in (4.15). It can be seen for hypotheses (iii) and (v) of Theorem 1 that the absolute value of each coefficient of wz or yw in $R(x, y, z, w)$ cannot exceed $D\varepsilon_i$ ($i=1$, 2 or 3). Thus, again using the inequalities

$$|yw|\leqq\tfrac{1}{2}(y^2+w^2),\qquad |zw|\leqq\tfrac{1}{2}(z^2+w^2),$$

we have that

$$|R(x, y, z, w)|\leqq D^*(\varepsilon_1+\varepsilon_2+\varepsilon_3)(y^2+z^2+w^2)$$

for some $D^*>0$. Thus, after substituting in (4.15), one obtains that

$$\begin{aligned}\dot V&\leqq-\left(\frac{\varepsilon}{4}w^2+\frac{\Delta_1}{8}z^2+\frac{\varepsilon d}{4}y^2\right)+D^*(\varepsilon_1+\varepsilon_2+\varepsilon_3)(y^2+z^2+w^2)\\&\leqq-\left(\frac{\varepsilon}{8}w^2+\frac{\Delta_1}{8}z^2+\frac{\varepsilon d}{8}y^2\right)\end{aligned}$$

if

$$D^*(\varepsilon_1+\varepsilon_2+\varepsilon_3)\leqq\frac{1}{8}\min\left(\varepsilon, \frac{\Delta_1}{2}, \varepsilon d\right).$$

The choice of D_6, D_7, D_8 is now clear, and the Lemma is proved for $w\neq 0$. The case $w=0$ is trivially dealt with.

5. Proof of Theorem 1. Consider the system

$$\begin{aligned}&\dot x=y,\quad \dot y=z,\quad \dot z=w,\quad \dot w=u,\\ &\dot u=-au-f_2(w)-cz-f_4(y)-f_5(x).\end{aligned}\tag{5.1}$$

From Lemma 1 and Lemma 2, we see that the V_1 defined in (4.1) satisfies

$$V_1(0, 0, 0, 0, 0) = 0, \quad V_1 \geqq D_1F_5(x) + D_2y^2 + D_3z^2 + D_4w^2 + D_5u^2; \tag{5.2}$$

and

$$\dot{V}_1 \leqq -D(y^2(t) + z^2(t) + w^2(t)), \tag{5.3}$$

for any solution $(x(t), y(t), z(t), w(t), u(t))$ of (5.1). Because $f_5(x) \operatorname{sgn} x > 0$, $x \neq 0$, and

$$F_5(x) \to \infty \quad \text{as } |x| \to \infty,$$

by (iv) we have that

$$V_1(x, y, z, w, u) \to +\infty \quad \text{as } x^2 + y^2 + z^2 + w^2 + u^2 \to +\infty. \tag{5.4}$$

Let $(x(t), y(t), z(t), w(t), u(t))$ be a solution of (5.1) with $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$, $w(0) = w_0$, $u(0) = u_0$ and denote this trajectory by γ; then by (5.3)

$$V_1(x, y, z, w, u) = V_1(x(t), y(t), z(t), w(t), u(t)) = V_1(t) \leqq V_1(0).$$

Also $V_1(t)$ is nonnegative and nonincreasing and therefore tends to a nonnegative limit $V_1(\infty)$ as $t \to \infty$. If $V_1(\infty) = 0$, the solution of (5.1) satisfying the initial condition tends to the trivial solution $x = y = z = w = u = 0$ as $t \to \infty$, so that we have the result.

If $V_1(\infty) > 0$, then the surface $V_1(x, y, z, w, u) = V_1(\infty)$ contains all the limit points of $(x(t), y(t), z(t), w(t), u(t))$. Let P_1 be a limiting point; then it is known that the solution through P_1 at $t = 0$ lies in the surface $V_1(x, y, z, w, u) = V_1(\infty)$ since $V_1(x, y, z, w, u) \geqq V_1(\infty)$. This implies that $\dot{V} = 0$ at all points of this solution. Also by (5.3) this is only possible if $y = z = w = 0$ and hence $u = 0$. It follows from (5.1) and hypothesis (ii) that $x = 0$. Thus the point $(0, 0, 0, 0, 0)$ lies on the surface $V_1(x, y, z, w, u) = V_1(\infty)$ and hence $V_1(\infty) = 0$. The contradiction proves the Theorem.

6. Proof of Theorem 2. The proof of Theorem 2 is a fifth order adaptation of an ingenious refinement by Ezeilo and Tejumola [8, § 5] of the well-known method of Antosiewicz [4, § V]. For the system

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = w, \quad \dot{w} = u, \quad \dot{u} = -au - f_2(w) - cz - f_4(y) - f_5(x) + p(t, x, y, z, w, u), \tag{6.1}$$

consider the function V_1 defined in (4.1). Because $f_5(0)$ is necessarily zero and only the mild restriction

$$f_5(x) \operatorname{sgn} x > 0 \quad \text{for } |x| \geqq 1$$

holds, we have only the following estimate in the proof of Lemma 1.

$$
\begin{aligned}
W_1 &= 2\varepsilon \int_0^x f_5(s)\,ds + \frac{2(ab-c)}{ad-e}\int_0^x f_5(s)\left[e - \frac{d}{\gamma}f_5'(s)\right]ds \\
&\quad - \frac{d}{\gamma}\left(\frac{ab-c}{ad-e}\right) f_5^2(0) \qquad (6.2)\\
&\geqq 2\varepsilon \int_0^x f_5(s)\,ds - D_9.
\end{aligned}
$$

Indeed

$$\int_0^x \mathrm{f}_5(s)\left[e - \frac{d}{\gamma}f_5'(s)\right]ds$$

is nonnegative for $|x| \geqq 1$ and is bounded for $|x| \leqq 1$ because of continuity, so that

$$\frac{2(ab-c)}{ad-e}\int_0^x f_5(x)\left[e - \frac{d}{\gamma}f_5'(s)\right]ds \geqq -D$$

for all x. It follows just as in Lemma 1 that

$$V_1 \geqq D_1F_5(x) + D_2y^2 + D_3z^2 + D_4w^2 + D_5u^2 - D_9,$$

and for a sufficiently small D_{10},

$$V_1 \geqq D_{10}(y^2 + z^2 + w^2 + u^2) + D_1F_5(x) - D_9. \qquad (6.3)$$

Once more, because $f_5(x)\operatorname{sgn} x > 0$, $|x| > 1$ and $f_5(x)$ continuous, there exists a D_{10}^* such that

$$F_5(x) \geqq D_{10}^*.$$

Therefore

$$V_1 \geqq D_{10}(y^2 + z^2 + w^2 + u^2) - D_{11}, \qquad (6.4)$$

where

$$D_{11} = D_1D_{10}^* + D_9.$$

Next suppose $(x(t), y(t), z(t), w(t), u(t))$ is any solution of (6.1) which satisfies the initial conditions (2.6). Set

$$V_1(t) = V_1(x(t), y(t), z(t), w(t), u(t)).$$

Then just as in Lemma 2,

$$\dot V_1 \leqq -D(y^2 + z^2 + w^2) + \left[u + aw + \frac{d(ab-c)}{ad-e}z + \delta y\right][p(t, x, y, z, w, u)],$$

so that

$$\dot V_1 \leqq D_{12}(|y| + |z| + |w| + |u|)\,|p(t, x, y, z, w, u)|,$$

where

$$D_{12} = \max\left[1,\ a,\ \delta\frac{d(ab-c)}{ad-e}\right].$$

It follows from (2.5) and the obvious inequalities

$$|y| \leqq 1+y^2,\quad |z| \leqq 1+z^2,\quad |w| \leqq 1+w^2,\quad |u| \leqq 1+u^2,$$
$$(|y|+|z|+|w|+|u|)^2 \leqq 4(y^2+z^2+w^2+u^2)$$

that

(6.5) $$\dot{V}_1 \leqq D_{12}[4A_0+(A_0+4)(y^2+z^2+w^2+u^2)]\psi(t).$$

On using (6.4) in (6.5), we deduce that

(6.6) $$\dot{V}_1 - D_{13}V_1(t)\psi(t) \leqq D_{14}\psi(t),$$

where

$$D_{13} = (A_0+4)D_{10}^{-1}D_{12}, \qquad D_{14} = (4A_0+(A_0+4)D_{11}D_{10}^{-1})D_{12}.$$

Thus on multiplying both sides by

$$\chi(t) = \exp\left(-D_{13}\int_0^t \psi(\tau)\,d\tau\right)$$

and integrating in $[0, t]$, we obtain

$$V_1(t)\chi(t) \leqq V_1(0) + D_{14}\int_0^t \psi(\tau)\chi(\tau)\,d\tau,$$

or on dividing both sides by $\chi(\tau)$,

(6.7) $$V_1(t) \leqq \chi^{-1}(t)\left[V_1(0) + D_{14}\int_0^t \psi(\tau)\chi(\tau)\,d\tau\right], \qquad t \geqq 0,$$

where $V_1(0) = V_1(x_0, y_0, z_0, w_0, u_0)$. This together with (6.4) shows that

$$y^2+z^2+w^2+u^2 \leqq D_{10}^{-1}\left[\chi^{-1}(t)\left\{V_1(0)+D_{14}\int_0^t \psi(\tau)\chi(\tau)\,d\tau\right\}+D_{11}\right],$$

from which (2.7) follows. The next result (2.8) follows at once from (6.7) and the deduction from (6.3) that

$$D_1F_5(x) \leqq V_1 + D_9.$$

Remark. We shall begin anew the numbering of the D's.

7. Yoshizawa technique. The proof of Theorem 3 depends on the existence of a continuous function $V(x, y, z, w, u)$ satisfying

(7.1) $$V(x, y, z, w, u) > -D_1 \quad \text{for all } x, y, z, w, u,$$

(7.2) $$V \to \infty \quad \text{as } x^2+y^2+z^2+w^2+u^2 \to \infty;$$

and also such that the limit

$$\dot{V}^{+}=\limsup_{h\to 0}$$

$$\left\{\frac{V(x(t+h),\, y(t+h),\, z(t+h),\, w(t+h),\, u(t+h))-V(x(t),\, y(t),\, z(t),\, w(t),\, u(t))}{h}\right\}$$

exists corresponding to any solution

$$(x(t),\, y(t),\, z(t),\, w(t),\, u(t))$$

of (1.2), and satisfies

$$\dot{V}^{+}\leqq -D_2 \quad \text{if } x^2(t)+y^2(t)+z^2(t)+w^2(t)+u^2(t)\geqq D_3. \tag{7.3}$$

Once the existence of such a V is established an appeal to Yoshizawa's argument [17] concludes the proof of Theorem 3. For the proof of the third order analogue of this see [1], [6]; and for the fourth, see [3] and [15]. The argument is used in [7], [8] and [16]. The required V is defined in the next section.

8. The function V. Consider the continuous function $V=V(x,\, y,\, z,\, w,\, u)$ defined by

$$V=V_1+V_2+V_3, \tag{8.1}$$

where V_1 is defined in (4.1),

$$V_2=\begin{cases}(A+2)(aw+u)\operatorname{sgn} x & \text{if } |x|\geqq|aw+u|,\\ (A+2)x\operatorname{sgn}(aw+u) & \text{if } |x|\leqq|aw+u|,\end{cases} \tag{8.2}$$

$$V_3=\begin{cases}-(A+1)w\operatorname{sgn} u & \text{if } |u|\geqq|w|,\\ -(A+1)u\operatorname{sgn} w & \text{if } |u|\leqq|w|.\end{cases} \tag{8.3}$$

V_1 defined in (4.1) is now being supplemented by two signum functions because it is an incomplete Yoshizawa function of type (u, x) as can easily be seen from the treatment in Chukwu [1].

We shall prove the following property of V.

LEMMA 3. *Subject to the conditions on a, c and* f_i $(i=2, 4, 5)$ *in Theorem* 3, *the function V defined in* (8.1) *satisfies*

$$2V\geqq D_4F_5(x)+D_5y^2+D_6z^2+D_7w^2+D_8u^2-D(|w|+|u|+1) \tag{8.4}$$

for all x, y, z, w and u provided

$$0<\varepsilon_0=\max[\varepsilon,\, \varepsilon_1,\, \varepsilon_2,\, \varepsilon_3]\leqq\varepsilon_4,$$

where ε_4 *depends only on a, b, c, d, e,* Δ, Δ_1 *and* Δ_2.

Proof. It is clear from their definitions that

$$|V_2|\leqq(A+1)(a|w|+|u|),$$

$$|V_3|\leqq(A+1)|w|,$$

so that

$$2V_2+2V_3\geqq -D(|u|+|w|). \tag{8.5}$$

It remains to verify that

(8.6) $$2V_1 \geqq D_4F_5(x) + D_5y^2 + D_6z^2 + D_7w^2 + D_8u^2 - D$$

provided that $0 < \varepsilon_0 \leqq \varepsilon_4$. The verification of (8.6) is only a slight modification of the proof of Lemma 1. Consider W_1 in (4.3):

$$W_1 = 2\varepsilon F_5(x) + \frac{ab-c}{ad-e}\left\{2e\int_0^x f_5(s)\,ds \frac{d}{\gamma} f_5^2(x)\right\}.$$

The term within the curly brackets can be put in the form

$$2e\int_0^x f_5(s)\,ds - \frac{d}{\gamma}f_5^2(x) = 2\int_0^x \left\{e - \frac{d}{\gamma}f_5'(x)\right\} f_5(s)\,ds - \frac{d}{\gamma}f_5^2(0),$$

from which, since the integral on the right-hand side is nonnegative by (iii) and since

(8.7) $$f_5(x)\operatorname{sgn} x > 0 \quad \text{for } |x| \geqq \sigma_0,$$

(where σ_0 is a fixed constant) deduced from hypothesis (i), it is clear that

$$\frac{ab-c}{ad-e}\left\{2e\int_0^x f_5(x)\,dx - \frac{d}{\gamma}f_5^2(x)\right\} \geqq -D$$

for some D. Hence

$$W_1 \geqq 2\varepsilon F_5(x) - D \quad \text{for all } x.$$

It follows from the continuity of $f_4(y)$ and the type of calculation that led to the estimate (4.7) that

$$W_2 \geqq \frac{e\Delta y^2}{2d(ad-e)} - D \quad \text{for all } y$$

provided that (4.8) holds, which we now assume. Also

$$W_3 \geqq \frac{\varepsilon}{a}w^2.$$

On collecting the estimates W_1, W_2 and W_3 into (4.3), we have that

$$\begin{aligned} 2V_1 \geqq & \left[u + aw + \frac{d(ab-c)z}{ad-e} + dy\right]^2 + \frac{d\Delta}{(ad-e)^2}\left[z + \frac{e}{d}y\right]^2 \\ & + \Delta_2[w + az]^2 + \frac{\varepsilon w^2}{a} + \frac{e\Delta y^2}{2d(ad-e)} \\ & + \frac{2\varepsilon(dc-be)}{(ad-e)}zy + 2\varepsilon F_5(x) - D, \end{aligned}$$

from which (8.6) follows provided (just as in § 4) (4.12) holds for ε.

9. The Property of $\dot{V}^+$. The property of $\dot{V}^+$ is required and is stated in Lemma 4.

LEMMA 4. *For any solution* $(x(t), y(t), z(t), w(t), u(t))$ *of* (1.2), *let*

(9.1)

$$\dot{V}^+ \equiv \limsup_{h\to 0+} \left\{\frac{V(x(t+h), y(t+h), z(t+h), w(t+h), u(t+h) - V(x(t), y(t), z(t), w(t), u(t))}{h}\right\},$$

where V *is the function defined in* (8.1). *Then, subject to the hypotheses of the Theorem* 3, $\dot{V}^+$ *exists and satisfies*

$$\dot{V}^+ \leqq -D_9 \tag{9.2}$$

provided

$$\rho^2(t) = x^2(t) + y^2(t) + z^2(t) + w^2(t) + u^2(t) \geqq D_{10}$$

whenever

$$0 < \varepsilon_0 = \max[\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3] \quad \textit{is sufficiently small.} \tag{9.3}$$

Proof. The existence part of the lemma is easily dealt with. It is clear that subject to the conditions on f_2, f_4 and f_5, V_1 has continuous first partial derivatives with respect to all its arguments. V_2 and V_3 are locally Lipschitzian in x, w and u, and w and u, respectively. Hence $V = V_1 + V_2 + V_3$ is, at least, locally Lipschitzian in x, y, z, w and u, and thus limit (9.1) exists.

Before proceeding to the actual verification of (9.2), we first note the following obvious consequences of the hypotheses of the Theorem which are useful later.

(i) For some constant D,

$$|f_5(x)| \leqq D(|x|+1), \quad |f_4(y)| \leqq D(y)+1), \quad |f_2(z)| \leqq D|z| \tag{9.4}$$

for all x, y, z;

(ii) There exists a constant $\sigma_0 > 0$ such that

$$f_5(x) \operatorname{sgn} x > 0 \quad \text{for } |x| \geqq \sigma_0. \tag{9.5}$$

Now let $(x(t), y(t), z(t), w(t), u(t))$ be any solution of (1.2). Then from (8.2) and (8.3) we have, on using (1.2), that

$$\dot{V}_2^+ = \begin{cases} -(A+2)(f_2(w) + cz + f_4(y) + f_5(x) - p) \operatorname{sgn} x & \text{if } |x| \geqq aw + u|, \\ (A+2)y \operatorname{sgn}(aw+u) & \text{if } |aw+u| \geqq |x|, \end{cases}$$

$$\dot{V}_3^+ = \begin{cases} -(A+1)|u| & \text{if } |u| \geqq |w|, \\ (A+1)(au + f_2(w) + cz + f_4(y) + f_5(x) - p) \operatorname{sgn} x & \text{if } |w| \geqq |u|, \quad |x| \geqq |aw+u|. \end{cases}$$

Hence utilizing (9.4), we obtain

$$\dot{V}_2^+ \leqq \begin{cases} -(A+2)f_5(x)\operatorname{sgn} x + D(|y|+|z|+|w|+1) & \text{if } |x| \geqq |aw+u|, \\ (A+2)|y| & \text{if } |x| \leqq |aw+u|, \end{cases} \tag{9.6}$$

$$\dot{V}_3 \leqq \begin{cases} -(A+1)|u| & \text{if } |u| \geqq |w|, \\ (A+1)|f_5(x)| + D(|y|+|z|+|w|+|u|+1) & \text{if } |w| \geqq |u|. \end{cases} \tag{9.7}$$

Note now from (8.1) that

$$\dot{V}^+ = \dot{V}_1 + \dot{V}_2^+ + \dot{V}_3^+. \tag{9.8}$$

It is convenient to first obtain an explicit expression for V_1 and its estimate. This is obtained from the calculations in Lemma 2. Indeed from (4.14) and (1.2), we have that

$$\begin{aligned} \dot{V}_1 &\leqq -(D_{11}y^2 + D_{12}z^2 + D_{13}w^2) \\ &\quad + \left[u + aw + \frac{d(ab-c)}{ad-e}z + \delta y\right] p(t, x, y, z, w, u) \\ &\leqq -(D_{11}y^2 + D_{12}z^2 + D_{13}w^2) \\ &\quad + \left[|u| + a|w| + \frac{d(ab-c)}{ad-e}|z| + \delta|y|\right] A, \end{aligned} \tag{9.9}$$

where we have used hypothesis (v). We now return to (9.8) and use (9.6), (9.7) and (9.9) to deduce that if $|w| \geqq |u|$,

$$\begin{aligned} \dot{V}^+ &\leqq -D_{14}(y^2+z^2+w^2+u^2) + A\left[|u| + a|w| + \frac{d(ab-c)}{ad-e}|z| + \delta|y|\right] \\ &\quad - (A+2)f_5(x)\operatorname{sgn} x + (A+1)|f_5(x)| + D(|y|+|z|+|w|+|u|+1) \end{aligned} \tag{9.10}$$

or

$$\begin{aligned} \dot{V}^+ &\leqq -D_{14}(y^2+z^2+w^2) + A\left[|u| + a|w| + \frac{d(ab-c)}{ad-e}|z| + \delta|y|\right] \\ &\quad + (A+2)|y| + (A+1)|f_5(x)| + D(|y|+|z|+|w|+|u|+1) \end{aligned} \tag{9.11}$$

according to whether

$$|x| \geqq |aw+u| \quad \text{or} \quad |aw+u| \geqq |x|.$$

We now see that the quantity

$$\mu(x) \equiv -\{(A+2)f_5(x)\operatorname{sgn} x - (A+1)|f_5(x)|\} < 0, \qquad |x| \geqq \sigma_0,$$

by (9.5), and

$$|\mu(x)| \leqq D \quad \text{if } |x| \leqq \sigma_0$$

from the continuity of $f_5(x)$. Hence using (9.4) in (9.11), we have at least

$$\dot{V}^+ \leqq -D_{14}(y^2+z^2+w^2) + D(|y|+|z|+|w|+1) \tag{9.12}$$

whenever $|w| \geqq |u|$.

Next consider the case $|w| \leqq |u|$; $\dot{V}^+$ satisfies

$$\dot{V}^+ \leqq -D_{14}(y^2+z^2+w^2)+A\left[|u|+a|w|+\frac{d(ab-c)}{ad-e}z+\delta|y|\right]$$
$$-(A+1)|u|-(A+2)f_5(x)\operatorname{sgn} x+D(|y|+|z|+|w|+1)$$

or

$$V^+ \leqq -D_{14}(y^2+z^2+w^2)+A\left[|u|+a|w|+\frac{d(ab-c)}{ad-e}|z|+\delta|y|\right]$$
$$-(A+1)|u|+(A+2)|y|$$

according to whether $|x| \geqq |aw+u|$ or $|aw+u| \geqq |x|$. It follows that, at least for $|u| \geqq |w|$,

(9.13) $$\dot{V}^+ \leqq -D_{14}(y^2+z^2+w^2)+D(|y|+|z|+|w|+1)-|u|;$$

so that $\dot{V}^+$ always satisfies (9.12). Hence there is a constant D_{15} such that

(9.14) $$V^+ \leqq -1 \quad \text{if } y^2+z^2+w^2 \geqq D_{15}^2.$$

We now show that the inequality (9.14) is still valid even if $y^2+z^2+w^2 \leqq D_{15}^2$ provided $x^2+u^2 \geqq D_{16}^2$ for some constant D_{16}; that is,

(9.15) $$\dot{V}^+ \leqq -1 \quad \text{when } y^2+z^2+w^2 \leqq D_{15}^2,$$

provided

$$x^2+u^2 \geqq D_{16}^2.$$

To prove this, let $y^2+z^2+w^2 \leqq D_{15}^2$ and assume initially that $|u| \geqq D_{15}$. Evidently $|u| \geqq |w|$, and as a consequence $\dot{V}^+$ satisfies (9.13):

$$\dot{V}^+ \leqq -D_{14}(y^2+z^2+w^2)+D(|y|+|z|+|w|+1)-|u|$$
$$\leqq -|u|+D \leqq -1,$$

if further, $|u| \geqq D_{17} \geqq D_{15}$ for some constant D_{17}. Hence

(9.16) $$\dot{V}^+ \leqq -1 \quad \text{if } y^2+z^2+w^2 \leqq D_{15}^2 \quad \text{but } |u| \geqq D_{17}.$$

Suppose on the other hand

$$y^2+z^2+w^2 \leqq D_{15}^2, \qquad |u| \leqq D_{17}.$$

Then let $|x| \geqq \max[\sigma_0, (a+1)D_{17}]$, where σ_0 is identified in (9.5). Clearly $|x| \geqq (a+1)D_{17}$ implies (since $D_{17} \geqq D_{15}$) that

$$|x| \geqq |aw+u|,$$

and as a consequence of (9.6),

$$\dot{V}_2^+ \leqq -(A+2)f_5(x)\operatorname{sgn} x+D.$$

Since $\dot{V}_1 \leqq D$ for the case under consideration, it is immediate that $\dot{V}^+$ satisfies

$$\dot{V}^+ \leqq -(A+2)f_5(x)\operatorname{sgn} x+D$$

for $|u| \geqq |w|$ or

$$\dot{V}^{+} \leqq -|f(x)| + D \quad \text{if } |w| \geqq |u|.$$

Thus whichever case we consider, we have at least

$$\dot{V}^{+} \leqq -D_{18}|f_5(x)| + D;$$

and as a direct result of hypothesis (i),

$$\dot{V}^{+} \leqq -1$$

provided

$$|x| \geqq D_{19} \geqq \max[\sigma_0, (a+1)D_{17}].$$

Concluding, we have

$$(9.17) \qquad \dot{V}^{+} \leqq -1 \quad \text{if } y^2 + z^2 + w^2 \leqq D_{15}^2 \quad \text{and} \quad |u| \leqq D_{17} \quad \text{but } |x| \geqq D_{19}.$$

Combining the two inequalities in (9.16) and (9.17), we deduce that

$$\dot{V}^{+} \leqq -1 \quad \text{when } y^2 + z^2 + w^2 \leqq D_{15}^2$$

provided

$$x^2 + u^2 \geqq D_{19}^2 + D_{17}^2,$$

and this is (9.15) with $D_{16}^2 = D_{19}^2 + D_{17}^2$.

Finally a combination of the estimates (9.14) and (9.15) yields that

$$\dot{V}^{+} \leqq -1 \quad \text{if } x^2 + y^2 + z^2 + w^2 + u^2 \geqq D_{15}^2 + D_{16}^2,$$

which proves (9.2) whenever ε is sufficiently small. The Lemma is proved, and as indicated in § 7, Theorem 3 follows at once.

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ERROR BOUNDS IN THE FINAL VALUE PROBLEM FOR THE HEAT EQUATION*

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Abstract. Consider the following problem. Given the positive constants δ, M, T and $f(x)$ in $L^2(\Omega)$, find all solutions of $u_t = \Delta u$ in $\Omega \times (0, T]$, $u = 0$ on $\partial\Omega \times (0, T]$, such that $\|u(\cdot, T) - f\|_{L^2} \leqq \delta$, $\|u(\cdot, 0)\|_{L^2} \leqq M$. It is known that if $u_1(x, t)$, $u_2(x, t)$ are any two solutions, then

$$\|u_1(\cdot, t) - u_2(\cdot, t)\|_{L^2} \leqq 2M^{(T-t)/T}\delta^{t/T}.$$

Let N be the dimension of Ω, q an integer $\geqq 0$, and let $\sigma > N/2 + q$. We show that there is a constant K such that for $0 < t < T$,

$$\max_{|\beta| \leqq q} \|D^\beta u_1(\cdot, t) - D^\beta u_2(\cdot, t)\|_\infty \leqq K\left\{t^{-\sigma/2} + (T-t)^{-\sigma/2} + \left(\frac{\log(M/\delta)}{T}\right)^{\sigma/2}\right\} M^{(T-t)/T}\delta^{t/T}.$$

1. Introduction. In a recent report [3], Buzbee and the author devised a new method for computing the solutions of linear parabolic equations backwards in time. This computational method is based on another equation altogether, the so-called "abstract backward beam equation" [4], [5]. An important feature of this approach is that it is not limited to parabolic equations for which one has an explicit formula for the solution operator. In fact, the method is applicable to problems with time-dependent coefficients and results in (sharp) logarithmic-convexity type error bounds in the L^2-norm [1], [3].

In the present note we consider this method as it applies to the simplest problem, the final value problem for the heat equation in a bounded domain Ω in R^N, with zero Dirichlet data on $\partial\Omega$, and we obtain new error bounds in Sobolev norms. We plan to treat problems with smooth variable coefficients in a later report. An immediate corollary of our results are some maximum norm stability estimates for the final value problem which appear to be new.

2. The final value problem. Let Ω be a bounded domain in R^N with a C^∞ boundary $\partial\Omega$. Consider the following problem. Given $f(x) \in L^2(\Omega)$ and the positive constants δ, M, T, find all solutions of

$$\begin{aligned} &u_t = \Delta u, \quad && x \in \Omega, \quad 0 < t < T, \\ &u = 0, \quad && x \in \partial\Omega, \quad 0 < t < T, \end{aligned} \tag{2.1}$$

such that

$$\|u(\cdot, T) - f\|_{L^2} \leqq \delta, \qquad \|u(\cdot, 0)\|_{L^2} \leqq M. \tag{2.2}$$

Physically, $f(x)$ represents a measurement of the temperature $u(x, t)$ at time T, which is known to be in error by at most δ in the L^2-norm. M is a given bound on the initial temperature obtained from physical considerations. The existence of solutions to (2.1), (2.2) hinges on the compatibility of M with δ. If $f(x)$ is not

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smooth and δ is small, M would have to be large in order to guarantee existence. There are in general infinitely many solutions.

If u is any solution of (2.1), $\log \|u(\cdot, t)\|_{L^2}$ is a convex function of t [1]. Hence, if $u_1(x, t)$ and $u_2(x, t)$ are any two solutions of (2.1), (2.2),

$$\|u_1(\cdot, t) - u_2(\cdot, t)\|_{L^2} \leqq 2(M)^{(T-t)/T}\delta^{t/T}, \qquad 0 \leqq t \leqq T. \tag{2.3}$$

Thus, the final value problem with a given bound is stable. These results are well known; see, e.g., [6] and the references therein.

3. The spaces $\dot{H}^s(\Omega)$, $s \geqq 0$. It will be convenient to make use of the spaces $\dot{H}^s$ introduced in [2]. Let $\{\lambda_m\}_{m=1}^{\infty}$ be the (positive) eigenvalues of the negative Laplacian in Ω, with zero Dirichlet data on $\partial\Omega$, and let $\{\varphi_m(x)\}_{m=1}^{\infty}$ be the corresponding sequence of L^2-orthonormal eigenfunctions. For $v \in L^2(\Omega)$, let $\{\beta_m\}_{m=1}^{\infty}$ be the sequence of Fourier coefficients of v relative to the $\{\varphi_m\}$. For each $s \geqq 0$, the Hilbert space $\dot{H}^s(\Omega)$ is the subspace of $L^2(\Omega)$ consisting of all v's for which the norm

$$\|v\|_s = \left(\sum_m \lambda_m^s |\beta_m|^2\right)^{1/2} < \infty. \tag{3.1}$$

$\dot{H}^{\infty}(\Omega) \equiv \bigcap_{s>0} \dot{H}^s$ is dense in every $\dot{H}^s$, and if $\partial\Omega \in C^{\infty}$,

$$\dot{H}^{\infty} = \{v | v \in C^{\infty}(\bar{\Omega}), \Delta^j v = 0 \text{ on } \partial\Omega, j = 0, 1, \cdots\} \tag{3.2}$$

If s is an integer, *and* if $v \in \dot{H}^s$, the s-norm of v is equivalent to the usual Sobolev norm,

$$\|v\|_{H^s} = \left(\sum_{|\alpha| \leqq s} \|D^{\alpha} v\|_{L^2}^2\right)^{1/2}. \tag{3.3}$$

Consider now the initial value problem for the heat equation

$$\begin{aligned} &u_t = \Delta u, && x \in \Omega, \quad t > 0, \\ &u(x, t) = 0 && \text{on } \partial\Omega, \quad t > 0, \\ &u(x, 0) = g(x), && x \in \Omega. \end{aligned} \tag{3.4}$$

Expanding in the eigenfunctions of $-\Delta$, one easily proves the following (see [2]).

THEOREM 3.1. *There exists a unique solution to* (3.4) *for arbitrary* $g \in L^2(\Omega)$. *Moreover,* $u(x, t) \in \dot{H}^{\infty}(\Omega)$ *for* $t > 0$. *For any* l *and* s *with* $0 \leqq s \leqq l$,

$$\|u(\cdot, t)\|_l \leqq C(t)^{-(l-s)/2}\|g\|_s, \qquad t > 0, \tag{3.5}$$

where C *is a positive constant depending only on* $l - s$.

4. The backward beam equation. We now turn to the problem

$$\begin{aligned} &u_{tt} = (\Delta + k)^2 u, && x \in \Omega, && 0 < t < T, \\ &u = \Delta u = 0, && x \in \partial\Omega, && t \in (0, T), \\ &u(x, 0) = g(x), && u(x, T) = f(x). && \end{aligned} \tag{4.1}$$

Here, k is a given positive number which may turn out to equal several eigenvalues of $-\Delta$. Note that (4.1) is an "initial-terminal value" problem for an evolution equation since data is specified both at $t = 0$ and $t = T$. We have the following theorem.

THEOREM 4.1. *There is a unique solution to* (4.1) *for any f, g in $L^2(\Omega)$, and, for any $l \geqq 0$,*

$$\|u(\cdot, t)\|_l \leqq \frac{T-t}{T}\|g\|_l + \frac{t}{T}\|f\|_l, \qquad 0 \leqq t \leqq T. \tag{4.2}$$

Moreover, $u(x, t)$ is in $\dot{H}^\infty(\Omega)$ for $0 < t < T$. If $r, s \geqq 0$ and $l \geqq \max(r, s)$, then

$$\begin{aligned} \|u(\cdot, t)\|_l \leqq {} & \left[k^{(l-s)/2}\frac{(T-t)}{T} + C(t)^{-(l-s)/2}\right]\|g\|_s \\ & + \left[k^{(l-r)/2}\frac{t}{T} + C(T-t)^{-(l-r)/2}\right]\|f\|_r, \end{aligned} \tag{4.3}$$

where C is a positive constant, depending only on l, r and s.

Proof. Write $g \sim \sum_m a_m\varphi_m, f \sim \sum_m b_m\varphi_m$ and construct $u(x, t) = \sum_m c_m(t)\varphi_m$. This leads to $\ddot{c}_m(t) = (\lambda_m - k)^2 c_m(t), 0 < t < T, c_m(0) = a_m, c_m(T) = b_m, m = 1, 2, \cdots$. Hence

$$\begin{aligned} c_m(t) &= a_m\left(\frac{T-t}{T}\right) + b_m\frac{t}{T} \quad \text{if } \lambda_m = k, \\ c_m(t) &= a_m\frac{\sinh(\lambda_m - k)(T-t)}{\sinh(\lambda_m - k)T} + b_m\frac{\sinh(\lambda_m - k)t}{\sinh(\lambda_m - k)T} \quad \text{if } \lambda_m \neq k. \end{aligned} \tag{4.4}$$

Since for any real β, $\sinh\beta t/\sinh\beta T$ and $(\sinh\beta(T-t)/\sinh\beta T$ are convex functions of t, it follows from (4.4) that

$$|c_m(t)| \leqq |a_m|\frac{(T-t)}{T} + |b_m|\frac{t}{T} \quad \forall m, \tag{4.5}$$

and this leads to (4.2). Actually, even if f and g are arbitrary L^2-functions, $\sum_m \lambda_m^l|c_m(t)|^2$ converges for every $0 < t < T$ and every $l \geqq 0$. From (4.4) and the Schwarz and triangle inequalities,

$$\begin{aligned} \left(\sum_m \lambda_m^l|c_m(t)|^2\right)^{1/2} \leqq {} & \left(\sum_{\lambda_m = k} k^{l-s}\frac{(T-t)^2}{T^2}k^s|a_m|^2\right)^{1/2} \\ & + \left(\sum_{\lambda_m \neq k} \lambda_m^{l-s}\frac{\sinh^2(\lambda_m - k)(T-t)}{\sinh^2(\lambda_m - k)T}\lambda_m^s|a_m|^2\right)^{1/2} \\ & + \left(\sum_{\lambda_m = k} k^{l-r}\left(\frac{t}{T}\right)^2 k^r|b_m|^2\right)^{1/2} \\ & + \left(\sum_{\lambda_m \neq k} \lambda_m^{l-r}\frac{\sinh^2(\lambda_m - k)t}{\sinh^2(\lambda_m - k)T}\lambda_m^r|b_m|^2\right)^{1/2}. \end{aligned} \tag{4.6}$$

Next, if $l \geqq \max(r, s)$ and $\lambda_m \neq k$,

$$\lambda_m^{l-s}\frac{\sinh^2(\lambda_m - k)(T-t)}{\sinh^2(\lambda_m - k)T} \leqq C(t)^{-(l-s)}(\lambda_m t)^{l-s} e^{-2|\lambda_m - k|t} \leqq C(t)^{-(l-s)}, \tag{4.7}$$

where C is a generic constant, independent of λ_m and t. Similarly,

$$\lambda_m^{l-r}\frac{\sinh^2(\lambda_m - k)t}{\sinh^2(\lambda_m - k)T} \leqq C(T-t)^{-(l-r)}. \tag{4.8}$$

Since Ω is bounded, $-\Delta$ has a compact inverse and so only finitely many of its eigenvalues can equal k. Hence the two sums in (4.6) involving those eigenvalues always converge. Using (4.7) and (4.8) in the remaining two sums, one obtains (4.3).

5. Application to the final value problem. With the proper choice of k in (4.1), one can use the backward beam equation to obtain a sharp L^2-approximation to the final value problem (2.1), (2.2). Futhermore, although the information in (2.2) is provided only in the L^2-norm, on $0 < t < T$, this approximation is a $C^\infty(\bar{\Omega})$-function of x which is close to all the solutions of (2.1), (2.2) in norms stronger than L^2.

With $f(x)$ as in (2.2), define $w(x, t)$ to be the unique solution of

$$\begin{array}{lll} w_{tt} = (\Delta + k)^2 w, & 0 < t < T, & x \in \Omega, \\ w = \Delta w = 0, & x \in \partial\Omega, & 0 < t < T, \\ w(x, 0) = 0, & w(x, T) = e^{kT} f(x), & x \in \Omega. \end{array} \tag{5.1}$$

We then have the following theorem.

THEOREM 5.1. *Set* $k = (1/T)\log(M/\delta)$ *in* (5.1). *Let* $u(x, t)$ *be any solution of* (2.1), (2.2). *Then*

$$\|e^{-kt}w(\cdot, t) - u(\cdot, t)\|_{L^2} \leqq M^{(T-t)/T}\delta^{t/T}, \qquad 0 \leqq t \leqq T. \tag{5.2}$$

For any $l > 0$, *there is a constant* C *such that, if* $0 < t < T$,

$$\|e^{-kt}w(\cdot, t) - u(\cdot, t)\|_l \leqq \left(C\{(t)^{-l/2} + (T-t)^{-l/2}\} + \left(\frac{\log(M/\delta)}{T}\right)^{l/2}\right) M^{(T-t)/T}\delta^{t/T}. \tag{5.3}$$

Hence, if q *is an integer* $\geqq 0$, *and* $\sigma > N/2 + q$, *we have for* $0 < t < T$,

$$\max_{|\beta| \leqq q} \|e^{-kt}D^\beta w(\cdot, t) - D^\beta u(\cdot, t)\|_\infty \leqq \text{const.}\left\{(t)^{-\sigma/2} + (T-t)^{-\sigma/2} + \left(\frac{\log(M/\delta)}{T}\right)^{\sigma/2}\right\} M^{(T-t)/T}\delta^{t/T}. \tag{5.4}$$

Finally, if in (2.2) *one has* $\|u(\cdot, T) - f\|_l \leqq \delta$, $\|u(\cdot, 0)\|_l \leqq M$ *for some* $l > 0$, *then*

$$\|e^{-kt}w(\cdot, t) - u(\cdot, t)\|_l \leqq M^{(T-t)/T}\delta^{t/T}. \tag{5.5}$$

Proof. If $u(x, t)$ is a solution of (2.1), (2.2), let $u(x, 0) = g(x)$. Then $\|g\|_{L^2} \leqq M$. By Theorem 3.1, $u(x, t) \in \dot{H}^\infty(\Omega)$ for $t > 0$, so that $u = \Delta u = 0$ on $\partial\Omega$. The same is true for $v = e^{kt}u$. In fact, $v(x, t)$ satisfies the system

$$
\begin{aligned}
&v_{tt} = (\Delta + k)^2 v, && 0 < t < T, && x \in \Omega, \\
(5.6)\qquad &v = \Delta v = 0, && 0 < t < T, && x \in \partial\Omega, \\
&v(x,0) = g(x), && v(x, T) = e^{kT}u(x, T). &&
\end{aligned}
$$

Let $\varepsilon(x, t) = w(x, t) - v(x, t)$. Then $\varepsilon(x, t)$ is a solution of the backward beam equation with $\|\varepsilon(\cdot, 0)\|_{L^2} \leqq M$, $\|\varepsilon(\cdot, T)\|_{L^2} \leqq e^{kT}\delta$. Using (4.2) with $l = 0$, it follows that

$$
(5.7)\qquad \|e^{-kt}\varepsilon(\cdot, t)\|_{L^2} \leqq \left(\frac{T - t}{T}\right) M\, e^{-kt} + \left(\frac{t}{T}\right) \delta\, e^{k(T-t)}.
$$

The choice $k = (1/T) \log (M/\delta)$ minimizes the right-hand side of (5.7) as a function of k and leads to (5.2). The proof of (5.5) is identical.

To estimate $e^{-kt}\varepsilon(x, t)$ in the l-norm for any $l > 0$, we use (4.3) with $r = s = 0$ and obtain (5.3). Finally, (5.4) follows from (5.3) and Sobolev's lemma.

COROLLARY. *Let $u_1(x, t)$, $u_2(x, t)$ be any two solutions of the final value problem* (2.1), (2.2). *Then for* $0 < t < T$, *and* $\sigma > N/2 + q$,

$$
(5.8)\qquad
\begin{aligned}
&\max_{|\beta| \leqq q} \|D^\beta(u_1(\cdot, t) - u_2(\cdot, t))\|_\infty \\
&\leqq \text{const.} \left\{ (t)^{-\sigma/2} + (T - t)^{-\sigma/2} + \left(\frac{\log (M/\delta)}{T}\right)^{\sigma/2} \right\} M^{T-t/T}\delta^{t/T}.
\end{aligned}
$$

Proof. This follows from (5.4) and the triangle inequality.

Remark. In actual numerical computation of the final value problem (2.1), (2.2), one uses finite differences and/or finite elements to approximate the related problem (5.1). See [3]. The error estimates in Theorem 5.1 are then modified by the inclusion of additional terms, which tend to zero as the mesh is refined, and constitute the discretization error of the numerical procedure for (5.1).

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A GENERALIZATION OF THE CARATHÉODORY–FEJÉR PROBLEM*

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Abstract. Best uniform approximation and interpolation by rational functions having a fixed number of free poles is investigated, and in some cases solutions to these problems can be calculated by means of an algorithm. Other results dealing with maximal convergence by polynomials are also given.

1. Introductory remarks and definitions. For a function f continuous on a compact set E in the plane let $\|f\|_E = \max_{z\in E} |f(z)|$ denote the uniform norm of f on E. A polynomial is said to be of degree n, $n = 0, 1, 2, \cdots$, if the highest power of z that appears is n or less. A rational function r is said to be of type (n, k) if $r(z) = p(z)/q(z)$, where p is a polynomial of degree n and q is a polynomial of degree k. A rational function b is said to be a Blaschke product of degree n if $b(z) = kp(z)/z^n\bar{p}(1/z)$, where k is a constant and p is a polynomial of degree n. It will also be helpful to let U denote the unit circle $\{|z| = 1\}$ and D the unit disc $\{|z| < 1\}$.

In their 1911 paper [1] Carathéodory and Fejér provided an algebraic method for determining the function analytic in the closed disc $\bar{D}$, of minimal norm on U, whose Taylor expansion about the origin has its first n coefficients prescribed. Since then, this work has been expanded and generalized by Nevanlinna, Schur, Pick, Walsh, Goluzin, this author and others. In particular, Akhiezer [10, p. 270] generalized the earlier work by considering the problem of approximating functions of the form $f(z)/z^n$, where $f(z)$ is analytic in $\bar{D}$ and has its first n Taylor coefficients prescribed by rational functions having no more than a fixed number of poles in D. These results are obtained using techniques similar to those introduced by Carathéodory and Fejér. Walsh [2] considered the problem of finding the analytic function(s) of minimal norm on U which interpolates n prescribed values at n prescribed points (multiplicities counted). His techniques required the use of an algorithm based evidently on one introduced by Nevanlinna [8], [9].

The main problem we consider here can be viewed as a direct generalization of Walsh's in that we allow the interpolating function to be analytic except for a prescribed number (or less) of poles in D. As we show, this problem is equivalent to approximating a given rational function by rational functions having a prescribed number of poles in D, on U. The approach is to prove existence and then to apply an algorithm (similar to that used by Walsh) to calculate the solution.

2. Chebyshev rational functions. We generalize the idea of Chebyshev polynomials in the complex plane so as to include also rational functions. In particular, let $\{a_k\}_{k=1}^n$ be a set of n distinct points in the open unit disc D, $\{A_k\}_{k=1}^n$ any set of n complex values, and $\mathcal{J}_K = \mathcal{J}_K[\{a_k\}_{k=1}^n, \{A_k\}_{k=1}^n]$ the set of all rational functions having K or fewer poles in D and such that if $r \in \mathcal{J}_K$, then $r(a_k) = A_k$ for $k = 1, \cdots, n$, where K is fixed, $0 \leqq K \leqq n-1$. We ask if there exists a function in $\mathcal{J}_K$ of minimal uniform norm on the unit circle U, and if there is, we ask whether it is unique and if it is possible to calculate it.

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Note that the restriction on K is natural; for if $K \geqq n$, then $\inf_{r\in\mathcal{J}_K} \|r\|_U = 0$ and there is obviously no existence. The proof of this fact follows by using induction on n and arguments essentially the same as those used in the proof of Lemma 1 following.

The case where $K=0$ has been settled for some time. Again, a good treatment can be found in Walsh's book [2, p. 286]. When $K=0$, there is a unique function r_0 of minimal norm in $\mathcal{J}_0$. Furthermore, r_0 is a finite Blaschke product of degree $n-1$, and there is an algorithm for its calculation.

The case where $K=n-1$ has similarly been settled, but only recently by this author [4]. Again when $K=n-1$, there is a unique function r_{n-1} of minimal norm on U in $\mathcal{J}_{n-1}$, and it can be calculated using virtually the same algorithm as in the case where $K=0$.

As we shall see, in the general case the situation is not quite so simple; in fact, a minimal function may not even exist. We can show however, that if a minimal function exists, it is a Blaschke product, and by putting a restriction on the values $\{A_k\}_{k=1}^n$, it will be unique, of degree $n-1$, and possible to calculate by way of an algorithm. Our results do not completely settle the general case, and we will discuss the remaining open questions as we proceed.

The motivation for considering this problem and one of the useful tools in its investigation can be found in the following theorem.

THEOREM 1. *Let $\mathcal{J}_K$ be as before, and suppose there exists a function $r_K \in \mathcal{J}_K$ such that*

$$\|r_K\|_U = \inf_{r\in\mathcal{J}_K} \|r\|_U.$$

Let $f(z)=\sigma(z)/\prod_{k=1}^n (z-a_k)$, where $\sigma(z)$ is the polynomial of degree $n-1$ with values

$$\sigma(a_k)=A_k \prod_{j=1}^n (1-\bar{a}_j a_k) \quad \text{for } k=1,2,\cdots,n. \tag{1}$$

It then follows that

$$R_K(z)=f(z)-r_K(z)\prod_{k=1}^n \left(\frac{1-\bar{a}_k z}{z-a_k}\right)$$

is a rational function having K or fewer poles in D of best uniform approximation to f on U.

Proof. The proof of Theorem 1 necessitates the following lemma.

LEMMA 1. *Let $\mathcal{J}_K$ be as before, and for $0\leqq M\leqq K$, let $\mathcal{J}_{K,M}$ denote the set of all rational functions having $K-M$ or fewer poles in D and such that if $r\in\mathcal{J}_{K,M}$, then $r(a_k)=A_k$ for $k=1,2,\cdots,n-M$. It follows that*

$$\inf_{r\in\mathcal{J}_{K,M}} \|r\|_U \geqq \inf_{r\in\mathcal{J}_K} \|r\|_U.$$

Proof of Lemma 1. We may assume that $M=1$. Let $\varepsilon>0$ and choose $r_{K,1}\in\mathcal{J}_{K,1}$ such that

$$\|r_{K,1}\|_U < \inf_{r\in\mathcal{J}_{K,1}} \|r\|_U + \varepsilon/2.$$

Let $\alpha_1, \alpha_2, \cdots, \alpha_m$ be the poles of $r_{K,1}$ $(m \leqq K-1)$, and let

$$r_K(z) = r_{K,1}(z) + \frac{\varepsilon' B - (z-a_n)q(z)}{(z-a_n+\varepsilon')\prod_{k=1}^m (z-\alpha_k)},$$

where $B = A_n - r_K(a_n)\prod_{k=1}^m (a_n - \alpha_k)$, and $q(z)$ is the polynomial of degree $n-2$ with values

$$q(a_k) = \varepsilon' B/(a_k - a_n) \quad \text{for } k = 1, 2, \cdots, n-1.$$

The function r_K is then in $\mathcal{J}_K$. Moreover, by choosing ε' sufficiently small, we obtain

$$\|r_K\|_U \leqq \|r_{K,1}\|_U + \varepsilon/2.$$

Consequently, $\inf_{r\in\mathcal{J}_K} \|r\|_U \leqq \|r_K\|_U \leqq \inf_{r\in\mathcal{J}_{K,1}} \|r\|_U + \varepsilon$, and so our lemma follows since ε can be made arbitrarily small.

Returning now to the proof of Theorem 1, let R_K^* be another rational function having K or fewer poles in D, and suppose

$$\|f - R_K^*\|_U < \|f - R_K\|_U.$$

Some of the poles of R_K^* may coincide with those of f, and so let us relabel the a_k's in such a way that those M poles that f and R_K^* have in common will be denoted by $a_{n-M+1}, \cdots, a_n$. Now set

$$r_{K,M}(z) = [f(z) - R_K^*(z)] \prod_{k=1}^n \left(\frac{z-a_k}{1-\bar{a}_k z}\right).$$

Now if we recall from (1) how f was defined and if we let $\mathcal{J}_{K,M}$ be as in Lemma 1, then we see that $r_{K,M} \in \mathcal{J}_{K,M}$. Now by Lemma 1, we have that

$$\|r_{K,M}\|_U \geqq \|r_K\|_U,$$

and so since

$$\|f - R_K^*\|_U = \|r_{K,M}\|_U \quad \text{and} \quad \|f - R_K\|_U = \|r_K\|_U,$$

our theorem follows by way of contradiction.

We now turn our attention to the existence of a function of minimal norm in $\mathcal{J}_K$. Let $\mathcal{J}_K^m$ denote those rational functions in $\mathcal{J}_K$ that are of type (m, m). If a function of minimal norm in $\mathcal{J}_K$ exists, then it is in $\mathcal{J}_K^m$ for some m. In fact, a stronger statement than this can be made. Namely,

$$\inf_{r\in\mathcal{J}_K^m} \|r\|_U = \inf_{r\in\mathcal{J}_K} \|r\|_U$$

if $m \geqq n+K-1$.

In order to demonstrate this last statement and more, let $m \geqq n+K-1$ and $\{S_j\}_{j=1}^\infty$ be a sequence in $\mathcal{J}_K^m$ such that $\lim_{j\to\infty} \|S_j\|_U = \inf_{r\in\mathcal{J}_K^m} \|r\|_U$. Since this sequence may be assumed to be uniformly bounded on U, it is possible to also assume [2, p. 348] that it converges uniformly on compact sets in the entire plane minus at most m points to a rational function $r_K^m(z)$ of type (m, m). The function $r_K^m(z)$ will have no poles on U or at any of the a_k's. There are at most K points in D where the sequence fails to converge to r_K^m. If we relabel the a_k's in such a way that the sequence converges to r_K^m at $a_1, a_2 \cdots, a_{n-M}$ but not at $a_{n-M+1}, \cdots, a_n$, it will then follow that r_K^m has at most $K-M$ poles in D.

Now let $\beta_1, \beta_2, \cdots, \beta_N$ $(N \leqq K-M)$ denote the poles of r_K^m in D,

$$\phi(z) = r_K^m(z) \prod_{j=1}^{N} \left(\frac{z-\beta_j}{1-\bar{\beta}_j z}\right), \qquad B_k = \phi(a_k)$$

for $k = 1, 2, \cdots, n-M$. We claim that $\phi(z)$ is the rational function having no poles in D with values $\phi(a_k) = B_k$ for $k = 1, 2, \cdots, n-M$, of minimal norm on U. Suppose it is not, and $\psi(z)$ is. Set

$$r_{K,M}(z) = \psi(z) \prod_{j=1}^{N} \left(\frac{1-\bar{\beta}_j z}{z-\beta_j}\right).$$

It then follows that $r_{K,M}$ is a rational function of type $(n+K-M-1, n+K-M-1)$ with $K-M$ or fewer poles in D and values $r_{K,M}(a_k) = A_k$ for $k = 1, 2, \cdots, n-M$. Furthermore,

$$\|r_{K,M}\|_U = \|\psi\|_U < \|\phi\|_U = \inf_{r \in \mathcal{T}_K^m} \|r\|_U.$$

Now using a similar argument to that used in Lemma 1, we can show that $\|r_{K,M}\|_U \geqq \inf_{r \in \mathcal{J}_K^m} \|r\|_U$. Consequently we are led to a contradiction, and so our claim about the function $\phi(z)$ must be true. As a result, $\phi(z)$ must be a Blaschke product of degree $n-M-1$, and so $r_K^m(z)$ is thus a Blaschke product of degree $n+K-M-1$. We can now state the following theorem.

THEOREM 2. *For some M, $0 \leqq M \leqq K$, there exists a Blaschke product $r_{K,M}$ of degree $n+K-M-1$ having $K-M$ or fewer poles in D with $r_{K,M}(a_k) = A_k$ for $k = 1, 2, \cdots, n-M$ (if we appropriately relabel the a_k's) and such that*

$$\|r_{K,M}\|_U = \inf_{r \in \mathcal{J}_K} \|r\|.$$

Furthermore, if we let $\mathcal{J}_{K,M}$ be as in Lemma 1, then $r_{K,M}$ is a function in $\mathcal{J}_{K,M}$ of minimal norm.

COROLLARY 1. *If there exists a function r_K of minimal norm in $\mathcal{J}_K$, then r_K must be a Blaschke product of degree $n+K-1$. Should r_K have K' poles in D where $K' < K$, then it is a Blaschke product of degree $n+K'-1$.*

Proof. The proof of Theorem 2 and Corollary 1 are an immediate consequence of our previous remarks and Lemma 1.

We will now show, via an example, that there is not in general a function of minimal norm on U in $\mathcal{J}_K$. We do this by appealing to the dual problem as described in Theorem 1.

Example 1. Let

$$f(z) = \frac{1}{z}\left(\frac{1-z^2/2}{z^2-1/2}\right) + \frac{1}{z-1/2} \quad \text{and} \quad R_1(z) = \frac{1}{z-1/2}.$$

Set $a_1 = 0$, $a_2 = 1/\sqrt{2}$, $a_3 = -1/\sqrt{2}$, $a_4 = 1/2$ and then $A_1 = -1/2$, $A_2 = (2-\sqrt{2})/(2+\sqrt{2}-1)$, $A_3 = (2+\sqrt{2})/(-2\sqrt{2}-1)$ and finally $A_4 = -4/35$. Using the notation previously introduced, let $n=4$, $K=1$ and $\mathcal{J}_1$ the set of rational functions corresponding to the above points and values. We claim that there does not exist a function of minimal norm on U in $\mathcal{J}_1$. Suppose this claim is false and r_1^* is such a function. By Theorem 1,

$$R_1^*(z)=f(z)-r_1^*(z)\left(\frac{1}{z}\right)\left(\frac{1-z^2/2}{z^2-1/2}\right)\left(\frac{1-z/2}{z-1/2}\right)$$

is a rational function having one pole (or none) in D of best uniform approximation to f on U. Thus,

$$\|f-R_1^*\|_U \leqq \|f-R_1\|_U. \tag{2}$$

There are two cases.

Case 1. Suppose there is strict inequality in (2). Then

$$\|f-R_1^*-(R_1-R_1)\|_U<\|f-R_1\|_U.$$

Since $f-R_1$ has constant modulus on U, as z traverses U, the argument of $(R_1^*-R_1)$ has the same net change as that of $f-R_1$, namely, -3. However, $R_1^*-R_1$ has at most 2 poles in D so this is impossible.

Case 2. Suppose now that there is equality in (2). Let

$$r_1(z)=[f(z)-R_1(z)]z\left(\frac{z^2-1/2}{1-z^2/2}\right)\left(\frac{z-1/2}{1-z/2}\right)=\left(\frac{z-1/2}{1-z/2}\right).$$

The function $r_1^*(z)-r_1(z)$ then has at least three distinct zeros in D and so also, three distinct zeros in $\{|z|>1\}$. By Theorem 2, $r_1^*(z)$ is a Blaschke product of degree 4 or less, and so $r_1^*(z)-r_1(z)$ can have at most 5 zeros. Our example is now complete.

Remark. A natural question to examine here is whether a minimal function for $\mathcal{J}_K$ is unique should it exist. As we shall see in the following, this is true provided the values $\{A_k\}_{k=1}^n$ satisfy a certain condition. This question remains open in the general case.

By imposing restrictions on the A_k's, $k=1,2,\cdots,n$, it is possible to strengthen Corollary 1.

THEOREM 3. *Let $\mathcal{J}_K$ be as before, and suppose that the A_k's, $k=1,2,\cdots,n$, have distinct moduli, i.e., $|A_i|\neq|A_j|$ if $i\neq j$. Also suppose that there exists a function $r_K\in\mathcal{J}_K$ of minimal norm on U. It then follows that r_K is unique in having this property, is a Blaschke product of degree $n-1$ and can be explicitly calculated by means of an algorithm.*

Theorem 3 has a converse, of sorts. It is given here as a theorem, and we shall find use for it in the proof of Theorem 3.

THEOREM 4. *Suppose there exists a Blaschke product $r_K(z)$ of degree $n-(K-K')-1$ having K' poles in D, $(0\leqq K'\leqq K)$ and such that $r_K\in\mathcal{J}_K$. It then follows that r_K is the unique function of minimal norm on U in $\mathcal{J}_K$.*

Proof of Theorem 4. Let $r_K(z)$ satisfy the hypothesis of our theorem, and suppose r_K^* is a different function in $\mathcal{J}_K$ such that $\|r_K^*\|_U\leqq\|r_K\|_U$. There are two cases.

Case 1. $\|r_K^*\|_U<\|r_K\|_U$. The proof in this case follows the same argument as those used in Example 1, Case 1, and is omitted.

Case 2. $\|r_K^*\|_U = \|r_K\|_U$. Let

$$S_K^*(z) = r_K^*(z) \prod_{k=1}^{n} \left(\frac{1-\bar{a}_k z}{z-a_k}\right).$$

If $|r_K^*(z)|$ is not constant on U, then $|S_K^*(z)|$ is not constant on U, and so [5] there exists a polynomial p such that

$$\|S_K^* - p\|_U < \|S_K^*\|_U = \|r_K^*\|_U.$$

If we let

$$r_K^{**}(z) = r_K^*(z) - p(z) \prod_{k=1}^{n} \left(\frac{z-a_k}{1-\bar{a}_k z}\right),$$

then $r_K^{**} \in \mathcal{J}_K$ and $\|r_K^{**}\|_U < \|r_K\|_U$, and we are back to Case 1. Thus we may assume that $|r_K^*(z)|$ is constant on U and consequently that r_K^* is a finite Blaschke product.

Let K'' denote the number of poles r_K^* has in D, and let $M''+K''$ denote its degree (as a Blaschke product). Also, let K' be the number of poles r_K has in D and let $M'+K'$ denote its degree. The equation

$$r_K(z)/r_K^*(z) = 1 \tag{I}$$

has n roots in D (at $a_1, a_2, \cdots, a_n$) and n roots in $\{|z|>1\}$ (at $1/\bar{a}_1, 1/\bar{a}_2, \cdots, 1/\bar{a}_n$). If we let $t = (M''-K'') - (M'-K')$, then (I) will also have $|t|$ roots on U. Now let $S = K''-K' \leqq K-K'$. Equation (I) then has a total of

$$\begin{aligned} M'+K'+M''+K'' &= 2(M'+K') + 2S + t \\ &\leqq 2(n-(K-K')-1) + 2(K-K') + t \\ &\leqq 2(n-1) + t \\ &< 2n + |t|. \end{aligned}$$

This leads to a contradiction and so the proof of Theorem 4 is complete.

The proof of Theorem 3 will now be given.

Proof of Theorem 3. Let $\lambda = 1/\|r_K\|_U$. By relabeling the A_k's, one of the two following cases must hold.

Case 1. $|A_n| < 1/\lambda$. Set

$$g_1(z) = \left(\frac{1-\bar{a}_n z}{z-a_n}\right)\left[\frac{\lambda r_K(z) - \lambda A_n}{1-\bar{A}_n \lambda^2 r_K(z)}\right].$$

Since r_K is a Blaschke product and since $r_K(a_n) = A_n$, it follows that g_1 is likewise a Blaschke product and its degree is less than that of r_K. Furthermore, if we let K' denote the number of poles r_K has in $D(K' \leqq K)$, then g_1 will likewise have K' poles in D. This can be easily demonstrated using Rouché's theorem since $|1/\bar{A}_n| > \|r_K\|_U$. Finally, if $h_1(z)$ is a rational function with K' or fewer poles in D and such that

$$h_1(a_k) = g_1(a_k) \quad \text{for } k = 1, 2, \cdots, n-1,$$

then $\|h_1\|_U \geqq \|g_1\|_U$. Should this not be true, set

$$S_K(z) = \frac{1}{\lambda}\left[\frac{\left(\frac{z-a_n}{1-\bar{a}_n z}\right)h_1(z)+\lambda A_n}{1+\lambda \bar{A}_n\left(\frac{z-a_n}{1-\bar{a}_n z}\right)h_1(z)}\right].$$

Then $S_K(z)$ will be a rational function having K or fewer poles in D, with values $S_K(a_k) = A_k$ for $k = 1, 2, \cdots, n$ and such that $\|S_K\|_U < \|r_K\|_U$. This of course contradicts our assumption about r_K.

It should also be noted that by our assumption concerning the moduli of the A_k's, we have that $|g_1(a_k)| = 1$ for at most one a_k. More precisely, $|\lambda A_k| < 1 (|\lambda A_k| > 1)$ if and only if $|g(a_k)| < 1 (|g(a_k)| > 1)$.

Case 2. $|A_n| > 1/\lambda$. Set

$$g_1(z) = \left(\frac{z-a_n}{1-\bar{a}_n z}\right)\left[\frac{1-\lambda^2 r_K(z)\bar{A}_n}{\lambda r_K(z) - \lambda A_n}\right].$$

Using arguments similar to those used in Case 1, it can be shown that g_1 is a Blaschke product, of degree less than that of r_K, with $K'-1$ poles in D (where K' is the number of poles r_K has in D). As before, if h_1 is any rational function having $K'-1$ poles in D and with values $h_1(a_k) = g_1(a_k)$ for $k = 1, 2, \cdots, n-1$, then $\|h_1\|_U \geqq \|g_1\|_U$. Again this follows by considering the function

$$S_K(z) = \frac{1}{\lambda}\left[\frac{1+\lambda A_n\left(\frac{1-\bar{a}_n z}{z-a_n}\right)h_1(z)}{\left(\frac{1-\bar{a}_n z}{z-a_n}\right)h_1(z)+\lambda \bar{A}_n}\right].$$

It should also be noted that here again at most one of the values $g_1(a_k)$, $k = 1, 2, \cdots, n-1$, has modulus one.

The proof that r_K is a Blaschke product of degree $n-1$ now consists of reapplying this treatment to the function g_1 and in this way defining a sequence of functions $g_1, g_2, \cdots, g_{n-1}$. Let K_j be the number of poles that g_j has in D for $j = 1, 2, \cdots, n-1$. For $j = 1, 2, \cdots, n-1$, if h_j is a rational function with K_j or fewer poles in D and with values $h_j(a_k) = g_j(a_k)$, $k = 1, 2, \cdots, n-j$ (after the a_k's have been appropriately relabeled), then $\|h_j\|_U \geqq \|g_j\|_U$. Furthermore, $\|g_j\|_U = 1$ for $j = 1, 2, \cdots, n-1$, and so $K_{n-1} = 0$. (This follows from Lemma 1.) Consequently, g_{n-1} is the analytic function of minimal norm on U which takes on the value $g_{n-1}(a_1)$ at a_1. Thus $g_{n-1}(z) \equiv g_{n-1}(a_1)$, i.e., g_{n-1} is a constant. Working backwards, and solving for r_K in terms of g_{n-1}, we see that r_K is in fact as desired, a Blaschke product of degree $n-1$.

An algorithm for the calculation of r_K (when it exists) will now be described in the case where the A_k's have distinct moduli. This algorithm will also determine whether r_K exists. The algorithm is based on the same idea used in the preceding part of this proof.

Suppose there exists a function r_K in $\mathcal{J}_K$ of minimal norm on U. We define a sequence of rational functions $\{g_k\}_{k=0}^{n-1}$ by setting

$$g_0(z)=\lambda r_K(z), \quad \text{where } \frac{1}{\lambda}=\|r_K\|_U$$

and

$$g_{k+1}(z)=\left(\frac{1-\bar{a}_{k+1}z}{z-a_{k+1}}\right)\left(\frac{g_k(z)-g_k(a_{k+1})}{1-g_k(z)\overline{g_k(a_{k+1})}}\right). \tag{3}$$

As was just demonstrated, r_K is a Blaschke product of degree $n-1$, so it follows that g_k is a Blaschke product of degree $n-k-1$ for $k=0, 1, \cdots, n-1$, and in particular, g_{n-1} is identically a constant. More precisely, $g_{n-1}(z)\equiv g_{n-1}(a_n)$, and $|g_{n-1}(a_n)|=1$. Now $g_{n-1}(a_n)$ is a rational function in λ whose coefficients depend on, and can be calculated in terms of the a_k's and the A_k's. By multiplying this constant, $g_{n-1}(a_n)$, by its conjugate and setting the resultant expression equal to unity, one obtains a polynomial $\Lambda(\lambda)$ in λ. Each positive real root of Λ yields a rational function $S_\lambda(z)$ which can be calculated using the inverse of (3). That is, each positive real root of Λ yields a value for $g_{n-1}(a_n)$. Setting $g_{n-1}(z)=g_{n-1}(a_n)$, we can solve for $g_{n-2}(z)$ in terms of g_{n-1} in (3). Proceeding in this way, we can finally obtain g_0, and then set $S_\lambda(z)=g_0(z)/\lambda$. If there exists a minimal r_K, then there exists precisely one positive real root λ_k of $\Lambda(\lambda)$ which yields $S_{\lambda_K}(z)$, a Blaschke product of degree $n-(K-K')-1$ having K' $(0\leqq K'\leqq K)$ poles in D. This follows by Theorem 4. Conversely, if none of the positive real roots of $\Lambda(\lambda)$ yield such a function, then there does not exist a minimal $r_K\in\mathcal{J}_K$.

If after calculating all of the positive real roots of $\Lambda(\lambda)$ and their associated functions $S_\lambda(z)$, it is found that there does not exist a function of minimal norm on U in $\mathcal{J}_K$, one can nevertheless calculate the $\inf_{r\in\mathcal{J}_K}\|r\|_U$. This can be done by attempting to calculate the rational function having $K-M$ or fewer poles in D of minimal norm on U, which takes on the given values at $n-m$ of the a_k's for each $0<M\leqq K$ and for each subset of $n-M$ of the a_k's. That is, by applying the algorithm to the class of functions $\mathcal{J}_{K,M}$ (see Lemma 1 for notation) for all $0<M\leqq K$ and for all possible relabelings of the a_k's. Notice that when $M=K$, there always exists a minimal function so the set of all such functions is nonempty. Now once this is carried out, and we have calculated this set of minimal functions and their norms on U, then that norm which is least is also $\inf_{r\in\mathcal{J}_K}\|r\|_U$. This follows from Theorem 2.

It should also be noted here that this algorithm may be applied to those situations that don't satisfy the conditions of Theorem 3. That is, even if the moduli of the A_k's are not all distinct, it is possible to apply this algorithm. The few such cases calculated do, in fact, yield a function of minimal norm in $\mathcal{J}_K$. One might, therefore, conjecture that Theorem 3 holds for whatever A_k's are chosen.

3. Best uniform approximation by rational functions with a fixed number of free poles. Let $f(z)=\sigma(z)/\prod_{k=1}^n(z-a_k)$, where $\sigma(z)$ is a polynomial and the a_k's are distinct and lie in D. The problem considered here is that of approximating f on U by rational functions having fewer than n poles in D. More precisely, for each K, $0\leqq K\leqq n-1$, we ask if there exists a rational function having K or fewer poles in D of best uniform approximation to f in U. Furthermore, we ask if such a function exists, then is it unique and can it be calculated.

Throughout this section let $A_k = \sigma(a_k)/\prod_{j=1}^n (1-\bar{a}_j a_k)$ for $k = 1, 2, \cdots, n$, $\mathcal{J}_K = \mathcal{J}_K(\{a_k\}_{k=1}^n, \{A_k\}_{k=1}^n)$ for $0 \leqq K \leqq n-1$, as before and, as in Lemma 1, for $0 \leqq M \leqq K$ let $\mathcal{J}_{K,M} = \mathcal{J}_{K,M}(\{a_k\}_{k=1}^{n-M}, \{A_k\}_{k=1}^{n-M}$. Also let $\mathcal{R}_K$ denote the set of all rational functions that have K or fewer poles in D, for $0 \leqq K \leqq n-1$.

Theorem 1 has already established a strong connection between this problem and that of investigating Chebyshev rational functions which was considered in § 1. A theorem which is a converse to Theorem 1 and which will play a central role in our subsequent discussion of this problem is now given.

THEOREM 5. *Let $f(z)$ be as above and suppose R_K is a best uniform approximant to f on U from $\mathcal{R}_K$. Suppose also that R_K has a pole (or poles) located at each of the points $a_{n-M+1}, \cdots, a_n$, but not at $a_1, a_2, \cdots, a_{n-M}$. Set*

$$r_{K,M}(z) = [f(z) - R_K(z)] \prod_{k=1}^n \left(\frac{z - a_k}{1 - \bar{a}_k z}\right).$$

The function $r_{K,M}$ is then a function of minimal norm on U in the class $\mathcal{J}_{K,M}$.

Proof. Suppose there exists a function $r_{K,M}^* \in \mathcal{J}_{K,M}$ such that $\|r_{K,M}^*\|_U < \|r_{K,M}\|_U$. Set

$$R_K^*(z) = f(z) - r_{K,M}^*(z) \prod_{k=1}^n \left(\frac{1 - \bar{a}_k z}{z - a_k}\right).$$

The function $R_{K,M}^*$ will then be rational with K or fewer poles in D and $\|f - R_K^*\|_U < \|f - R_K\|_U$, thus yielding a contradiction.

The existence of a best uniform approximant to f on U from the class $\mathcal{R}_K$ now follows easily. By Lemma 1 and the preceding argument, we see that

$$\inf_{R_K \in \mathcal{R}_k} \|f - R_K\|_U = \inf_{r \in \mathcal{J}_K} \|R\|_U. \tag{4}$$

By Theorem 2, for some $0 \leqq M \leqq K$ and some appropriate relabeling of the a_k's, there exists a function $r_{K,M}$ of minimal norm on U in $\mathcal{J}_{K,M}$ such that

$$\|r_{K,M}\|_U = \inf_{r \in \mathcal{J}_K} \|r\|_U.$$

If we let

$$R_K = f(z) - r_{K,M}(z) \prod_{k=1}^n \left(\frac{1 - \bar{a}_k z}{z - a_k}\right),$$

then R_K will then be a best uniform approximant to f on U from the class $\mathcal{R}_K$. Moreover, by Theorem 2, the deviation $f - R_K$ will be a Blaschke product of degree $2n + K - 1$ (or less).

The question of the existence of an algorithm for the calculation of a best approximant to f from $\mathcal{R}_K$ also follows from our work in § 1, subject to a restriction on the moduli of the values $\sigma(a_k)$, $k = 1, \cdots, n$.

THEOREM 6. *Let f and $\mathcal{R}_K$ be as before and suppose that the values $A_k = \sigma(a_k)/\prod_{j=1}^n (1 - \bar{a}_j a_k)$ have distinct moduli for $k = 1, 2, \cdots, n$. There then exists a unique best uniform approximant R_K to f on U from $\mathcal{R}_K$. Furthermore, $f - R_K$*

is a Blaschke product of degree $2n-1$ and R_K can be calculated explicitly by means of an algorithm.

Proof. By (4) and our previous remarks, the $\mathcal{J}_{K,M}$ can be relabeled so that for some $0 \leqq M \leqq K$, there exists a function $r_{K,M}$ of minimal norm on U in $\mathcal{J}_{K,M}$ such that

$$(5) \quad \begin{aligned} &\text{(i)} \qquad \|r_{K,M}\|_U = \inf_{R \in R_K} \|f-R\|_U, \\ &\text{(ii)} \qquad R_K = f(z) - r_{K,M}(z) \prod_{k=1}^{n} \left(\frac{1-\bar{a}_k z}{z-a_k}\right) \end{aligned}$$

is a best approximant to f on U from $\mathcal{R}_K$. Furthermore, by assumption about the moduli of the A_K's we have, by Theorem 3, an algorithm for the calculation of $r_{K,M}$ and hence for R_K.

The method for calculating R_K is now clear. For all possible relabelings of the a_k's and for each M, $0 \leqq M \leqq K$, we attempt to calculate $r_{K,M}$, the function of minimal norm on U in $\mathcal{J}_{K,M}$. Of all such functions that exist we choose one of least norm U. This function will then yield (5) a best approximant R_K to f on U from $\mathcal{R}_K$. The existence of at least one such $r_{K,M}$ is guaranteed and by Theorem 3 and $f-R_K$ will be a Blaschke product of degree $2n-1$.

Since there is no guarantee that there is no more than one function $r_{K,M}$ of minimal norm on U, there is likewise no guarantee that R_K is unique. Only in the case where there exists a function of minimal norm in $\mathcal{J}_K$ does this analysis insure the uniqueness of the best approximant in $\mathcal{R}_K$. It is thus an open question as to when R_K is unique. Attempts to prove the uniqueness of R_K seem to lead to the following problem which also arises when we attempt to prove Theorem 3 without any assumptions about the moduli of the A_k's. That is, if $B(z)$ is a Blaschke product with N zeros and P poles in D where $P-N=2K \geqq 0$ and if R_K is a rational function with exactly K poles in D, is R_K the unique best approximant to $g(z)=B(z)+R_K(z)$ on U from $\mathcal{R}_K$? It should be noted here that if $K=0$ (and so $R_K = R_0$ is the set of all polynomials), then the answer to this question is no [5]. If the answer is no for all K, then it will follow that R_K will always be unique and Theorem 3 will hold for any choice of the A_k's.

Remark. Let us briefly look at generalizations of this work that have been heretofore overlooked. To begin with, the initial assumption that the a_k's be distinct can be dispensed with. That is, one can prescribe not only the value but also the first m derivatives at a_k and obtain the same results given in § 1. The results in § 2 can likewise be generalized.

It is also possible to extend the results in §§ 1 and 2 by replacing the unit circle by an arbitrary closed Jordan curve. However, in this case, the corresponding Chebyshev rational functions and best rational approximants can be calculated only in terms of the mapping function for the interior of that curve. The details of the generalization can be found in [6].

An example of how our algorithm can be applied to a specific case will not be given, for a single example will provide little instructive information. However, a systematic set of calculations could provide information dealing with the locus of the poles of the best approximant as a function of the locus of the poles (and their weights) of the functions being approximated (we refer to § 3). Such information is nonexistent at present and may prove to be useful in other areas that use rational approximations. The author hopes to carry out such an investigation.

4. Best maximal convergence. For a continuous function f defined on E, a compact set, for each $n=0,1,2,\cdots$, let $p_n\ (f, E)$ denote the polynomial of degree n of best uniform approximation to f on E and let

$$\rho_n(f, E)=\|f-p_n(f, E)\|_E.$$

As a special case of Walsh's theorem [2, p. 75], we have that if f is analytic in the open disc, $\{|z|<R\}$, $R>1$, and meromorphic in $\{|z|\leqq R\}$ (i.e., $f(z)$ has a finite number of poles on $\{|z|=R\}$), then the sequence $\{R^n\rho_n(f, U)\}_{n=0}^{\infty}$ is bounded above. Our final application of the Nevanlinna–Walsh algorithm is thus to show that this sequence does not converge and that it is possible to determine the set of all of its limit points which may, in fact, include a continuum.

The main tool used here is the algorithm developed in § 2 applied to a very special case. It may be less confusing to use Walsh's book [2, p. 286] or this author's paper [6] as a reference here. Before constructing our example we first give two lemmas.

LEMMA 2. *Let* $f(z;\theta)=(z-r)^{-1}+e^{i\theta}(z+r)^{-1}$, *where* $0<r<1$, *and let* $\lambda(\theta)=[\lim_{n\to\infty}\rho_n\ (f(z;\theta), U)]^{-1}$. *Then* $\lambda(\theta)$ *is a nonconstant function of* θ.

Proof. Let $f_1(z)=((z^2-r^2)/(1-r^2z^2))f(z)$, and let $F(z)$ be the function analytic in $\{|z|\leqq 1\}$ and of minimal uniform norm on U which interpolates the values

$$F(r)=f_1(r) \quad\text{and}\quad F(-r)=f_1(-r).$$

It then follows that $\|F\|_U=\lambda(\theta)^{-1}$. Furthermore, if we set

$$F_1(z)=\left(\frac{1-rz}{z-r}\right)\left[\frac{\lambda(\theta)F(z)-\lambda(\theta)F(r)}{1-\lambda(\theta)^2\overline{F(r)}F(z)}\right],$$

then $|F_1(-r)|=1$. This yields the following equation for $\lambda(\theta)$:

$$\left|\left(\frac{1+r^2}{-2r}\right)\left[\frac{-\lambda(\theta)A\,e^{i\theta}-\lambda(\theta)A}{1+\lambda(\theta)^2A^2\,e^{i\theta}}\right]\right|=1,$$

where $A=2r/(1-r^4)$. This may be written as,

$$\lambda(\theta)|e^{i\theta}+1|=(1-r^2)|\,1+\lambda(\theta)^2A^2\,e^{i\theta}|.$$

Squaring both sides, this expression yields,

$$(1-r^2)^2A^4\lambda(\theta)^4+2[(1-r^2)^2A^2\cos\theta-(1+\cos\theta)]\lambda(\theta)^2+(1-r^2)^2=0.$$

If we now treat r and hence A as constants and differentiate with respect to θ, we obtain the following derivative:

$$\lambda'(\theta)=\frac{[(1-r^2)^2A^2\sin\theta-\sin\theta]}{2(1-r^2)^2A^4\lambda(\theta)^3+2[(1-r^2)^2A^2\cos\theta-(1+\cos\theta)]\lambda(\theta)}.$$

It is easy to show that the denominator of this expression is nonzero and that the numerator vanishes only when $\sin\theta$ does, that is, when $\theta=0$ or π. As a result, we see that $\lambda(\theta)$ is not a constant function of θ and our lemma follows. Furthermore, a closer examination of this expression indicates that the function $\lambda(\theta)$ attains its maximum value, $(1-r^4)/2r$, when $\theta=\pi$ and its minimum value, $(1-r^4)/2$, when $\theta=0$.

LEMMA 3. *Let $f(z;\theta)$ and $\lambda(\theta)$ be as before. Then, given any $\varepsilon>0$, there exists an N such that if $n>N$,*

$$|\rho_n(f(z;\theta), U)-\lambda(\theta)^{-1}|<\varepsilon \quad \text{for all } \theta\in[0,2\pi].$$

Proof. Assume that the above conclusion is false. There then exists an $\varepsilon>0$, an increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ and a sequence of arguments $\{\theta_k\}_{k=1}^{\infty}$, such that

$$|\rho_{n_k}(f(z,\theta_k), U)-\lambda(\theta_k)^{-1}|>\varepsilon \quad \text{for } k=1,2,\cdots.$$

We may assume that the sequence $\{\theta_k\}_{k=1}^{\infty}$ converges to some θ_0. It will then follow that the sequence of functions $\{f(z;\theta_k)\}_{k=1}^{\infty}$ converges uniformly on U to $f(z;\theta_0)$. Consequently $\lim_{k\to\infty}\lambda(\theta_k)=\lambda(\theta_0)$.

Setting $p_k=p_{n_k}(f(z;\theta_0), U)$, we can write

$$\begin{aligned}\rho_{n_k}(f(z;\theta_0), U) &= \|f(z;\theta_0)-p_{n_k}(z)\|_U\\ &\geqq \|f(z;\theta_k)-p_{n_k}(z)\|_U+\|f(z;\theta_0)-f(z;\theta_k)\|_U\\ &\geqq \lambda(\theta_k)^{-1}+\varepsilon-\delta_k,\end{aligned}$$

where $\delta_k\to 0$ as $k\to\infty$. However, $\lambda(\theta_k)\to\lambda(\theta_0)$ as $k\to\infty$, and so we have that

$$\lim_{k\to\infty}\rho_{n_k}(f(z;\theta_0), U)\geqq\lambda(\theta_0)^{-1}+\varepsilon.$$

This is a contradiction since $\lim_{k\to\infty}\rho_{n_k}(f(z;\theta_0), U)=\lambda(\theta_0)^{-1}$, and so our lemma follows.

We are now able to construct an example that will substantiate our main contentions.

Example 2. Let $g(z)=(1-rz)^{-1}+(1+rz)^{-1}$ where, as before, $0<r<1$. Then the sequence $\{\rho_n(g, U)/r^n\}_{n=0}^{\infty}$ does not converge.

Proof. Let $p_n=p_n(g, U)$. We can then write

$$\begin{aligned}\rho_n(g, U) &= \left\|\frac{1}{1-rz}+\frac{1}{1+rz}-p_n(z)\right\|_U\\ &= \left\|\frac{z}{z-r}+\frac{z}{z+r}-p_n\left(\frac{1}{z}\right)\right\|_U\\ &= \left\|\frac{z^{n+1}}{z-r}+\frac{z^{n+1}}{z+r}-z^n p_n\left(\frac{1}{z}\right)\right\|_U.\end{aligned}$$

However, we have that

$$\frac{z^{n+1}}{z-r}=\frac{r^{n+1}}{z-r}+S_n(z),$$

$$\frac{z^{n+1}}{z+r}=\frac{(-r)^{n+1}}{z+r}+t_n(z),$$

where S_n and t_n are polynomials of degree n, and so we have that

$$\rho_n(g, U)/r^n = r\rho_n(f(z;(n+1)\pi), U),$$

where $f(z;(n+1)\pi)$ is, as before, the function $(z-r)^{-1}+(-1)^{n+1}(z+r)^{-1}$. In light of Lemmas 1 and 2, we have that the sequence $\{\rho_n(g, U)/r^n\}_{n=0}^{\infty}$ does not converge and does, in fact, have precisely two limit points, namely, $2r/(1-r^4)$ and $2r^2/(1-r^4)$.

The same analysis can be carried out on functions of the form $g(z) = (1-a_1z)^{-1}+\mathrm{Re}^{i\theta}(1-a_2z)^{-1}+h(z)$, where $|a_1|=|a_2|=r^{-1}$ and $h(z)$ is analytic in $\{|z|\leqq r^{-1}\}$. In this case, one finds that sequence $\{\rho_n(g, U)/r^n\}_{n=0}^{\infty}$ has as many limit points as does the sequence $\{(a_1/a_2)^n\}_{n=0}^{\infty}$. It is also possible to determine the range of these limit points.

In light of the available examples of best uniform polynomial approximation [7], it seems reasonable to guess that the only functions f, as described in the introduction for which the sequence $\{R^n\rho_n(f, U)\}_{n=0}^{\infty}$, does converge, would be those of the form

$$f(z) = \frac{\sigma z^k}{z^m - e^{iv}R^m} + h(z),$$

where $k, m \in \{0, 1, 2, \cdots\}$, $0 \leqq k \leqq m$ and $h(z)$ is analytic in $\{|z| \leqq R\}$.

Finally, the methods employed here can be extended in a straightforward manner so as to allow one to calculate explicitly the range of any sequence $\{R^n\rho_n(f, U)\}_{n=0}^{\infty}$, where f and R are as in the introduction of this section.

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ASYMPTOTIC INTEGRATION OF LINEAR DIFFERENTIAL EQUATIONS SUBJECT TO INTEGRAL SMALLNESS CONDITIONS INVOLVING ORDINARY CONVERGENCE*

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Abstract. The problem of asymptotic behavior of solutions of an nth order linear differential equation is reconsidered, and a result obtained by Hartman under integral smallness conditions requiring absolute integrability is shown to hold with most of the conditions stated in terms of ordinary integrability. Results of Fubini and Halanay for linear perturbations of nonoscillatory second order equations are similarly extended.

1. Introduction. We study the behavior as $t \to \infty$ of solutions of the scalar equation

$$x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = 0, \qquad t > 0, \tag{1}$$

with $n \geqq 2$. Except when stated otherwise, all functions are permitted to be complex-valued; t is a real variable throughout.

Our main result is the following theorem.

THEOREM 1. *If* $p_1, \cdots, p_n \in C[0, \infty)$,

$$\int_0^\infty |p_1(t)|t^q \, dt < \infty, \tag{2}$$

and the integrals

$$\int_0^\infty p_k(t)t^{q+k-1} \, dt, \qquad 2 \leqq k \leqq n, \tag{3}$$

converge—perhaps conditionally—for some $q > 0$, *then* (1) *has solutions* $x_0, \cdots, x_{n-1}$ *which satisfy*

$$x_r^{(j)}(t) = \begin{cases} \dfrac{t^{r-j}}{(r-j)!}(1 + o(t^{-q})), & 0 \leqq j \leqq r, \\ o(t^{r-j-q}), & r + 1 \leqq j \leqq n - 1. \end{cases} \tag{4}$$

Hartman [4, Thm. 17.1, p. 315] has shown that the conclusion of Theorem 1 holds if

$$\int_0^\infty |p_k(t)|t^{q+k-1} \, dt < \infty, \qquad 1 \leqq k \leqq n,$$

for some $q \geqq 0$, and Hartman and Wintner [5] had earlier obtained the result for $q = 0$. (For a history of the problem with $q = 0$, see [4, p. 321].) The contribution here is that ordinary—rather than absolute—convergence is sufficient in (3) if $q > 0$.

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A special case of Hartman's result, due to Hille [6, part of Thm. 3], is that if $\int_0^\infty t^2|p(t)|\,dt < \infty$, then

$$y'' + p(t)y = 0$$

has solutions x_0 and x_1 such that $\lim_{t\to\infty} x_0(t) = 1$ and $\lim_{t\to\infty}(x_1(t) - t) = 0$. Theorem 1 shows that this conclusion holds even if $\int_0^\infty t^2 p(t)\,dt$ converges conditionally.

2. Proof of Theorem 1. To avoid unnecessary subscripts, we let r be a fixed integer $(0 \leqq r \leqq n-1)$ throughout. For convenience, let

$$Mx = \sum_{k=1}^{n} p_k x^{(n-k)} \tag{5}$$

(thus, (1) can be written as $x^{(n)} + Mx = 0$) and define the transformation $y = Tx$ by

$$y(t) = \frac{t^r}{r!} + \int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!}(Mx)(s)\,ds \tag{6}$$

if $0 \leqq r \leqq q$, or by

$$y(t) = \frac{t^r}{r!} + \int_{t_0}^t \frac{(t-\lambda)^{r-[q]-1}}{(r-[q]-1)!}\,d\lambda \int_\lambda^\infty \frac{(\lambda - s)^{n-r+[q]-1}}{(n-r+[q]-1)!}(Mx)(s)\,ds \tag{7}$$

if $q < r \leqq n-1$. Here $[q]$ is the integer part of q and $t_0 \geqq 0$.

Under the hypotheses of Theorem 1, we will show that T maps the space $V[t_0, \infty)$, consisting of functions in $C^{n-1}[t_0, \infty)$ and satisfying

$$x^{(j)}(t) = O(t^{r-j}), \qquad 0 \leqq j \leqq n-1, \tag{8}$$

and

$$(t^{j-r}x^{(j)}(t))' = O(t^{-q-1}), \qquad 0 \leqq j \leqq n-2, \tag{9}$$

into itself, and is a contraction mapping with respect to the norm

$$\|x\| = \sup_{t \geqq t_0}\left\{\sum_{j=0}^{n-1} t^{j-r}|x^{(j)}(t)| + t^{q+1}\sum_{j=0}^{n-2}|(t^{j-r}x^{(j)}(t))'|\right\} \tag{10}$$

if t_0 is sufficiently large. (The condition (8) is partially redundant, since (9) and the condition that $x(t) = O(t^r)$ imply (8); however, it is convenient in the following proof to define $\|x\|$ as in (10).) Since $V[t_0, \infty)$ is a Banach space under this norm, it will then follow from the contraction mapping principle [1, p. 11] that T has a fixed point (function) which, we will show, is essentially the solution of (1) which satisfies (4).

Throughout the rest of the paper, it is to be understood that all estimates hold for $t \geqq t_0$.

The following lemma is the key to the proof of Theorem 1.

LEMMA 1. *Suppose the hypotheses of Theorem* 1 *hold and* $x \in V[t_0, \infty)$. *Then*

$$\left|\int_t^\infty \frac{(t-s)^i}{i!}(Mx)(s)\,ds\right| \leqq \|x\| m(t) t^{i-n-q+r+1}, \qquad 0 \leqq i \leqq n+q-r-1, \tag{11}$$

where m is continuous on $(0, \infty)$, *decreases monotonically to zero as* $t \to \infty$, *and does not depend on* x *or* t_0.

Proof. First observe that

$$\left| \int_t^\infty s^j p_1(s) x^{(n-1)}(s)\, ds \right| \leqq \|x\| E_1(t) t^{j-n-q+r+1}, \qquad 0 \leqq j \leqq n+q-r-1, \tag{12}$$

where

$$E_1(t) = \int_t^\infty s^q |p_1(s)|\, ds,$$

which exists, because of (2). For $k = 2, \cdots, n-1$, define

$$e_k(t) = \int_t^\infty s^{k+q-1} p_k(s)\, ds,$$

which exists because of the assumed convergence of (3). Then

$$\int_t^{t_1} s^{n+q-r-1} p_k(s) x^{(n-k)}(s)\, ds = -\int_t^{t_1} e_k'(s) s^{n-k-r} x^{(n-k)}(s)\, ds,$$

which, by integration by parts, equals

$$-e_k(s) s^{n-k-r} x^{(n-k)}(s) \Big|_t^{t_1} + \int_t^{t_1} e_k(s)(s^{n-k-r} x^{(n-k)}(s))'\, ds.$$

This converges to a finite limit as t_1 approaches ∞, since

$$|e_k(t_1) t_1^{n-k-r} x^{(n-k)}(t_1)| \leqq \|x\|\, |e_k(t_1)|,$$
$$|e_k(s)(s^{n-k-r} x^{(n-k)}(s))'| \leqq \|x\|\, |e_k(s)| s^{-q-1},$$

and $\lim_{t\to\infty} e_k(t) = 0$. Therefore, the integral

$$\begin{aligned} I_k(t) &= \int_t^\infty s^{n+q-r-1} p_k(s) x^{(n-k)}(s)\, ds \\ &= e_k(t) t^{n-k-r} x^{(n-k)}(t) + \int_t^\infty e_k(s)(s^{n-k-r} x^{(n-k)}(s))'\, ds \end{aligned}$$

converges and satisfies

$$|I_k(t)| \leqq \|x\| E_k(t), \tag{13}$$

where

$$E_k(t) = (1 + t^{-q}/q) \sup_{s \geqq t} |e_k(s)|,$$

because, from (10),

$$|t^{n-k-r} x^{(n-k)}(t)| \leqq \|x\|$$

and

$$\left| \int_t^\infty e_k(s)(s^{n-k-r} x^{(n-k)}(s))'\, ds \right| \leqq \|x\| \left(\sup_{s \geqq t} |e_k(s)|\right) \int_t^\infty s^{-q-1}\, ds.$$

(Here we need the assumption that $q > 0$.)

Now, if $0 \leqq j < n + q - r - 1$,

$$\int_t^\infty s^j p_k(s) x^{(n-k)}(s)\,ds = -\int_t^\infty s^{j-n-q+r+1} I_k'(s)\,ds$$

$$= I_k(t) t^{j-n-q+r+1} + (j - n - q + r + 1)\int_t^\infty I_k(s) s^{j-n-q+r}\,ds$$

and, because of (13) and the obvious monotonicity of E_k,

$$\left|\int_t^\infty s^j p_k(s) x^{(n-k)}(s)\,ds\right| \leqq 2\|x\| E_k(t) t^{j-n-q+r+1}, \quad 0 \leqq j < n + q - r - 1. \tag{14}$$

This inequality also holds for $j = n + q - r - 1$, because then the integral on the left is just $I_k(t)$ (cf. (13)). Now, from (5), (12) and (14),

$$\left|\int_t^\infty s^j (Mx)(s)\,ds\right| \leqq \|x\| \left(E_1(t) + 2\sum_{k=2}^n E_k(t)\right) t^{j-n-q+r+1},$$
$$0 \leqq j \leqq n + q - r - 1,$$

and so (11) holds, with

$$m(t) = 2^{n+q-r-1}\left(E_1(t) + 2\sum_{k=2}^n E_k(t)\right).$$

Since $E_1, \cdots, E_n$ all decrease monotonically to zero as $t \to \infty$, this completes the proof of Lemma 1.

Returning to the proof of Theorem 1, we consider two cases.

Case 1. Suppose $0 \leqq r \leqq q$. Then Lemma 1 implies that the integral

$$H(t) = \int_t^\infty \frac{(t - s)^{n-1}}{(n - 1)!}(Mx)(s)\,ds \tag{15}$$

is defined whenever $x \in V[t_0, \infty)$, and that

$$|H^{(j)}(t)| \leqq \|x\| m(t) t^{r-j-q}, \qquad 0 \leqq j \leqq n - 1. \tag{16}$$

Since (6) can be rewritten as

$$y(t) = \frac{t^r}{r!} + H(t),$$

(16) implies that

$$y^{(j)}(t) = O(t^{r-j}), \qquad 0 \leqq j \leqq n - 1. \tag{17}$$

Moreover,

$$(t^{j-r} y^{(j)}(t))' = (t^{j-r} H^{(j)}(t))' = t^{j-r-1}[(j - r)H^{(j)}(t) + tH^{(j+1)}(t)],$$
$$0 \leqq j \leqq n - 2,$$

so (16) also implies that

$$|(t^{j-r}y^{(j)}(t))'| = |(t^{j-r}H^{(j)}(t))'| \leqq \|x\|(|j-r|+1)m(t)t^{-q-1}, \quad 0 \leqq j \leqq n-2, \tag{18}$$

which, with (17), implies that $y \in V[t_0, \infty)$; thus, T maps $V[t_0, \infty)$ into itself. If $\tilde{x}, \tilde{\tilde{x}} \in V[t_0, \infty)$, then

$$T\tilde{x}(t) - T\tilde{\tilde{x}}(t) = \int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!}[M(\tilde{x} - \tilde{\tilde{x}})](s)\,ds$$

and, by setting $x = \tilde{x} - \tilde{\tilde{x}}$ in (15) and using (16), (18) and the monotonicity of m, we find that

$$\|T\tilde{x} - T\tilde{\tilde{x}}\| \leqq \|\tilde{x} - \tilde{\tilde{x}}\| m(t_0)\left(nt_0^{-q} + \sum_{j=0}^{n-2}(|j-r|+1)\right).$$

Since $m(t) = o(1)$, this implies that

$$\|T\tilde{x} - T\tilde{\tilde{x}}\| < \tfrac{1}{2}\|\tilde{x} - \tilde{\tilde{x}}\|$$

if t_0 is sufficiently large. Hence, T is a contraction mapping of $V[t_0, \infty)$ into itself, and therefore has a unique fixed point (function) x_r such that $Tx_r = x_r$; i.e.,

$$x_r(t) = \frac{t^r}{r!} + \int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!}(Mx_r)(s)\,ds.$$

Clearly, x_r satisfies (1) on (t_0, ∞), and it can therefore be extended as a solution of (1) over $(0, \infty)$. That x_r satisfies (4) can be seen from (16), with $x = x_r$ in (15).

Case 2. Suppose $q < r$. Then Lemma 1 implies that the integral

$$g(t) = \int_t^\infty \frac{(t-s)^{n-r+[q]-1}}{(n-r+[q]-1)!}(Mx)(s)\,ds \tag{19}$$

is defined whenever $x \in V[t_0, \infty)$, and that

$$|g^{(i)}(t)| \leqq \|x\| m(t) t^{[q]-q-i}, \quad 0 \leqq i \leqq n-r+[q]-1. \tag{20}$$

Now (7) can be rewritten as

$$y(t) = \frac{t^r}{r!} + G(t),$$

where

$$G^{(j)}(t) = \frac{1}{(r-[q]-j-1)!}\int_{t_0}^t (t-\lambda)^{r-[q]-j-1} g(\lambda)\,d\lambda, \quad 0 \leqq j \leqq r-[q]-1, \tag{21}$$

and

$$G^{(j)}(t) = g^{(j-r+[q])}(t), \quad r-[q] \leqq j \leqq n-1. \tag{22}$$

From (20) (with $i = 0$), (21) and the monotonicity of m,

$$|G^{(j)}(t)| \leqq \frac{t^{r-[q]-j-1}}{(r-[q]-j-1)!}\int_{t_0}^{t} |g(\lambda)|\, d\lambda$$

$$(23) \qquad \leqq \frac{\|x\| m(t_0) t^{r-[q]-j-1}}{(r-[q]-j-1)!}\int_0^t \lambda^{[q]-q}\, d\lambda$$

$$= \frac{\|x\| m(t_0) t^{r-j-q}}{(1+[q]-q)(r-[q]-j-1)!}, \qquad 0 \leqq j \leqq r-[q]-1.$$

From (20) (with $i = j - r + [q]$) and (22),

$$(24) \qquad |G^{(j)}(t)| \leqq \|x\| m(t) t^{r-j-q}, \qquad r-[q] \leqq j \leqq n-1.$$

Using (23) and (24) and a computation similar to that of Case 1, it is straightforward to verify that y, as defined by (7), is in $V[t_0, \infty)$ and that T is a contraction mapping if t_0 is sufficiently large. The function left fixed by T satisfies

$$(25) \qquad x_r(t) = \frac{t^r}{r!} + \int_{t_0}^{t} \frac{(t-\lambda)^{r-[q]-1}}{(r-[q]-1)!}\, d\lambda \int_{\lambda}^{\infty} \frac{(\lambda-s)^{n-r+[q]-1}}{(n-r+[q]-1)!}(Mx_r)(s)\, ds,$$

and so is a solution of (1) on (t_0, ∞), and can be extended as such over $(0, \infty)$. Since the integral on the right of (25) is $G(t)$ (cf. (21)), with $x = x_r$ in (19), it is clear from (24) that x_r satisfies (4) for $r - [q] \leqq j \leqq n - 1$. The same conclusion cannot be obtained from (23) for $0 \leqq j \leqq r - [q] - 1$, since the last member of (23) is $O(t^{r-j-q})$ rather than $o(t^{r-j-q})$; hence, a different analysis is needed for this case, as follows. Again let $x = x_r$ in (19). From (20) (with $i = 0$) and (21),

$$|G^{(j)}(t)| \leqq \frac{\|x_r\| t^{r-[q]-j-1}}{(r-[q]-j-1)!}\int_{t_0}^{t} m(\lambda)\lambda^{[q]-q}\, d\lambda,$$

$$0 \leqq j \leqq r-[q]-1;$$

hence

$$(26) \qquad |t^{j-r+q} G^{(j)}(t)| \leqq \frac{\|x_r\| t^{q-[q]-1}}{(r-[q]-j-1)!}\int_{t_0}^{t} m(\lambda)\lambda^{[q]-q}\, d\lambda,$$

which shows that (4) also holds if $0 \leqq j \leqq r - [q] - 1$, since the right side of (26) approaches zero as $t \to \infty$. (This is obvious if the integral converges, and it follows from l'Hôpital's rule if it diverges, since $m(t) = o(1)$.)

This completes the proof of Theorem 1.

3. A related result.

THEOREM 2. *Suppose* $p_1, \cdots, p_n \in C[0, \infty)$ *and* r *is a fixed integer*, $0 \leqq r \leqq n - 2$. *Then* (1) *has a solution* x_r *satisfying* (4) *if*

$$(27) \qquad \int_0^{\infty} |p_1(t)|\, dt < \infty,$$

$$(28) \qquad \int_0^{\infty} p_k(t) t^{k-1}\, dt \quad \textit{exists for } 2 \leqq k \leqq n-r-1,$$

and, for some $q > 0$,

$$\int_0^\infty p_k(t)t^{q+k-1}\,dt \quad \textit{exists for } n - r \leqq k \leqq n. \tag{29}$$

We omit the proof of this theorem, which is very similar to that of Theorem 1. The essential difference is the need to restrict further the domain $V[t_0, \infty)$ of the transformation T, by defining $V[t_0, \infty)$ to be the subset of $C^{n-1}[t_0, \infty)$ consisting of functions x which satisfy (9) and

$$x^{(j)}(t) = \begin{cases} O(t^{r-j}), & 0 \leqq j \leqq r, \\ O(t^{r-j-q}), & r+1 \leqq j \leqq n-1, \end{cases}$$

instead of (8), and defining $\|x\|$ by

$$\|x\| = \sup_{t \geqq t_0} \left\{ \sum_{j=0}^{r} t^{j-r}|x^{(j)}(t)| + \sum_{j=r+1}^{n-1} t^{j-r+q}|x^{(j)}(t)| + t^{q+1}\sum_{j=0}^{n-2} |(t^{j-r}x^{(j)}(t))'| \right\},$$

instead of (10). The other changes required to adapt the proof of Theorem 1 to that of Theorem 2 stem naturally from these and the differences between the hypotheses of the two theorems.

Hartman [4, p. 315] has shown that the conclusions of Theorem 2 hold if the integrals in (27), (28) and (29) all converge absolutely.

The essential difference between the conclusions of Theorems 1 and 2 is this: the former states that (1) has a fundamental system $\{x_0, \cdots, x_{n-1}\}$ consisting of functions which satisfy (4), while the latter implies that (1) has a "partial" system of $r + 1$ $(<n)$ solutions $\{x_0, \cdots, x_r\}$ such that

$$x_i^{(j)}(t) = \begin{cases} \dfrac{t^{i-j}}{(i-j)!}(1 + o(t^{-q})), & 0 \leqq j \leqq i, \\ o(t^{i-j-q}), & i+1 \leqq j \leqq n-1, \end{cases}$$

for $0 \leqq i \leqq r$.

4. Linear perturbations of a nonoscillatory second order equation. We now apply Theorem 1 to obtain a result on the asymptotic behavior of solutions of

$$(r(t)x')' + g(t)x = 0, \qquad t > 0, \tag{30}$$

considered as a perturbation of

$$(r(t)y')' + f(t)y = 0, \qquad t > 0, \tag{31}$$

which is assumed to be nonoscillatory. In this case it is known [4, p. 355] that (31) has solutions y_0 and y_1 such that

$$y_0(t) > 0 \quad \text{and} \quad y_1(t) > 0, \qquad t \geqq \bar{t} \text{ (for some } \bar{t}), \tag{32}$$

$$r(y_0y_1' - y_0'y_1) = 1 \tag{33}$$

and

$$\lim_{t\to\infty} \frac{y_1(t)}{y_0(t)} = \infty. \tag{34}$$

THEOREM 3. *Suppose r, f, and g are continuous, $r > 0$, and f is real-valued on* $[0, \infty)$. *Let* (31) *be nonoscillatory on* $(0, \infty)$, *suppose y_0 and y_1 are solutions of* (31) *which satisfy* (32), (33) *and* (34), *and suppose*

$$\int_0^\infty (g(t) - f(t))(y_1(t))^{q+1}(y_0(t))^{-q+1}\, dt$$

converges—perhaps conditionally—for some $q > 0$. Then (30) *has solutions x_0 and x_1 such that*

$$\begin{aligned} x_0(t) &= y_0(t)(1 + o(s^{-q})), \\ x_0'(t) &= y_0'(t)(1 + o(s^{-q})) + y_1'(t)o(s^{-q-1}), \end{aligned} \tag{35}$$

and

$$\begin{aligned} x_1(t) &= y_1(t)(1 + o(s^{-q})), \\ x_1'(t) &= y_1'(t)(1 + o(s^{-q})) + y_0'(t)o(s^{-q+1}), \end{aligned} \tag{36}$$

where

$$s = s(t) = \frac{y_1(t)}{y_0(t)}, \qquad t > \bar{t}. \tag{37}$$

Proof. From (33),

$$s'(t) = \frac{1}{r(t)(y_0(t))^2} > 0, \qquad t \geqq \bar{t}, \tag{38}$$

so (34) implies that $s = s(t)$ maps $[\bar{t}, \infty)$ one-to-one onto $[s(\bar{t}), \infty)$. By rewriting (30) as

$$(r(t)x')' + f(t)x + (g(t) - f(t))x = 0$$

and making the change of variables $s = s(t)$ and $u(s) = x(t)/y_0(t)$, it is straightforward to verify that (30) is equivalent to

$$\frac{d^2u}{ds^2} + p(s)u = 0, \tag{39}$$

with

$$p(s) = r(t)(y_0(t))^4(g(t) - f(t)), \qquad (s = s(t)).$$

From (37) and (38),

$$\int_{s(\bar{t})}^\infty s^{q+1}p(s)\, ds = \int_{\bar{t}}^\infty (g(t) - f(t))(y_1(t))^{q+1}(y_0(t))^{-q+1}\, dt,$$

which exists for some $q > 0$, by assumption; hence Theorem 1 implies that (39) has solutions u_0 and u_1 such that

$$u_0(s) = 1 + o(s^{-q}), \qquad \frac{du_0(s)}{ds} = o(s^{-q-1}),$$

and

$$u_1(s) = s(1 + o(s^{-q})), \qquad \frac{du_1(s)}{ds} = 1 + o(s^{-q}).$$

Now let $x_i(t) = y_0(t)u_i(s(t))$ $(i = 0, 1)$; then x_0 and x_1 are solutions of (30), and elementary manipulations (which make use of (33), (37) and (38)) show that they satisfy (35) and (36).

Halanay [3] obtained the conclusion of Theorem 3 for $r \equiv 1$ and $q = 1$ under the stronger assumption that

$$\int_0^\infty |g(t) - f(t)|(y_1(t))^2\,dt < \infty.$$

He also obtained the conclusions of Theorem 3 for $r \equiv 1$ and $q = 0$ by assuming that

$$\int_0^\infty |g(t) - f(t)|y_0(t)y_1(t)\,dt < \infty; \tag{40}$$

of course, Theorem 3 does not improve on this, because it applies only if $q > 0$. (Hartman and Wintner obtained a similar result for $q = 0$, under an assumption weaker than (40); cf. [4, Thm. 9.1, p. 379].)

By considering

$$x'' - x + P(t)x = 0 \tag{41}$$

as a perturbation of $y'' - y = 0$ and taking $a = 2q$, we obtain the following corollary to Theorem 3.

COROLLARY 1. *If* $P \in C[t_0, \infty)$ *and* $\int_0^\infty P(t)\,e^{at}\,dt$ *converges—perhaps conditionally—for some* $a > 0$, *then* (41) *has solutions* x_0 *and* x_1 *such that*

$$\begin{aligned} x_0^{(j)}(t) &= (-1)^j e^{-t}(1 + o(e^{-at})), \\ x_1^{(j)}(t) &= e^t(1 + o(e^{-at})), \end{aligned} \qquad j = 0, 1. \tag{42}$$

This corollary contains a result obtained by Fubini [2] for $a = 2$, under the stronger assumption that $\int_0^\infty |P(t)|\,e^{2t}\,dt < \infty$; however, Fubini did not specify the order of convergence in (42).

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A PRIORI INEQUALITIES AND THE DIRICHLET PROBLEM FOR A PSEUDO-PARABOLIC EQUATION*

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Abstract. Inequalities are developed which can be used to give norm error bounds for approximate solutions of the Dirichlet problem for a pseudo-parabolic equation. Alternatively the inequalities can be used as the basis of a method for obtaining approximate solutions with computable error bounds.

1. Introduction. In this paper we derive two inequalities which give norm error bounds for *any* approximate solution to the Dirichlet problem for a pseudo-parabolic partial differential equation. Alternatively, the inequalities can be used as the basis of a method to compute approximate solutions, with associated error bounds [1], [2].

The first inequality is an explicit a priori inequality which bounds $\int_D w^2\,dV$, (w is an arbitrary $C^3(D)$ function) in terms of the "data" of the initial-boundary value problem (IBVP)

$$Lu \equiv \Delta(u + u_t) - u_t = f(x, t) \quad \text{in } D. \tag{1}$$

$$u = g(x) \quad \text{on } B. \tag{2}$$

$$u = h(x, t) \quad \text{on } S. \tag{3}$$

The second inequality is a bound on $\int_D (u - w)^2\,dV$, where u is the solution of the semilinear equation

$$Lu = f(x, t, u) \tag{4}$$

with initial-boundary values (2) and (3). Here D is the three-dimensional space-time cylinder $D = B \times (0, \tau)$, $0 < \tau < \infty$, and B is a bounded region in the $x = (x_1, x_2)$-plane. The sides of this cylinder will be denoted by $S = \partial B \times [0, \tau)$, and the top by $B_\tau = D \cap \{t = \tau\}$. The outward pointing unit normal vector on the surface of D will be denoted by $n = (n_1, n_2, n_t)$. Evidently, $n_t = 0$ on S while $n_t = -1$ on B and $n_t = +1$ on B_τ. The notation $u_{,i}$ will be used to mean $\partial u/\partial x_i$ and when a Latin index is repeated in a single term, summation over that index from one to two is assumed. Greek indices appear occasionally and when they are repeated in a single term, summation is assumed to go from one to three, the third variable being the time t. The restriction to two space variables is not necessary and, in fact, the results are immediately generalizable to higher space dimensions. We have chosen to limit the results to two spatial dimensions because the physical problems which are described by the IBVP (1)–(3) are one and two-dimensional in nature.

Equation (1) is a member of a class of equations referred to as pseudo-parabolic [3]. It has recently begun to receive attention because of its appearance in a variety of such important physical processes as the nonsteady flow of second

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order fluids [4]; the seepage of homogeneous fluids through fissured rock [5]; the diffusion of "imprisoned" resonant radiation through a gas [6], [7], [8] (which has applications in the analysis of certain laser systems [9]) and finally in the two-temperature theory of heat conduction of Chen and Gurtin [10].

Because of the relationship between (1) and the heat operator $\Delta - \partial/\partial t$, many of the results for pseudo-parabolic equations bear a close resemblance to well-known results for the heat equation [3], [11], [12], [13]. The work reported herein is related to the author's earlier work [14].

2. The a priori inequality. In this section we derive the following a priori inequality:

If w is any $C^3(D)$ function (a condition which can be weakened somewhat with respect to the t variable), then

$$\int_D w^2\,dV \leqq \alpha_1 \int_S (w^2 + w_t^2)\,d\sigma + \alpha_2 \int_B w^2\,dx + \alpha_3 \int_D (Lw)^2\,dV,$$

where α_1, α_2, and α_3 are explicitly determined constants which are independent of w.

The development of the inequality begins with the introduction of the auxiliary function u which satisfies

$$L^*u = \Delta(u - u_t) + u_t = w \quad \text{in } D, \tag{5}$$

$$u = 0 \quad \text{on } B_\tau \cup \bar{S}. \tag{6}$$

For results concerning the existence of such functions, see [3], [11], [15] and references cited therein.

Then, using (5) and (6) and the divergence theorem, we obtain

$$\int_D w^2\,dV = \int_S (w + w_t)(\partial u/\partial n)\,d\sigma + \int_D uLw\,dV + \int_B (\Delta u - u)w\,dx,$$

from which follows

$$\begin{aligned} \int_D w^2\,dV \leqq & \left\{ \int_S (w^2 + w_t^2)\,d\sigma \int_S (\partial u/\partial n)^2\,d\sigma \right\}^{1/2} \\ & + \left\{ \int_B [u^2 + (\Delta u)^2]\,dx + \lambda_1 \int_D u^2\,dV \right\}^{1/2} \\ & \cdot \left\{ 2 \int_B w^2\,dx + \lambda_1^{-1} \int_D (Lw)^2\,dV \right\}^{1/2}, \end{aligned} \tag{7}$$

by the Schwarz inequality. Here λ_1 is the lowest fixed membrane eigenvalue for B. The object now is to bound the integrals containing u on the right-hand side of (7) by $\int_D w^2\,dV$.

To obtain the desired bound on $\int_B [u^2 + (\Delta u)^2]\, dx$, we proceed as follows:

$$\begin{aligned}\int_B [u^2 + (\Delta u)^2]\, dx &= \int_D (\partial[u^2 + (\Delta u)^2]/\partial t)\, dV \\ &= 2\int_D (u - \Delta u)L^* u\, dV + 2\int_D u[\Delta(u_t - u)]\, dV \\ &\quad + 2\int_D \Delta u(\Delta u + u_t)\, dV.\end{aligned} \tag{8}$$

But since

$$\int_D u^2\, dV \leqq \lambda_1^{-1}\int_D u_{,i}u_{,i}\, dV \tag{9}$$

(see (3.10) of [14]), we obtain, again using the divergence theorem,

$$\begin{aligned}\int_B [u^2 + (\Delta u)^2]\, dx + \lambda_1 \int_D u^2\, dV &\leqq \int_B [u^2 + (\Delta u)^2]\, dx \\ &\quad + 2\int_D u_{,i}u_{,i}\, dV - \lambda_1 \int_D u^2\, dV,\end{aligned} \tag{10}$$

and substitution of (8) into (10) yields

$$\begin{aligned}\int_B [u^2 + (\Delta u)^2]\, dx + \lambda_1 \int_D u^2\, dV &\leqq -2\int_D (u - \Delta u)L^* u\, dV - 2\int_D u[\Delta(u_t - u)]\, dV \\ &\quad - 2\int_D \Delta u(\Delta u + u_t)\, dV + 2\int_D u_{,i}u_{,i}\, dV \\ &\quad - \lambda_1 \int_D u^2\, dV.\end{aligned} \tag{11}$$

Now by (6) and the divergence theorem,

$$-2\int_D (\Delta u)u_t\, dV = -2\int_D u\, \Delta u_t\, dV = -\int_B u_{,i}u_{,i}\, dx \leqq 0. \tag{12}$$

Thus, if we drop the negative terms indicated by (12), recall that $\int_D u\, \Delta u\, dV = -\int_D u_{,i}u_{,i}\, dV$, and use an appropriately weighted arithmetic–geometric mean inequality we obtain,

$$\int_B [u^2 + (\Delta u)^2]\, dx + \lambda_1 \int_D u^2\, dV \leqq \left\{\frac{\lambda_1 + 2}{2\lambda_1}\right\}\int_D w^2\, dV. \tag{13}$$

Before proceeding to bound the second term on the right-hand side of (7), we derive three useful inequalities. We start off with

$$\int_D (\Delta u)^2\, dV = \int_D \Delta u L^* u\, dV + \int_D \Delta u\, \Delta u_t\, dV - \int_D u_t\, \Delta u\, dV,$$

which, by the divergence theorem and (6), becomes

$$\int_D (\Delta u)^2\, dV = \int_D \Delta u L^* u - \frac{1}{2}\int_B (\Delta u)^2\, dx - \frac{1}{2}\int_B u_{,i}u_{,i}\, dx \tag{14}$$
$$\leqq \int_D \Delta u L^*\, dV.$$

An application of the Schwarz inequality now yields

$$\int_D (\Delta u)^2\, dV \leqq \int_D (L^* u)^2\, dV = \int_D w^2\, dV. \tag{15}$$

The second inequality we need follows from (14) also, for we have

$$\frac{1}{2}\int_B (\Delta u)^2\, dx \leqq \int_D \Delta u L^* u\, dV,$$

which yields, by the Schwartz inequality and (15),

$$\frac{1}{2}\int_B (\Delta u)^2\, dx \leqq \int_B w^2\, dV. \tag{16}$$

To obtain the third inequality we use the fact that $u = 0$ on S to write

$$\int_D u_{,i}u_{,i}\, dV = -\int_D u\, \Delta u\, dV = -\int_D uL^* u\, dV - \int_D u\, \Delta u_t\, dV + \int_D uu_t\, dV. \tag{17}$$

But by (12) and since $\int_D uu_t\, dV = -\frac{1}{2}\int_B u^2\, dx \leqq 0$, we have $\int_D u_{,i}u_{,i}\, dV \leqq -\int_D uL^* u\, dV$, which, by (9) and the Schwarz inequality, yields

$$\int_D u_{,i}u_{,i}\, dV \leqq \lambda_1^{-1}\int_D w^2\, dV. \tag{18}$$

Now to bound the second term on the right-hand side of (7): we begin with the inequality

$$\int_D f^\alpha u_{,\alpha} L^* u\, dV = \int_S f^\alpha u_{,\alpha}(\partial u/\partial n) d\sigma - \int_D f^\alpha_{,i} u_{,i} u_{,\alpha}\, dV - \frac{1}{2}\int_S f^\alpha n_\alpha u_{,i}u_{,i}\, d\sigma \tag{19}$$
$$- \frac{1}{2}\int_{B\cup B_\tau} f^\alpha n_\alpha u_{,i}u_{,i}\, dx + \frac{1}{2}\int_D f^\alpha_{,\alpha} u_{,i}u_{,i}\, dV$$
$$+ \int_D f^\alpha u_{,\alpha}(u_t - \Delta u_t)\, dV,$$

where f^α denotes the αth component of a piecewise continuously differential vector field. But since $u = 0$ on S, $u_{,i} = n_i\, \partial u/\partial n$ and thus the first and last terms of the right-hand side of (19) combine to give $\frac{1}{2}\int_S f^\alpha n_\alpha(\partial u/\partial n)\, d\sigma$. Furthermore, by

(6), $u_{,i} = 0$ on B_τ and thus we can write (19) as

$$\frac{1}{2}\int_S f^\alpha n_\alpha(\partial u/\partial n)^2\,d\sigma = \int_D f^\alpha u_{,\alpha}L^*u\,dV + \int_D f^\alpha_{,i}u_{,i}u_{,\alpha}\,dV + \frac{1}{2}\int_B f^\alpha n_\alpha u_{,i}u_{,i}\,dx - \frac{1}{2}\int_D f^\alpha_{,\alpha}u_{,i}u_{,i}\,dV - \int_D f^\alpha u_{,\alpha}(u_t - \Delta u_t)\,dV,$$

Now choose the f^l, $(l = 1, 2)$, such that $f^l n_l$ has a positive minimum, say p_m,[1] on S, to obtain

$$\begin{aligned}(20)\quad p_m/2\int_S (\partial u/\partial n)^2\,d\sigma \leqq{}& \int_D f^\alpha u_{,\alpha}L^*u\,dV + \int_D f^3_{,i}u_{,i}u_t\,dV + \int_D f^j_{,i}u_{,i}u_{,j}\,dV\\ &- \frac{1}{2}\int_B f^3u_{,i}u_{,i}\,dx - \frac{1}{2}\int_D f^\alpha_{,\alpha}u_{,i}u_{,i}\,dV - \int_D f^3(u_t)^2\,dV\\ &- \int_D f^iu_{,i}u_t\,dV + \int_D f^3u_t\,\Delta u_t\,dV + \int_D f^iu_{,i}\,\Delta u_t\,dV.\end{aligned}$$

Consider the last two terms; we have

$$\int_D f^3u_t\,\Delta u_t\,dV = -\int_D f^3u_{,it}u_{,it}\,dV, \quad \text{if} \quad f^3_{,i} = 0, \qquad i = 1, 2,$$

and

$$\int_D f^iu_{,i}\,\Delta u_t\,dV = -\int_B f^iu_{,i}\,\Delta u\,dx - \int_D f^iu_{,it}\,\Delta u\,dV, \quad \text{if} \quad f^i_t = 0, \qquad i = 1, 2.$$

Using these results in (20) (along with the arithmetic-geometric mean inequality) we have

$$\begin{aligned}\frac{1}{2}p_m\int_S (\partial u/\partial n)^2\,d\sigma \leqq{}& \int_D f^3(L^*u)^2\,dV + \frac{1}{4}\int_D f^3(u_t)^2\,dV + \frac{1}{2}\int_D \frac{[f^3_{,i}f^3_{,i}]}{\beta}(u_t)^2\,dV\\ &+\frac{1}{2}\int_D \beta u_{,i}u_{,i}\,dV + \int_D f^j_{,i}u_{,i}u_{,j}\,dV - \frac{1}{2}\int_B f^3u_{,i}u_{,i}\,dx\\ &-\frac{1}{2}\int_D f^\alpha_{,\alpha}u_{,i}u_{,i}\,dV - \int_D f^3(u_t)^2\,dV - \int_D f^3u_{,it}u_{,it}\,dV\\ &+ \int_D \frac{f^jf^j}{f^3}u_{,i}u_{,i}\,dV + \frac{1}{4}\int_D f^3(u_t)^2\,dV - \int_B f^iu_{,i}\,\Delta u\,dx\\ &- \int_D f^iu_{,it}\,\Delta u\,dV + \int_D f^iu_{,i}L^*u\,dV,\end{aligned}$$

where we have imposed the further condition that $f^3 \geqq 0$ in $\bar{D}$.

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[1] Such vector fields are usually easily obtained. For instance if B is star-shaped with respect to the origin we may choose $f^i = x_i$. For more complicated boundaries see [16].

If we now choose $\beta = 2f^3_{,i}f^3_{,i}/f^3$ we obtain that

$$\frac{1}{2}p_m\int_S (\partial u/\partial n)^2\,d\sigma \leqq \left\{\left|\frac{f^3_{,j}f^3_{,j} + f^jf^j}{f^3}\right|_M\right\}\int_D u_{,i}u_{,i}\,dV + |f^3|_M\int_D (L^*u)^2\,dV$$

$$\tag{21} + \int_D f^j_{,i}u_{,i}u_{,j}\,dV - \frac{1}{2}\int_D f^\alpha_{,\alpha}u_{,i}u_{,i}\,dV - \frac{1}{2}\int_B f^3u_{,i}u_{,i}\,dx$$

$$- \int_D f^3u_{,it}u_{,it}\,dV - \int_B f^iu_{,i}\,\Delta u\,dx - \int_D f^iu_{,it}\,\Delta u\,dV$$

$$+ \int_D f^iu_{,i}L^*u\,dV,$$

where $|\cdot|_M$ denotes the maximum of the absolute value of the quantity. Now, straightforward applications of the arithmetic–geometric and Schwarz inequalities results in the following:

$$- \int_B f^iu_{,i}\,\Delta u\,dx \leqq \frac{1}{2}\int_B f^3u_{,i}u_{,i}\,dx + \frac{1}{2}\int_B \frac{f^if^i}{f^3}(\Delta u)^2\,dx,$$

$$- \int_D f^iu_{,i|}\,\Delta u\,dV \leqq \int_D f^3u_{,it}u_{,it}\,dV + \frac{1}{4}\int_D \frac{f^if^i}{f^3}(\Delta u)^2\,dV,$$

$$\int_D f^j_{,i}u_{,i}u_{,j}\,dV \leqq |f^l_{,j}f^l_{,j}|_M^{1/2}\int_D u_{,i}u_{,i}\,dV,$$

$$\int_D f^iu_{,i}L^*u\,dV \leqq \left|\frac{f^lf^l}{2}\right|\int_D u_{,i}u_{,i}\,dV + \frac{1}{2}\int_D (L*u)^2\,dV.$$

Combining these inequalities with (21) yields

$$\frac{1}{2}p_m\int_S (\partial u/\partial n)^2\,d\sigma \leqq \left\{\left|\frac{f^3_{,j}f^3_{,j} + f^jf^j}{f^3}\right|_M + |f^l_{,j}f^l_{,j}|_M^{1/2} + \tfrac{1}{2}|f^\alpha_{,\alpha}|_M\right.$$

$$\tag{22} \left.+\left|\frac{f^lf^l}{2}\right|_M\right\}\int_D u_{,i}u_{,i}\,dV + \frac{1}{2}\left|\frac{f^if^i}{f}\right|_M\int_B (\Delta u)^2\,dx$$

$$+ \frac{1}{4}\frac{f^if^i}{f^3}{}_M\int_D (\Delta u)^2\,dV + (|f^3|_M + \tfrac{1}{2})\int_D (L^*u)^2\,dV,$$

and the desired bound is now obtained from (22) using (15), (16), and (18), i.e.,

$$\tag{23} \int_S (\partial u/\partial n)^2 \leqq K\int_D w^2\,dV,$$

where

$$K = \frac{2}{p_m}\left[\lambda_1^{-1}\left\{\left|\frac{f^3_{,j}f^3_{,j} + f^jf^j}{f^3}\right|_M + |f^l_{,j}f^l_{,j}|_M^{1/2} + \tfrac{1}{2}|f^\alpha_{,\alpha}|_M + \tfrac{1}{2}|f^lf^l|_M\right\}\right.$$

$$\left.+ \frac{5}{4}\left|\frac{f^if^i}{f^3}\right|_M + |f^3|_M + \tfrac{1}{2}\right].$$

Finally the a priori inequality follows from (7), (13) and (23) via the arithmetic-geometric inequality. We obtain

$$\alpha_1 = 2K, \quad \alpha_2 = 2(1 + 2\lambda_1^{-1}), \quad \alpha_3 = (\lambda_1 + 2)\lambda_1^{-2}.$$

3. The semilinear case. Let v be a solution of the semilinear equation $Lv = f(x, t, v)$ where f satisfies a Lipschitz condition in v with Lipschitz constant M. Let $\psi = v - \phi$ where ϕ is an arbitrary $C^3(D)$ function. We now derive the following inequality:

$$\int_D \psi^2 \, dV \leqq \alpha_1 \int_B \psi^2 \, dx + \alpha_2 \int_S (\psi^2 + \psi_t^2) \, d\sigma + \alpha_3 \int_D F^2 \, dV,$$

where $F(x, t) = f(x, t, \phi) - L(\phi)$ and $\alpha_1, \alpha_2, \alpha_3$ are explicitly determined constants depending on the domain geometry, M and a positive constant b to be introduced shortly.

The derivation follows closely that of § 2 after a few preliminaries. Introduce the function

$$w = \psi \, e^{b(\tau - t)},$$

where b is a positive constant. Then

$$L(\psi) = (Lw - bw) \, e^{-b(\tau - t)}, \tag{24}$$

and we now want to bound

$$\int_D e^{2b(\tau - t)} \psi^2 \, dV = \int_D w^2 \, dV.$$

To do this introduce the function u which satisfies the IBVP

$$L^* u - bu = w \quad \text{in } D,$$
$$u = 0 \quad \text{on } S \cup B_\tau.$$

Proceeding as in § 2 we obtain

$$\int_D w^2 \, dV \leqq \left\{ \int_S (w^2 + w_t^2) \, d\sigma \int_S (\partial u / \partial n)^2 \right\}^{1/2} + \left\{ \int_B [u^2 + (\Delta u)^2] \, dx + (b + \lambda_1) \int_D u^2 \, dV \right\}^{1/2} \left\{ 2 \int_B w^2 \, dx + \lambda_1^{-1} \int_D (Lw - bw)^2 \, dV \right\}^{1/2}, \tag{7'}$$

and the two important inequalities corresponding to (13) and (23) are

$$\int_B [u^2 + (\Delta u)^2] \, dx + (b + \lambda_1) \int_D u^2 \, dV \leqq \left\{ \frac{2 + b + \lambda_1}{2(b + \lambda_1)} \right\} \int_D w^2 \, dV \tag{13'}$$

and

$$\int_S (\partial u / \partial n)^2 \, d\sigma \leqq K' \int_D w^2 \, dV, \tag{23'}$$

where $K' = K + b |f^{\alpha}_{,\alpha}|_M / 2\lambda_1^2$.

To complete the bound we write

$$|L(\psi)| \leqq M |\psi| + F \tag{25}$$

so that

$$\int_D (Lw - bw)^2 \, dV \leqq 2M^2 \int_D e^{2b(\tau - t)} \psi^2 \, dV + 2 \int_D e^{2b(\tau - t)} F^2 \, dV \tag{26}$$

by (24) and (25). The bound then follows from (7′), (13′), (23′) and (26), and we obtain

$$\alpha_1 = \frac{2(2 + b + \lambda_1)}{\bar{K}^2 \cdot (b + \lambda_1)},$$

$$\alpha_2 = 2K'/\bar{K}^2,$$

$$\alpha_3 = \frac{2(2 + b + \lambda_1)\, e^{2b\tau}}{\bar{K}^2 \cdot (b + \lambda_1)},$$

where $\bar{K} = 1 - M(2 + b + \lambda_1)^{1/2}/(b + \lambda_1)$ and b is chosen so that $\bar{K} > 0$.

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LIE THEORY AND SEPARATION OF VARIABLES FOR THE EQUATION

$$iU_t+\Delta_2 U-\left(\frac{\alpha}{x_1^2}+\frac{\beta}{x_2^2}\right)U=0^*$$

CHARLES P. BOYER†

Abstract. This work constitutes a detailed study of the symmetries of the time-dependent Schrödinger equation in two spatial dimensions with an added inverse square potential of the form $((\alpha/x_1^2)+(\beta/x_2^2))$. An intimate connection between these symmetries and the coordinate systems in which the equation separates is established. It is shown that there is a 1–1 correspondence between orbits of commuting pairs of symmetry operators—one taken from the Lie algebra of the symmetry group G and the other a second order symmetry operator—and G—inequivalent separable coordinate systems for the equation. The spectral analysis for all the basis functions corresponding to the different separable coordinate systems is computed. Then, making use of the symmetry group G, many addition and expansion theorems for the basis functions are derived. In this way, we find many special function identities involving Jacobi and Gegenbauer polynomials, Laguerre and Hermite polynomials, Whittaker functions, Bessel functions, parabolic cylinder functions, Airy functions, anharmonic oscillator functions, generalized spheroidal wave functions, generalized Ince functions and others. Many of these relations appear to be new.

Introduction. In [3] (hereafter referred to as VI) the authors gave a detailed investigation of the nine-parameter symmetry group for the free particle time-dependent Schrödinger equation in two spatial dimensions. There it was found that this equation separates in 26 coordinate systems and that to each coordinate system there corresponds a commuting pair of orbit representatives (under the action of the Galilei group $\mathcal{G}_2$) from the set of second order symmetry operators for the equation. One member of the pair of commuting symmetry operators is taken from the Lie algebra of the symmetry group.

In this article, we consider the analogous problem when one adds an inverse square potential of the form $((\alpha/x_1^2)+(\beta/x_2^2))$. The problem of separation of variables carries over from VI with only slight modifications and is treated in § 2. It is found that the equation

$$iU_t+U_{x_1x_1}+U_{x_2x_2}-\left(\frac{\alpha}{x_1^2}+\frac{\beta}{x_2^2}\right)U=0 \tag{A}$$

separates in 25 coordinate systems when $\alpha=0$ and in 15 coordinate systems when $\alpha\neq 0$. Moreover, for each separable coordinate system there is a pair of commuting second order symmetry operators of (A) which describe the separation. In contradistinction to VI, the second order symmetry operators, found in § 1, are *not* members of the universal enveloping algebra of the symmetry algebra $\mathcal{G}$ of (A). However, this is not a disadvantage since we prove a lemma which states that the set of second order symmetry operators $\mathcal{S}$ forms a representation space for the symmetry group G of (A). Then in § 3 it is seen that there is a 1–1 correspondence

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between all G-inequivalent separable coordinate systems of (A) and commuting pairs of orbit representatives (K, S) under the action of G where $K \in \mathcal{G}$ and $S \in \mathcal{S}$.

In § 4 the spectral analysis of all basis functions corresponding to the separable coordinate systems is performed first in a simpler two variable model (i.e., $t = 0$) of the problem and then in one of the systems labeled by superscript (1) which appear in Tables 1 and 2. In this way, we are able to derive, in some cases, new integral identities for the basis functions. Furthermore, in § 5 the overlap functions are calculated between many basis functions in the simple two variable model. This then allows us to derive many apparently new addition and expansion theorems satisfied by the various special functions which occur as basis functions. Our method also provides simple derivations of generalized eigenfunction expansion and inversion formulas some of which involve multiparameter eigenvalue problems.

This work is a continuation of a program of studying the connection between Lie theory and the separation of variables for partial differential equations of mathematical physics initiated a few years ago by Winternitz and collaborators [14], [23], [24] in connection with investigating the scattering amplitudes occurring in high energy physics. While the present paper should certainly have some relevance to nonrelativistic physical problems, it is in the spirit of the work of Miller [16]–[18], Kalnins and Miller [10]–[12] and Kalnins, Miller and the present author [3], [4] (references [12], [3], [4] are hereafter referred to as V, VI and VII, respectively) that the paper is written. Its purpose is to establish the intimate connection between Lie theory and the theory of special functions and from this to derive special function identities many of which appear to be new. In this regard we mention also the work of Kalnins [9], Patera and Winternitz [19] and Koornwinder [13].

1. Symmetries of the equation. Let X denote the partial differential operator

$$X = i\,\partial_t + \partial_{x_1x_1} + \partial_{x_2x_2} - \left(\frac{\alpha}{x_1^2} + \frac{\beta}{x_2^2}\right) \tag{1.1}$$

acting on the space C^∞ of locally infinitely differentiable functions of the real variables x_i, t, with $-\infty < t < \infty$, $0 < x_i < \infty$ (when $\alpha = 0$, $-\infty < x_1 < \infty$). The maximal invariance algebra $\mathcal{G}$ for the equation

$$Xu = iu_t + u_{x_1x_1} + u_{x_2x_2} - \left(\frac{\alpha}{x_1^2} + \frac{\beta}{x_2^2}\right)u = 0 \tag{1.2}$$

was determined in [2]. Here we omit the details and merely present a brief review.

Now an infinitesimal generator for a group of space time transformations takes the form

$$L = a(\mathbf{x}, t)\partial_t + b_i(\mathbf{x}, t)\partial_{x_i} + c(\mathbf{x}, t),$$

where a, b_i, $c \in C^\infty$. A necessary and sufficient condition for L to be an infinitesimal generator of a group of symmetries for (1.2) is

$$[L, X] = \lambda(x_i, t)X, \tag{1.3}$$

where $\lambda(x_i, t) \in C^\infty$. The condition (1.3) completely determines [2] both L and λ; the results are the following generators:

Case 1. $\alpha = 0$.

$$K_2 = -t^2\,\partial_t - tx_i\,\partial_{x_i} - t + \frac{i}{4}(x_1^2 + x_2^2), \qquad K_{-2} = \partial_t,$$

$$D = x_i\,\partial_{x_i} + 2t\,\partial_t + 1, \qquad E = i, \tag{1.4}$$

$$P_1 = \partial_{x_1}, \qquad B_1 = -t\,\partial_{x_1} + ix_1/2.$$

Case 2. $\alpha \neq 0$. The generators are (1.4) with P_1 and B_1 missing and E is irrelevant (i.e., can be removed by a change of basis).

The commutation relations are given by

$$[D, K_{\pm 2}] = \pm 2K_{\pm 2}, \quad [D, B_1] = B_1, \quad [D, P_1] = -P_1,$$

$$[K_2, K_{-2}] = D, \qquad [P_1, B_1] = \tfrac{1}{2}E, \tag{1.5}$$

$$[K_{-2}, B_1] = -P_1, \qquad [K_2, P_1] = -B_1,$$

where E is in the center of $\mathcal{G}$ and all other commutators vanish. Hence we have the structure $\mathcal{G} \approx s\ell(2, R) \oplus w_1$ for Case 1 and $\mathcal{G} \approx s\ell(2, R)$ for Case 2, where w_1 denotes the one-dimensional Heisenberg–Weyl algebra generated by P_1, B_1 and E, and is an invariant subalgebra of $\mathcal{G}$; $\oplus$ denotes the semidirect sum.

As in VI we can exponentiate $\mathcal{G}$ to obtain a local Lie group G of operators acting on C^∞ and mapping solutions of (1.2) into solutions. The action of the Weyl group W_1 in Case 1 given by the representation

$$T(w, z, e) = e^{wB_1}\, e^{zP_1}\, e^{eE}$$

is

$$[T(w, z, e)f](\mathbf{x}, t) = e^{(i/4)(2\mathbf{x}\cdot\mathbf{w} - t\mathbf{w}\cdot\mathbf{w} + 4e)} f(\mathbf{x} - t\mathbf{w} + \mathbf{z}, t), \tag{1.6}$$

where the vectors $\mathbf{w}$ and $\mathbf{z}$ are $(w, 0)$ and $(z, 0)$, respectively. The action of the representation $T(A)$ of $SL(2, R)$ is

$$[T(A)f](\mathbf{x}, t) = \exp\left[\frac{ib(x_1^2 + x_2^2)}{4(d + tb)}\right](d + tb)^{-1}$$

$$\cdot f\left[(d + tb)^{-1}\mathbf{x}, \frac{c + ta}{d + tb}\right], \tag{1.7}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, R).$$

The adjoint action of $SL(2, R)$ on W is

$$T^{-1}(A)T(\mathbf{w}, \mathbf{z}, e)T(A) = T(\mathbf{w}A, \mathbf{z}A, e'),$$

where $e' = e + (1/4)(\mathbf{w}\cdot\mathbf{z} - \mathbf{w}A \cdot \mathbf{z}A)$.

Now the group G acts on the Lie algebra $\mathcal{G}$ through the adjoint representation

$$K^g = T(g)KT^{-1}(g) \tag{1.8}$$

for $g \in G$, $K \in \mathcal{G}$ and splits $\mathcal{G}$ into G-orbits. We classify the orbit structure of the factor algebra $\mathcal{G}' \approx \mathcal{G}/E$. Writing a general element of the $s\ell(2, R)$ subalgebra as $A_2K_2 + A_0D + A_{-2}K_{-2}$, it is easily seen that $I = A_2A_{-2} + A_0^2$ is invariant under (1.8). Thus we have the following orbit representatives for $\mathcal{G}$: In Case 1 ($\alpha = 0$),

	(i)	$(I<0)$	$K_{-2} - K_2$,
(1.9a)	(ii)	$(I>0)$	cD, $\quad 0 \neq c \in R$
	(iii)	$(I=0)$	$K_2 + P_1$, K_2, P_1,

In Case 2 ($\alpha \neq 0$), where P_1 and B_1 do not appear, we have the usual orbits for $s\ell(2, R)$, viz.,

	(i)	$(I<0)$	$K_{-2} - K_2$,
(1.9b)	(ii)	$(I>0)$	cD,
	(iii)	$(I=0)$	K_2.

As in VI we need to determine the second order symmetry operators of (1.2) in order to establish the connection with the separation of variables. We only consider operators S which are first order in ∂_t since this can be related through (1.2) to a second order operator in the spatial derivatives. More precisely, we look for all operators S which take the form

$$S = a(\mathbf{x}, t)\,\partial_t + b_i(\mathbf{x}, t)\,\partial_{x_i} + c_{ij}(\mathbf{x}, t)\,\partial_{x_ix_j} + d(\mathbf{x}, t) \tag{1.10}$$

and satisfy the equation

$$[S, X] = RX, \tag{1.11}$$

where it follows from (1.2) and (1.10) that R can at most be a first order operator.

LEMMA. *The space $\mathcal{S}$ of symmetry operators having the form* (1.10) *and satisfying* (1.11) *forms a finite-dimensional vector space and the adjoint action of the symmetry group G on $\mathcal{S}$ defines an automorphism on $\mathcal{S}$.*

Proof. The vector space property of $\mathcal{S}$ is clear, and the finite dimensionality will be seen explicitly below. By expanding the group action infinitesimally and by using the Jacobi identity, it can be seen that $[L, S] \in \mathcal{S}$. Exponentiating to the group we get the desired result. We notice that as long as $\mathcal{S}$ is finite-dimensional, the proof is independent of the explicit form of the potential $V(x_1, x_2)$ and its symmetry group G.

Now before finding an explicit basis for $\mathcal{S}$, it is convenient to simplify (1.10) somewhat. We do this by constructing the factor space

$$\tilde{\mathcal{S}} \approx \mathcal{S}/\{X\}, \quad \text{where } \{X\} = \{Q \in \mathcal{S} : Q = \phi X,\ \phi \in C^\infty\},$$

under the equivalence relation $S' \simeq S$ if $S' = S + \phi X$ form some $\phi \in C^\infty$. We can always choose ϕ such that no ∂_t terms appear in S.

As in VI, a straightforward but somewhat tedious calculation yields, modulo the above equivalence relation, the following second order symmetries:

Case 1. $\alpha=0$. A basis for $\mathcal{S}$ is given by the six operators (1.4) plus

$$(1.12)\quad \begin{aligned} S_1&=t^2(\partial_{x_1x_1}-\partial_{x_2x_2})-it(x_1\,\partial_{x_1}-x_2\,\partial_{x_2})+\frac{\beta}{x_2^2}t^2-\tfrac{1}{4}(x_1^2-x_2^2),\\ S_2&=2t(\partial_{x_1x_1}-\partial_{x_2x_2})-i(x_1\,\partial_{x_1}-x_2\,\partial_{x_2})+\frac{2\beta t}{x_2^2},\\ S_3&=(\partial_{x_1x_1}-\partial_{x_2x_2})+\frac{\beta}{x_2^2},\\ S_4&=2t(x_1\,\partial_{x_2x_2}-x_2\,\partial_{x_1x_2})+ix_2(x_2\,\partial_{x_1}-x_1\,\partial_{x_2})-t\,\partial_{x_1}-\frac{i}{2}x_1-\frac{2\beta x_1 t}{x_2^2},\\ S_5&=2(x_1\,\partial_{x_2x_2}-x_2\,\partial_{x_1x_2})-\partial_{x_1}-\frac{2\beta x_1}{x_2^2},\\ S_6&=x_2^2\,\partial_{x_1x_1}+x_1^2\,\partial_{x_2x_2}-2x_1x_2\,\partial_{x_1x_2}-(x_1\,\partial_{x_1}+x_2\,\partial_{x_2})-\frac{\beta x_1^2}{x_2^2}.\end{aligned}$$

Hence, in Case 1, $\mathcal{S}$ defines a 12-dimensional vector space.

Case 2. $\alpha=0$. A basis for $\mathcal{S}$ is given by the four operators in (1.4) not including P_1 and B_1 (here E is irrelevant) plus

$$(1.13)\quad \begin{aligned} S_1'&=t^2(\partial_{x_1x_1}-\partial_{x_2x_2})-it(x_1\,\partial_{x_1}-x_2\,\partial_{x_2})-\left(\frac{\alpha}{x_1^2}-\frac{\beta}{x_2^2}\right)t^2-\tfrac{1}{4}(x_1^2-x_2^2),\\ S_2'&=2t(\partial_{x_1x_1}-\partial_{x_2x_2})-i(x_1\,\partial_{x_1}-x_2\,\partial_{x_2})-2\left(\frac{\alpha}{x_1^2}-\frac{\beta}{x_2^2}\right)t,\\ S_3'&=\partial_{x_1x_1}-\partial_{x_2x_2}-\left(\frac{\alpha}{x_1^2}-\frac{\beta}{x_2^2}\right),\\ S_4'&=x_2^2\,\partial_{x_1x_1}+x_1^2\,\partial_{x_2x_2}-2x_1x_2\,\partial_{x_1x_2}-x_1\,\partial_{x_1}-x_2\,\partial_{x_2}-\left(\frac{\alpha x_2^2}{x_1^2}+\frac{\beta x_1^2}{x_2^2}\right).\end{aligned}$$

In Case 2, $\mathcal{S}$ defines an 8-dimensional vector space. Notice that for $\alpha=0$, we have $S_1'=S_1$, $S_2'=S_2$, $S_3'=S_3$ and $S_4'=S_6$. We remark that in contradistinction to VI the second order symmetries are not members of the universal enveloping algebra of the symmetry algebra (1.4). In fact they are more closely related to the underlying group of motions.

We consider next the orbits of $\tilde{\mathcal{S}}\approx\mathcal{S}/\{X\}$ under the action of G. In fact we discuss the orbit structure of the factor space $\tilde{\mathcal{S}}/\{E\}$. For Case 2, which is simpler, the representation of $G\approx SL(2,R)$ on $\tilde{\mathcal{S}}/\{E\}$ is reducible and splits into three irreducible components $\tilde{\mathcal{S}}/\{E\}\approx\mathcal{D}_{ad}+(\mathcal{D}_3+\mathcal{D}_1)$, where the representation $\mathcal{D}_{ad}$ is the adjoint representation of G on its Lie algebra $\mathcal{G}$, $\mathcal{D}_3$ is a three-dimensional representation spanned by S_1', S_2', S_3', and $\mathcal{D}_1$ is a one-dimensional representation with S_4' as its basis. It is a straightforward computation to show that the orbits of G in $\mathcal{D}\equiv\mathcal{D}_3+\mathcal{D}_1$ lead to precisely one of the orbit representatives given by S_4'; $S_4'+a(S_3'+S_1')$, $S_4'+aS_2'$, $S_4'+aS_1'$; S_2', S_1', $S_3'+S_1'$. Moreover, it will be seen later that there is a 1–1 correspondence between the set of all commuting pairs (K, S), where K is one of the orbit representatives of $\mathcal{G}$ from (1.9b) and S is one of the above representatives of $\mathcal{D}$, and the set of all G-inequivalent separable coordinate systems for (1.2).

Case 1 is more complicated owing to the fact that the representation of $G \approx SL(2, R) \otimes W_1$ on $\tilde{\mathcal{S}}/\{E\}$ is indecomposable. Our procedure then is to search for all members of $\tilde{\mathcal{S}}/\{E\}$, which commute with a given orbit representative of $\mathcal{G}/\{E\}$ given by (1.9a). Once this is done for each orbit in $\mathcal{G}/\{E\}$, we augment our definition of equivalence in $\tilde{\mathcal{S}}/\{E\}$ by considering two elements S_a, $S_b \in \mathcal{S}$ to be equivalent if they differ by the member $K \in \mathcal{G}$ we have chosen; i.e., $S_a \approx S_b$ if $S_a \approx S_b + cK$, $c \in R$. Then with this extended definition of equivalence, we will see later that there is a 1–1 correspondence between commuting orbit pairs (K, S), $K \in \mathcal{G}/\{E\}$, $S \in \tilde{\mathcal{S}}/\{\sim\}$ and G-inequivalent separable coordinate systems for (1.2) with $\alpha = 0$. These results will be presented later in Table 2.

2. Separation of variables for the equation. The separation of variables for (1.2), proceeds in the same manner as in VI with only slight modification. We present only a brief sketch here and refer to VI for the details. The results are given in Table 1. In general, we are interested in R-separability. That is, we consider a change of variables

$$x_1 = G(v_1, v_2, v_3), \quad x_2 = H(v_1, v_2, v_3), \quad t = F(v_1, v_2, v_3) \tag{2.1}$$

such that the solution $u(x_1, x_2, t)$ of (1.2) can be written as

$$u(G, H, F) = e^{iS(v_1,v_2,v_3)} A(v_1)B(v_2)C(v_3), \tag{2.2}$$

where $S(v_1, v_2, v_3)$ is a function determined from the analysis and the functions $A(v_1)$, $B(v_2)$ and $C(v_3)$ reduce (1.2) to ordinary differential equations in the corresponding arguments. When the function S is a sum of functions of each of the individual variables v_i, we have ordinary separation (this is equivalent to $S = 0$). Furthermore, the separation process always allows us to take $t = F = v_3$, so this will be assumed in what follows.

Now the classification of separable coordinate systems is actually a classification of equivalence classes of coordinates. Any two coordinates which can be related by rotations, translations, or inhomogeneous Galilei transformations of the underlying space $(\mathbf{x}, t)$ are considered to be equivalent; i.e., two systems which lie on the same orbit under a transformation in the extended Galilei group $\mathcal{G}_2$ are equivalent. Finally, we mention that our notation is the same as VI. There the coordinate systems are labeled by two letters and a superscript. The first (capital letter) indicates the type of Hamiltonian to which the system in some sense corresponds, $F \leftrightarrow$ free particle, $L \leftrightarrow$ linear potential, $0 \leftrightarrow$ harmonic oscillator, and $R \leftrightarrow$ repulsive harmonic oscillator; of course, now we have an extra centrifugal type potential added in each case, but we will retain the notation. The second (small letter) indicates which of the separable coordinate systems for the two-dimensional Helmholtz equation appears, i.e., $c \leftrightarrow$ Cartesian, $r \leftrightarrow$ radial (polar), $p \leftrightarrow$ parabolic, and $e \leftrightarrow$ elliptic. The superscript (1) or (2) indicates two separable coordinate systems which are equivalent under an $SL(2, R)$ transformation, but not under the above described equivalence (more will be said about this shortly). Suffice it now to say that the system with superscript (2) corresponds to precisely one of the physical Hamiltonians mentioned above, while the superscript (1) indicates a system whose spectral analysis is simple (simpler than that of (2)). The "subgroup coordinates", i.e., system 22–25 in Table 1 are only labeled by the type of Hamiltonian and correspond to the equation $u_{x_2x_2} + iu_t - (\beta/x_2^2)u = 0$ treated in V.

3. The operator characterization of separation of variables. In this section, we give a characterization of the separation of variables of the preceding section in terms of symmetry operators. Corresponding to each of the coordinate systems listed in Table 1 there is a commuting pair of symmetry operator (K, S) in $\mathscr{S}$. The first operator of the pair K is a first order symmetry operator and is a member of the Lie algebra $\mathscr{G}$. The second member S is a second order operator in $\tilde{\mathscr{S}}$. In contradistinction to VI, the second order elements in $\mathscr{S}$ are *not* members of the universal enveloping algebra of $\mathscr{G}$. However, this is no disadvantage. What is important is that the adjoint action of the group G map $\tilde{\mathscr{S}}$ into itself, which is assured us by the Lemma. This action as seen previously splits $\tilde{\mathscr{S}}$ into G-orbits, and it is straightforward to classify all commuting pair (K, S) in the way described in § 1. However, as mentioned previously, there are in general from the point of view of separation of variables, two points on each G-orbit of pairs (K, S) which correspond to separable coordinate systems. Nevertheless, in the next section we will use explicitly this *G-orbit* structure of the pairs (K, S) by performing the spectral analysis and expansions at a simple point $(\mathscr{K}, \mathscr{S})$ on the orbit pairs. Then by an arbitrary action of the symmetry group, the pair $(\mathscr{K}, \mathscr{S})$ is transformed to a pair (K^g, S^g) where the spectral analysis and expansion theorems are much more difficult. This is the power of the group theoretical technique in deriving special function identities.

The separation of variables is thus characterized by three simultaneous differential equations

$$Xu = 0, \quad Ku = \lambda u, \quad Su = \mu u, \tag{3.1}$$

$K \in \mathscr{G}$, $S \in \mathscr{S}$. From the discussion of the preceding section it is straightforward to determine the commuting operators K and S corresponding to each of the separable coordinate systems listed in Table 1. The results are presented in Table 2, and it is seen that the list exhausts all commuting pairs of G-orbits. Systems which differ only by the superscript correspond to the same G-orbit but different separable coordinates. We mention that for the second order operators the unprimed S's refer to Case 1 equation (1.12) while the primed S's refer to Case 2 equation (1.13). When no primed S's appear it means that Case 2 does not admit that particular separable coordinate system.

As just noted above, the coordinate systems denoted by $Ab^{(1)}$ and $Ab^{(2)}$ in Tables 1 and 2 are equivalent under the action of the symmetry group G. Including this equivalence then, there are 16 inequivalent separable coordinate systems in Case 1 and 9 inequivalent separable coordinate systems in Case 2. As in

TABLE 1

Separable coordinate systems for the equation

Coordinate system	Coordinates	Multiplier S	Remarks
1. $Fc^{(1)}$	$x_1 = v_1 v_3,\ x_2 = v_2 v_3$	$\frac{1}{4}(v_1^2 + v_2^2)v_3$	
2. $Fc^{(2)}$	$x_1 = v_1,\ \ x_2 = v_2$	0	
3. $Fr^{(1)}$	$x_1 = v_1 v_3 \cos v_2,\ x_2 = v_1 v_3 \sin v_2$	$\frac{1}{4}v_1^2 v_3$	
4. $Fr^{(2)}$	$x_1 = v_1 \cos v_2,\ x_2 = v_1 \sin v_2$	0	
5. $Fp^{(1)}$	$x_1 = \frac{1}{2}v_3(v_1^2 - v_2^2),\ x_2 = v_1 v_2 v_3$	$\frac{1}{16}(v_1^2 + v_2^2)^2 v_3$	$\alpha = 0$
6. $Fp^{(2)}$	$x_1 = \frac{1}{2}(v_1^2 - v_2^2),\ x_2 = v_1 v_2$	0	$\alpha = 0$
7. $Fe^{(1)}$	$x_1 = v_3 \cosh v_1 \cos v_2,\ x_2 = v_3 \sinh v_1 \sin v_2$	$\frac{1}{4}(\sinh^2 v_1 + \cos^2 v_2)v_3$	
8. $Fe^{(2)}$	$x_1 = \cosh v_1 \cos v_2,\ x_2 = \sinh v_1 \sin v_2$	0	
9. $Lc^{(1)}$	$x_1 = v_1 v_3 + \dfrac{a}{v_3},\ x_2 = v_2 v_3$	$\frac{1}{4}(v_1^2 + v_2^2)v_3 - \dfrac{1}{2v_3}av_1$	$\alpha = 0$
10. $Lc^{(2)}$	$x_1 = v_1 + av_3^2,\ x_2 = v_2$	$av_1 v_3$	$\alpha = 0$
11. $Lp^{(1)}$	$x_1 = \frac{1}{2}v_3(v_1^2 - v_2^2) + \dfrac{a}{v_3},\ x_2 = v_1 v_2 v_3$	$\frac{1}{16}(v_1^2 + v_2^2)^2 v_3 - \dfrac{a}{4v_3}(v_1^2 - v_2^2)$	$\alpha = 0$
12. $Lp^{(2)}$	$x_1 = \frac{1}{2}(v_1^2 - v_2^2) + av_3^2,\ x_2 = v_1 v_2$	$\dfrac{av_3}{2}(v_1^2 - v_2^2)$	$\alpha = 0$
13. Oc	$x_1 = v_1\sqrt{1 + v_3^2},\ x_2 = v_2\sqrt{1 + v_3^2}$	$\frac{1}{4}(v_1^2 + v_2^2)v_3$	
14. Or	$x_1 = \sqrt{1 + v_3^2}\,v_1 \cos v_2,\ x_2 = \sqrt{1 + v_3^2}\,v_1 \sin v_2$	$\frac{1}{4}v_1^2 v_3$	
15. Oe	$x_1 = \sqrt{1 + v_3^2} \cosh v_1 \cos v_2,\ x_2 = \sqrt{1 + v_3^2} \sinh v_1 \sin v_2$	$\frac{1}{4}(\sinh^2 v_1 + \cos^2 v_2)v_3$	
16. $Rc^{(1)}$	$x_1 = v_1 v_3^{1/2},\ x_2 = v_2 v_3^{1/2}$	0	

TABLE 1 (*continued*)

Coordinate system	Coordinates	Multiplier S	Remarks
17. $\mathrm{Rc}^{(2)}$	$x_1 = v_1\sqrt{\lvert v_3^2-1\rvert},\ x_2 = v_2\sqrt{\lvert v_3^2-1\rvert}$	$\frac{\varepsilon}{4}(v_1^2+v_2^2)v_3$, $\varepsilon = \pm$ for $v_3 \gtrless 1$	
18. $\mathrm{Rr}^{(1)}$	$x_1 = v_1 v_3^{1/2}\cos v_2,\ x_2 = v_2 v_3^{1/2}\sin v_2$	0	
19. $\mathrm{Rr}^{(2)}$	$x_1 = \sqrt{\lvert v_3^2-1\rvert}\,v_1\cos v_2,\ x_2 = \sqrt{\lvert v_3^2-1\rvert}\,v_1\sin v_2$	$\frac{\varepsilon}{4}v_1^2 v_3$, $\varepsilon = \pm$ for $v_3 \gtrless 1$	
20. $\mathrm{Re}^{(1)}$	$x_1 = v_3^{1/2}\cosh v_1\cos v_2,\ x_2 = v_3^{1/2}\sinh v_1\sin v_2$	0	
21. $\mathrm{Re}^{(2)}$	$x_1 = \sqrt{\lvert v_3^2-1\rvert}\cosh v_1\cos v_2,\ x_2 = \sqrt{\lvert v_3^2-1\rvert}\sinh v_1\sin v_2$	$\frac{\varepsilon}{4}(\sinh^2 v_1+\cos^2 v_2)v_3$ $\varepsilon = \pm$ for $v_3 \gtrless 1$	
22. F1	$x_1 = v_1,\ x_2 = v_2 v_3$	$\frac{1}{4}v_3 v_2^2$	$\alpha = 0$
23. O1	$x_1 = v_1,\ x_2 = v_2\sqrt{1+v_3^2}$	$\frac{1}{4}v_3 v_1^2$	$\alpha = 0$
24. R1	$x_1 = v_1,\ x_2 = v_2 v_3^{1/2}$	0	$\alpha = 0$
25. R2	$x_1 = v_1,\ x_2 = v_2\sqrt{\lvert v_3^2-1\rvert}$	$\frac{\varepsilon}{4}v_2^2 v_3$, $\varepsilon = \pm$ for $v_3 \gtrless 1$;	$\alpha = 0$

VI, this equivalence can be described by considering the operator $J = \exp(\pi/4)(K_2 - K_{-2})$. Then using (1.7) we find that

TABLE 2
Symmetry operators associated with variable separation

Coordinate System	1st Order Symmetry K	2nd Order Symmetry S
1. $\mathrm{Fc}^{(1)}$	K_2	S_1, S_1'
2. $\mathrm{Fc}^{(2)}$	K_{-2}	S_3, S_3'
3. $\mathrm{Fr}^{(1)}$	K_2	S_6, S_4'
4. $\mathrm{Fr}^{(2)}$	K_{-2}	S_6, S_4'
5. $\mathrm{Fp}^{(1)}$	K_2	S_4
6. $\mathrm{Fp}^{(2)}$	K_{-2}	S_5
7. $\mathrm{Fe}^{(1)}$	K_2	$S_6+\frac{1}{2}S_1, S_4'+\frac{1}{2}S_1'$
8. $\mathrm{Fe}^{(2)}$	K_{-2}	$S_6+\frac{1}{2}S_3, S_4'+\frac{1}{2}S_3'$
9. $\mathrm{Lc}^{(1)}$	K_2+2aP_1	S_1+2iaP_1
10. $\mathrm{Lc}^{(2)}$	$K_{-2}-2aB_1$	S_3-2iaB_1
11. $\mathrm{Lp}^{(1)}$	K_2+2aP_1	$S_4+a(iK_{-2}+S_3)$
12. $\mathrm{Lp}^{(2)}$	$K_{-2}+2aB_1$	$S_5+a(iK_2-S_1)$
13. Oc	$K_{-2}-K_2$	$S_3+S_1, S_3'+S_1'$
14. Or	$K_{-2}-K_2$	S_6, S_4'
15. Oe	$K_{-2}-K_2$	$S_6-(S_3+S_1), S_4'-(S_3'+S_1')$
16. $\mathrm{Rc}^{(1)}$	D	$2S_2, 2S_2'$
17. $\mathrm{Rc}^{(2)}$	$K_{-2}+K_2$	$S_3-S_1, S_3'-S_1'$
18. $\mathrm{Rr}^{(1)}$	D	S_6, S_4'
19. $\mathrm{Rr}^{(2)}$	$K_{-2}+K_2$	S_6, S_4'
20. $\mathrm{Re}^{(1)}$	D	$S_6+\frac{1}{4}S_2, S_4'+\frac{1}{4}S_2'$
21. $\mathrm{Re}^{(2)}$	$K_{-2}+K_2$	$S_6+\frac{1}{8}(S_3-S_1), S_4'+\frac{1}{8}(S_3'-S_1')$
22. F1	P_1	S_3
23. O1	P_1	$-i(K_2-K_{-2})+S_1$
24. R1	P_1	$iD+S_2$
25. R2	P_1	$-i(K_2+K_{-2})+S_1$

$$Jf(\mathbf{x}, t) = \frac{\sqrt{2}}{1+t} \exp\left[\frac{i}{4}(1+t)^{-1}\mathbf{x}\cdot\mathbf{x}\right] f\left(\sqrt{2}(1+t)^{-1}\mathbf{x}, \frac{t-1}{t+1}\right). \tag{3.2}$$

Then it follows that

$$\begin{aligned} J^2 f(\mathbf{x}, t) &= t^{-1} \exp\left[\frac{i}{4t}\mathbf{x}\cdot\mathbf{x}\right] f(t^{-1}\mathbf{x}, -t^{-1}), \\ J^4 f(\mathbf{x}, t) &= -f(-\mathbf{x}, t), \\ J^8 f(\mathbf{x}, t) &= f(\mathbf{x}, t). \end{aligned} \tag{3.3}$$

It is easy to see that $J(K_2+K_{-2})J^{-1} = D$, $J(S_3-S_1)J^{-1} = 2S_2$, $JS_6J^{-1} = S_6$ and similarly for the primed pairs; hence, the three systems $Rc^{(2)}$, $Rr^{(2)}$ and $Re^{(2)}$ are equivalent under J to $Rc^{(1)}$, $Rr^{(1)}$ and $Re^{(1)}$, respectively. Furthermore, denoting $\tilde{K}^2 = J^2KJ^{-2}$, we see that

$$\tilde{P}_1 = -B_1, \quad \tilde{B}_1 = P_1, \quad \tilde{K}_{-2} = -K_2, \quad \tilde{K}_2 = -K_{-2},$$
$$\tilde{D} = -D, \quad \tilde{S}_3 = S_1, \quad \tilde{S}_1 = S_3, \quad \tilde{S}_5 = S_4, \quad \tilde{S}_2 = -S_2$$

and similarly for the primed S's. As a result we find the six pairs $Fc^{(2)}$, $Fr^{(2)}$, $Fp^{(2)}$, $Fe^{(2)}$, $Lc^{(2)}$, $Lp^{(2)}$ are equivalent under J^2 to $Fc^{(1)}$, $Fr^{(1)}$, $Fp^{(1)}$, $Fe^{(1)}$, $Lc^{(1)}$, $Lp^{(1)}$, respectively.

4. Basis functions: Two and three variable models. We now wish to construct unitary representations of G on the Hilbert space $\mathcal{L}_2(R_2)$ of Lebesque square-integrable functions on the half-plane, R_2^+, $-\infty < x_1 < \infty$, $0 \leqq x_2 < \infty$, in Case 1 and the quadrant R_2^{++}, $0 \leqq x_1, x_2 < \infty$, in Case 2. First consider for Case 1 the space of functions $\mathcal{F}$ with compact support in the upper half-plane away from the x_1-axis, and for Case 2, the space of functions $\mathcal{F}'$ with compact support in the upper right-hand quadrant away from both axes. By introducing the inner product

$$(f_1, f_2) = \iint_{R_2} dx_1\, dx_2\, \bar{f}_1(\mathbf{x}) f_2(\mathbf{x}), \tag{4.1}$$

where R_2 denotes R_2^+ in Case 1 and R_2^{++} in Case 2, and completing $\mathcal{F}$ and $\mathcal{F}'$, respectively, with respect to the norm $\|f\| = (f, f)^{1/2}$, we obtain the Hilbert spaces denoted by $\mathcal{L}^2(R_2^+)$ and $\mathcal{L}^2(R_2^{++})$, respectively. Hereafter, it should be understood that R_2 denotes R_2^+ in Case 1 and R_2^{++} in Case 2.

Now we describe the Lie algebra $\mathcal{G}$ with basis (1.4) as a subset of $\mathcal{S}$ by setting $t = 0$ and replacing ∂_t by $i[(\partial_{x_1x_1} + \partial_{x_2x_2}) - (\alpha/x_1^2) - (\beta/x_2^2)]$, viz.,

$$\begin{aligned} &\mathcal{K}_2 = \frac{i}{4}(x_1^2 + x_2^2), \qquad \mathcal{K}_{-2} = i\left(\partial_{x_1x_1} + \partial_{x_2x_2} - \frac{\alpha}{x_1^2} - \frac{\beta}{x_2^2}\right), \\ &\mathcal{D} = x_1\, \partial_{x_1} + x_2\, \partial_{x_2} + 1, \qquad \mathcal{E} = i, \\ &\mathcal{P}_1 = \partial_{x_1}, \qquad \mathcal{B}_1 = \frac{i}{2} x_1 \quad \text{for } \alpha = 0. \end{aligned} \tag{4.2}$$

The script letters in (4.2) correspond to the block letters in (1.4) under the transformation $K = e^{t\mathcal{K}_{-2}} \mathcal{K} e^{-t\mathcal{K}_{-2}}$. Now it is clear that the generators (4.2) are skew-symmetric in $\mathcal{F}$ and $\mathcal{F}'$ with respect to the $\mathcal{L}^2(R_2)$ norm. Moreover, we can find skew-adjoint extensions in the usual manner. The only operator for which there is any difficulty is $\mathcal{K}_{-2}$. However, when $\alpha, \beta \geqq \frac{3}{4}$ or $\alpha = 0$, $\beta \geqq \frac{3}{4}$ there is a unique self-adjoint extension [6, Chap. 13], [21]. For simplicity in what follows we make these restrictions. Thus the map $Ad\, e^{t\mathcal{K}_{-2}}$ is an isometric isomorphism of (4.2) onto (1.4).

The integrated group action of the operators (4.2) is given by

$$[U(w, z, e)f](x) = e^{i(\alpha + (1/2)wx_1)} f(x_1 + z_1 x_2), \tag{4.3}$$

where $U(w, z, e) = e^{wB_1}\, e^{zP_1}\, e^{eE}$ (for Case 1 only).

$$(e^{c\mathcal{D}} f)(x) = e^c f(e^c x)\; e^{b\mathcal{K}_2} f)(x)\; e^{i(b/4)x^2} f(x). \tag{4.4}$$

The action of $e^{t\mathcal{K}_{-2}}$ is more complicated since $\mathcal{K}_{-2}$ is a second order operator. We have the Green's function problem.

$$(e^{t\mathcal{K}_{-2}} f)(x) = \text{l.i.m.} \iint_{R_2} dk\, G(x, y, t) f(y) \tag{4.5}$$

with the boundary condition

$$(4.6)\qquad \underset{t\to 0}{\text{l.i.m.}}\, G(\mathbf{x}, \mathbf{y}; t) = \delta(\mathbf{x}-\mathbf{y}) = \delta(x_1-y_1)\,\delta(x_2-y_2).$$

We find then the following:

Case 1.

$$(4.7a)\qquad G(\mathbf{x}, \mathbf{y}, t) = \frac{e^{\mp(i\pi/2)(1+\nu)}}{4|t|\sqrt{\pi i t}}\, e^{-((x_1-y_1)^2+x_2^2+y_2^2)/4it}(y_2x_2)^{1/2} J_\nu\left(\frac{y_2x_2}{2|t|}\right);$$

Case 2.

$$(4.7b)\qquad G(\mathbf{x}, \mathbf{y}, t) = \frac{e^{\mp i\pi(1+(\mu+\nu)/2)}}{4t}(x_1x_2y_1y_2)^{1/2}\, e^{i(\mathbf{x}^2+\mathbf{y}^2)/4t} J_\mu\left(\frac{x_1y_1}{2|t|}\right) J_\nu\left(\frac{y_2x_2}{2|t|}\right),$$

where $J_\mu(z)$ is a Bessel function [7], $\mp$ is taken for $t \gtrless 0$, respectively, and $\alpha = \mu^2 - \frac{1}{4}$, $\beta = \nu^2 - \frac{1}{4}$. Similarly or by the group composition, we can give the action for a general transformation in $SL(2, R)$; however, we only have need for the transformation (4.5), so we omit writing the general form.

We obtain the full vector space of second order symmetries $\mathcal{S}$ in our t-independent formalism by simply putting $t = 0$ for the operators in (1.12) and (1.13); i.e., we define the operators $\mathcal{S}_i = S_i(t=0)$ and similarly for the primed operators. It is clear that the operators $\mathcal{S}_i(\mathcal{S}_i')$ are skew-symmetric in $\mathcal{L}^2(R_2)$ when defined with the domain $\mathcal{F}(\mathcal{F}')$, respectively.

The spectral analysis for the different bases proceeds as follows: We first perform the spectral analysis of the pairs of commuting operators corresponding to the point on each orbit where $t = 0$, i.e., the script operators just discussed. Then by performing the unitary transformation $e^{tAd\mathcal{K}_{-2}}$ given explicitly by (4.5) and (4.7), we obtain the spectral analysis for the separable solutions of (1.2) with superscript (1) listed in Table 1 corresponding to the operator pairs (with superscript (1)) appearing in Table 2. One important consideration here is that the spectral analysis for the script operators always reduces to two separate Sturm–Liouville problems (in general irregular), whereas the operator pairs listed in Table 2 *do not* always do so. This type of multiparameter eigenvalue problem occurs in parabolic and elliptic coordinates. Hence, our procedure provides a simple resolution of this problem when it occurs. Another important consideration is that in many cases the transformation (4.5) provides us with new integral equations for the basis functions. This happens when the integral in (4.5) cannot be computed by previously known results. Then since we know that the transformed basis functions satisfy certain ordinary differential equations modulo the multiplier function e^{iS}, we know the solution up to a normalization constant which can be computed by inserting fixed values of the arguments of the functions.

A third point is that since we are dealing with continuous as well as discrete spectrum problems, many of our eigenfunctions and transformations are to be interpreted in the generalized sense. The procedure for obtaining expansion and inversion formulas for a general irregular Sturm–Liouville problem is, of course, well known [6, Chap. 13], [21]. It is only noted here that we construct our (generalized) eigenfunctions to be normalized in the generalized sense, i.e.,

$$(A'b'_{\lambda_1,\lambda_2}, Ab_{\lambda_1,\lambda_2}) = (a'b'_{\lambda_1,\lambda_2}, ab_{\lambda_1,\lambda_2}) = \delta(\lambda'_1, \lambda_1)\delta(\lambda'_2, \lambda_2), \tag{4.8}$$

where $\delta(\lambda, \lambda')$ means $\delta_{\lambda\lambda}$ when the spectrum is discrete, and $\delta(\lambda - \lambda')$ when it is continuous. When the spectrum is continuous, we can apply a unitary transformation of the group interpreted in the generalized sense to any one of the generalized eigenfunctions preserving the orthonormality property (4.8) as well as the Parseval identity [21], $f \in \mathcal{L}^2$,

$$\int\int_{R_2} dx_1\, dx_2 |f(x)|^2 = \int\int_{S_p} d\lambda_1\, d\lambda_2 |(ab_{\lambda_1,\lambda_2}, f)|^2, \tag{4.9}$$

where S_p denotes the spectrum of the pair of commuting operators under consideration. We note that when computing integrals involving generalized eigenfunctions, it is customary to perform a contour integral. This will be done in many places in what follows without further mention. It is noted that the systems labeled with the superscript (2) can be treated by applying the transformation (3.2) or (3.3).

Finally, when computing the spectral analysis, we treat Case 2 first (with S's denoted by primes) since then Case 1 appears as a special case.

Fc system.

$$-i\mathcal{K}_2 f = \frac{k^2}{4} f, \qquad \mathcal{S}'_1 f = \frac{k_2^2 - k_1^2}{4} f.$$

The generalized eigenfunctions are

$$fc_{k_1,k_2}(\mathbf{x}) = \delta(x_1 - k_1)\,\delta(x_2 - k_2). \tag{4.10}$$

In the three variable model we have

$$Fc_{k_1,k_2(\mathbf{x},t)} = e^{t\mathcal{K}_{-2}} fc_{k_1,k_2}(\mathbf{x}), \qquad e^{t\mathcal{K}_{-2}}(-i\mathcal{K}_2, \mathcal{S}'_1)\, e^{-t\mathcal{K}_{-2}} = (-iK_2, S'_1)$$

with the generalized eigenfunctions obtained trivially from direct integration as the Green's functions (4.7), i.e., $Fc_{k_1,k_2}(\mathbf{x}, t) = G(\mathbf{x}, \mathbf{k}, t)$.

Fr system.

$$-i\mathcal{K}_2 f = \frac{k^2}{4} f, \qquad \mathcal{S}'_1 f = -sf$$

with the generalized eigenfunctions

$$fr_{k,m}(\mathbf{x}) = N_m(\mu, \nu)\frac{\delta(r-k)}{\sqrt{k}} \sin^{\nu+1/2}\theta \cos^{\mu+1/2}\theta P_m^{(\nu,\mu)}(\cos 2\theta), \tag{4.11a}$$

where $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $s + \alpha + \beta = (2m + \mu + \nu + 1)^2$, m a nonnegative integer and the $P_m^{(\mu,\nu)}(z)$ are Jacobi polynomials [7] with

$$N_m(\mu, \nu) = \left[\frac{2m!\Gamma(m+\mu+\nu+1)(2m+\mu+\nu+1)}{\Gamma(m+\mu+1)(m+\nu+1)}\right]^{1/2}. \tag{4.11b}$$

The transformed pair $(-iK_2, S_4')$ gives rise to the generalized eigenfunctions

$$Fr_{k,m}(x) = e^{\mp i\pi(m+1+(\mu+\nu)/2)} N_m^{(\mu,\nu)} \frac{k^{1/2}\, e^{i(r^2+k^2)/4t}}{2t} J_{2m+\mu+\nu+1}\left(\frac{kr}{2|t|}\right) \sin^{\nu+1/2}\theta \cdot \cos^{\mu+1/2}\theta\, P_m^{(\nu,\mu)}(\cos 2\theta). \tag{4.12}$$

Then making use of the transform (4.5) we find a generalization of Sonine's finite integral [22]

$$\int_0^{\pi/2} d\theta' \sin^{\nu+1}\theta' \cos^{\mu+1}\theta' J_\mu(z\cos\theta\cos\theta') J_\nu(z\sin\theta\sin\theta') P_n^{(\nu,\mu)}(\cos 2\theta') = (-1)^n z^{-1} J_{2n+\mu+\nu+1}(z) \sin^\nu\theta \cos^\mu\theta\, P_n^{(\nu,\mu)}(\cos 2\theta). \tag{4.13}$$

Similarly one can apply the inverse transform on the basis (4.12) and express the double integral of a product of three Bessel functions and a Jacobi polynomial in terms of the functions (4.11). This integral however, collapses with aid of (4.13) to the completeness statement for Bessel functions.

Now in Case 1 ($\alpha = 0$), we have a special case of the preceding. The basis functions for the pair $(-\mathcal{K}_2, \mathcal{S}_6)$ with eigenvalues $(k^2/4, -s)$ are given in terms of Gegenbauer polynomials [7] as

$$fr_{k,m}(\mathbf{x}) = N_m(\nu) \frac{\delta(r-k)}{\sqrt{k}} \sin^{\nu+1/2}\theta\, C_m^{(\nu+1/2)}(\cos\theta), \tag{4.14a}$$

where

$$N_m(\nu) = \left[\frac{n!(n+\nu+\frac{1}{2})}{\pi 2^{-2\nu}\Gamma(n+2\nu+1)}\right]^{1/2} \Gamma(\nu+\tfrac{1}{2}) \tag{4.14b}$$

and $s+\beta = (m+\nu+1/2)^2$ with m a nonnegative integer. In this case there is a further symmetry. We see that the eigenfunctions have definite parity under $\theta \to \pi - \theta$. When this is to be emphasized, we write the positive parity eigenfunctions as $fr_{k,2n}(\mathbf{x})$ and the negative parity ones as $fr_{k,2n+1}(\mathbf{x})$.

Then transforming to the three-variable model, we have

$$Fr_{k,m}(\mathbf{x}) = \frac{e^{\mp(i\pi/2)(m+\nu+(3/2))} N_m(\nu)}{2t} e^{(i(r^2+k^2)/4t)} k^{1/2} J_{m+\nu+1/2}\left(\frac{kr}{2|t|}\right) \sin^{\nu+1/2}\theta \cdot C_m^{(\nu+1/2)}(\cos\theta). \tag{4.15}$$

Using the transform (4.5) now with the kernel (4.7a) to transform (4.14) into (4.15) leads directly to Gegenbauer's finite integral [22].

Fp system. Only for Case 1 ($\alpha = 0$) do we have separability. The pair $(-i\mathcal{K}_2, \mathcal{S}_4)$ with eigenvalues $(k^2/4, -ks)$, respectively, have generalized eigenvectors as in VI and [16] except the domain of the variable 2 is half of that in VI and consequently the multiplicity of the continuous spectrum is one. We have the orthonormal eigenvectors

$$fp_{k,s}(\mathbf{x}) = \frac{\delta(r-k)}{\sqrt{\pi k}} (1+\cos\theta)^{is-(1/4)} (1-\cos\theta)^{-is-(1/4)} \tag{4.16}$$

with $-\infty < s < \infty$. Then for the transformed pair $(-iK_2, S_4)$, we find

$$Fp_{k,s}(\mathbf{x}, t) = \frac{e^{\mp(i\pi/2)(1+\nu)}\Gamma\left(\frac{1+\nu}{2}+is\right)\Gamma\left(\frac{1+\nu}{2}-is\right)}{2\pi i^{1/2}t[\Gamma(1+\nu)]^2(v_1v_2)^{1/2}} e^{(ik^2)/(4t)} e^{(i(v_1^2+v_2^2)^2t)/16}$$

$$\cdot M_{is,\nu/2}\left(\frac{ik}{2}v_1^2\right)M_{is,\nu/2}\left(-\frac{ik}{2}v_2^2\right), \tag{4.17}$$

where $x_1 = (t/2)(v_1^2 - v_2^2)$, $x_2 = tv_1v_2$ and $M_{is,\nu/2}(z)$ is a Whittaker function [7]. In this case, the integral is known [8].

Fe system. In Case 2 we have the pair $(-i\mathcal{K}_2, \mathcal{S}_4' + \frac{1}{2}\mathcal{S}_1')$ with eigenvalues $(k^2/4, -s)$. The first equation is again trivial whereas the second equations yields after some algebra

$$(1-z^2)g_{zz} - 2zg_z + \left[\frac{s+\alpha+\beta-1}{4} - \frac{\mu^2/2}{1+z} - \frac{\nu^2/2}{1-z} - \frac{k^2}{32}z\right]g = 0, \tag{4.18}$$

where the eigenfunctions of $\mathcal{S}_4' + \frac{1}{2}\mathcal{S}_1'$ are related to the functions $g(z)$ through

$$fe(z) = (1-z^2)^{1/4}g(z), \qquad z = \cos 2\theta.$$

We shall refer to the solutions of (4.18) as generalized spheroidal wave functions since they are related to the ordinary spheroidal wave functions [7], [14] in the same way Jacobi polynomials are related to associated Legendre polynomials. The spectral analysis of (4.18) is quite similar to that of the Jacobi equation [6, Chap. 13]. Indeed for $\mu, \nu \geqq 1$ the deficiency indices are (1, 1) near each endpoint (Weyl's limit point), thus (0, 0) for the entire interval and there exists a unique self-adjoint extension of $\mathcal{S}_4' + \frac{1}{2}\mathcal{S}_1'$. Moreover, it follows from the general theory [6, Chap. 13], [21] that the spectrum is discrete, bounded from below and simple with eigenvalues

$$\lambda_n^{\mu,\nu}(k^2) = \frac{s+\alpha+\beta-1}{4}$$

assumed ordered as $\lambda_0 < \lambda_1 < \lambda_2 \cdots$ for fixed μ, ν and k. Such solutions of (4.18) are denoted by $\sin^\nu \theta \cos^\mu \theta Ps_n^{(\nu,\mu)}(\theta, k^2)$ generalizing the notation of Meixner and Schäfke [15] for ordinary spheroidal wave functions. The normalized generalized eigenvectors are taken as

$$fe_{k,n}(\mathbf{x}) = \frac{\delta(r-k)}{\sqrt{k}} \sin^{\nu+1/2}\theta \cos^{\mu+1/2}\theta Ps_n^{(\nu,\mu)}(\theta, k^2). \tag{4.19}$$

It is convenient to expand the functions $Ps_n^{(\mu,\nu)}$ in a Jacobi series

$$Ps_n^{(\mu,\nu)}(\theta, k^2) = \sum_{m=0}^{\infty} A_{n,m}^{(\mu,\nu)}(k^2)P_m^{(\mu,\nu)}(\cos 2\theta), \tag{4.20}$$

then the normalization condition implies upon choosing the $A_{n,m}^{(\mu,\nu)}$ to be real

$$\sum_{m=0}^{\infty}\left[\frac{A_{n,m}^{(\mu,\nu)}(k^2)}{N_m(\mu, \nu)}\right]^2 = 1. \tag{4.21}$$

Now the eigenvectors of the transformed pair $(-ik_2, S_4'+\frac{1}{2}S_1')$ take the form

$$(4.22)\quad Fe_{k,n}(\mathbf{x},t)=\frac{k^{1/2}e^{-i\pi(1+(\mu+\nu)/2)}}{2t}\gamma_n^{(\nu,\mu)}(k^2)\exp\left[\frac{it}{4}(\sinh^2\rho+\cos^2\sigma)+\frac{ik^2}{4t}\right]$$
$$\cdot(\sinh\rho\sin\sigma)^{\nu+(1/2)}(\cosh\rho\cos\sigma)^{\mu+(1/2)}Ps_n^{(\nu,\mu)}(i\rho,k^2)Ps_n^{(\nu,\mu)}(\sigma,k^2),$$

where $x_1=t\cosh\rho\cos\sigma$, $x_2=t\sinh\rho\sin\sigma$, and $\gamma_n^{(\nu,\mu)}(k^2)$ is a normalization constant. Then using the transform (4.5) we obtain the analogue of (4.13) for generalized spheroidal wave functions, viz.,

$$\int_0^{\pi/2}d\theta\, J_\mu\left(\frac{k}{2}\cosh\rho\cos\sigma\cos\theta\right)J_\nu\left(\frac{k}{2}\sinh\rho\sin\sigma\sin\theta\right)\cos^{\mu+1}\theta\sin^{\nu+1}\theta\, Ps_n^{(\nu,\mu)}(\theta,k)$$
$$(4.23)\quad =\gamma_n^{(\nu,\mu)}(k^2)\cosh^\mu\rho\sinh^\nu\rho\cos^\mu\sigma\sin^\nu\sigma\, Ps_n^{(\nu,\mu)}(i\rho,k^2)Ps_n^{(\nu,\mu)}(\sigma,k^2).$$

By multiplying both sides by $(\cosh\rho\cos\sigma)^{-\mu}(\sinh\rho\sin\sigma)^{-\nu}$ and evaluating at $\rho=0$, $\sigma=\pi/2$, we find

$$(4.24)\quad \gamma_n^{(\mu,\nu)}(k^2)=\frac{2^{-2\mu-2\nu-1}k^{\mu+\nu}A_{n,0}^{(\mu,\nu)}(k^2)}{\Gamma(\mu+\nu+2)Ps_n^{(\mu,\nu)}(0,k^2)Ps_n^{(\mu,\nu)}(\pi/2,k^2).}$$

A similar analysis for Case 1 yields the normalized eigenvectors

$$(4.25)\quad fe_{k,n}(\mathbf{x},t)=\frac{\delta(r-k)}{\sqrt{k}}\sin^{\nu+1/2}\theta\, Cs_n^{\nu+1/2}(\theta,k^2)$$

which again have definite parity and eigenvalues $\lambda_{2n}^\nu(k^2)=s+\beta-\frac{1}{4}$ and $\lambda_{2n+1}^\nu=s+\beta-\frac{1}{4}$ for $\pm$ parity states, respectively.

Here we have introduced a type of spheroidal wave function Cs_n^ν which is related to Gegenbauer polynomials through

$$(4.26)\quad Cs_{2n}^\nu(\theta,k^2)=\sum_{m=0}^{\infty}A_{2n,2m}^\nu(k^2)C_{2m}^{\nu+1/2}(\cos\theta),$$

where a similar relation holds for the odd parity functions with $2n$, $2m$ replaced by $2n+1$, $2m+1$, respectively. The Cs_n^ν are related to the usual spheroidal wave functions [7], [15] by

$$(4.27a)\quad \sin^{\nu+1/2}\theta Cs_{2n}^\nu(\theta;k^2)=2^\nu e^{-(i\pi\nu)/2}\Gamma(\nu+1)ps_\nu^{-\nu}(\cos\theta,k^2)$$

and

$$(4.27b)\quad \sin^{\nu+1/2}\theta Cs_{2n+1}^\nu(\theta;k^2)=2^\nu e^{-(i\pi\nu)/2}\Gamma(\nu+1)ps_{\nu+1}^{-\nu}(\cos\theta;k^2)$$

with the proper identification of the coefficients occurring in the expansions. Normalization conditions similar to (4.21) can be worked out from (4.25) and (4.26).

Now in the three variable model the eigenvectors of the transformed pair $(-iK_2, S_6+\frac{1}{2}S_1)$ are

$$(4.28)\quad Fe_{k,n}(\mathbf{x},t)=\frac{e^{-i\pi(1+\nu)/2}\sqrt{k}}{2t\sqrt{\pi i}}\gamma_n^\nu(k^2)\, e^{(it)/4[(\sinh^2\rho+\cos^2\sigma)+k^2]}\sin^{\nu+1/2}\sigma\sinh^{\nu+1/2}\rho$$
$$\cdot\, Cs_n^{\nu+1/2}(i\rho,k^2)Cs_n^{\nu+1/2}(\sigma,k^2).$$

The transform with the kernel (4.7a) will then give an integral formula closely related to but not identical with one given by Meixner and Schäfke [15, (33), p. 314], viz.,

$$\tag{4.29}\begin{aligned}\int_0^\pi d\theta\, e^{-(ik/2)\cosh\rho\cos\sigma\cos\theta}\sin^{\nu+1}\theta J_\nu\Big(\frac{k}{2}\sin\sigma\sinh\rho\sin\theta\Big)\, Cs_n^{\nu+1/2}(\theta,k^2)\\ =\frac{2\gamma_n^\nu(k^2)}{k}(\sinh\rho\sin\sigma)^\nu\, Cs_n^{\nu+1/2}(i\rho,k^2)\, Cs_n^{\nu+1/2}(\sigma,k^2).\end{aligned}$$

The normalization constant γ_n^ν can be computed as in (4.24)

Lc system. Both the *Lc* and *Lp* systems are separable only for Case 1. The eigenvalue equations in the two variable model are

$$i(2a\mathcal{P}_1+\mathcal{K}_2)f=\lambda f,\qquad (\mathcal{S}_1-i\mathcal{K}_2)f=\frac{k_2^2}{2}f$$

with $-\infty<\lambda<\infty$ $0<k_2<\infty$ and normalized generalized eigenfunctions

$$\tag{4.30} lc_{\lambda,k_2}(\mathbf{x})=\frac{\delta(x_2-k_2)}{2\sqrt{\pi|a|}}\exp-\frac{i}{2a}\Big(\lambda x_1+\frac{k_2^2}{4}x_1+\frac{x_1^3}{12}\Big).$$

Then in the three variable model the eigenfunctions of the pair (iK_2+2aiP_1, S_1-iK_2) with eigenvalues $(\lambda, k_2^2/2)$, respectively, are

$$\tag{4.31}\begin{aligned}Lc_{\lambda,k_2}(\mathbf{x},t)=\frac{a^{1/3}\,e^{\pm(i\pi/2)(1+\nu)}}{2\sqrt{i|a|t}}\sqrt{k_2v_2}\,J_\nu\Big(\frac{k_2|v_2|}{2}\Big)\exp i\Big[\frac{v_1^2+v_2^2}{4}t-\frac{v_1a}{2t}-\frac{\lambda}{t}-\frac{a^2}{12t^3}\Big]\\ \cdot\operatorname{Ai}\Big[a^{1/3}\Big(v_1+\frac{\lambda}{a}+\frac{k_2^2}{4a}\Big)\Big],\end{aligned}$$

where $x_1=v_1t+(a/t)$, $x_2=v_2t$ and Ai (z) is an Airy function [7].

Lp system. Here we have the eigenvalue problem for the pair $(i\mathcal{K}_2+2ai\mathcal{P}_1,\ \mathcal{S}_4+a(i\mathcal{K}_{-2}+\mathcal{S}_3))$ with eigenvalues $(\lambda,-\mu)$, respectively. The first operator yields the same functions as the previous case, whereas the second operator gives rise to the equation

$$\tag{4.32} h_{x_2x_2}-\Big(\frac{\mu}{2a}+\frac{\lambda}{4a^2}x_2^2+\frac{x_2^4}{16a^2}+\frac{\beta}{x_2^2}\Big)h=0.$$

This is the equation of the anharmonic oscillator with an inverse square potential. It also appears in the Stark effect problem in the hydrogen atom (see [21, vol. 2, p. 134]). The spectrum is discrete, simple and bounded from below [20], and so we assume the eigenvalues μ_n to be ordered as $\mu_0<\mu_1<\mu_2<\cdots$. The normalized eigenfunctions are written as

$$lp_{\lambda,n}(x)=\frac{e^{-(i/2a)(\lambda x_1+(x_1x_2/4)+x_1^3/12)}}{2\sqrt{\pi|a|}}h_n^{\nu,\theta}\left(x_2;\frac{\lambda}{4a^2};(4a)^{-2}\right). \tag{4.33}$$

Then in the three variable model corresponding to the pair $(iK_2+2aiP_1, S_4+a(iK_{-2}+S_3))$ with eigenvalues $(\lambda,-\mu)$, we have the generalized eigenfunctions

$$\begin{aligned}Lp_{\lambda,n}(\mathbf{x},t)=&\frac{e^{\mp(i\pi/2)(1+\nu)}\gamma^{\nu}_{n,a}(\lambda)}{t}\exp i\left[\frac{(v_1^2+v_2^2)^2}{16}t-\frac{a}{4t}(v_1^2-v_2^2)-\frac{\lambda}{k}-\frac{a^2}{12t^2}\right]\\&\cdot h_n^{\nu}(v_1;\lambda,a)h_n^{\nu}(iv_2;\lambda,-a),\end{aligned} \tag{4.34}$$

where $x_1=(t/2)(v_1^2-v_2^2)+(a/t)$, $x_2=v_1v_2t$ and $\gamma^{\nu}_{n,a}(\lambda)$ is a normalization constant.

Now upon writing the transformation (4.5) explicitly with the kernel (4.7a), we obtain the integral equation for anharmonic oscillator wave functions

$$\begin{aligned}&\int_0^{\infty}dy\,y^{1/2}J_{\nu}\left(\frac{v_1v_2}{2}y\right)\mathrm{Ai}\left[a^{1/3}\left(\frac{v_1^2-v_2^2}{2}-\frac{\lambda}{a}+\frac{y^2}{4a}\right)\right]h_n^{\nu}\left(y;\frac{\lambda}{4a^2},(4a)^{-2}\right)\\&=\frac{2\sqrt{i|a|}}{a^{1/3}}\gamma^{\nu}_{\lambda,n}(a)(v_1v_2)^{-1/2}h_n^{\nu}(v_1;\lambda,a)h_n^{\nu}(iv_2;\lambda,-a).\end{aligned} \tag{4.35}$$

We remark that at best we can express the normalization constant $\gamma^{\nu}_{\lambda,n}(a)$ as an integral of a product of h_n^{ν} functions.

Oc system. For Case 2 we have the pair $(i(\mathcal{K}_{-2}-\mathcal{K}_2), \mathcal{S}'_3+\mathcal{S}'_1)$ with eigenvalues (λ,λ') and orthonormal eigenvectors

$$\begin{aligned}oc_{n,n_2}(\mathbf{x})=&2^{-(\mu+\nu-1)/2}\left[\frac{n_1!n_2!}{\Gamma(n_1+\mu+1)\Gamma(n_2+\nu+1)}\right]^{1/2}e^{-(x_1^2+x_2^2)/4}x_1^{\mu+1/2}x_2^{\nu+1/2}\\&\cdot L^{\mu}_{n_1}\left(\frac{x_1^2}{2}\right)L^{\nu}_{n_2}\left(\frac{x_2^2}{2}\right),\end{aligned} \tag{4.36}$$

where $\lambda-\lambda'=2n_1+\mu+1$, $\lambda+\lambda'=2n_2+\nu+1$, and $L_n^{\mu}(z)$ is an associated Laguerre function [7]. Then for the transformed pair $(-i(K_{-2}-K_2), S'_3+S'_1)$ with the respective eigenvalues, we find

$$\begin{aligned}Oc_{n_1n_2}(\mathbf{x},t)=&\frac{(n_1!n_2!)^{1/2}(1+it)^{-1}((1-it)/(1+it))^{n_1+n_2+(\mu+\nu+1)/2}}{2^{(\mu+\nu-1)/2}[\Gamma(n_1+\mu+1)\Gamma(n_2+\nu+1)]^{1/2}}\\&\cdot v_1^{\mu+1/2}v_2^{\nu+1/2}e^{-((v_1^2+v_2^2)/4)(1-it)}L^{\mu}_{n_1}\left(\frac{v_1^2}{2}\right)L^{\nu}_{n_2}\left(\frac{v_2^2}{2}\right),\end{aligned} \tag{4.37}$$

where $x_1=\sqrt{1+t^2}\,v_1$, $x_2=\sqrt{1+t^2}\,v_2$.

Similarly for Case 1 we find

$$oc_{n_1n_2}(\mathbf{x})=\frac{(n_2!)^{1/2}x_2^{\nu+1/2}\,e^{-(x_1^2+x_2^2)/4}}{[n_1!\Gamma(n_2+\nu+1)]^{1/2}2^{(n_1+\nu)/2-(1/4)}\pi^{1/4}}H_{n_1}\left(\frac{x_1}{\sqrt{2}}\right)L_{n_2}^{\nu}\left(\frac{x_2^2}{2}\right), \tag{4.38}$$

where $H_n(z)$ is a Hermite polynomial [7]. In the three variable model then we have

$$\begin{aligned} Oc_{n_1n_2}(\mathbf{x},t)&=\frac{(n_2!)^{1/2}((1-it)/(1+it))^{(n_1/2)+n_2+(\nu/2)+(1/4)}v_2^{\nu+1/2}}{[n_1!\Gamma(n_2+\nu+1)]^{1/2}2^{(n_1+\nu)/2-(3/4)}\pi^{1/4}}\frac{e^{-((v_1^2+v_2^2)/4)(1-it)}}{(1+it)}\\ &\quad\cdot H_{n_1}\left(\frac{v_1}{\sqrt{2}}\right)L_{n_2}^{\nu}\left(\frac{v_2^2}{2}\right). \end{aligned} \tag{4.39}$$

Or system. In Case 2 we have the pair $(i(\mathcal{K}_{-2}-\mathcal{K}_2),\ \mathcal{S}_4')$ with eigenvalues $(\lambda,-s)$, respectively. The eigenvalue problem for $\mathcal{S}_4'$ was solved for the *Fr* system, and the operator $i(\mathcal{K}_{-2}-\mathcal{K}_2)$ in radial coordinates with $\mu,\nu\geqq 1$ as indicated previously has a unique self-adjoint extension. In all we have the orthonormal eigenfunctions

$$\begin{aligned} Or_{n,m}(\mathbf{x})&=\frac{(n!)^{1/2}N_m(\mu,\nu)\,e^{-r^2/4}r^{2m+\mu+\nu+1}}{[\Gamma(n+2m+\mu+\nu+2)]^{1/2}2^{m+1+(\mu+\nu)/2}}\\ &\quad\cdot L_n^{(2m+\mu+\nu+1)}(r^2/2)\sin^{\nu+1/2}\theta\cos^{\mu+1/2}\theta P_m^{(\nu,\mu)}(\cos 2\theta), \end{aligned} \tag{4.40}$$

where $n, m=0,1,\cdots,\lambda=2n+2m+\mu+\nu+2$ and $N_m(\mu,\nu)$ is given by (4.11b). Then for the transformed pair $(i(K_{-2}-K_2), S_4')$, we calculate the integral explicitly using the identity (4.13) to obtain

$$\begin{aligned} Or_{n,m}(\mathbf{x},t)&=N_m^n(\mu,\nu)\left(\frac{1-it}{1+it}\right)^{n+m+(\mu+\nu+1)/2}(1+it)^{-1}\cos^{\mu+1/2}\theta\sin^{\nu+1/2}\theta\\ &\quad\cdot P_m^{(\nu,\mu)}(\cos 2\theta)\,e^{-(v_1^2/4)(1-it)}v_1^{2m+\mu+\nu+1}L_n^{(2m+\mu+\nu+1)}\left(\frac{v_1^2}{2}\right), \end{aligned} \tag{4.41a}$$

where $v_1=r/\sqrt{1+t^2}$ and

$$N_m^n(\mu,\nu)=\frac{(n!)^{1/2}N_m(\mu,\nu)}{2^{m+2+(\mu+\nu)/2}[\Gamma(n+2m+\mu+\nu+2)]^{1/2}}. \tag{4.41b}$$

Similarly for Case 1 we find

$$\begin{aligned} or_{n,m}^{\pm}(\mathbf{x})&=\frac{(n!)^{1/2}N_m(v)\,e^{-r^2/4}r^{m+\nu+1/2}}{[\Gamma(n+m+v+3/2)]^{1/2}2^{(m+\nu)/2}}L_n^{(m+\nu+1/2)}(r^2/2)\\ &\quad\cdot\sin^{\nu+1/2}\theta C_m^{(\nu+1/2)}(\cos\theta), \end{aligned} \tag{4.42}$$

where $n, m = 0, \cdots, \lambda = 2n+m+\nu+3/2$ and $\pm$ is taken for $m \binom{\text{even}}{\text{odd}}$, respectively. Again transforming to the three variable model we have using Gegenbauer's finite integral [22]

$$Or_{n,m}(\mathbf{x}, t) = N_m^n(\nu)\left(\frac{1-it}{1+it}\right)^{n+(m+\nu)/2+1/4} (1+it)^{-1} \sin^{\nu+1/2}\theta\, C_m^{\nu+1/2}(\cos\theta)$$

$$\cdot e^{-(v_1/4)(1-it)} v_1^{m+\nu+1/2} L_n^{(m+\nu+1/2)}(v_{1/2}^2), \tag{4.43a}$$

where again $v_1 = r\sqrt{1+t^2}$ and

$$N_m^n(\nu) = \frac{(n!)^{1/2} N_m(\nu)}{2^{(m+\nu)/2+1}[\Gamma(n+m+\nu+3/2)]^{1/2}}. \tag{4.43b}$$

Oe system. For Case 2 we have the pair $(i(\mathcal{K}_{-2}-\mathcal{K}_2), \mathcal{S}_4'-(\mathcal{S}_3'+\mathcal{S}_1'))$ with eigenvalues (λ, η), respectively. This gives rise to the eigenvectors

$$oe_{n,\eta}(\mathbf{x}) = hp_{\lambda,\eta}^{(\mu,\nu)}(i\rho, \tfrac{1}{2})\, hp_{\lambda,\eta}^{(\mu,\nu)}(\sigma, \tfrac{1}{2}), \tag{4.44}$$

where $x_1 = \cosh\rho\cos\sigma$, $x_2 = \sinh\rho\sin\sigma$, $0 < \sigma < \pi/2$ and $hp_{\lambda,\eta}^{(\mu,\nu)}(\sigma, \frac{1}{2})$ is a solution of the equation

$$h_{\sigma\sigma} + \left[\left(\eta + \frac{\lambda}{2} - \frac{1}{32}\right) - \frac{\lambda}{2}\cos 2\sigma + \frac{1}{32}\cos 4\sigma - \frac{2\alpha}{1+\cos 2\sigma} - \frac{2\beta}{1-\cos 2\sigma}\right] h = 0. \tag{4.45}$$

This equation is a generalization of both the Whittaker–Hill equation (hence Ince's equation) [1] and the generalized spheroidal wave equation (4.18). We know from the previous analysis that $\lambda = 2n+\mu+\nu+2$, $n = 0, 1, \cdots$ and the eigenvalues η_m will form a discrete set. A detailed analysis of this equation will be given in a forthcoming work using an appropriate Hilbert space of analytic functions in analogy with VII. We know however, from our general analysis here that in the three-variable model the pair $(i(K_{-2}-K_2), S_4'-S_3'-S_1')$ has the corresponding eigenfunctions

$$Oe_{n,m}(\mathbf{x}, t) = \gamma_n^{(\mu,\nu)}(\eta_m) e^{it(\sinh^2 v_1+\cos^2 v_2)/4}\left(\frac{1-it}{1+it}\right)^{(n+\mu+\nu+1)/2} (1+it)^{-1} hp_{n,m}^{(\mu,\nu)}(iv_1, \tfrac{1}{2})$$

$$\cdot hp_{n,m}^{(\mu,\nu)}(v_2, \tfrac{1}{2}), \tag{4.46}$$

where $x_1 = \sqrt{1+t^2}\cosh v_1 \cos v_2$, $x_2 = \sqrt{1+t^2}\sinh v_1, \sin v_2$ and $\gamma_\lambda^{(\mu,\nu)}(\eta)$ is a normalization constant which can be calculated by evaluating the transform (4.7b) at special values of the arguments. In this case the transform can only be written as a double integral so we omit writing it explicitly. Since Case 1 reduces to a special case (i.e., $\alpha = 0$) with σ extended to $0 < \sigma < \pi$ and we have no further need of the functions in this article, we omit writing them explicitly.

Rc system. For Case 2 we have the pair $((-i\mathcal{D} + \mathcal{S}_2')/2, (-i\mathcal{D} - \mathcal{S}_2')/2)$ with eigenvalues (λ_1, λ_2) and generalized eigenfunctions given by the two-dimensional Mellin transform kernel

$$rc_{\lambda_1,\lambda_2}(\mathbf{x}) = \frac{1}{2\pi} x_1^{i\lambda_1 - 1/2} x_2^{i\lambda_2 - 1/2} \tag{4.47}$$

with $-\infty < \lambda_1, \lambda_2 < \infty$. Then in the three variable model the eigenfunctions of the pair $((-iD + S_2')/2, (-iD - S_2')/2)$ are

$$\begin{aligned} Rc_{\lambda_1\lambda_2}(\mathbf{x}, t) = {} & \frac{e^{\mp i\pi(1+(\mu+\nu)/2)} e^{-\pi(\lambda_1+\lambda_2)/4}}{\pi\Gamma(\mu+1)\Gamma(\nu+1)} \Gamma\left(\frac{i\lambda_1+\mu+1}{2}\right)\Gamma\left(\frac{i\lambda_2+\nu+1}{2}\right) 2^{i(\lambda_1+\lambda_2)-1} \\ & \cdot \frac{t^{-1/2+i(\lambda_1+\lambda_2)/2} e^{i(v_1^2+v_2^2)/8}}{(v_1 v_2)^{1/2}} M_{i\lambda_1/2,\mu/2}\left(\frac{iv_1^2}{4}\right) M_{i\lambda_2/2,\mu/2}\left(\frac{iv_2^2}{4}\right), \end{aligned} \tag{4.48}$$

where $x_1 = v_1 t^{1/2}$, $x_2 = v_2 t^{1/2}$ and $\mp$ indicate $t \gtrless 0$, respectively. For Case 1 the eigenfunctions of the pair $((-i\mathcal{D} + \mathcal{S}_2)/2, (-i\mathcal{D} - \mathcal{S}_2)/2$ are

$$rc^{\varepsilon}_{\lambda_1\lambda_2'}(\mathbf{x}) = \frac{1}{2\pi} x_{1\varepsilon}^{i\lambda_1 - 1/2} x_2^{i\lambda_2 - 1/2}, \tag{4.49}$$

where $-\infty < \lambda_1, \lambda_2 < \infty$ and $\varepsilon = \pm$ with

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0, \\ 0, & x < 0, \end{cases} \qquad x_-^\lambda = \begin{cases} 0, & x > 0, \\ (-x)^\lambda, & x < 0. \end{cases}$$

Transforming to the three variable model we find

$$\begin{aligned} R^\varepsilon c_{\lambda_1\lambda_2}(\mathbf{x}, t) = {} & \frac{e^{\mp i\pi(1+\nu)/2} e^{-\pi(\lambda_1+\lambda_2)/4 + i\pi/8}}{\pi^{3/2}\Gamma(\nu+1) v_2^{1/2}} 2^{i\lambda/2 + i\lambda_2 - 7/4} \Gamma(i\lambda_1 + 1/2) \\ & \cdot \Gamma\left(\frac{i\lambda_2+\nu+1}{2}\right) t^{i(\lambda_1+\lambda_2)/2} e^{i(v_1^2+v_2^2)/8} D_{-i\lambda_1-1/2}\left(\frac{\varepsilon e^{i3\pi/4} v_1}{\sqrt{2}}\right) \\ & \cdot M_{i\lambda_2/2,\nu/2}\left(\frac{iv_2^2}{4}\right), \end{aligned} \tag{4.50}$$

where $D_\lambda(z)$ is a parabolic cylinder function [7].

Rr system. In Case 2 the pair $(-i\mathcal{D}, \mathcal{S}_4')$ with eigenvalues $(\lambda, -s)$ gives rise to the generalized eigenfunctions

$$rr_{\lambda,m}(\mathbf{x}) = \frac{N_m(\mu, \nu)}{\sqrt{2\pi}} r^{i\lambda-1} \sin^{\nu+1/2}\theta \cos^{\mu+1/2}\theta\, P_m^{(\nu,\mu)}(\cos 2\theta) \tag{4.51}$$

with $s+\alpha+\beta = (2m+\mu+\nu+1)^2$ and $-\infty < \lambda < \infty$. Transforming to the three-variable model the eigenfunctions of $(-iD, S_4')$ are

$$\begin{aligned} Rr_{\lambda,m}(\mathbf{x}, t) = {}& \frac{e^{\mp i(m+1+(\mu+\nu)/2)}\, e^{-\pi\lambda/4}}{\sqrt{\pi}\,\Gamma(2m+\mu+\nu+2)} N_m(\mu, \nu) 2^{i\lambda-1/2} \\ & \cdot \Gamma\left(\frac{i\lambda+\mu+\nu}{2}+m+1\right) t^{(i\lambda-1)/2} \sin^{\nu+1/2}\theta \cos^{\mu+1/2}\theta\, P_m^{(\nu,\mu)}(\cos 2\theta) \\ & \cdot \frac{e^{iv^2/8}}{v} M_{i\lambda/2, m+(\mu+\nu+1)/2}\left(\frac{iv^2}{4}\right), \end{aligned} \tag{4.52}$$

where $r = vt^{1/2}$. Similarly for Case 1 the eigenfunctions of the pair $(-i\mathcal{D}, \mathcal{S}_6)$ with eigenvalues $(\lambda, -s)$ are

$$rr_{\lambda,n}(\mathbf{x}) = \frac{N_m(\nu)}{\sqrt{2\pi}} r^{i\lambda-1} \sin^{\nu+1/2}\theta\, C_m^{(\nu+1/2)}(\cos\theta) \tag{4.53}$$

with $-\infty < \lambda < \infty$, even, odd parity corresponding to m even, odd, respectively, and $s+\beta = (m+\nu+1/2)^2$. Upon transforming to the three variable model we find

$$\begin{aligned} Rr_{\lambda,m}(\mathbf{x}, t) = {}& \pi^{-1/2} e^{\mp(i\pi/2)(m+\nu+3/2)} e^{-\pi\lambda/4} N_m(\nu) 2^{i\lambda-1/2} \Gamma\left(\frac{i\lambda+\nu+m}{2}+\frac{3}{4}\right) t^{i\lambda/2} \\ & \cdot \sin^{\nu+1/2}\theta\, C_m^{(\nu+1/2)}(\cos\theta) \frac{e^{iv^2/8}}{v} M_{i\lambda/2, m+v/2+1/4}\left(\frac{iv^2}{4}\right), \end{aligned} \tag{4.54}$$

where $\pm$ indicates $t \gtrless 0$, respectively, and $r = vt^{1/2}$.

Re system. For Case 2 we consider the pair $(-i\mathcal{D}, \mathcal{S}_4 - \xi\mathcal{S}_2')$ with eigenvalues $(\lambda, -\eta)$. Here we consider the arbitrary real parameter ξ for purposes of generality. As before, the first operator gives the Mellin transform kernel in the variable r, whereas the second equation takes the form

$$\begin{aligned} 0 = g_{\theta\theta} + i\xi \sin 2\theta g_\theta + \Big[& (\eta+\alpha+\beta+\xi(\lambda+i)\cos 2\theta) - \frac{2\alpha}{1+\cos 2\theta} \\ & - \frac{2\beta}{1-\cos 2\theta}\Big] g. \end{aligned} \tag{4.55}$$

Notice the following special cases of (4.55): When $\alpha=\beta=0$ we have Ince's equation [1] treated in VI; when $\xi=0$ (4.55) reduces to the Jacobi equation, and in the limit $\lambda\to\infty$, $\xi\to 0$ such that $\lambda\xi$ is fixed (4.55) becomes the generalized spheroidal wave equation (4.18). The spectral analysis of (4.55) is similar to that of the Jacobi equation. For α, β restricted as previously, there is a unique self-adjoint extension with a simple discrete spectrum with eigenvalues $\eta_n^{(\mu,\nu)}(\lambda,\xi)$ assumed ordered as $\eta_0<\eta_1<\eta_2<\cdots$. The normalized solutions of (4.55) are denoted by $\cos^{\mu+1/2}\theta\sin^{\nu+1/2}\theta Ge_n^{(\nu,\mu)}(\theta,\xi,\lambda)$ and we call the $Ge_n^{(\nu,\mu)}$ associated Ince functions. It is convenient to express these in a series of Jacobi polynomials, viz.,

$$Ge_n^{(\mu,\nu)}(\theta,\xi,\lambda)=\sum_{r=0}^{\infty} D_{n,m}^{(\mu,\nu)}(\xi,\lambda)P_m^{(\mu,\nu)}(\cos 2\theta), \tag{4.56}$$

where a normalization condition similar to (4.21) holds. It is a straightforward calculation to show that the coefficients $D_{n,m}^{(\mu,\nu)}$ satisfy a three term recursion relation; however, it is somewhat complicated and since we make no further use of it we do not give it explicitly.

Thus the generalized eigenfunctions are

$$re_{\lambda,n}(\mathbf{x})=\frac{r^{i\lambda-1}}{\sqrt{2\pi}}\cos^{\mu+1/2}\theta\sin^{\nu+1/2}\theta\, Ge_n^{(\nu,\mu)}(\theta,\tfrac14,\lambda), \tag{4.57}$$

where we have chosen $\xi=\frac14$ for convenience. Then in the three variable model we find the solutions

$$\begin{aligned} Re_{\lambda,n}(\mathbf{x},t)&=e^{\mp i\pi(1+(\mu+\nu)/2)}K_n^{(\nu,\mu)}(\lambda)t^{(i\lambda-1)/2}(\cosh v_1\cos v_2)^{\mu+1/2}(\sinh v_1\sin v_2)^{\nu+1/2}\\ &\quad\cdot Ge_n^{(\nu,\mu)}(iv_1,\tfrac14,\lambda)\,Ge_n^{(\nu,\mu)}(v_2,\tfrac14,\lambda), \end{aligned} \tag{4.58}$$

where $x_1=t^{1/2}\cosh v_1\cos v_2$, $x_2=t^{1/2}\sinh v_1\sin v_2$, and $K_n^{(\mu,\nu)}(\lambda)$ is a normalization constant. Using the transform (4.7b) explicitly we obtain the integral equation

$$\begin{aligned} &\int_0^{\pi/2} d\theta'\cos^{2\mu+1}\theta'\sin^{2\nu+1}\theta' A_\lambda^{(\mu,\nu)}(v_1,v_2;\theta')Ge_n^{(\nu,\mu)}(\theta',\tfrac14,\lambda)\\ &\qquad=K_n^{(\nu,\mu)}(\lambda)e^{-i(\sinh^2 v_1+\cos^2 v_2)/4}\,Ge_n^{(\nu,\mu)}(iv_1,\tfrac14,\lambda)Ge_n^{(\nu,\mu)}(v_2,\tfrac14,\lambda), \end{aligned} \tag{4.59a}$$

where the kernel $A(v_1,v_2;\theta')$ is given by

$$\begin{aligned} A_\lambda^{(\mu,\nu)}(v_1,v_2;\theta)&=\frac{e^{-\pi\lambda/4}}{2^{\mu+\nu-i\lambda+3/2}\Gamma(\nu+1)\sqrt{\pi}}\sum_{m=0}^{\infty}\Gamma(m+1+(\mu+\nu+i\lambda)/2)\\ &\quad\cdot\left[\frac{-i\cosh^2 v_1\cos^2 v_2\cos^2\theta}{4}\right]^m\\ &\quad\cdot F(-m,-\mu-m;\nu+1;th^2 v_1\tan^2 v_2\tan^2\theta) \end{aligned} \tag{4.59b}$$

and $F(a,b;c;z)$ is a hypergeometric function [7] which in the above case is a polynomial of its argument. We can compute the constant $K_n^{(\nu,\mu)}(\lambda)$ by evaluating the above expression at $v_1=0$, $v_2=\pi/2$.

A similar analysis for Case 1 gives the normalized functions which are solutions of (4.55) with $\alpha = 0$ and $0 < \theta < \pi$:

$$re_{\lambda,n}(\mathbf{x}) = \frac{r^{i\lambda-1}}{\sqrt{2\pi}} \sin^{\nu+1/2} \theta \, Ge_n^{\nu}(\theta, \tfrac{1}{4}, \lambda), \tag{4.60}$$

where the functions with $2n$, $2n+1$ are even and odd, respectively, under $\theta \to \pi - \theta$. In this case we have an expansion in a series of Gegenbauer polynomials, viz.,

$$Ge_{2n}^{\nu}(\theta, \tfrac{1}{4}, \lambda) = \sum_{r=0}^{\infty} D_{2n,2m}^{\nu}(\lambda) C_{2m}^{(\nu+1/2)}(\cos\theta) \tag{4.61}$$

with a similar relation holding for odd parity functions with $2n$ and $2m$ replaced by $2n+1$, $2m+1$, respectively. Again the coefficients $D_{n,m}^{\nu}$ satisfy a three term recursion relation and a normalization condition. Again in the three variable model we find the eigenfunctions

$$\begin{aligned} Re_{\lambda,n}(\mathbf{x}, t) = {} & e^{\mp(i\pi/2)(1+\nu)} K_n^{\nu}(\lambda) t^{(i\lambda-1)/2} (\sinh v_1 \sin v_2)^{\nu+1/2} Ge_n^{\nu}(iv_1, \tfrac{1}{4}, \lambda) \\ & \cdot Ge_n^{\nu,\pm}(v_2, \tfrac{1}{4}, \lambda). \end{aligned} \tag{4.62}$$

This gives rise to the integral equation

$$\begin{aligned} & \int_0^{\pi/2} d\theta \sin^{2\nu+1} \theta A^{(-1/2,\nu)}(v_1, v_2; \theta) Ge_{2n}^{\nu}(\theta, \tfrac{1}{4}, \lambda) \\ & \qquad = K_{2n}^{\nu}(\lambda) Ge_{2n}^{\nu}(iv_1, \tfrac{1}{4}, \lambda) Ge_{2n}^{\nu}(v_2, \tfrac{1}{4}, \lambda), \end{aligned} \tag{4.63a}$$

whereas the odd functions satisfy

$$\begin{aligned} & \int_0^{\pi/2} d\theta \sin^{2\nu+1}\theta \cos\theta A^{(1/2,\nu)}(v_1, v_2; \theta) Ge_{2n+1}^{\nu}(\theta, \tfrac{1}{4}, \lambda) \\ & \quad = -K_{2n+1}^{\nu}(\lambda)(\cosh v_1 \cos v_2)^{-1} e^{-1/4(\sinh^2 v_1 + \cos^2 v_2)} Ge_{2n+1}^{\nu}(iv_1, \tfrac{1}{4}, \lambda) \\ & \quad \cdot Ge_{2n+1}^{\nu}(v_2, \tfrac{1}{4}, \lambda). \end{aligned} \tag{4.63b}$$

Again the constant K_n^{ν} can be computed by evaluating the integrals at special values of v_1 and v_2.

5. Overlap functions and expansion theorems. In this section we compute the overlap functions between the different bases given in the last section and apply them to the derivation of various expansion theorems. We concentrate on giving only those overlap functions which are readily calculable in closed form. Moreover only those expansion formulas which appear to be new and which can be written as a single sum or integral are given explicitly; the double sum and integral expansions, however, are straightforward to calculate and only one such expansion is written explicitly.

The overlap functions are always easiest to calculate on the point on each orbit which corresponds to the two variable model where the generalized eigenfunctions are written as $ab_{\lambda_1,\lambda_2}(\mathbf{x})$, where $a = f, l, o, r$ and $b = c, r, p, e$ as discussed previously. The important point is that the overlap functions are invariant under an arbitrary transformation of the symmetry group G, viz.,

$$(5.1)\qquad (U_g a'b'_{\lambda_1',\lambda_2'}, U_g ab_{\lambda_1,\lambda_2}) = (a'b'_{\lambda_1',\lambda_2'}, ab_{\lambda_1,\lambda_2}).$$

In particular when $U_g = e^{+\mathcal{K}_{-2}}$, the functions $U_g ab_{\lambda_1,\lambda_2}(x, t)$ are written explicitly in the last section. Thus using the spectral theorem we can derive expansion formulas written in the form

$$(5.2a)\qquad A'b'_{\lambda_1',\lambda_2'}(v, t) = \int d\sigma_1(\lambda_1)\, d\sigma_2(\lambda_2)(ab_{\lambda_1,\lambda_2}, a'b'_{\lambda_1',\lambda_2'})Ab_{\lambda_1,\lambda_2}(x, t),$$

and conversely

$$(5.2b)\qquad Ab_{\lambda_1,\lambda_2}(v, t) = \int d\sigma_1(\lambda_1')\, d\sigma_2(\lambda_2')(a'b'_{\lambda_1',\lambda_2'}, ab_{\lambda_1\lambda_2})A'b'_{\lambda_1',\lambda_2'}(x, t).$$

It should be mentioned that the above formulas in general are relations between generalized eigenfunctions, so the corresponding expansion formulas are to be interpreted in the generalized sense. The classical type expansion formulas, however, can always be obtained by an appropriate analytic continuation.

One further point which should be clear is that in all cases where the right-hand side of (5.1) is evaluated, (5.1) provides us with the evaluation of the double integral on the left-hand side which in many cases is far from trivial to compute directly. We do not, however, write such integrals explicitly. Even in the case when the right-hand side cannot be found from known results, in most cases it can be reduced to a single integral. Thus (5.1) gives an evaluation of a double integral in terms of a single one.

We begin with the overlap functions relating the Fc system to an arbitrary system Ab (i.e., Fc-Ab). Owing to simplicity of the functions $fc_{k_1,k_2}(x)$ of (4.9), all such overlap functions are calculable, viz.,

$$(5.3)\qquad (fc_{k_1,k_2}, ab_{\lambda_1\lambda_2}) = ab_{\lambda_1,\lambda_2}(\mathbf{k})$$

and can be obtained explicitly from the previous section. We will only repeat writing them explicitly in the case that the corresponding expansions appear to be new. For future convenience the vector $\mathbf{k}$ is written with components in either Cartesian or polar form as $k_1 = k\cos\phi$, $k_2 = k\sin\phi$. A simple example of an expansion using (5.3) is the Fc-Fr system which leads directly to Bateman's expansion formula involving Bessel functions and Jacobi polynomials; see [22, p. 370].

Fr-Fp.

$$(5.4)\qquad \begin{aligned}(fr_{k',m}, fp_{k,s}) &= \frac{\delta(k-k')N_m(\nu)2^\nu\Gamma\left(-is+\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(is+\frac{\nu}{2}+\frac{1}{2}\right)\Gamma(m+2\nu+1)}{\sqrt{\pi}\, m!\,\Gamma(2\nu+1)\Gamma(\nu+1)}\\ &\quad\cdot {}_3F_2\left[\begin{matrix}-m, & m+2\nu+1, & -is+(\nu-1)/2\\ \nu+1, & \nu+1 & \end{matrix};1\right],\end{aligned}$$

where $N_n(\nu)$ is given by (4.14b) and ${}_pF_q$ is a generalized hypergeometric function [7]. Equation (5.4) leads directly to the expansion

$$\sum_{m=0}^{\infty} e^{-\pi m/2}\frac{N_m^2(\nu)}{m!}(2\nu+1)_m\, {}_3F_2\begin{bmatrix} -m, & m+2\nu+1, & -is+(\nu-1)/2 & ;1 \\ \nu+1, & \nu+1 & \end{bmatrix}$$

(5.5)
$$\cdot\, C_m^{\nu+1/2}\left(\frac{v_1^2-v_2^2}{v_1^2+v_2^2}\right)J_{m+\nu+1/2}\left(\frac{k(v_1^2+v_2^2)}{4}\right)$$

$$=\frac{(v_1^2+v_2^2)^{\nu+1/2}M_{is,\nu/2}\left(\frac{ikv_1^2}{2}\right)M_{is,\nu/2}\left(\frac{-ikv_2^2}{2}\right)}{\Gamma(\nu+1)2^{2\nu+1/2}k^{1/2}(v_1v_2)^{\nu+1}},$$

where $(a)_n$ is Pochhammer's symbol [7]. From the inverse expansion we find equation (6) [5, p. 158].

Fr-Fe.

(5.6)
$$(fr_{k',m}, fe_{k,n})=\delta(k-k')\frac{A_{n,m}^{(\nu,\mu)}(k)}{N_m(\mu,\nu)}.$$

This leads directly to the expansions

$$\sum_{m=0}^{\infty} A_{n,m}^{(\nu,\mu)}(k^2)P_m^{(\nu,\mu)}(\cos 2\theta)J_{2m+\mu+\nu+1}\left(\frac{k}{2}\sqrt{\sinh^2\rho+\cos^2\sigma}\right)$$

(5.7)
$$=\gamma_n^{(\nu,\mu)}(k^2)\left(\frac{\sinh\rho\sin\sigma^{\nu+1/2}}{\sin\theta}\right)$$

$$\cdot\left(\frac{\cosh\rho\cos\sigma}{\cos\theta}\right)^{\mu+1/2}Ps_n^{(\nu,\mu)}(i\rho, k^2)Ps_n^{(\nu,\mu)}(\sigma, k)$$

and

$$\sum_{m=0}^{\infty}\gamma_n^{(\nu,\mu)}(k^2)A_{n,m}^{(\nu,\mu)}(k^2)Ps_n^{(\nu,\mu)}(i\rho, k^2)Ps_n^{(\nu,\mu)}(\sigma, k^2)$$

(5.8)
$$=N_m^2(\mu,\nu)\left(\frac{\sin\theta}{\sinh\rho\sin\sigma}\right)^{\nu+1/2}\left(\frac{\cos\theta}{\cosh\rho\cos\sigma}\right)^{\mu+1/2}$$

$$\cdot\, P_m^{(\nu,\mu)}(\cos 2\sigma)J_{2m+\mu+\nu+1}\left(\frac{k}{2}\sqrt{\sinh^2\rho+\cos^2\sigma}\right),$$

where here $\tan\theta=\tanh\rho\tan\sigma$. Alternatively we can derive (5.7) from (4.13), (4.20) and (4.23).

Fc-Fe.

(5.9)
$$(fc_{k_1,k_2}, fe_{k',n})=fe_{k',n}(\mathbf{k})$$

given explicitly by (4.19). This leads to the expansion

$$\sum_{n=0}^{\infty} \gamma_n^{(\nu,\mu)}(k^2) Ps_n^{(\nu,\mu)}(i\rho, k^2) Ps_n^{(\nu,\mu)}(\sigma, k^2) Ps_n^{(\nu,\mu)}(\phi, k^2)$$

$$(5.10)\qquad = \frac{k}{2}(\sin\phi \sinh\rho \sin\sigma)^{-\nu}(\cos\phi\cosh\rho\cos\sigma)^{-\mu} J_\mu\left(\frac{k}{2}\cos\phi\cosh\rho\cos\sigma\right) \cdot J_\nu\left(\frac{k}{2}\sin\phi\sinh\rho\sin\sigma\right).$$

We remark that (5.10) can be obtained by multiplying (5.8) by a Jacobi polynomial, summing over m and using Bateman's expansion [22]. Similar expressions can be found for the Case 1 basis functions, which are actually special cases of the above.

Fr-Lc.

$$(5.11a)\qquad (fr_{k,2m} lc_{\lambda,k_2}) = \frac{\sqrt{2} N_{2m}(\nu)\sin^{\nu+1/2}\phi}{\sqrt{\pi}|a|k|\cos\phi|} C_{2m}^{\nu+1/2}(\cos\phi) \cdot \cos\left[\frac{1}{2a}\left(\lambda k_1 + \frac{k_2^2 k_1}{4} + \frac{k_1^3}{12}\right)\right] \quad \text{for } k^2 = k_1^2 + k_2^2,$$

$$(5.11b)\qquad (fr_{k,2m+1}, lc_{\lambda,k_2}) = \frac{\sqrt{2} N_{2m+1}(\nu)\sin^{\nu+1/2}\phi}{i\sqrt{\pi}|a|k|\cos\phi|} C_{2m+1}^{\nu+1/2}(\cos\phi) \cdot \sin\left[\frac{1}{2a}\left(\lambda k_1 + \frac{k_2^2 k_1}{4} + \frac{k_1^3}{12}\right)\right] \quad \text{for } k^2 = k_1^2 + k_2^2.$$

When $k^2 \neq k_1^2 + k_2^2$ the above overlaps vanish. These overlap functions lead to integrals given in the Bateman project, equations (12) and (13) of [8, vol. 2, p. 44], while the inverse expansion collapses by using a special case of Bateman's expansion [22].

Fp-Lc.

$$(5.12)\qquad (fp_{k,s}, lc_{\lambda,k_2}) = \begin{cases} \dfrac{\sqrt{k}}{\pi\sqrt{|a|}\,|k_1|} \operatorname{Re}\left[(1+\cos\phi)^{is-1/4}(1-\cos\phi)^{-is-1/4} \cdot \exp\left[-\dfrac{i}{2a}\left(\lambda k_1 + \dfrac{k_2^2}{4}k_1 + \dfrac{k_1^3}{12}\right)\right]\right] & \text{for } k_1^2 + k_2^2 = k^2, \\ 0 & \text{otherwise,} \end{cases}$$

where Re $[z]$ indicates the real part of z. From (5.12) we obtain the integral

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)}} J_\nu\left(\frac{kv_1v_2}{2}x\right)\cos\left[\ln\left(\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right)^s - \frac{k}{4}(v_1^2-v_2^2)\sqrt{1-x^2}\right] = \frac{\left|\Gamma\left(is+\frac{\nu+1}{2}\right)\right|^2}{2k[\Gamma(1+\nu)]^2 v_1v_2} M_{is,\nu/2}\left(\frac{ik}{2}v_1^2\right) M_{is,\nu/2}\left(-\frac{ik}{2}v_2^2\right) \tag{5.13}$$

while the converse expansion reduces, using (5), of [5, p. 158] to the integral representation of an Airy function.

Lc-Lp.

$$(lc_{\lambda',k_2}, lp_{\lambda,n}) = \frac{\delta(\lambda-\lambda')}{\pi\sqrt{2}} h_n^\nu\left(k_2, \frac{\lambda}{2a^2}, (4a)^{-2}\right). \tag{5.14}$$

We obtain the expansion formula, for anharmonic oscillator wave functions

$$\sum_{n=0}^\infty \gamma_{n,a}^\nu(\lambda) h_n^\nu\left(k_2; \frac{\lambda}{2a^2}, (4a)^{-2}\right) h_n^\nu(v_1; \lambda, a) h_n^\nu(iv_2; \lambda, -a) = \frac{a^{1/3}}{\sqrt{2|a|}} e^{-i\pi/4}(k_2v_1v_2)^{1/2} J_\nu\left(\frac{k_2v_1v_2}{2}\right) \mathrm{Ai}\left[a^{1/3}\left(\frac{v_1^2-v_2^2}{2}+\frac{\lambda}{a}+\frac{k_2^2}{4a}\right)\right]. \tag{5.15}$$

From the inverse expansion one finds again (4.35).

Fr-Or.

$$(fr_{k,m'}, or_{n,m}) = \frac{\delta_{mm'}(n!)^{1/2} e^{-k^2/4} k^{2m+\mu+\nu+3/2}}{[\Gamma(n+2m+\mu+\nu+2)]^{1/2} 2^{m+1+(\mu+\nu)/2}} L_n^{(2m+\mu+\nu+1)}(k^2/2). \tag{5.16}$$

From this overlap one can derive the Hille–Hardy formula [7], [18], as well as a known integral [9] involving a product of a Bessel function and an associated Laguerre polynomial.

Fr-Oc.

$$(fr_{k,m}, oc_{n_1n_2}) = \frac{N_m(\mu,\nu)}{2^{(\mu+\nu+3)/2}}[n_1!n_2!\Gamma(n_1+\mu+1)\Gamma(n_2+\nu+1)]^{1/2} e^{-k_2/4} k^{\mu+\nu+3/2} \cdot \sum_{j_1,j_2=0}^{n_1,n_2} \frac{(-k^2/2)^{j_1+j_2} {}_3F_2\left[\begin{matrix}-m, & \mu+\nu+m+1, & j_2+\nu+1 \\ \nu+1, & j_1+j_2+\mu+\nu+2 & \end{matrix}; 1\right]}{j_1!j_2! \quad \Gamma(n_1-j_1+1)\Gamma(n_2-j_2+1)\Gamma(j_1+j_2+\mu+\nu+2)}. \tag{5.17}$$

In spite of the complicated form of (5.17), it gives rise to expansions which can be reduced to known results. Similar results hold for Case 1.

Fp-Or.

$$(fp_{k,s}, or_{n,m}) = \frac{(n!)^{1/2} e^{-k^2/4} k^{m+\nu} L_n^{(m+\nu+1/2)}(k^2/2)\overline{(fp_{k,s}, fr_{k,m})}}{\sqrt{\pi}[\Gamma(n+m+\nu+3/2)]^{1/2} 2^{(m+\nu)/2}}, \tag{5.18}$$

where we have introduced the notation of designating an overlap function modulo $\delta(k-k')$ with a tilde, e.g., $(fp_{k,s}, fr_{k',m}) = \delta(k-k')(\widetilde{fp_{k,s}, fr_{k,m}})$.

In the above case we can use the Hille–Hardy formula to rederive (5.5), and for the converse expansion using (6) of [5, p. 158] yields a known integral.

Fe-Or.

$$(5.19)\qquad (fe_{k,l}, or_{n,m}) = \frac{A_{l,n}^{(\nu,\mu)}(k^2)}{N_m(\mu,\nu)}(fr_{k,m}, or_{n,m}).$$

Here the expansions reduce to a combination of Hille–Hardy, (5.7) and (5.8).

Fc-Oe.

$$(5.20)\qquad (fc_{k_1,k_2}, oe_{n,m}) = hp_{n,m}^{(\mu,\nu)}(i\rho, \tfrac{1}{2})hp_{n,m}^{(\mu,\nu)}(\sigma, \tfrac{1}{2}),$$

where $k_1 = \cosh\rho\cos\sigma$, $k_2 = \sinh\rho\sin\sigma$. We obtain in this case the only double expansion formula which is given explicitly, viz.,

$$\sum_{n,m} \gamma_{n,}^{(\mu,\nu)}(\eta_m) hp_{n,m}^{(\mu,\nu)}(i\rho, \tfrac{1}{2}) hp_{n,m}^{(\mu,\nu)}(\sigma, \tfrac{1}{2}) hp_{n,m}^{(\mu,\nu)}(iv_1, \tfrac{1}{2}) hp_{n,m}^{(\mu,\nu)}(v_2, \tfrac{1}{2}) z^n$$

$$(5.21)\qquad = -\left(\frac{1-z}{1+z}\right)^2 (-z)^{-(\mu+\nu)/2} (\sinh 2v_1 \sin 2v_2 \sinh 2\rho\ \sin 2\sigma)^{1/2}$$

$$\cdot \exp\left[-\frac{1}{4}\left(\frac{1+z}{1-z}\right)(\sinh^2 v_1 + \cos^2 v_2 + k^2)\right]$$

$$\cdot J_\mu\left(\frac{i\sqrt{z}\cosh v_1 \cos v_2 \cosh\rho\cos\sigma}{1-z}\right) J_\nu\left(\frac{i\sqrt{z}\sinh v_1 \sin v_2 \sinh\rho\sin\sigma}{1-z}\right).$$

As previously mentioned the functions $hp_{n,m}^{\mu,\nu}$ will be studied in much more detail in a forthcoming work. There we will also treat the overlap functions for the different harmonic oscillator systems *Oa-Ob*.

Lc-Or.

$$(5.22)\qquad (oc_{n_1 n_2}, lc_{\lambda,k}) = \frac{(n_2!)^{1/2} e^{-k^2/4} k^{\nu+1/2} L_{n_2}^{\nu}(k^2/2) C_{n_1}(\lambda + k^2/4, a)}{[n_1! \Gamma(n_2+\nu+1)]^{1/2} |a|^{1/2} \pi^{3/4} 2^{(n_1+\nu)/2+3/4}}$$

where following V and VI, we can define C_n by the generating function

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} C_n(y, a)$$

$$(5.23)\qquad = 4\pi a^{1/3} \exp\left[\frac{2a^2}{3} + y - z^2 + 2\sqrt{2}iaz\right] \mathrm{Ai}\left[a^{1/3}\left(\frac{y}{a} + a + i\sqrt{2}z\right)\right].$$

The overlap function (5.22) also gives us the integral representation for C_n,

$$(5.24)\qquad C_n(y, a) = \int_{-\infty}^{\infty} dx \exp\left[-\frac{i}{2a} yx - \frac{x^2}{4} - \frac{ix^3}{24a}\right] H_n\left(\frac{x}{\sqrt{2}}\right),$$

which can also be viewed as a Fourier transform. Using these results and

Hille–Hardy, one of the expansions coming from (5.22) can be reduced to

$$\sum_{n=0}^{\infty} \frac{(\sqrt{z}/2)^n}{n!} H_n\left(\frac{v_1}{\sqrt{2}}\right) C_n(y, a)$$

(5.25)
$$= \frac{4\pi a^{1/3}}{(1-z)^{1/2}} \exp\left[\frac{z}{1-z}\frac{v_1}{2} + \frac{2ai(i+z)\sqrt{z}}{(i-z)^2} v_1 + \left(\frac{1+z}{1-z}\right) y + \frac{2a^2}{3}\left(\frac{1+z}{1-z}\right)^3\right]$$
$$\cdot \operatorname{Ai}\left[a^{1/3}\left(\frac{2i\sqrt{z}}{1-z} v_1 + a\left(\frac{1+z}{1-z}\right)^2 + \frac{y}{a}\right)\right].$$

We remark that (5.25) can be obtained also from the integral representation (5.24) and the use of Mehler's generating function for a product of Hermite polynomials [7]. Some further properties of the function C_n are $\overline{C_n}(y, a) = C_n(y, -a) = (-1)^n C_n(y, a)$. The converse expansion leads to

(5.26)
$$\int_{-\infty}^{\infty} dy \operatorname{Ai}\left[a^{1/3}\left(u_1 + \frac{y}{a}\right)\right] C_n(y, a)\, e^{y(1+z)/(1-z)} = \frac{\sqrt{2\pi}|a|}{a^{1/3}} (i-z)^{1/2} z^{n/2} H_n\left(\frac{v_1}{\sqrt{2}}\right),$$

where $v_1 = [u_1 - a((1+z)/(1-z))](1-z)/2\sqrt{z}$. All of the above results for this case can be obtained from the results of Kalnins and Miller in V.

Rc-Rr.

(5.27)
$$(rr_{\lambda,m}, rc_{\lambda_1,\lambda_2}) = \frac{\delta(\lambda-\lambda_1-\lambda_2) N_m(\mu, \nu)}{\sqrt{\pi}\, 2^{3/2} \Gamma\left(\frac{\mu+\nu+i\lambda}{2}+1\right)} \Gamma\left(\frac{\nu+i\lambda_2+1}{2}\right) \Gamma\left(\frac{\mu+i\lambda_1+1}{2}\right)$$
$$\cdot {}_3F_2\left[\begin{matrix} -m, & m+\mu+\nu+1 & (\nu+i\lambda_2+1)/2 \\ \nu+1 & (\mu+\nu+i\lambda)/2+1 & \end{matrix} ; 1\right].$$

This leads to the expansions

(5.28)
$$\sum_{m=0}^{\infty} \frac{N_m^2(\mu, \nu)}{\Gamma(2m+\mu+\nu+2)} \left(\frac{\mu+\nu+i\lambda}{2}+1\right)_m$$
$$\cdot {}_3F_2\left[\begin{matrix} -m, & m+\mu+\nu+1, & (\nu+i\lambda_2+1)/2 \\ \nu+1, & (\mu+\nu+i\lambda)/2+1 & \end{matrix} ; 1\right]$$
$$\cdot P_m^{(\nu,\mu)}(\cos 2\theta) M_{i\lambda/2, m+(\mu+\nu+1)/2}(ir^2)$$
$$= \frac{2(\sin\theta)^{-\nu-1}(\cos\theta)^{-\mu-1}}{\Gamma(\mu+1)\Gamma(\nu+1)} M_{i(\lambda-\lambda_2)/2, \mu/2}(ir^2\cos^2\theta) M_{i\lambda_2/2, \nu/2}(ir^2\sin^2\theta),$$

and conversely,

$$\begin{aligned}(5.29)\quad &\int_{-\infty}^{\infty} d\sigma \left|\Gamma\left(\frac{\mu+i\lambda-i\sigma+1}{2}\right)\Gamma\left(\frac{i\sigma+\nu+1}{2}\right)\right|^2 \\ &\cdot {}_3F_2\left[\begin{matrix}-m, & m+\mu+\nu+1, & (\nu+i\sigma+1)/2 \\ \nu+1, & (\mu+\nu-i\lambda)/2+1 & \end{matrix};\ 1\right] \\ &\cdot M_{i(\lambda-\sigma)/2,\mu/2}(ir^2\cos^2\theta) M_{i\sigma/2,\nu/2}(ir^2\sin^2\theta) \\ &= \frac{4\pi^2\Gamma(\mu+1)\Gamma(\nu+1)\left|\Gamma\left(\frac{\mu+\nu-i\lambda}{2}+1\right)\right|^2\left(\frac{\mu+\nu+i\lambda}{2}+1\right)_m}{\Gamma(2m+\mu+\nu+2)}\sin^{\nu+1}\theta\cos^{\mu+1}\theta \\ &\quad\cdot P_m^{(\nu,\mu)}(\cos 2\theta) M_{i\lambda/2,m+(\mu+\nu+1)/2}(ir^2).\end{aligned}$$

Similarly expansions for Case 1 can be obtained which are special cases of the above.

Fr-Rr.

$$(5.30)\qquad (fr_{k,n'}, rr_{\lambda,n}) = \frac{\delta_{nn'}}{\sqrt{2\pi}} k^{i\lambda-1/2}.$$

From (5.30) we can derive equation (7.5.19) of [8, vol. 1].

Fr-Rc.

$$(5.31)\qquad (fr_{k,m}, rc_{\lambda_1,\lambda_2}) = \frac{k^{i(\lambda_1+\lambda_2)-1/2}}{\sqrt{2\pi}} \overline{(rr_{\lambda_1+\lambda_2,m}, rc_{\lambda_1,\lambda_2})}.$$

Here we can use the Mellin transform of $Fr_{k,m}(x,t)$ to reduce one expansion to (5.28), while use of (5.29) reduces the converse again to equation (7.5.19) of [8].

Fp-Rr.

$$(5.32)\qquad (fp_{k,s}, rr_{\lambda,m}) = \sqrt{\pi} k^{i\lambda-1/2} \overline{(fp_{k,s}, fr_{k,m})}.$$

Here we can use the Mellin transform of a Whittaker function to rederive (5.5) and (6) of [5, p. 158] to obtain the Mellin transform of the Fr system.

Fp-Rc.

$$\begin{aligned}(5.33a)\quad (fp_{k,s}, rc^{+}_{\lambda_1\lambda_2}) &= \frac{k^{i(\lambda_1+\lambda_2)-1/2}}{(2\pi)^{3/2}} \frac{\Gamma(i\lambda_1+1/2)\Gamma(is+i\lambda_2/2)}{\Gamma(i\lambda_1+is+i\lambda_2/2-1/2)} \\ &\quad\cdot {}_2F_1\left(is+\frac{i\lambda_2}{2}-1,\ i\lambda_1+1/2;\ is+i\lambda_1+\frac{i\lambda_2}{2}-\frac{1}{2};\ -1\right)\end{aligned}$$

and

$$(5.33b)\qquad (fp_{k,s}, rc^{-}_{\lambda_1,\lambda_2}) = (fp_{k,-s}, rc^{+}_{\lambda_1,\lambda_2}).$$

In this case one expansion can be reduced by using the integral representation for the overlap functions and the inverse Mellin transforms of a Whittaker function and a parabolic cylinder function, while the inverse expansion gives a rederivation of (5.13).

Fe-Rr.

$$(fe_{k,n}, rr_{\lambda,m}) = \frac{k^{i\lambda-1}}{\sqrt{2\pi}} \frac{A_{n,m}^{(\nu,\mu)}(k^2)}{N_m(\mu,\nu)}. \tag{5.34}$$

The corresponding expansions can be reduced with the aid of equation (7.5.18) of [8, vol. 1], (5.7), and (5.8).

Lc-Rc.

$$(lc_{\gamma,k}, rc^{\pm}_{\lambda_1,\lambda_2}) = \frac{k^{i\lambda_2-1/2}}{2\pi\sqrt{|a|}} C_{\lambda_1}\left(\gamma + \frac{k^2}{4}, \pm a\right), \tag{5.35a}$$

where $C_{\gamma,k}(\lambda_1)$ can be written as a Mellin transform

$$C_{\lambda_1}(y, a) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dx\, x^{i\lambda_1-1/2}\, e^{(i/2a)(yx+x^3/12)}. \tag{5.35b}$$

By taking the inverse Mellin transform we arrive at the continuous generating function for $C_\lambda(y, a)$ discussed by Kalnins and Miller in V. In fact both expansions can be obtained from the results of V. It is easily seen by using the expansions themselves or directly from the integral (5.35b) that we can obtain the continuous version of (5.25) and (5.26) involving parabolic cylinder instead of Hermite polynomials and C_λ instead of C_n. We omit the details.

Oc-Rc. Again there is no more content here than in V.

$$\begin{aligned}(oc_{n_1n_2}, rc_{\lambda_1\lambda_2}) &= \frac{2^{i\lambda_1+i\lambda_2+(\mu+\nu-3)/2}[\Gamma(n_1+\mu+1)\Gamma(n_2+\nu+1)]^{1/2}}{\pi\sqrt{n_1!n_2!}\Gamma(\mu+1)\Gamma(\nu+1)} \\ &\cdot \Gamma\left(\frac{\mu+i\lambda_1+1}{2}\right)\Gamma\left(\frac{\nu+i\lambda_2+1}{2}\right)F\left(-n_1, \frac{\mu+i\lambda_1+1}{2}; \mu+1; 2\right) \\ &\cdot F\left(-n_2, \frac{\nu+i\lambda_2+1}{2}; \nu+1; 2\right).\end{aligned} \tag{5.36}$$

Indeed, the results of V give the generating function

$$\begin{aligned}&\sum_{n=0}^{\infty} F\left(-n, \frac{\mu+i\lambda+1}{2}, \mu+1; 2\right) L_n^{\mu}\left(\frac{v_1^2}{2}\right) z^n \\ &= \frac{e^{-i\pi(1+\mu)/2}}{v_1^{\mu+1}} z^{-(\mu+1)/2}\left(\frac{1-z}{1+z}\right)^{i\lambda/2} e^{-(v_1^2/2)z(2-z)/(1-z^2)} M_{i\lambda/2,\mu/2}\left(\frac{e^{i\pi}zv_1^2}{(1-z^2)}\right)\end{aligned} \tag{5.37}$$

and the continuous generating function

$$\begin{aligned}&\int_{-\infty}^{\infty} d\sigma \left|\Gamma\left(i\sigma + \frac{\mu+1}{2}\right)\right|^2 F\left(-n, -i\sigma + \frac{\mu+1}{2}; \mu+1; 2\right) M_{i\sigma,\mu/2}\left(\frac{iv^2}{4}\right)(it)^{i\sigma} \\ &= \frac{\pi n! [\Gamma(\mu+1)]^2 \left(\frac{1-it}{1+it}\right)^n (e^{i\pi}t)^{(\mu+1)/2}}{\Gamma(n+\mu+1)2^{\mu+1/2}(1+it)^{\mu+1}} v_1^{\mu+1} e^{-v_1^2(1-it)/8(1+it)} L_n^{\mu}\left(\frac{tv_1^2}{2(1+t^2)}\right);\end{aligned} \tag{5.38}$$

similar results can be derived for Case 1 which are special cases of the above.

Or-Rr.

$$(or_{n,m'}, rr_{\lambda,m}) = \frac{\delta_{mm'}[\Gamma(n+2m+\mu+\nu+2)]^{1/2} 2^{m+i\lambda+(\mu+\nu-1)/2}\Gamma(m+1+(\mu+\nu+i\lambda)/2)}{\sqrt{\pi}(n!)^{1/2}\Gamma(2m+\mu+\nu+2)} \tag{5.39}$$
$$\cdot F\left(-n, m+1+\frac{\mu+\nu+i\lambda}{2}, 2m+\mu+\nu+2; 2\right).$$

These expansions are equivalent to (5.37) and (5.38).

Or-Rc.

$$(or_{n,m}, rc_{\lambda_1,\lambda_2}) = (or_{n,m}, rr_{\lambda_1+\lambda_2,m})(\widetilde{rr_{\lambda_1+\lambda_2,m} rc_{\lambda_1,\lambda_2}}). \tag{5.40}$$

Using (5.37) one expansion reduces to (5.28) while using (5.29) reduces the converse to (5.38).

Rr-Re.

$$(rr_{\lambda',m}, re_{\lambda,n}) = \delta(\lambda-\lambda')\frac{D^{(\nu,\mu)}_{n,m}(\lambda)}{N_m(\mu,\nu)}. \tag{5.41}$$

From (5.41) we find the expansions

$$\sum_{m=0}^{\infty} \frac{D^{(\nu,\mu)}_{n,m}(\lambda)\Gamma\left(\frac{\mu+\nu+i\lambda}{2}+m+1\right)}{} \cdot P^{(\nu,\mu)}_m(\cos 2\theta) M_{i\lambda/2, m+(\mu+\nu+1)/2}\left(\frac{i}{4}(\sinh^2 v_1+\cos^2 v_2)\right)$$
$$= \frac{\sqrt{\pi}\, e^{\lambda/4} K^{(\nu,\mu)}_m(\lambda)}{2^{i\lambda-1/2}}(\sinh^2 v_1+\cos^2 v_2)^{(\mu+\nu)/2+1} e^{-i(\sinh^2 v_1+\cos^2 v_2)/8} \tag{5.42}$$
$$\cdot Ge^{(\nu,\mu)}_n(iv_1, \tfrac{1}{4}, \lambda) Ge^{(\nu,\mu)}_n(v_2, \tfrac{1}{4}, \lambda),$$

where $\tan\theta = \tanh v_1 \tan v_2$, and conversely, we have

$$\sum_{n=0}^{\infty} D^{(\nu,\mu)}_{n,m}(\lambda) K^{(\nu,\mu)}_n(\lambda) Ge^{(\nu,\mu)}_n(iv_1, \tfrac{1}{4}, \lambda) Ge^{(\nu,\mu)}_n(v_2, \tfrac{1}{4}, \lambda)$$
$$= \frac{e^{-\pi\lambda/4} N^2_m(\mu,\nu)\Gamma(m+1+(\mu+\nu+i\lambda)/2)\, e^{i(\sinh^2 v^1+\cos^2 v_2)/8}}{\sqrt{\pi}\Gamma(2m+\mu+\nu+2) 2^{1/2-i\lambda}(\sinh^2 v_1+\cos^2 v_2)^{(\mu+\nu)/2+1}} P^{(\nu,\mu)}_m(\cos 2\theta) \tag{5.43}$$
$$\cdot M_{i\lambda/2, m+(\mu+\nu+1)/2}\left(\frac{i}{4}(\sinh^2 v_1+\cos^2 v_2)\right).$$

Fr-Re.

$$(fr_{k,m}, re_{\lambda,n}) = \frac{k^{i\lambda-1}}{\sqrt{2\pi k}} \frac{D^{(\nu,\mu)}_{n,m}(\lambda)}{N_m(\mu,\nu)}. \tag{5.44}$$

Or-Re.

$$(or_{n,m}, re_{\lambda,l}) = (or_{n,m}, rr_{\lambda,m})\frac{D^{(\nu,\mu)}_{l,m}(\lambda)}{N_m(\mu,\nu)}. \tag{5.45}$$

The expansions from both (5.44) and (5.45) can be reduced by using (5.42) and (5.43). Again Case 1 yields special cases of the above.

Further expansions can be obtained by suitable manipulations of the above results. It should also be clear that many more results can be obtained by analyzing in detail the most general overlap functions, that is, cross-basis matrix elements of G as well as the ordinary matrix elements. This problem will be studied in the future.

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INCLUSION THEOREMS FOR THE ZEROS OF CERTAIN RECURSIVELY GENERATED POLYNOMIALS*

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Abstract. Consider the sequence $\{\varphi_n\}$ of monic polynomials generated by the recurrence relation $\varphi_0 = 1$, $\varphi_1 = z - b_0$, $\varphi_{n+1} = (z - b_n)\varphi_n - c_n\varphi_{n-1}$, $n \geqq 1$, b_n, c_n complex and independent of z, $c_n \neq 0$. This paper contains two theorems of the inclusion type for zeros of the polynomials φ_n. The results are obtained by applying majorization techniques to certain tridiagonal matrices associated with such polynomial sequences. If each b_n is real and each $c_n > 0$, the polynomials are orthogonal with respect to some distribution on some set of points of the real line, and the zeros are real. Two additional theorems deal with the smallest interval containing these zeros, i.e., the true interval of orthogonality.

1. Introduction. Let $\{\varphi_n\}$ be a sequence of monic polynomials generated by the recurrence relation

$$\varphi_0 = 1,$$

$$\varphi_1 = z - b_0, \tag{1}$$

$$\varphi_{n+1} = (z - b_n)\varphi_n - c_n\varphi_{n-1},$$

$n \geqq 1$, where $c_n \neq 0$ and b_n, c_n are independent of z. The importance of such sequences is well known, especially in the theory of polynomials orthogonal with respect to some distribution on some set of points of the real line (in which case b_n is real and $c_n > 0$), but there are other significant examples as well, e.g., the Bessel polynomials.

Of particular usefulness in studying these polynomials is the familiar fact that, for each n, φ_n is the characteristic polynomial for the indecomposable tridiagonal matrix

$$C_n = \begin{pmatrix} b_0 & c_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & b_1 & c_2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & b_2 & c_3 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & b_{n-2} & c_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 1 & b_{n-1} \end{pmatrix}, \tag{2}$$

and also, if b_n is real and $c_n > 0$, for the real symmetric matrix

$$S_n = \begin{pmatrix} b_0 & \sqrt{c_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{c_1} & b_1 & \sqrt{c_2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{c_2} & b_2 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & b_{n-2} & \sqrt{c_{n-1}} \\ 0 & 0 & 0 & 0 & 0 & \cdots & \sqrt{c_{n-1}} & b_{n-1} \end{pmatrix}. \tag{3}$$

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Consequently, matrix methods, particularly those dealing with characteristic values, are available for use in studying recursively generated polynomials. For instance, it follows from (3) that if each b_k is real and each $c_k > 0$, then the zeros of φ_n are real, and those of φ_{n-1} interlace those of φ_n (cf. [6, pp. 103–104]).

This paper is concerned with theorems of the inclusion type for the zeros of the polynomials generated by (1). Our approach is based largely on matrix theory, and in order to facilitate the proofs, we first present some necessary supporting theorems and lemmas.

2. Results.

THEOREM (Ky Fan [2]). *If $C=(c_{ij})$ is a matrix of order n with nonnegative elements such that $|a_{ij}| \leqq c_{ij} (i, j = 1, 2, \cdots, n;\ i \neq j)$, and if $\mu \geqq 0$ is the maximal characteristic value of C, then every characteristic value of the matrix $A = (a_{ij})$ lies in at least one of the n circular disks $|z - a_{ii}| \leqq \mu - c_{ii}$, $i = 1, \cdots, n$.*

Let $\{P_n\}$ be the sequence of polynomials generated by

$$P_0 = 1, \qquad P_1 = z, \qquad P_{n+1} = zP_n - \delta_n P_{n-1}, \tag{4}$$

$n \geqq 1$, where $\delta_n > 0$, and suppose the sequence is orthogonal on the true interval of orthogonality $(-a, a)$. The nonnegative indecomposable tridiagonal matrix

$$\Delta_n = \begin{pmatrix} 0 & \delta_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \delta_2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \delta_3 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & \delta_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{5}$$

then has P_n as its characteristic polynomial, and the following lemmas apply.

LEMMA 1. *For each n, Δ_n has a positive maximal characteristic value μ_n (which is the largest zero of P_n), and a corresponding positive characteristic vector $X_n = (x_1, \cdots, x_n)$.*

Proof. The proof is immediate by the Perron–Frobenius Theorem (see e.g., [5, p. 124]).

LEMMA 2. *The sequence $\{\mu_n\}$ is monotone increasing, and* $\lim_{n\to\infty} \mu_n = a$. (cf. [1]).

LEMMA 3. *Let $X_n = (x_1, \cdots, x_n)$ be the positive characteristic vector associated with μ_n, as noted in Lemma 1. Then $x_{k+1}/x_k = P_k(\mu_n)/\delta_k P_{k-1}(\mu_n)$, $k = 1, \cdots, n-1$.* (cf. [3, p. 253]).

Proof. By hypothesis $\Delta_n X_n = \mu_n X_n$, or

$$\delta_1 x_2 = \mu_n x_1, \tag{i}$$

$$x_{i-1} + \delta_i x_{i+1} = \mu_n x_i, \qquad i = 2, \cdots, n-1, \tag{ii}$$

$$x_{n-1} = \mu_n x_n. \tag{iii}$$

It is clear from (i) that $x_2/x_1 = \mu_n/\delta_1 = P_1(\mu_n)/\delta_1 P_0(\mu_n)$. Also, from (ii), if $2 \leqq i \leqq n-1$ and the assertion holds for $k \leqq i-1$, then

$$\begin{aligned} x_{i+1}/x_i &= \mu_n/\delta_i - x_{i-1}/\delta_i x_i \\ &= [\mu_n - \delta_{i-1}P_{i-2}(\mu_n)/P_{i-1}(\mu_n)]/\delta_i \\ &= P_i(\mu_n)/\delta_i P_{i-1}(\mu_n), \end{aligned}$$

so the assertion holds for $k = i, i \leqq n-1$. The statements (ii) with $i = n-1$ and (iii) together assert that $\mu_n P_{n-1}(\mu_n) - \delta_{n-1}P_{n-2}(\mu_n) = 0$, which being equivalent to $P_n(\mu_n) = 0$, is certainly true.

LEMMA 4. *For fixed k and $n > k$ the sequence $\{P_{k+1}(\mu_n)/P_k(\mu_n)\}$ is monotone increasing, and* $\lim_{n\to\infty} P_{k+1}(\mu_n)/P_k(\mu_n) = P_{k+1}(a)/P_k(a)$.

Proof. For all $n > k$, $P_k(\mu_n) \neq 0$. Also, for $x > \mu_k$, $d/dx(P_{k+1}(x)/P_k(x)) = [P'_{k+1}(x)P_k(x) - P_{k+1}(x)P'_k(x)]/P_k^2(x)$, and the numerator is positive by the Christoffel–Darboux identity. The limit assertion follows from Lemma 2.

Let $C = (c_{ij})$ be any complex $n \times n$ matrix. The spread of C, denoted by $s(C)$, is defined to be $s(C) = \max_{i,j} |\tau_i - \tau_j|$, where τ_i, τ_j are characteristic values of C (cf. [5, pp. 167–8]). For any hermitian matrix C, $s(C) = \tau_1 - \tau_n$, if $\tau_1, \cdots, \tau_n$ is a listing of the characteristic values of C in nonincreasing order. For the matrix S_n in (3), we have the following result.

LEMMA 5. *Given S_n in* (3), *let $\tilde{S}_n$ be the matrix obtained from S_n by replacing the entries on the main diagonal with zeros. Then $s(S_n) \geqq s(\tilde{S}_n)$.*

Proof. Let $s_1, \cdots, s_n, \tilde{s}_1, \cdots, \tilde{s}_n$ be the characteristic values, respectively, of S_n and $\tilde{S}_n$ arranged in nonincreasing order. Let $X_n = (x_1, \cdots, x_n)$ be the unit vector having the property that $X_n^T\tilde{S}_nX_n = \tilde{s}_1$. If $Y_n = (y_1, \cdots, y_n)$ where $y_i = (-1)^{i-1}x_i$, $i = 1, \cdots, n$, then it is easy to verify that $Y_n^T\tilde{S}_nY_n = \tilde{s}_n = -\tilde{s}_1$ and $s(\tilde{S}_n) = 2\tilde{s}_1$. If we now let $D_n = \operatorname{diag} S_n$, then we have $X_n^TS_nX_n = X_n^T\tilde{S}_nX_n + X_n^TD_nX_n = \tilde{s}_1 + \sum_{i=1}^n x_i^2 b_{i-1}$, and correspondingly, $Y_n^TS_nY_n = -\tilde{s}_1 + \sum_{i=1}^n x_i^2 b_{i-1}$. But $s_1 = \max Z_n^TS_nZ_n$ and $s_n = \min Z_n^TS_nZ_n$, where the maximum and minimum are taken over all unit vectors (cf. [6, p. 99]). Hence the inequalities $s_1 \geqq \tilde{s}_1 + \sum_{i=1}^n x_i^2 b_{i-1}$, $s_n \leqq -\tilde{s}_1 + \sum_{Ji=1}^n x_i^2 b_{i-1}$ hold, and so $s_1 - s_n = s(S_n) \geqq 2\tilde{s}_1 = s(\tilde{S}_n)$.

We are now in position to state and prove the main results. The first of these was suggested by a study of Fan's theorem noted above, and the proof is an adaptation of his proof.

THEOREM 1. *Let $\{\varphi_k\}$ be the sequence generated by the recurrence* (1), *with b_k, c_k complex, $c_k \neq 0$, and let C_n be the matrix associated with φ_n as defined in* (2). *Let $\{P_k\}$ be the sequence generated by the recurrence* (4) *and having the properties noted there, so that Lemmas* 1–4 *apply. Then for arbitrary fixed n ($n \geqq 1$), the zeros of φ_n lie in the union of the n disks*

$$(6) \qquad |z - b_k| \leqq \mu_n - (1 - |c_{k+1}|/\delta_{k+1})P_{k+1}(\mu_n)/P_k(\mu_n), \qquad k = 0, 1, \cdots, n-1.$$

Proof. Let $X_n = (x_1, \cdots, x_n)$ be the positive characteristic vector associated with μ_n, as already noted in Lemmas 1 and 3, and let $D_n = \operatorname{diag} X_n$. By Gershgorin's Theorem applied to the matrix $D_n^{-1} C_n D_n$, the characteristic values of C_n lie in the union of the disks

$$|z - b_0| \leqq |c_1| x_2/x_1,$$
$$|z - b_k| \leqq x_k/x_{k+1} + |c_{k+1}| x_{k+2}/x_{k+1}, \qquad k = 1, \cdots, n-2,$$
$$|z - b_{n-1}| \leqq x_{n-1}/x_n.$$

But by Lemma 3, these are the disks

$$\begin{aligned}
|z - b_0| &\leqq |c_1| P_1(\mu_n)/P_0(\mu_n) \\
&= \mu_n - (1 - |c_1|/\delta_1) P_1(\mu_n)/P_0(\mu_n), \\
|z - b_k| &\leqq \delta_k P_{k-1}(\mu_n)/P_k(\mu_n) \\
&\quad + |c_{k+1}| P_{k+1}(\mu_n)/\delta_{k+1} P_k(\mu_n) \\
&= [\mu_n P_k(\mu_n) - P_{k+1}(\mu_n)]/P_k(\mu_n) \\
&\quad + |c_{k+1}| P_{k+1}(\mu_n)/\delta_{k+1} P_k(\mu_n) \\
&= \mu_n - (1 - |c_{k+1}|/\delta_{k+1}) P_{k+1}(\mu_n)/P_k(\mu_n), \quad k = 1, 2, \cdots, n-2 \\
|z - b_{n-1}| &\leqq \mu_n \\
&= \mu_n - (1 - |c_n|/\delta_n) P_n(\mu_n)/P_{n-1}(\mu_n) \qquad (\text{since } P_n(\mu_n) = 0).
\end{aligned}$$

Theorem 1 is of particular interest when $|c_{k+1}| < \delta_{k+1}$, $k = 0, 1, \cdots, n-2$. For then, since $P_{k+1}(\mu_n)/P_k(\mu_n) > 0$ as noted in Lemma 3, the inequalities (6) provide a better estimate than either Gershgorin's theorem applied to the matrix C_n or Fan's theorem applied to the matrices C_n and Δ_n. However, the radii of the disks in (6) are not readily computed, nor is their dependence on n easily assessed, and so it is desirable to have a more practical set of estimates, even at the cost of some loss of precision. Theorem 2 which follows provides such estimates.

THEOREM 2. *Given the sequences* $\{\varphi_k\}$ *and* $\{P_k\}$ *as in Theorem* 1, *suppose for arbitrary n that the following inequalities hold*:

$$\text{(i)} \qquad |c_{k+1}| < \delta_{k+1}$$

and

$$\text{(ii)} \qquad |b_k| < (1 - |c_{k+1}|/\delta_{k+1}) P_{k+1}(a)/P_k(a),$$

$k = 0, \cdots, n-1$. *Then for each* K, $1 \leqq K \leqq n$, *the zeros of* φ_K *lie in the union of the* K *disks* $|z - b_k| < a - |b_k|$, $k = 0, \cdots, K-1$.

Proof. Because of (i), (ii) and Lemma 4, there is an integer $N > n$ such that

$$\begin{aligned}
|b_k| &< (1 - |c_{k+1}|/\delta_{k+1}) P_{k+1}(\mu_N)/P_k(\mu_N) \\
&< (1 - |c_{k+1}|/\delta_{k+1}) P_{k+1}(a)/P_k(a), \qquad k = 0, \cdots, n-1.
\end{aligned} \tag{7}$$

Consider the polynomials $\hat{\varphi}_n$ generated by

$$\hat{\varphi}_0 = 1,$$
$$\hat{\varphi}_1 = z - |b_0|,$$
$$\hat{\varphi}_{k+1} = (z - |b_k|)\hat{\varphi}_k - |c_k|\hat{\varphi}_{k-1}, \qquad k = 1, \cdots, n-1,$$

and

$$\hat{\varphi}_{k+1} = z\hat{\varphi}_k - \delta_k \hat{\varphi}_{k-1}, \qquad k = n, n+1, \cdots, N-1.$$

By Theorem 1 the zeros of $\hat{\varphi}_N$ lie in the union of the disks

$$|z - |b_k|| \leqq \mu_N - (1 - |c_{k+1}|/\delta_{k+1}) P_{k+1}(\mu_N)/P_k(\mu_N),$$

$k = 0, \cdots, n-1$, and the disk $|z| \leqq \mu_N$. Hence these zeros also lie in the union of the disks

$$|z| \leqq \mu_N - [(1 - |c_{k+1}|/\delta_{k+1}) P_{k+1}(\mu_n)/P_k(\mu_n) - |b_k|],$$

$k = 0, \cdots, n-1$, and the disk $|z| \leqq \mu_N$. By (7) and the hypotheses, this union is clearly $|z| \leqq \mu_N$, which is contained in $|z| < a$. Since $|c_k| > 0$ and $|b_k|$ is real, the zeros of $\hat{\varphi}_N$ are real, and hence they lie in the interval $(-a, a)$. By the interlacing property, the zeros of $\hat{\varphi}_1, \cdots, \hat{\varphi}_{N-1}$ also lie in $(-a, a)$. If now we apply Fan's theorem to the matrices

$$C_K = \begin{pmatrix} b_0 & c_1 & 0 & 0 & \cdots & 0 \\ 1 & b_1 & c_2 & 0 & \cdots & 0 \\ 0 & 1 & b_2 & 0 & \cdots & 0 \\ & \cdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & c_{K-1} \\ 0 & 0 & 0 & \cdots & 1 & b_{K-1} \end{pmatrix}, \quad \hat{C}_K = \begin{pmatrix} |b_0| & |c_1| & 0 & 0 & \cdots & 0 \\ 1 & |b_1| & |c_2| & 0 & \cdots & 0 \\ 0 & 1 & |b_2| & 0 & \cdots & 0 \\ & \cdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & |c_{K-1}| \\ 0 & 0 & 0 & \cdots & 1 & |b_{K-1}| \end{pmatrix},$$

where $1 \leqq K \leqq n$, we obtain the desired conclusion, since the maximal positive characteristic value for $\hat{C}_K$ is less than a.

Theorem 2 is easier to apply than Theorem 1 because of the relative ease of computing $P_{k+1}(a)/P_k(a)$ rather than $P_{k+1}(\mu_n)/P_k(\mu_n)$. For instance, one suitable choice for the sequence $\{P_k\}$ consists of the monic Legendre polynomials generated by $P_0 = 1$, $P_1 = z$, $P_{n+1} = zP_n - [n^2/(4n^2 - 1)]P_{n-1}$. Here $a = 1$, $P_{k+1}(1)/P_k(1) = (k+1)/(2k+1)$, and the hypotheses (i) and (ii) are readily checked.

Note that Theorem 2 implies also that all the zeros of $\varphi_1, \cdots, \varphi_n$ lie in the disk $|z| < a$. If each b_k is real and each $c_k > 0$, then these zeros are real and lie in the interval $(-a, a)$. The remaining two theorems deal with the question of the smallest such interval.

THEOREM 3. *Let the sequences of polynomials* $\{Q_n\}$, $\{\tilde{\varphi}_n\}$ *and* $\{P_n\}$ *be generated, respectively, by* (i) $Q_0 = 1$, $Q_1 = z$, $Q_{n+1} = zQ_n - \gamma_n Q_{n-1}$; (ii) $\tilde{\varphi}_0 = 1$, $\tilde{\varphi}_1 = z$, $\tilde{\varphi}_{n+1} = z\tilde{\varphi}_n - c_n\tilde{\varphi}_{n-1}$; (iii) $P_0 = 1$, $P_1 = z$, $P_{n+1} = zP_n - \delta_n P_{n-1}$; $n \geqq 1$. *Suppose the sequence* $\{Q_n\}$ *is orthogonal on the true interval of orthogonality* $(-a, a)$ *and that all the zeros of each* P_n *lie in* $(-a, a)$. *If* $\gamma_n \leqq c_n \leqq \delta_n$ *for* $n \geqq 1$, *then all the zeros of each* $\tilde{\varphi}_n$ *also lie in* $(-a, a)$, *and this is the smallest such interval for the sequence* $\{\tilde{\varphi}_n\}$ *as well as the sequence* $\{P_n\}$.

Proof. Because of the orthogonality hypothesis, $\gamma_n > 0$ for each n. As before let μ_n denote the positive maximal characteristic value for Δ_n, and let σ_n, r_n denote, respectively, the positive maximal characteristic values for the nonnegative indecomposable tridiagonal matrices

$$(8)\Gamma_n = \begin{pmatrix} 0 & \gamma_1 & 0 & 0 & \cdots & & 0 \\ 1 & 0 & \gamma_2 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \gamma_3 & \cdots & & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & & \gamma_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad \tilde{C}_n = \begin{pmatrix} 0 & c_1 & 0 & 0 & \cdots & & 0 \\ 1 & 0 & c_2 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & c_3 & \cdots & & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & & c_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

whose corresponding characteristic polynomials are, respectively, the polynomials Q_n, $\tilde{\varphi}_n$. By Fan's theorem, the zeros of Q_n lie in the disk $|z| \leqq r_n$, while those of $\tilde{\varphi}_n$ lie in the disk $|z| \leqq \mu_n$. It follows that $\sigma_n \leqq r_n \leqq \mu_n < a$. But $\sigma_n \to a$ as $n \to \infty$; hence a is the l.u.b. for the set of maximal characteristic values of the matrices Δ_n, $\tilde{C}_n$, Γ_n, $n \geqq 1$. The corresponding conclusion holds at $-a$ because of the symmetry of Q_n, $\tilde{\varphi}_n$ and P_n. The zeros in question are real since γ_n, c_n and δ_n are all positive.

As an illustration, let $\gamma_n = 1/4$, $\delta_n = n^2/(4n^2 - 1)$, $n \geqq 1$. Then the polynomials Q_n are the Chebyshev polynomials of the second kind normalized to be monic, while the polynomials P_n are the monic Legendre polynomials already noted. Both these sets are orthogonal on the true interval of orthogonality $(-1, 1)$. Consequently, if $1/4 \leqq c_n \leqq n^2/(4n^2 - 1)$, $n \geqq 1$, then the zeros of each $\tilde{\varphi}_n$ all lie in $(-1, 1)$, and this is the smallest such interval. A more significant example involves the associated polynomials [4, pp. 43–50]. If the polynomials $\{\varphi_n(z)\}$ are generated by (1), then the associated polynomials $\{\varphi_n(z, \eta)\}$ are generated by

$$\begin{aligned} \varphi_0(z, \eta) &= 1, \\ \varphi_1(z, \eta) &= z - b_\eta, \\ \varphi_{n+1}(z, \eta) &= (z - b_{n+\eta})\varphi_n(z, \eta) - c_{n+\eta}\varphi_{n-1}(z, \eta), \end{aligned}$$

$n \geqq 1$. The following corollary to Theorem 3 is easily proved.

COROLLARY 1. *In Theorem* 3 *let* $\gamma_n = a^2/4$ *and suppose the sequence* $\{c_n\}$ *is monotone nonincreasing, with* $c_n \geqq a^2/4$. *If the sequence* $\{\tilde{\varphi}_n\}$ *is orthogonal on the true interval of orthogonality* $(-a, a)$, *then for every* η *the associated sequence* $\{\tilde{\varphi}_n(z, \eta)\}$ *also has* $(-a, a)$ *as the true interval of orthogonality.*

Proof. It suffices to note that the polynomials Q_n with $\gamma_n = a^2/4$ are orthogonal on the true interval $(-a, a)$, since aside from an appropriate linear change of variable they are the monic Chebyshev polynomials of the second kind.

The monic Jacobi polynomials $P_n^{(\alpha,\alpha)}$ generated by

$$\begin{aligned} &P_0^{(\alpha,\alpha)} = 1, \qquad P_1^{(\alpha,\alpha)} = z, \\ &P_{n+1}^{(\alpha,\alpha)} = zP_n^{(\alpha,\alpha)} - [n(n+2\alpha)/(4(n+\alpha)^2 - 1)]P_{n-1}^{(\alpha,\alpha)}, \end{aligned}$$

$n \geqq 1$, satisfy the hypotheses of Corollary 1 for $-1/2 < \alpha \leqq 1/2$, with $a = 1$, and so all the associated polynomials $P_n^{(\alpha,\alpha)}(z, \eta)$ for the indicated range of α are orthogonal on the true interval of orthogonality $(-1, 1)$.

Theorem 3 and Corollary 1 dealt with the smallest interval containing all the zeros for the symmetric case, i.e., all $b_n = 0$. Theorem 4 below is concerned with the nonsymmetric case, but it also includes the symmetric case.

THEOREM 4. *Let the sequences $\{Q_n\}$ and $\{\varphi_n\}$ be generated, respectively, by* (i) *of Theorem* 3 *and* (1), *with b_n real, $c_n > 0$. Suppose also that b_n, c_n satisfy any conditions which insure that all the zeros of each φ_n lie in $(-a, a)$* (e.g., *conditions* (i) *and* (ii) *of Theorem* 2 *hold for every n, or in the symmetric case, the hypothesis $c_n \leqq \delta_n$ of Theorem* 3 *holds for every n*). *Then if $\gamma_n \leqq c_n$ for $n \geqq 1$, the interval $(-a, a)$ is the smallest interval containing all the zeros of each φ_n.*

Proof. Let S_n, $\tilde{S}_n$ be the matrices of Lemma 5, and define the matrix G_n by

$$G_n = \begin{pmatrix} 0 & \sqrt{\gamma_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{\gamma_1} & 0 & \sqrt{\gamma_2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{\gamma_2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \sqrt{\gamma_{n-1}} \\ 0 & 0 & 0 & 0 & 0 & \cdots & \sqrt{\gamma_{n-1}} & 0 \end{pmatrix}.$$

Then Q_n is the characteristic polynomial for both Γ_n and G_n, and similarly $\tilde{\varphi}_n$ is the characteristic polynomial for the matrices $\tilde{C}_n$, $\tilde{S}_n$ (Γ_n and $\tilde{C}_n$ as defined in (8)). Thus the maximal positive characteristic values for G_n and $\tilde{S}_n$ are, respectively, σ_n and r_n, with $\sigma_n \leqq r_n$ as shown in Theorem 3. It follows that $s(\tilde{S}_n) = 2r_n \geqq 2\sigma_n = s(G_n)$. By Lemma 5 $s(S_n) \geqq s(\tilde{S}_n)$ and hence $s(S_n) \geqq 2\sigma_n \to 2a$ *as* $n \to \infty$. But all the characteristic values of each S_n lie in $(-a, a)$ and so $s(S_n) \leqq 2a$ for each n.

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THE CHARACTERIZATION OF SOLUTIONS OF $\nabla^2\phi + \lambda^2\phi = 0$ BY A FUNCTIONAL EQUATION*

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Abstract. If $E^n (n \geqq 2)$ denotes real n-dimensional Euclidean space, then the following result is known. If f is a solution of $\nabla^2\phi + \lambda^2\phi = 0$ throughout $E^n - \{\mathbf{0}\}$, then

$$\frac{1}{S(r)}\int_{S(r)} f(\mathbf{x}+\mathbf{y})\sigma_r = \Gamma(\nu+1)\left(\frac{2}{\lambda r}\right)^{\nu} J_\nu(\lambda r) f(\mathbf{x}), \qquad 0 < r = \|\mathbf{y}\| < \|\mathbf{x}\|.$$

Here $\nu = (1/2)n - 1$ and $S(r)$ denotes both the $(n-1)$-dimensional manifold $\|\mathbf{y}\| = r$ and its volume.

The present paper treats the converse problem and proves that if f is continuous in $\|\mathbf{x}\| > 0$, if $\nabla^2\phi = f$ is soluble in $\|\mathbf{x}\| > 0$ and if $\lim_{r\to 0+} r^{-2}\{H(0+) - H(r)\}$ exists $(= \lambda^2/2n$, say), then the functional equation

$$\frac{1}{S(r)}\int_{S(r)} f(\mathbf{x}+\mathbf{y})\sigma_r = H(r) f(\mathbf{x}), \qquad 0 < r = \|\mathbf{y}\| < \|\mathbf{x}\|,$$

implies that $\nabla^2 f + \lambda^2 f = 0$ throughout $E^n - \{\mathbf{0}\}$ and that $H(r) = \Gamma(\nu+1)(2/(\lambda r))^\nu J_\nu(\lambda r)$.

In conclusion, it is noted that if one assumes that f has spherical symmetry, then the present result reduces to a special case of the author's earlier theorem on Bessel functions.

1. Introduction. With $n \geqq 2$ let E^n denote real n-dimensional Euclidean space and let f be a real-valued function defined on $E^n - \{\mathbf{0}\}$. Then an immediate consequence of a result proved in [1] (see Theorem 2 there) is the following theorem.

THEOREM A. *If f is a solution of $\nabla^2\phi + \lambda^2\phi = 0$ throughout $E^n - \{\mathbf{0}\}$, then*

$$\frac{1}{S(r)}\int_{S(r)} f(\mathbf{x}+\mathbf{y})\sigma_r = \Gamma(\nu+1)\left(\frac{2}{\lambda r}\right)^{\nu} J_\nu(\lambda r) f(\mathbf{x}), \quad 0 < r = \|\mathbf{y}\| < \|\mathbf{x}\|. \tag{1.1}$$

Here $\nu = (1/2)n - 1$, $S(r)$ denotes both the $(n-1)$-dimensional manifold $\|\mathbf{y}\| = r$ and its volume while σ_r is the volume element in $S(r)$.

If we write $H(r) = \Gamma(\nu+1)(2/(\lambda r))^\nu J_\nu(\lambda r)$ it is seen at once that $H(0+)$ exists and has the value 1. Furthermore $\lim_{r\to 0+} (1/r^2)\{H(0+) - H(r)\}$ exists and equals $\lambda^2/(2n)$.

It is the purpose of the present note to consider the converse problem, that is, to study the functional equation

$$\frac{1}{S(r)}\int_{S(r)} f(\mathbf{x}+\mathbf{y})\sigma_r = H(r) f(\mathbf{x}), \qquad 0 < r = \|\mathbf{y}\| < \|\mathbf{x}\|, \tag{1.2}$$

subject to the side condition that the $\lim_{r\to 0+} (1/r^2)\{H(0+) - H(r)\}$ exists. It will appear later that this side condition is quite weak and cannot, apparently, be discarded.

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For (1.2) to be meaningful at all we must impose some condition on f to ensure the existence of the integral, and we shall assume throughout that f is continuous when $\|\mathbf{x}\|>0$. In fact, it appears necessary to impose another condition on f, and this is stated in our main result which is the following.

THEOREM 1. *Suppose that f is continuous when $\|\mathbf{x}\|>0$ and has the property that $\nabla^2\phi=f$ is soluble in $\|\mathbf{x}\|>0$. Let the functions f and H satisfy* (1.2) *and in addition let the limit $\lim_{r\to 0+}(1/r^2)\{H(0+)-H(r)\}$ exist $(=\lambda^2/(2n)$, say). Then f satisfies $\nabla^2\phi+\lambda^2\phi=0$ throughout $E^n-\{\mathbf{0}\}$ and*

$$H(r)=\Gamma(\nu+1)\left(\frac{2}{\lambda r}\right)^{\nu}J_\nu(\lambda r).$$

It should be noted that both here and in (1.1) above we have treated λ as being nonzero and we shall continue to do this. However, little modification is needed to treat the zero case also, and indeed the results obtained are precisely those one gets by letting $\lambda\to 0$ formally in the various formulas.

We introduce an operator A defined by

$$(\mathrm{A}g)(\mathbf{x})=\lim_{r\to 0+}\frac{2n}{r^2S(r)}\int_{S(r)}\{g(\mathbf{x}+\mathbf{y})-g(\mathbf{x})\}\sigma_r, \qquad r=\|\mathbf{y}\|,$$

and the next section is devoted to proving two properties of this operator which will be needed.

2. Properties of the operator *A*. We shall prove the following lemmas.

LEMMA 1. *Let $\nabla^2 g$ be continuous for $\|\mathbf{x}\|>0$. Then Ag exists there and $(Ag)(\mathbf{a})=(\nabla^2 g)(\mathbf{a})$ whenever $\|\mathbf{a}\|>0$.*

LEMMA 2. *Let g be continuous for $\|\mathbf{x}\|>0$ and let $Ag=0$ there. Then $\nabla^2 g=0$ in $\|\mathbf{x}\|>0$.*

Proof of Lemma 1. If $\|\mathbf{a}\|>0$, then

$$G(r)=\frac{2n}{S(r)}\int_{S(r)}g(\mathbf{a}+\mathbf{y})\sigma_r$$

is defined for all sufficiently small $r>0$ and we shall define $G(0)=G(0+)$. Then, provided that the limit exists, we have

$$(Ag)(\mathbf{a})=\lim_{r\to 0+}\frac{1}{r^2}\{G(r)-G(0)\}.$$

The hypothesis on g ensures that G' exists in $(0,r)$ for sufficiently small $r>0$, and so by Cauchy's mean value theorem we have

$$\frac{1}{r^2}\{G(r)-G(0)\}=\frac{1}{2\xi r}G'(\xi r) \quad \text{for some } \xi\in(0,1).$$

Hence, provided that the limit exists, we have

$$(Ag)(\mathbf{a})=\lim_{r\to 0+}\frac{1}{2r}G'(r).$$

Since $S(r)=2r^{n-1}\pi^{n/2}/\Gamma(n/2)$, then $G(r)$ and $W(r)$, defined by

$$W(r)=r^{1-n}\int_{S(r)} g(\mathbf{a}+\mathbf{y})\sigma_r,$$

differ only by a multiplicative constant, and $W'(r)$ has already been calculated in [1].
In fact we have

$$G(r)=\frac{n\Gamma(n/2)}{\pi^{n/2}}\,W(r),$$

and from [1],

$$W'(r)=r^{1-n}\int_{S(r)} (^*dg)(\mathbf{a}+\mathbf{y}).$$

(Regarding the notation here, we refer the reader to [2].) Hence, using Stokes' theorem in E^n, we find

$$\frac{1}{2r}\,G'(r)=\frac{n}{rS(r)}\int_{S(r)} (^*dg)(\mathbf{a}+\mathbf{y})=\frac{n}{rS(r)}\int_{\|\mathbf{y}\|\leqq r} (d^*dg)(\mathbf{a}+\mathbf{y}).$$

But $d^*dg=(\nabla^2 g)\omega$ where ω is the volume element in the ball $\|\mathbf{y}\|\leqq r$. Hence

$$\frac{1}{2r}\,G'(r)=\frac{n}{rS(r)}\int_{\|\mathbf{y}\|\leqq r} (\nabla^2 g)(\mathbf{a}+\mathbf{y})\omega. \tag{2.1}$$

Now $(rS(r))/n$ is simply the volume of the ball $\|\mathbf{y}\|\leqq r$. Since $\nabla^2 g$ is continuous at $\mathbf{a}$ we can let $r\to 0+$ in (2.1) obtaining the required result, namely, that $(Ag)(\mathbf{a})=(\nabla^2 g)(\mathbf{a})$.

Proof of Lemma 2. It is sufficient to prove that g is harmonic in any open ball which excludes the origin. Let $B=\{\mathbf{x}:\|\mathbf{x}-\mathbf{a}\|<\delta\}$ with $\|\mathbf{a}\|>\delta$ be such a ball.

First we remark that there exist unique functions ω and Ω satisfying

$$\nabla^2\omega=0 \quad \text{in } B, \qquad \omega(\mathbf{x})=g(\mathbf{x}) \quad \text{on } \delta B,$$
$$\nabla^2\Omega=1 \quad \text{in } B, \qquad \Omega(\mathbf{x})=0 \quad \text{on } \delta B.$$

Furthermore we will have $\Omega(\mathbf{x})\leqq 0$ in B. This can be seen from § 5(c) of [1], for example.

We write $\Phi(\mathbf{x})=\omega(\mathbf{x})-g(\mathbf{x})$ and if $\Phi(\mathbf{x})=0$ for all $\mathbf{x}\in B$ we are finished. If not, then Φ will assume a nonzero value there, say at $\mathbf{c}$, which we may suppose to be positive.

Choose $\varepsilon>0$ to be so small that if $\psi(\mathbf{x})=\Phi(\mathbf{x})+\varepsilon\Omega(\mathbf{x})$, then $\psi(\mathbf{c})>0$. Now ψ is zero on δB and $\psi(\mathbf{c})>0$ with $\mathbf{c}\in B$. So ψ attains its supremum, which is positive, in B, say at $\mathbf{b}$. Next if $r>0$ is sufficiently small, we will have

$$\int_{S(r)} \{\omega(\mathbf{b}+\mathbf{y})-\omega(\mathbf{b})\}\sigma_r=0$$

because ω is harmonic in B and

$$\int_{S(r)} \{\Omega(\mathbf{b}+\mathbf{y})-\Omega(\mathbf{b})\}\sigma_r = \frac{r^2}{2n} S(r)$$

by virtue of § 5(b) in [1]. Hence we find that

$$\frac{2n}{r^2 S(r)} \int_{S(r)} \{\psi(\mathbf{b}+\mathbf{y})-\psi(\mathbf{b})\}\sigma_r = \frac{2n}{r^2 S(r)} \int_{S(r)} \{g(\mathbf{b})-g(\mathbf{b}+\mathbf{y})\}\sigma_r + \varepsilon. \tag{2.2}$$

Letting $r \to 0+$ and noting that $(Ag)(\mathbf{b})=0$ by hypothesis, we obtain a contradiction because $\varepsilon > 0$ while the limit of the left-hand side of (2.2) must be nonpositive. This completes the proof of Lemma 2.

3. The proof of Theorem 1. If we choose a value $\mathbf{x}$ for which $f(\mathbf{x}) \neq 0$ and let $r \to 0+$ in (1.2), we find $H(0+)=1$. Next it follows from (1.2) that

$$\frac{1}{S(r)} \int_{S(r)} \{f(\mathbf{x}+\mathbf{y})-f(\mathbf{x})\}\sigma_r = \{H(r)-1\}f(\mathbf{x}). \tag{3.1}$$

Multiplying this by $2n/r^2$ and letting $r \to 0+$, we obtain

$$(Af)(\mathbf{x}) = -\lambda^2 f(\mathbf{x}). \tag{3.2}$$

By the second hypothesis on f, we can find a function F such that

$$(\nabla^2 F)(\mathbf{x}) = f(\mathbf{x}), \qquad \|\mathbf{x}\| > 0.$$

$\nabla^2 F$ is continuous in $\|\mathbf{x}\|>0$ and so, by Lemma 1, $(\nabla^2 F)(\mathbf{x}) = (AF)(\mathbf{x})$, and so

$$(AF)(\mathbf{x}) = f(\mathbf{x}), \qquad \|\mathbf{x}\| > 0. \tag{3.3}$$

By (3.2) and (3.3) we have

$$A(f+\lambda^2 F) = 0 \quad \text{in } \|\mathbf{x}\|>0.$$

Then by Lemma 2 it follows that

$$\nabla^2(f+\lambda^2 F) = 0 \quad \text{in } \|\mathbf{x}\|>0,$$

and so

$$\nabla^2 f + \lambda^2 f = 0 \quad \text{in } \|\mathbf{x}\|>0.$$

This is the required result so far as the function f is concerned. Since f satisfies this differential equation it follows from Theorem A that

$$\frac{1}{S(r)} \int_{S(r)} f(\mathbf{x}+\mathbf{y})\sigma_r = \Gamma(\nu+1)\left(\frac{2}{\lambda r}\right)^{\nu} J_\nu(\lambda r) f(\mathbf{x}), \quad 0 < r = \|\mathbf{y}\| < \|\mathbf{x}\|.$$

Comparing this right-hand side with that of (1.2), we find that $H(r) = \Gamma(\nu+1)(2/(\lambda r))^{\nu} J_\nu(\lambda r)$, and the proof of the theorem is complete.

It may be mentioned here that the assumption concerning the existence of the $\lim_{r\to 0+} r^{-2}\{H(0+)-H(r)\}$ can be seen, by (3.1), to be equivalent to assuming that there exists a single point $\mathbf{a} \neq \mathbf{0}$, with $f(\mathbf{a}) \neq 0$ at which $Af(\mathbf{a})$ exists. It appears, then, that this assumption is indeed a weak one, and it is by no means clear how it could be discarded.

4. Conclusion. If we had assumed at the outset that f had spherical symmetry so that $f(\mathbf{x})=g(x)$, say, where we have written x for $\|\mathbf{x}\|$, then by taking an appropriate choice of polar coordinates in E^n, (1.2) would have read

$$\frac{1}{2\pi}\int_0^{2\pi} g(x_{r,\theta})|\sin\theta|^{2\nu}\,d\theta=K(r)g(x),\qquad 0<r<x. \tag{4.1}$$

Here we have written $x_{r,\theta}\equiv[x^2+r^2-2rx\cos\theta]^{1/2}$, $K(r)\equiv(1/\sqrt{\pi})(\Gamma(\nu+\frac{1}{2})/\Gamma(\nu+1))H(r)$ and as before $\nu=(n/2)-1(n\geqq 2)$. If we assume then that $g\in C(0,\infty)$ and that the $\lim_{r\to 0} r^{-2}[K(0+)-K(r)]$ exists $(=\Gamma(\nu+\frac{1}{2})/(\sqrt{\pi}\Gamma(\nu+2))(\lambda^2/4)$, say), we find from Theorem 1 that the only solutions of (4.1) are given by $g(x)=(\lambda x)^{-\nu}\{AJ_\nu(\lambda x)+BY_\nu(\lambda x)\}$, $K(r)(1/\sqrt{\pi})\Gamma(\nu+\frac{1}{2})(2/(\lambda r))^\nu J_\nu(\lambda r)$. This is a special case of a result proved previously in [3] where it was shown to hold generally for $\nu\geqq 0$. Obviously the restriction on ν in the present instance is due to the fact that $2\nu+1=n$ is the dimension of the space. On the other hand, the present theorem is more general in the sense that we are not now restricted to cases of spherical symmetry.

In the paper [3] cited above there appeared a short description of similar investigations carried out by other authors. We shall not repeat this here but merely refer the reader to that source.

Note added in proof. Professor T. Koornwinder has brought the interesting reference [4] to my attention. On page 399 of that book, the functional equation

$$\int_K f(x\,|\,y)\,dk=f(x)f(y)$$

on a symmetric space G/K is considered. The method there, which would seem to require spherical symmetry, is quite different from the one given in the present paper and our right-hand side in (1.2) is somewhat more general.

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TEMPORALLY INHOMOGENEOUS SCATTERING THEORY. II: APPROXIMATION THEORY AND SECOND ORDER EQUATIONS*

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Abstract. Of concern is scattering theory for a pair of abstract Schrödinger equations

$$du/dt = iH_j(t)u, \qquad j = 0, 1.$$

Sufficient conditions are given for the existence and completeness of the wave operators and for the unitarity of the scattering operator. Approximation theorems are established; these show that the wave and scattering operators depend continuously on the Hamiltonians $H_j(t)$. These abstract results are applied to quantum mechanical potential scattering with time-dependent potentials. Finally a scattering theory is developed for certain classes of second order evolution equations. Examples include scattering theory for the pair

$$\partial^2/\partial t^2 = \Delta u,$$

$$\partial^2 u/\partial t^2 + ir(t, x)\,\partial u/\partial t = \Delta u = q(t, x)u,$$

where either q is independent of t or $r \equiv 0$.

1. Introduction. Let H_0, H_1 be self-adjoint operators on a complex Hilbert space $\mathcal{H}$, and let $U_j = \{U_j(t) = \exp(itH_j) : t \in \mathbb{R} = (-\infty, \infty)\}$ denote the unitary group generated by iH_j. Of fundamental importance in scattering theory are the wave operators

$$(1.1) \qquad \Omega_\pm = \Omega_\pm(H_1, H_0) = \operatorname*{s\text{-}lim}_{t\to\pm\infty} U_1(-t)U_0(t).$$

Here s-lim means limit in the strong operator topology.

If the Hamiltonian $H_j = H_j(\cdot)$ is a self-adjoint operator-valued function of time ($t \in \mathbb{R}$), then H_j (in many cases) determines a family of unitary evolution operators, and wave operators can be defined by a formula similar to (1.1). (A precise formulation of this is given in § 2). *Temporally inhomogeneous scattering theory* refers to scattering theory in this context of time-dependent Hamiltonians.

There is much motivation for studying temporally inhomogeneous scattering theory. Problems involving time-dependent Hamiltonians arise in laser physics, solid state physics, magnetic resonance, and quantum field theory (cf., e.g., [28]). Scattering theory for the Dirac equation with time-dependent potentials may be able to explain certain creation and annihilation phenomena [27]. Finally, in Remark 2.9 below we will indicate how temporally inhomogeneous scattering theory subsumes long range potential scattering with modified wave operators by introducing a time dependent *unperturbed* Hamiltonian.

Temporally inhomogeneous scattering theory was studied in [26]. The main results of [26] were theorems guaranteeing the existence and completeness of the

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temporally inhomogeneous wave operators. These results are strengthened and greatly simplified in § 2. An important problem not dealt with in [26] is the following approximation problem: Show that the temporally inhomogeneous wave operators depend "continuously" (in a suitable sense) on the Hamiltonians H_1 and H_0. A result of this nature is established in § 3, and it is applied to the case of potential scattering with time-dependent potentials.

Section 4 deals with temporally inhomogeneous scattering theory for a class of second order evolution equations. The existence and completeness of the wave operators are established, as is the "continuous dependence" of the wave operators on the perturbation. Examples are included. Some of the results were presented at the Colloquium for Analysis at the Federal University of Rio de Janeiro in August, 1972 [9] and at the N.A.T.O. Advanced Study Institute on Scattering Theory at the University of Denver in June, 1973.

2. Summary of previous work. Let $\mathscr{H}$ be a complex Hilbert space. For $j=0, 1, t\in\bar{\mathbb{R}}=[-\infty, \infty]$, let $H_j(t)$ be a self-adjoint operator on $\mathscr{H}$.

Assume the following:

(A1) (i) $\mathscr{D}$, the domain of $H_j(t)$, is independent of $j=0$, 1 and $t\in\bar{\mathbb{R}}$;

(ii) $H_j(\cdot)(iI-H_j(0))^{-1}$ is piecewise strongly continuously differentiable on $\mathbb{R}$ for $j=0, 1$.

This means that each compact interval in $\mathbb{R}$ can be written as a finite union of subintervals $[a,b]$ such that for each of these subintervals $[a, b]$ there is a strongly continuously differentiable function Q on $[a, b]$ such that $Q(t)=H_j(t)(iI-H_j(0))^{-1}$ for $a<t<b$. When (A1) holds, the Cauchy problem

$$u'(t)=iH_j(t)u(t), \qquad u(s)=f\in\mathscr{D}$$

is well-posed and admits a unique (strongly) continuous and piecewise continuously differentiable solution. This follows by an easy modification of a classical result of Kato [17]. Writing the solution as $u(t)=U_j(t, s)f$, the family $U_j=\{U_j(t, s): t, s\in\mathbb{R}\}$ is called the family of evolution operators determined by iH_j. Each $U_j(t, s)$ is unitary.

Let (A1) hold. The (*temporally inhomogeneous*) *wave operators* are defined to be

$$W_\pm(\sigma)=W_\pm(\sigma; H_1, H_0)=\operatorname*{s-lim}_{t\to\pm\infty} U_1(\sigma, t)U_0(t, \sigma), \qquad \sigma\in\mathbb{R}.$$

We shall also call $W_\pm(\sigma)$ the *wave operator with initial time* σ. These wave operators (when they exist) are isometries. Note that if H_j does not depend on t, then $W_\pm(\sigma; H_1, H_0)=\Omega_\pm(H_1, H_0)$ for all $\sigma\in\mathbb{R}$ (see (1.1)).

The next assumption is that $H_j(\pm t)$ tends to $H_j(\pm\infty)$ in a suitable sense as $t\to\infty$.

(A2) There is a $\tau>0$ such that for $t>\tau$, $H_j(\pm t)-H_j(\pm\infty)$ is a bounded operator and

$$\int_\tau^\infty \|H_j(\pm t)-H_j(\pm\infty)\|\,dt<\infty \quad \text{for } j=0, 1.$$

(A3) The wave operators $\Omega_\pm(H_1(s), H_0(s))$ exist and are complete for $s = \pm\infty$.

THEOREM 2.1. *Let (A1)–(A3) hold. Then the wave operators $W_\pm(\sigma) = W_\pm(\sigma; H_1, H_0)$ exist and are unitary, and the (temporally inhomogeneous) scattering operators $S(\sigma) = W_+(\sigma)^* W_-(\sigma)$ are unitary operators on $\mathcal{H}$ for each $\sigma \in \mathbb{R}$.*

This is a variant of a result in [26]. We now sketch a proof of Theorem 2.1 which is considerably simpler than the proof given in [26].

First consider the special case where $H_1(\pm\infty) = H_0(\pm\infty)$. Fix $f \in \mathcal{D}$, $\sigma \in \mathbb{R}$. Then

$$(d/dt)U_1(\sigma, t)U_0(t, \sigma)f = -iU_1(\sigma, t)[H_1(t) - H_0(t)]U_0(r, \sigma)f$$

for all but countably many values of t, whence

$$\begin{aligned} &\|U_1(\sigma, t)U_0(t, \sigma)f - U_1(\sigma, r)U_0(r, \sigma)f\| \\ &\quad = \left\| -i\int_r^t U_1(\sigma, x)[H_1(x) - H_0(x)]U_0(x, \sigma)f\,dx \right\| \\ &\quad \leqq \left| \int_r^t \|H_1(x) - H_0(x)\|\,dx \right| \|f\|. \end{aligned} \tag{2.1}$$

Using (A2) we conclude that $\{U_1(\sigma, t)U_0(t, \sigma)f\}$ is Cauchy as $t \to \pm\infty$, whence $W_\pm(\sigma; H_1, H_0)$ exists.

For the general case, let

$$H_{k+2}(t) = \begin{cases} H_k(+\infty) & \text{for } 0 \leqq t \leqq \infty, \\ H_k(-\infty) & \text{for } 0 > t \geqq -\infty, \end{cases}$$

$k = 0, 1$. Using (A3) it is easy to see that $W_\pm(\sigma; H_3, H_2)$ exists. By the above paragraph, $W_\pm(\sigma; H_1, H_3)$ and $W_\pm(\sigma; H_2, H_0)$ exist. By the chain rule (see [25], [26]), $W_\pm(\sigma; H_1, H_0)$ exists and

$$W_\pm(\sigma; H_1, H_0) = W_\pm(\sigma; H_1, H_3)W_\pm(\sigma; H_3, H_2)W_\pm(\sigma; H_2, H_0). \tag{2.2}$$

The conclusions are symmetric in the subscripts 0, 1, since the hypotheses are, so that $W_\pm(\sigma; H_0, H_1)$ exists. The chain rule implies that $W_\pm(\sigma; H_1, H_0)$ is unitary and

$$S(\sigma) = W_\pm(\sigma; H_1, H_0)^* W_-(\sigma; H_1, H_0) = W_+(\sigma; H_0, H_1)W_-(\sigma; H_1, H_0)$$

is unitary. □

Remark 2.2. The above proof is essentially a Banach space argument, and one can formulate a Banach space version of Theorem 2.1.

Now let H_0, H_1 be self-adjoint on $\mathcal{H}$. It is customary to work with the generalized wave operators

$$\omega_\pm = \omega_\pm(H_1, H_0) = \operatorname*{s-lim}_{t\to\pm\infty} U_1(-t)U_0(t)P_0,$$

where P_0 is the orthogonal projection onto the subspace of absolute continuity of H_0 (cf. [19, pp. 516–517, 535 ff.]). The next result is a variant of Theorem 2.1 in the context of generalized wave operators. For this purpose we introduce, in addition to (A1), the following condition.

(A4) For $j=0, 1$, $s=\pm\infty$, the subspace M_j of absolute continuity of $H_j(s)$ does not depend on s; and the orthogonal projection P_j onto M_j commutes with $\exp(irH_j(t))$ for all $r, t \in \mathbb{R}$.

Let (A1), (A4) hold. The *generalized* (*temporally inhomogeneous*) *wave operators* are

$$\tilde{W}_\pm(\sigma) = \tilde{W}_\pm(\sigma; H_1, H_0) = \operatorname*{s-lim}_{t\to\pm\infty} U_1(\sigma, t)U_0(t, \sigma)P_0, \qquad \sigma \in \mathbb{R}.$$

When they exist, these operators are partial isometries with initial set M_0. They are called *complete* when their range is M_1. For this case (A3) can be weakened to the following:

(A3′) The generalized wave operators $\omega_\pm(H_1(s), H_0(s))$ exist and are complete for $s = \pm\infty$.

THEOREM 2.3. *Let* (A1), (A2), (A3′), (A4) *hold. Then the generalized wave operators* $W_\pm(\sigma) = W_\pm(\sigma; H_1, H_0)$ *exist and are complete, and the* (*temporally inhomogeneous*) *scattering operators* $S(\sigma) = W_+(\sigma)^* W_-(\sigma)$ *are unitary operators on* M_0 *for each* $\sigma \in \mathbb{R}$.

This is a generalization of [26]. In [26] (A4) was replaced by the stronger hypothesis that the subspace of absolute continuity of $H_j(t)$ was independent of $t \in \bar{\mathbb{R}}$. This stronger hypothesis is unnecessarily restrictive.

The proof of Theorem 2.3 is similar to that of Theorem 2.1, except that the chain rule part of the proof is more delicate and (A4) is used at that point. We omit the details.

The hypotheses of Theorem 2.3 hold in some cases when the hypotheses of Theorem 2.1 fail to hold. However, we shall use Theorem 2.1 in the applications since we have been unable to find any significant application of Theorem 2.3 not covered by Theorem 2.1.

Remark 2.4. There are many known conditions, for instance those involving the trace class, which can be used to check that (A3′) holds. However, since the subspace $\mathcal{H}_{ac}(H)$ of absolute continuity of a self-adjoint operator H tends to be quite unstable with regard to perturbing H, it seems difficult to check (A3) when $H_1(\infty)$ and $H_0(\infty)$ are different operators unless $\mathcal{H}_{ac}(H_1(\infty)) = \mathcal{H}_{ac}(H_0(\infty)) = \mathcal{H}$, in which case (A3′) is the same as (A3) (if the same is true for $H_1(-\infty)$ and $H_0(-\infty)$). (An exception to this is the smoothness criterion of Kato [16].)

We will verify (A3) in the applications in one of two ways: (i) $H_1(s) = H_0(s)$ for $s = \pm\infty$, or (ii) (A3′) holds and $\mathcal{H}_{ac}(H_j(s)) = \mathcal{H}$ for $j = 0, 1$ and $s = \pm\infty$.

Remark 2.5. From the intertwining relation

$$S(\sigma) = U_0(\sigma, 0)S(0)U_0(0, \sigma),$$

we see that there is in effect only one scattering operator; i.e. $S(0)$ and U_0 determine $S(\sigma)$ for all real σ.

Remark 2.6. If $H_j(t)$ and $H_k(s)$ commute for $j, k \in \{0, 1\}$ and all $t, s \in \mathbb{R}$ (in the sense that $\exp(irH_j(t)) = \exp(iuH_k(s))$ for all $r, u \in \mathbb{R}$), then (A2) can be replaced by the weaker condition

(A4′)
$$\int_0^\infty \|(H_j(\pm t) - H_j(\pm\infty))f\|\, dt < \infty$$

for all f in a dense subset D_j of $\mathscr{H}$, $j = 0, 1$. In this case (2.1) becomes

$$\begin{aligned}\|U_1(\sigma, t)U_0(t, \sigma)f - U_1(\sigma, r)U_0(r, \sigma)f\| &= \left\| -i \int_r^t U_1(\sigma, x)U_0(x, \sigma)[H_1(x) - H_0(x)]f\, dx \right\| \\ &\leqq \left| \int_r^t \|(H_1(x) - H_0(x))f\|\, dx \right|\end{aligned}$$

for $f \in D_0$, and the rest of the proof of Theorem 2.1 applies. As an example, we may take $H_j(t)$, a differential operator of the form

$$H_j(t) = \sum_{l,m=1}^{n} a^j_{lm}(t)\, \partial^2/\partial x_l\, \partial x_m + b^j(t),$$

acting on $\mathscr{H} = L^2(\mathbb{R}^n)$, where the coefficients depend on t but not on x. (Cf. [26].)

Remark 2.7. The proof of Theorems 2.1 and 2.3 are much simpler than the proofs in [26]. However, the proofs here seem to depend crucially on the linearity of $H_j(t)$, whereas the complicated proofs given in [26] can be extended to a nonlinear situation; see Wichnoski [33].

Remark 2.8. An alternate proof of Theorem 2.1 (and Theorem 2.3) can be based on the following observations.

$$\begin{aligned}(d/d\sigma)U_1(\sigma, t)f &= iH_1(t)U_1(\sigma, t)f \\ &= iH_0(t)U_1(\sigma, t)f + i[H_1(t) - H_0(t)]U_0(\sigma, t)f,\end{aligned}$$

whence by the variation of parameters formula,

$$U_1(\sigma, t)f = U_0(\sigma, t)f + i\int_t^\sigma U_0(\sigma, r)[H_1(r) - H_0(r)]U_0(\sigma, t)f\, dr.$$

This Volterra equation can be solved by iteration, and with the aid of (A2), one gets a series expansion for

$$W_\pm(\sigma) = \operatorname*{s-lim}_{t\to\pm\infty} U_1(\sigma, t)U_0(t, \sigma)$$

from which the conclusions of Theorem 2.1 can be shown to follow. The series expansion for $W_\pm(\sigma)$ appears to be known to physicists [29, p. 331].

Remark 2.9. We point out a motivation for considering a time-dependent unperturbed Hamiltonian. This observation was pointed out to us by Amrein and Piron [4]. In studying long range potential scattering, Dollard, Buslaev, Matveev, Alsholm, Kato and others (cf. [3], [10] and the references therein) defined modified wave operators

(2.3)
$$w_\pm = \operatorname*{s-lim}_{t\to\pm\infty} U_1(-t)U_0(t)Y_0(t),$$

where $\mathscr{H} = L^2(\mathbb{R}^n)$, $H_0 = \Delta$, $H_1 = H_0 - V(x)$, $V(x)$ being the potential, and $Y_0(t)$ being a certain unitary operator which depends on the long range part of $V(x)$ and which commutes with $U_0(s)$ for each $s \in \mathbb{R}$. If we set $U_0(t, 0) = U_0(t)Y_0(t)$, then formally we get

$$(d/dt)U_0(t, 0)f = (-iH_0 + Y_0'(t))U_0(t, s)f,$$

and so (2.3) can be regarded as defining the wave operator $w_\pm = W_\pm(0)$ for the pair H_1, $H_0(t) = H_0 - iY_0'(t)$. Thus, formally at least, long range potential scattering with modified wave operators is a special case of temporally inhomogeneous scattering theory.

3. Approximation theory. Suppose we have a sequence $H_j(t; n)$ of self-adjoint operator-valued functions of time. The main result of this section can be roughly stated as follows. If for $j = 0, 1$, $H_j(\cdot\,; n)$ satisfies the hypotheses of Theorem 2.1 uniformly in n, if $H_j(t; n)$ converges to $H_j(t; \infty)$ (in a suitable sense), and if the wave operators for $H_j(\pm\infty; n)$ converge to the wave operators for $H_j(\pm\infty; \infty)$, then $W_\pm(\sigma; H_1(\cdot\,; n), H_0(\cdot\,; n))$ converges to $W_\pm(\sigma; H_1(\cdot\,; \infty), H_0(\cdot\,; \infty))$ as $n \to \infty$ for each real σ.

Let $\mathbb{N}_0 = \{1, 2, \cdots, \infty\}$. The following hypotheses will be made.

(B1) For $j = 0, 1$, $n \in \mathbb{N}_0$, $H_j(\cdot\,; n)$ is a self-adjoint operator-valued function on $\bar{\mathbb{R}}$ such that (A1)–(A3) hold with $\mathscr{D}(n) = \operatorname{dom}(H_j(t; n))$ possibly depending on n.

(B2) The strong derivative of $H_j(t; n)(iI - H_j(0))^{-1}$ is bounded independently of $t \in K$, $n \in \mathbb{N}_0$, $j = 0, 1$, where K is an arbitrary compact interval in $\mathbb{R}$.

(B3) (Uniform version of (A2)). There is a $\tau > 0$ and a $\psi \in L^1[\tau, \infty)$ such that

$$\|H_j(\pm t; n) - H_j(\pm\infty; n)\| \leqq \psi(t)$$

for all $t \geqq \tau$, $n \in \mathbb{N}_0$, $j = 0, 1$.

(B4) For $t = \pm\infty$,

$$\operatorname*{s-lim}_{n\to\infty} \Omega_\pm(H_1(t; n), H_0(t; n)) = \Omega_\pm(H_1(t; \infty), H_0(t; \infty)).$$

(B5) For all $\alpha \in \mathbb{R}\backslash\{0\}$, $t \in \mathbb{R}$, $j = 0, 1$,

$$\operatorname*{s-lim}_{n\to\infty} \{iI - \alpha H_j(t; n)\}^{-1} = \{iI - \alpha H_j(t; \infty)\}^{-1}.$$

THEOREM 3.1. *Let* (B1)–(B5) *hold. Then*

$$\operatorname*{s-lim}_{n\to\infty} W_\pm(\sigma; H_1(\cdot\,; n), H_0(\cdot\,; n)) = W_\pm(\sigma; H_1(\cdot\,; \infty), H_0(\cdot\,; \infty))$$

for all $\sigma \in \mathbb{R}$. *Moreover,* $\operatorname{s-lim}_{n\to\infty} S_n(\sigma) = S_\infty(\sigma)$ *for all* $\sigma \in \mathbb{R}$, *where* $S_n(\sigma)$ *is the scattering operator for the pair* $H_1(\cdot\,; n)$, $H_0(\cdot\,; n)$ *with initial time* σ.

Proof. We shall treat the case of W_+; the proof for the case of W_- is analogous and will be omitted.

Let $H_{k+2}(t; n) = H_k(\infty; n)$ or $H_k(-\infty; n)$ according as $0 \leqq t \leqq \infty$ or $0 > t \geqq -\infty$, for $n \in \mathbb{N}_0$, $k = 0, 1$. By (B1), Theorem 2.1, and (2.2),

$$W_+(\sigma; H_1(\cdot\,; n), H_0(\cdot\,; n)) = W_+(\sigma; H_1(\cdot\,; n), H_3(\cdot\,; n)) \tag{3.1}$$
$$\cdot W_+(\sigma; H_3(\cdot\,; n), H_2(\cdot\,; n))\, W_+(\sigma; H_2(\cdot\,; n), H_0(\cdot\,; n)),$$

and a straightforward calculation using (B4) shows that

$$\operatorname*{s-lim}_{n\to\infty} W_+(\sigma; H_3(\cdot; n), H_2(\cdot; n)) = W_+(\sigma; H_3(\cdot; \infty), H_2(\cdot; \infty)).$$

Consequently, as was the case in the proof of Theorem 2.1, we may assume, without loss of generality, that $H_1(t; n) - H_0(t; n)$ tends to zero as $t \to \pm\infty$ for all $n \in \mathbb{N}_0$. Let $\{U_j(t, s; n)\}$ be the family of evolution operators determined by $iH_j(\cdot\,; n)$ for $n \in \mathbb{N}_0$, $j = 0, 1$. By (B1), (B2), (B5) and a known approximation theorem for evolution operators (Goldstein [7, p. 571], Kato [21]),

$$\operatorname*{s-lim}_{n\to\infty} U_j(t, r; n) = U_j(t, r; \infty) \tag{3.2}$$

for all $t, r \in \mathbb{N}$, $j = 0, 1$. Also, (2.1) and (B3) imply

$$\begin{gathered} \|U_1(\sigma, t; n)U_0(t, \sigma; n)f - U_1(\sigma, r; n)U_0(r, \sigma; n)f\|, \\ \|U_1(\sigma, t; n)U_0(t, \sigma; n)f - W_+(\sigma; H_1(\cdot\,; n), H_0(\cdot\,; n))f\| \\ \leqq \int_t^\infty |\psi(x)|\, dx \|f\| \end{gathered} \tag{3.3}$$

for all $n \in \mathbb{N}_0$, $f \in \mathcal{H}$, and $t \geqq \tau$. Writing $W_+(\sigma; n) = W_+(\sigma; H_1(\cdot\,; n), H_0(\cdot\,; n))$, we have that

$$\|W_+(\sigma; n)f - W_+(\sigma; \infty)f\| \leqq J_1 + J_2 + J_3,$$

where

$$J_1 = \|W_+(\sigma; n)f - U_1(\sigma, t; n)U_0(t, \sigma; n)f\|,$$
$$J_2 = \|U_1(\sigma, t; n)U_0(t, \sigma; n)f - U_1(\sigma, t; \infty)U_0(t, \sigma; \infty)f\|,$$
$$J_3 = \|U_1(\sigma, t; \infty)U_0(t, \sigma; \infty)f - W_+(\sigma; \infty)f\|.$$

Let $\varepsilon > 0$ be given. By (3.3) and (B3) we can choose t so large that $J_1 < \varepsilon/3$ and $J_3 < \varepsilon/3$. Fix this t. Next, using (3.2), we can find $N = N(\varepsilon, f)$ such that $J_2 < \varepsilon/3$ for all $n > N$. It follows that

$$\|W_+(\sigma; n)f - W_+(\sigma; \infty)f\| < \varepsilon$$

for all $n > N$. This proves the first assertion of the theorem.

Next, $S_n(\sigma) = W_+(\sigma; n)^* W_-(\sigma; n)$ for each $\sigma \in \mathbb{R}$ and each $n \in \mathbb{N}_0$. But $W_+(\sigma; n)^* = W_+(\sigma; n)^{-1}$, and $W_+(\sigma; n)^{-1} = W_+(\sigma; H_0(\cdot\,; n), H_1(\cdot\,; n))$ by the chain rule. By the first part of Theorem 3.1,

$$\operatorname*{s-lim}_{n\to\infty} W_+(\sigma; H_0(\cdot\,; n), H_1(\cdot\,; n)) = W_+(\sigma; H_0(\cdot\,; \infty), H_1(\cdot\,; \infty))$$

since the subscripts 0, 1 are interchangeable in the hypotheses. Thus s-$\lim_{n\to\infty} W_-(\sigma; n) = W_-(\sigma; \infty)$ and s-$\lim_{n\to\infty} W_+(\sigma; n)^* = W_+(\sigma; \infty)^*$, whence s-$\lim_{n\to\infty} S_n(\sigma) = S_\infty(\sigma)$. □

An alternate proof of the above theorem can be based on Remark 2.8. Also, a variant of Theorem 3.1 can be proved by basing the proof on Theorem 2.3 rather than on Theorem 2.1.

We shall apply Theorem 3.1 to potential scattering with time-dependent potentials. Let $\mathcal{H} = L^2(\mathbb{R}^l)$. Let H_0 be the self-adjoint realization of Δ, the Laplacian. Let $\mathcal{D}$ denote the domain of H_0. (Thus $\mathcal{D}$ is the Sobolev space $W^{2,2}(\mathbb{R}^l)$.)

(C1) Let $p_1, \cdots, p_m$ satisfy $2 \leqq p_j < \infty$, $p_j > l/2$ for $j = 1, \cdots, m$ and let $p_0 = \infty$. For each $t \in \bar{\mathbb{R}}$, let $q(t) = \sum_{j=0}^m q_j(t)$ where $q_j(t)$ is real-valued and in $L^{p_j}(\mathbb{R}^l)$, and let the maps $q_j\colon \mathbb{R} \to L^{p_j}(\mathbb{R}^l)$ be piecewise (strongly) continuously differentiable.

(C2) (i) $q(\pm\infty, x) = O(|x|^{-\alpha})$ for some $\alpha > 1$ as $|x| \to \infty$;
(ii) $q(\pm\infty, x) \geqq 0$.

(C3) There is a $\tau > 0$ such that

$$\int_\tau^\infty \|q(\pm t) - q(\pm\infty)\|_\infty \, dt < \infty,$$

where $\|\cdot\|_\infty$ is the $L^\infty(\mathbb{R}^l)$ norm.

THEOREM 3.2. *Let* (C1)–(C3) *hold. Set* $H_0 = \Delta$ (*independent of* t), $H_1(t) = \Delta - q(t)$ *for* $t \in \bar{\mathbb{R}}$. *Then the wave operators* $W_\pm(\sigma; H_1, H_0)$ *exist and are complete, and the scattering operators* $S(\sigma)$ *are unitary on* $L^2(\mathbb{R}^l)$ *for each* $\sigma \in \mathbb{R}$.

This is a refinement of results in [26], [9], and the proof is similar to proofs in those papers. Briefly, (C1) together with a result of Kato [15] imply that (A1) holds. Using (C2), a result of Agmon [2] and Lavine [24] implies that $\mathcal{H}_{sc}(H_1(\pm\infty)) = \{0\}$, and a result of Kato [18] implies $H_1(\pm\infty)$ has no negative eigenvalues. Since $q(\pm\infty) \geq 0$, $H_1(\pm\infty)$ has no nonnegative eigenvalues. Hence $H_1(\pm\infty)$ is spectrally absolutely continuous, and (A3′) holds with $P_0 = P_1 = I$; hence (A3) holds. (C3) implies (A2), and (A3) holds by (C2) and a result of Kato [22]. Theorem 3.2 now follows from Theorem 2.1.

That (C1) implies (A1) (ii) was stated in [9] without proof, so we give a proof here. We must show that if J is an interval on which each q_j is continuously differentiable, then $q(\cdot)f$ is strongly continuously differentiable on J for each $f \in \mathcal{D}$. By replacing 3 by l in some elementary calculations with Fourier transforms (cf. [19, p. 301], [6, p. 84]) and by using the Hausdorff–Young theorem [34, p. 251], we see that $\mathcal{D} \subset L^r(\mathbb{R}^l)$ when $r^{-1} = 2^{-1} - p^{-1}$ for each $p > l/2$. Hence (since $p_j \geqq 2$, $p_j > l/2$),

$$\begin{aligned}\left\|\sum_{j=0}^m [h^{-1}\{q_j(t+h) - q_j(t)\} - q_j'(t)]f\right\|_2 \\ \leqq \|h^{-1}\{q_0(t+h) - q_0(t)\} - q_0'(t)\|_\infty \|f\|_2 \\ + \sum_{j=1}^m \|h^{-1}\{q_j(t+h) - q_j(t)\} - q_j'(t)\|p_j'.\end{aligned}$$

$\|f\|_{r_j} \to 0$ as $h \to 0$ for $t, t+h \in J, f \in \mathcal{D}$, where $r_j^{-1} = 2^{-1} - p_j^{-1}$. The desired conclusion follows. □

Condition (C2) can be weakened. Agmon [2] and Lavine [24] have given general conditions on q for the Schrödinger operator $H=\Delta-q$ on $L^2(\mathbb{R}^l)$ to satisfy $\mathcal{H}_{sc}(H)=\{0\}$. Agmon [1] has given general conditions for H to have no negative eigenvalues. Weidmann [32] has given conditions for $\mathcal{H}_d(H)=\{0\}$ which allow q to take negative values.

Again, let $\mathbb{N}_0=\{1, 2, \cdots, \infty\}$.

(D1) For $n\in\mathbb{N}_0$, $t\in\bar{\mathbb{R}}$, let $q(t; n)$ be such that (C1)–(C3) hold.

(D2) Write $q(t;n)=\sum_{j=0}^m q_j(t;n)$ as in (C1). Then for $0\leqq j\leqq m$, $\sup\{\|(d/dt)q_j(t;n)\|_{p_j} : |t|\leqq T, m\in\mathbb{N}_0\}<\infty$ and $\sup\{\|q_j(t;n)\|_{p_j} : |t|\leqq T, 0\leqq j\leqq m, n\in\mathbb{N}_0\}<\infty$ for each $T\in\mathbb{R}$.

(D3) There is a $\tau>0$ and a $\psi\in L^1[\tau,\infty)$ such that

$$\|q(\pm t; n)-q(\pm\infty; n)\|_\infty\leqq\psi(t)$$

for all $t\geqq\tau$, $n\in\mathbb{N}_0$. Here $\|\cdot\|_\infty$ is the $L^\infty(\mathbb{R}^l)$ norm.

(D4) $q(\pm\infty; n)\in L^1(\mathbb{R}^l)\cap L^2(\mathbb{R}^l)$ and

$$\lim_{n\to\infty}\|q(\pm\infty; n)-q(\pm\infty;\infty)\|_1=0.$$

(D5)
$$\lim_{n\to\infty}\|q_j(t;n)-q_j(t;\infty)\|_{p_j}=0$$

for all $t\in\mathbb{R}$ and $j=0, 1, \cdots, m$.

THEOREM 3.3. *Let* (D1)–(D5) *hold, and suppose* $l\leqq 3$. *Let* $H_0=\Delta$, $H_1(t;n)=\Delta-q(t;n)$ *for* $t\in\bar{\mathbb{R}}$, $n\in\mathbb{N}_0$. *Then*

$$\text{s-}\lim_{n\to\infty} W_\pm(\sigma; H_1(\cdot\,;\infty), H_0)=W_\pm(\sigma; H_1(\cdot\,;\infty), H_0)$$

for all $\sigma\in\mathbb{R}$. *Moreover,* $\text{s-lim}_{n\to\infty} S_n(\sigma)=S_\infty(\sigma)$ *for all* $\sigma\in\mathbb{R}$, *where* $S_n(\sigma)$ *is for* $n\in\mathbb{N}_0$ *the scattering operator for the pair* $H_1(\cdot\,;n)$, H_0 *with initial time* σ.

Proof. We must check that the hypotheses of Theorem 3.1 are satisfied. First (D1) implies (B1) with $\mathcal{D}(n)=\mathcal{D}$ $(=W^{2,2}(\mathbb{R}^l))$ and $\mathcal{H}=L^2(\mathbb{R}^l)$.

Note that

$$H_j(t;n)(iI-H_0)^{-1}=(\Delta-q(t;n))(iI-\Delta)^{-1}$$
$$=\Delta(iI-\Delta)^{-1}-\sum_{j=0}^m q_j(t;n)(iI-\Delta)^{-1}.$$

Let J be a compact interval in $\mathbb{R}$, and let C be a bound for $\|q_j'(t;n)\|_{p_j}$, for $0\leqq j\leqq m$, $t\in J$, $n\in\mathbb{N}_0$ (by the first part of (D2)). Then

$$\|q_0'(t;n)(iI-\Delta)^{-1}f\|_2\leqq C\|(iI-\Delta)^{-1}\|\,\|f\|_2.$$
$$\|q_j'(t;n)(iI-\Delta)^{-1}f\|_2\leqq C\|(iI-\Delta)^{-1}f\|_\infty\leqq CK\|f\|_2$$

for $1\leqq j\leqq m$, $f\in\mathcal{H}$, $t\in J$, $n\in\mathbb{N}_0$, where K is the norm of $(iI-\Delta)^{-1}$ as a mapping from $L^2(\mathbb{R}^l)$ to $L^\infty(\mathbb{R}^l)$; $K<\infty$ by the closed graph theorem since $l\leqq 3$. It follows that (B2) holds.

Clearly (D3) implies (B3). Assumption (D4) and the last part of (D2) imply (B4) by a result of Kuroda [23, p. 452] since $l\leqq 3$. It remains to show that

(B5) holds for $j=1$. For this we use (D5). Let $f\in\mathscr{H}$, $\alpha\in\mathbb{R}\backslash\{0\}$, $t\in\mathbb{R}$, and set $g=(iI-\alpha H_1(t;\infty))^{-1}f$. Recall that $g\in\bigcap\{L^r(\mathbb{R}^l):2\leqq r\leqq\infty\}$. Thus

$$
\begin{aligned}
&\|(iI-\alpha H_1(t;n))^{-1}f-(iI-\alpha H_1(t;\infty))^{-1}f\|_2\\
&\quad\leqq\|f-(iI-\alpha H_1(t;n))(iI-\alpha H_1(t;\infty))^{-1}f\|_2\\
&\quad=|\alpha|\,\|(H_1(t;n)-H_1(t;\infty))g\|_2\\
&\quad\leqq|\alpha|\left\{\sum_{j=1}^{m}\|q_j(t;n)-q_j(t;\infty)\|_{p_j}\|g\|_{r_j}+\|q_0(t;n)-q_0(t;\infty)\|_\infty\|g\|_2\right\}\\
&\quad\to 0\quad\text{as } n\to\infty,
\end{aligned}
$$

where $r_j^{-1}+p_j^{-1}=2^{-1}$. Thus (B5) holds. Theorem 3.3 now follows from Theorem 3.1. □

Theorem 3.3 can be modified so that it is valid with $l>3$. Rather than using Kuroda's result [23], one can use Stankevič's more general results [30].

4. A class of second order evolution equations. The scattering theory discussed thus far can be termed scattering theory for the pair of Schrödinger equations

$$u'(t)=iH_j(t)u(t),\qquad ('=d/dt),\quad j=0,1.$$

In this section we discuss scattering theory for second order evolution equations of the form

$$u''(t)+iB_j(t)u'(t)+A_j^2u(t)=0,\qquad j=0,1. \tag{4.1}$$

Here A_j, $B_j(t)$ are self-adjoint operators on a complex Hilbert space $\mathscr{K}$ with $B_j(t)$ bounded for $|t|$ sufficiently large. Writing

$$U(t)=\begin{bmatrix}A_ju(t)\\ u'(t)\end{bmatrix},\qquad G_j(t)=\begin{bmatrix}0 & A_j\\ -A_j & -iB_j(t)\end{bmatrix},$$

the Cauchy problem for (4.1) becomes equivalent to the Cauchy problem for

$$U'(t)=G_j(t)U(t),\qquad j=0,1$$

in the Hilbert space $\mathscr{H}=\mathscr{K}\oplus\mathscr{K}$. $H_j(t)=-iG_j(t)$ with domain $\mathscr{D}(H_j(t))=\mathscr{D}(A_j)\oplus\mathscr{D}(A_j)$ will turn out to be a self-adjoint operator on $\mathscr{H}$. The wave operators $W_\pm(\sigma)=W_\pm(\sigma;H_1,H_0)$ will be termed the wave operators for (4.1). The following assumptions will be made.

(E1) (i) For $j=0$, 1, $t\in\bar{\mathbb{R}}$, A_j and $B_j(t)$ are self-adjoint on $\mathscr{K}$ with A_j invertible and $\mathscr{D}(A_0)=\mathscr{D}(A_1)$; there is a $\tau\geqq0$ such that $B_j(t)$ is bounded for $|t|\geqq\tau$ and $B_j(\cdot)$ is piecewise strongly continuously differentiable on $\{t\in\mathbb{R}:|t|\geqq\tau\}$. (ii) For $|t|\leqq\tau$, $\mathscr{D}(B_j(t))\supset\mathscr{D}(A_j)$ and there are constants $a(t)<1$, $b(t)\geqq0$ such that

$$\|B_j(t)f\|\leqq a(t)\|A_jf\|+b(t)\|f\|$$

for each $f\in\mathscr{D}(A_j)$, and $B_j(\cdot)(iI-A_j)^{-1}$ is piecewise strongly continually differentiable on $[-\tau,\tau]$.

(E2) $\int_\tau^\infty\|B_j(\pm t)-B_j(\pm\infty)\|\,dt<\infty$, $j=0,1$.

(E3) $\Omega_{\pm}(H_1(s), H_0(s))$ exist and are complete for $s = \pm\infty$.

THEOREM 4.1. *Let* (E1)–(E3) *hold. Then the wave operators* $W_{\pm}(\sigma)$ *for the equations* (4.1) *exist and are complete, and the scattering operators are unitary on* $\mathcal{H}$ *for each* $\sigma \in \mathbb{R}$.

Proof. Assumptions (A1) and (A2) follow from (E1) by a known result (Goldstein [8, Cor. 5]). Assumption (E3) implies (A3), (E2) implies (A4), and (E4) implies (A5). The result now follows from Theorem 2.1. □

Remark 4.2. (i) In many cases of interest, $B_j(t)$ will be bounded for each $t \in \mathbb{R}$, in which case one can take $\tau = 0$ in (E1)(i) and (E1) (ii) can be omitted.

(ii) As noted previously (see Remark 2.4), it is important (and usually nontrivial) to find criteria for (E3) to hold. The following condition is a variant of (E3).

(E4) (i) $\omega_{\pm}(H_1(s), H_0(s))$ exist and are complete for $s = \pm\infty$. (ii) $\mathcal{H}_{sc}(H_j(\pm\infty)) = \{0\}$, $j = 0, 1$. (iii) $H_j(s)$ has no eigenvalues for $s = \pm\infty$, $j = 0, 1$.

Taken together, (E4) (ii) and (iii) say that $\mathcal{H}_{ac}(H_j(\pm\infty)) = \mathcal{H}$, $j = 0, 1$. Concerning (E4) (ii), there are some known results concerning the absence of the subspace of singular continuity. (Cf., e.g., Agmon [2], Lavine [24].) The following two propositions give conditions for (E4) (ii)–(iii) and for (E4) (ii) to hold.

Two self-adjoint operators $A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda)$ and $B = \int_{-\infty}^{\infty} \lambda \, dF(\lambda)$ will be said to *commute* if $F(\lambda)E(\mu) = E(\mu)F(\lambda)$ for all $\lambda, \mu \in \mathbb{R}$.

PROPOSITION 4.3. *Suppose that for* $j = 0, 1$, $B_j(\pm\infty)$ *and* A_j *commute. Then* $\mathcal{H}_{ac}(H_j(s)) = \mathcal{H}$ *for* $s \in \{-\infty, \infty\}$ *if and only if* $\mathcal{H}_{ac}(C_{\pm}(s; j)) = \mathcal{H}$ *for* $s \in \{-\infty, \infty\}$, *where*

$$C_{\pm}(s; j) = -B_j(s) \pm (B_j(s)^2 + 4A_j^2)^{1/2}.$$

PROPOSITION 4.4. *Assumption* (E4) (iii) *holds if and only if* λ^2 *is not an eigenvalue of* $A_j^2 - \lambda B_j(s)$ *for all* $\lambda \in \mathbb{R}$, $j = 0, 1$, $s = \pm\infty$.

Proof of Proposition 4.3. Let

$$N_{\pm}(s; j) = A_j^2 + 4^{-1}C_{\pm}(s; j)^2 = 2A_j^2 - 2^{-1}C_{\pm}(s; j)B_j(s);$$

the last equality holds since $X = 2^{-1}C_{\pm}(s; j)$ is easily seen to satisfy $X^2 + XB_j(s) - A_j^2 = 0$. (We are using the commutativity hypothesis.) Let

$$U_j(s) = \begin{bmatrix} iA_jN_+(s; j)^{-1/2} & -iA_jN_-(s; j)^{-1/2} \\ 2^{-1}C_-(s; j)N_+(s; j)^{-1/2} & -2^{-1}C_+(s; j)N_-(s; j)^{-1/2} \end{bmatrix}.$$

Note that by the operational calculus associated with the spectral theorem, each entry in the matrix $U_j(s)$ is a bounded operator, even if $N_{\pm}(s; j)$ is not invertible.

Some straightforward but rather messy computations, which we omit, show that for each $s = \pm\infty$ and $j = 0, 1$, $U_j(s)$ is unitary and

$$U_j(s)^* H_j(s) U_j(s) = -2^{-1} \begin{bmatrix} C_+(s; j) & 0 \\ 0 & C_-(s; j) \end{bmatrix}.$$

Denote this last operator by $Q_j(s)$. Then since $U_j(s)$ is unitary,

$$U_j(s)^* \mathcal{H}_{ac}(H_j(s)) = \mathcal{H}_{ac}(Q_j(s)) = \mathcal{H}_{ac}(C_+(s; j)) \oplus \mathcal{H}_{ac}(C_-(s; j))$$

and the proposition follows. □

Proof of Proposition 4.4. By (E1), $H_j(s)$ acting on $\mathcal{D}(A_j)\oplus\mathcal{D}(A_j)$ is self-adjoint for $j=0, 1, s=\pm\infty$. $H_j(s)f=\lambda f$ means

$$(4.2) \qquad iA_jf_2=-\lambda f_1, \qquad -iA_jf_1+B_j(s)f_2=-\lambda f_2,$$

where

$$f=\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

For $\lambda\neq 0$, this is equivalent to $(A_j^2-\lambda B_j(s))f_2=\lambda^2 f_2$ and $f_1=-i\lambda^{-1}A_jf_2$. Thus $\lambda\in\mathbb{R}\backslash\{0\}$ is an eigenvalue for $H_j(s)$ implies λ^2 is an eigenvalue for $A_j^2-\lambda B_j(s)$, and conversely. For $\lambda=0$, $H_j(s)f=0$ means $iA_jf_2=iA_jf_1+B_j(s)f_2$ by (4.2), which is equivalent to $f_1=f_2=0$ since A_j is invertible. The proposition is thus proved. □

Remark 4.5. In connection with Proposition 4.3, we note that temporally inhomogeneous scattering theory is greatly simplified when one makes the (physically unrealistic) assumption that the Hamiltonians $H_j(t)$ and $H_j(s)$ commute for all real values of t and s and $j=0, 1$. (Cf. Remark 2.6.) Note that the commutativity hypothesis of Proposition 4.3 does not imply that $H_j(+\infty)$ commutes with $H_j(-\infty)$.

We next establish an approximation theorem in the context of Theorem 4.1.

(F1) For $j=0, 1$, $t\in\bar{\mathbb{R}}$, $n\in\mathbb{N}_0$ let $A_j(n)$ and $B_j(t; n)$ be self-adjoint operators on $\mathcal{K}$. Suppose that (E1)–(E3) hold for each $n\in\mathbb{N}_0$, with τ independent of n.

(F2) The strong derivative of $B_j(\cdot\,; n)(iI-A_j(n))^{-1}$ satisfies

$$\sup\{\|(d/dt)B_j(t; n)(iI-A_j(n))^{-2}\| : n\in\mathbb{N}_0, |t|\leqq T\}<\infty$$

for each $T\in\mathbb{R}^+$, $j=0, 1$.

(F3) There is a $\psi\in L^1[\tau,\infty)$ such that

$$\|B_j(\pm t; n)-B_j(\pm\infty; n)\|\leqq\psi(t)$$

for each $t\geqq\tau$, $n\in\mathbb{N}_0$.

(F4) Same as (B4).

(F5) For all $\alpha\in\mathbb{R}\backslash\{0\}$ and $j=0, 1$,

$$\operatorname*{s-lim}_{n\to\infty}(iI-\alpha A_j(n))^{-1}=(iI-\alpha A_j(\infty))^{-1},$$

$$\operatorname*{s-lim}_{n\to\infty}B_j(t; n)=B_j(t;\infty) \quad \text{for } |t|\geqq\tau,$$

$$\operatorname*{s-lim}_{n\to\infty}B_j(t; n)(iI-A_j(n))^{-1}=B_j(t;\infty)(iI-A_j(\infty))^{-1} \quad \textit{for } |t|\leqq\tau.$$

THEOREM 4.6. *Let* (F1)–(F5) *hold. Then the conclusions of Theorem* 3.1 *hold.*

This is proved by showing that (F1)–(F5) imply (B1)–(B5) and applying Theorem 3.1. We omit the details.

Example 4.7. Consider the equation

$$(4.3) \qquad \partial^2u/\partial t^2+ir_j(t, x)\,\partial u/\partial t=\Delta u-q_j(x)u, \qquad j=0.\ 1.$$

Take $\mathcal{K}=L^2(\mathbb{R}^3)$, $\mathcal{H}=\mathcal{K}\oplus\mathcal{K}$, $A_j^2=-\Delta+q_j(x)$, and $(B_j(t)f)(x)=r_j(t, x)f(x)$. With these identifications, (4.4) is seen to be a special case of (4.1). The following hypotheses on the coefficients will be made.

(G1) For $j = 0, 1$, q_j satisfies (C1)–(C3).

(G2) For $j = 0, 1$, r_j satisfies (C1) and (C3) with $r_{j,k}(t) = 0$ for all $t \in \mathbb{R}$, $k = 1, \cdots, m$.

(G3) $r_j(\pm\infty) = 0$ for $j = 0, 1$.

If (G1)–(G3) *hold, then* (E1), *then* (E1)–(E3) *hold, so the conclusions of Theorem* 4.1 *apply.*

Proof. Assumptions (G1) and (G2) imply that (E1) holds with $\tau = 0$. Assumption (E2) follows from (G2) and (G3). Assumption (E4) (ii) and (iii) follow from Proposition 4.8 below and (G3). Assumption (E4) (i) holds by (G1) and a result of Kato [20] (cf. also Thoe [31]). □

Note that $B_j(\pm\infty)$ and A_j will not commute if $r_j(\pm\infty) \neq 0$. In this case, it is easy to apply Proposition 4.4 and conclude that (E4)(iii) holds.

Whenever $B_j(\pm\infty) = 0$, we have the following consequence of Proposition 4.3.

COROLLARY 4.8. *Let A on $\mathcal{K}$ be a nonnegative self-adjoint operator. Then*

$$M = \begin{bmatrix} 0 & iA \\ -iA & 0 \end{bmatrix}$$

on $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$, with domain $\mathcal{D}(M) = \mathcal{D}(A) \oplus \mathcal{D}(A)$, is self-adjoint, and $\mathcal{H}_{ac}(M) = \mathcal{H}$ if and only if $\mathcal{K}_{ac}(A) = \mathcal{K}$.

(H1) For $j = 0, 1$ and $t \in \bar{\mathbb{R}}$ let $A_j(t)$ be a self-adjoint operator on $\mathcal{K}$ satisfying $A_j(t) \geqq \varepsilon(t) I$ where $\varepsilon : \bar{\mathbb{R}} \to (0, \infty)$ is measurable; $\mathcal{D} = \mathcal{D}(A_j(t))$ is independent of j and t; and $A_j(\cdot) A_j(0)^{-1}$ is piecewise strongly continuously differentiable.

(H2) There is a $\tau > 0$ *such that for* $t > \tau$, $A_j(\pm\tau, A_j(\pm t) - A_j(\pm\infty)$ is bounded and $\int_\tau^\infty \|A_j(\pm t) - A_j(\pm\infty)\| \, dt < \infty$, $j = 0, 1$.

(H3) The wave operators $\Omega_\pm(A_1(s), A_0(s))$ exist and are complete for $s = \pm\infty$.

THEOREM 4.9. *Let* (H1)–(H3) *hold. Then the wave operators governing the equations*

$$u''(t) - A_j'(t) A_j(t)^{-1} u'(t) + A_j(t)^2 u(t) = 0, \qquad j = 0, 1, \tag{4.4}$$

exist and are complete, and the associated scattering operators are unitary on $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$.

Proof. Write

$$w(t) = \begin{bmatrix} u(t) \\ A(t)^{-1} u'(t) \end{bmatrix}.$$

Then u is a solution of (4.4) in $\mathcal{K}$ if and only if w is a solution of

$$w'(t) = \begin{bmatrix} 0 & A_j(t) \\ -A_j(t) & 0 \end{bmatrix} w(t), \qquad j = 0,1, \tag{4.5}$$

in $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$.

$$H_j(t) = -i \begin{bmatrix} 0 & A_j(t) \\ A_j(t) & 0 \end{bmatrix},$$

with domain $\mathcal{D}(H_j(t)) = \mathcal{D} \oplus \mathcal{D}$ (cf. (H1)), is self-adjoint on $\mathcal{H}$, and by the wave operators $W_\pm(\sigma)$ governing (4.4) we mean the wave operators $W_\pm(\sigma; H_1, H_0)$ in the sense of § 2. Assumption (A1) holds by (H1). Assumption (H2) implies (A2). Finally, (H3) together with a computation involving the Birman–Kato invariance principle show that (A5) holds; cf. Kato [20]. Theorem 4.9 now follows from Theorem 2.1. □

Example 4.10. Let us specialize (4.4). If $A_j(\cdot)$ is piecewise constant (so that $A_j'(t) = 0$ wherever it exists), then (4.4) becomes

$$u''(t) + A_j(t)u(t) = 0, \qquad j = 0, 1. \tag{4.6}$$

More specifically, let $\{J_k : K \in \mathbb{N}_0\}$ be a collection of pairwise disjoint intervals whose union is $\mathbb{R}$, and with the property that each compact interval meets only finitely many of the J_k. Suppose $A_j(t) = B_{jk}$ for $t \in J_k$, and suppose that (H1)–(H4) hold. Then Theorem 4.9 applies to (4.6).

In particular, the wave operators exist and are complete for the pair of wave equations

$$\partial^2 u/\partial t^2 = \Delta u - q(t, x)u,$$

$$\partial^2 u/\partial t^2 = \Delta u,$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, $q(t, x) = q_k(x)$ for $t \in J_k$, and where (C1)–(C3) hold. In other words, the conditions which imply the existence and completeness of the wave operators for the Schrödinger equation with a time-dependent potential also imply the existence and completeness of the wave operators for the wave equation with the same time-dependent potential, if we assume that as a function of time the potential is piecewise constant.

Remark 4.11. (i) It is easy to formulate approximation theorems in the context of Examples 4.7 and 4.9 and Corollary 4.8. We omit the details.

(ii) After a draft of this paper was completed, we received the interesting preprints of A. Inoue [12], [13], [14]. Inoue's results are in the same spirit as the present paper and our earlier work [9], [26], but his results and methods are different. J. Howland [11] also (independently) considered temporally inhomogeneous scattering theory, but using a stationary approach as opposed to a time-dependent approach.

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QUADRATIC TRANSFORMATIONS OF APPELL FUNCTIONS*

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Abstract. A double integral average of x^t with two rows and two columns has eight quadratic transformations into itself, each with two free parameters and two independent variables. Three of the transformations change double hypergeometric series of order two into series of order three, and one of these (modified by a linear transformation applicable to polynomial cases) permits a very direct proof of the addition theorem and related results for Gegenbauer polynomials. Two others transform Appell's F_2 into F_2 or F_1, and two transform F_1 into F_1. One of the latter contains Landen's transformation of the first and second incomplete elliptic integrals, and the other contains Bartky's transformation of the third complete elliptic integral.

1. Introduction. Let v and v' be complex numbers with positive real parts, and define a complex measure $m_{(v,v')}$ on the interval $0 \leqq u \leqq 1$ by

$$dm_{(v,v')}(u) = \frac{\Gamma(v + v')}{\Gamma(v)\Gamma(v')} u^{v-1}(1 - u)^{v'-1}\, du. \tag{1.1}$$

Let $\mathbb{C}_0 = \{\xi \in \mathbb{C} : \xi \neq 0,\ |\arg \xi| < \pi\}$ denote the complex plane cut along the negative real axis. The R-function [2] is defined for any complex t as an average of ξ^t over a line segment in $\mathbb{C}_0$ with endpoints x and y,

$$R_t(v, v'; x, y) = \int_0^1 [ux + (1 - u)y]^t\, dm_{(v,v')}(u). \tag{1.2}$$

The $\mathscr{R}$-function [4] is similarly defined as a double average over a quadrilateral in $\mathbb{C}_0$ with vertices x, y, z, w,

$$\begin{gathered}\mathscr{R}_t(\mu, \mu'; Z; v, v') = \int_0^1 \int_0^1 (\mathbf{u} \cdot Z \cdot \mathbf{v})^t\, dm_{(\mu,\mu')}(u)\, dm_{(v,v')}(v),\\ Z = \begin{bmatrix} x & y \\ z & w \end{bmatrix}, \qquad \mathbf{u} \cdot Z \cdot \mathbf{v} = uxv + uy(1 - v) + (1 - u)zv + (1 - u)w(1 - v).\end{gathered} \tag{1.3}$$

If the rows (or columns) of Z are identical, $\mathscr{R}$ reduces to R. Also, by (1.2),

$$\mathscr{R}_t(\mu, \mu'; Z; v, v') = \int_0^1 R_t[\mu, \mu'; vx + (1 - v)y, vz + (1 - v)w]\, dm_{(v,v')}(v). \tag{1.4}$$

The R-function is known [2] to have a holomorphic continuation to all v, v' such that $v + v' \neq 0, -1, -2, \cdots$ and to all $x, y \in \mathbb{C}_0$ (even if the line segment with endpoints x and y intersects the cut). The $\mathscr{R}$-function has a holomorphic continuation in the parameters provided neither $\mu + \mu'$ nor $v + v'$ is zero or a negative integer. Despite an assertion in [4, p. 421], it has not been established that x, y, z, w can have arbitrary positions with respect to the cut. Hence the formulas

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in this paper will be understood to apply when all entries of Z lie in the right half-plane or in any larger domain to which continuation is possible. The parameters will be understood to satisfy the restrictions stated above.

The R-function in two variables is equivalent to the Gauss hypergeometric function,

$$R_t(v, v'; x, y) = y^t \, {}_2F_1(-t, v; v + v'; 1 - x/y). \tag{1.5}$$

Both sides are homogeneous of degree t and are unchanged by simultaneously interchanging x with y and v with v'. The $\mathscr{R}$-function also is homogeneous and is unchanged by permutation of rows or columns together with their associated parameters, μ and μ' being associated respectively with the first and second rows and v and v' with the columns. It is unchanged also by interchanging (μ, μ') with (v, v') and replacing Z by its transpose. We shall be concerned with four special cases of $\mathscr{R}$:

$$\begin{gathered} \mathscr{R}_t(\mu, \mu'; Y_1; v, v') = w^{-t} R_t(\mu, \mu'; y, w) R_t(v, v'; z, w), \\ Y_1 = \begin{bmatrix} yz/w & y \\ z & w \end{bmatrix} \quad (Y_1 \text{ singular}); \end{gathered} \tag{1.6}$$

$$\begin{gathered} \mathscr{R}_t(\mu, \mu'; Y_2; v, v') = w^t F_2\left(-t, \mu, v; \mu + \mu', v + v'; 1 - \frac{y}{w}, 1 - \frac{z}{w}\right), \\ Y_2 = \begin{bmatrix} y + z - w & y \\ z & w \end{bmatrix} \quad \text{(parallelogram condition)}; \end{gathered} \tag{1.7}$$

$$\begin{gathered} \mathscr{R}_t(\mu, \mu'; Y_3; v, v') = z^t \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-t)_{m+n}(\mu)_{m+n}(v)_m(v')_n}{(\mu + \mu')_{m+n}(v + v')_{m+n} m! n!} \left(1 - \frac{x}{z}\right)^m \left(1 - \frac{y}{z}\right)^n, \\ Y_3 = \begin{bmatrix} x & y \\ z & z \end{bmatrix} \quad \text{(triangle condition)}, \quad \left|1 - \frac{x}{z}\right| < 1, \quad \left|1 - \frac{y}{z}\right| < 1; \end{gathered} \tag{1.8}$$

$$\begin{gathered} \mathscr{R}_{-v-v'}(\mu, \mu'; Y_4; v, v') = z^{-v} w^{-v'} F_1\left(\mu, v, v'; \mu + \mu'; 1 - \frac{x}{z}, 1 - \frac{y}{w}\right), \\ Y_4 = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \quad \text{(condition on parameters)}. \end{gathered} \tag{1.9}$$

Equations (1.6) and (1.7) follow respectively from [4, (6.5)] and [4, (3.3)]. The entries of Y_2 are the vertices of a parallelogram in the complex plane, and F_2 is an Appell function. To prove (1.8), we may assume $z = 1$ (by homogeneity), expand $\mathscr{R}$ in series [4, (6.2)], and use [7, (2.6), (2.1)]. In Horn's classification of double hypergeometric series, the right side of (1.8) is a series of order three. It reduces to Appell's F_1 if $\mu = v + v'$. Equation (1.9) follows from [4, § 4 and (5.7)].

The many quadratic transformations of ${}_2F_1$ into itself [8, § 2.11] can be deduced from two independent transformations, which we write using (1.5) as

$$R_{2t}(v, v; x, y) = R_t\left[v + t, \tfrac{1}{2} - t; \left(\frac{x+y}{2}\right)^2, xy\right], \tag{1.10}$$

$$R_t(v, v; x^2, y^2) = R_t\left[2v + t, \tfrac{1}{2} - v - t; \left(\frac{x+y}{2}\right)^2, xy\right]. \tag{1.11}$$

All members are symmetric and homogeneous of degree $2t$ in x and y, and the arguments on the right sides are squares of arithmetic and geometric means. In the case of double hypergeometric series, there are many quadratic transformations connecting different series of order two [8, § 5.11], [9]. The only ones previously known [1] which transform a double series into another series of the same type (Appell's F_1) have a single free parameter. One of these [1, (5.10)] is a generalization of Landen's transformation of incomplete elliptic integrals.

2. Quadratic transformations of $\mathcal{R}$. In the following list of eight quadratic transformations of $\mathcal{R}$ into itself, the first connects a series of order three with a product of ${}_2F_1$-series, as one sees from (1.6) and (1.8). The next five connect F_2 with various series (in order, a product of ${}_2F_1$-series, a series of order three, a series of order three, another F_2, and F_1). The seventh and eighth connect F_1 with itself. There are essentially only two independent variables (because of homogeneity) and two free parameters in each case. (Four similar transformations with three free parameters have only one independent variable.) Erdélyi [9, (11.4), (11.5)] found the second transformation in 1948. The first transformation is modified for polynomials in (3.5) and subsequently applied to Gegenbauer polynomials. The fifth and sixth are restated in Appell's notation later in this section, and the seventh and eighth in § 4.

$$\begin{gathered}\mathcal{R}_t(\mu, \mu; Z_1; \mu, \mu) = \mathcal{R}_t(2\mu + t, \tfrac{1}{2} - \mu - t; W_1; \mu, \mu),\\ Z_1 = \begin{bmatrix} x^2 & y^2 \\ z^2 & w^2 \end{bmatrix}, \quad xw = yz, \quad W_1 = \begin{bmatrix} \left(\frac{x+w}{2}\right)^2 & \left(\frac{y+z}{2}\right)^2 \\ xw & yz \end{bmatrix};\end{gathered} \tag{2.1}$$

$$\begin{gathered}\mathcal{R}_{(1/2)-\mu-v}(\mu, \mu; Z_2; v, v) = \mathcal{R}_{(1/2)-\mu-v}(\mu, \mu; W_2; v, v),\\ Z_2 = \begin{bmatrix} x^2 & y^2 \\ z^2 & w^2 \end{bmatrix}, \quad xw = yz \quad \text{or} \quad x^2 + w^2 = y^2 + z^2,\\ W_2 = \begin{bmatrix} (\lambda - x)^2 & (\lambda - y)^2 \\ (\lambda - z)^2 & (\lambda - w)^2 \end{bmatrix}, \quad 2\lambda = x + y + z + w,\end{gathered} \tag{2.2}$$

(W_2 satisfies the parallelogram condition if Z_2 is singular, and vice versa);

$$\mathscr{R}_t(2v+t,\tfrac{1}{2}-v-t;Z_3;v,v)=\mathscr{R}_t(2v+t,\tfrac{1}{2}-v-t;W_3;v,v),$$

(2.3)
$$Z_3=\begin{bmatrix} x^2 & y^2 \\ x^2-z^2 & y^2-z^2 \end{bmatrix},\quad W_3=\begin{bmatrix} \left(\dfrac{x+y}{2}\right)^2 & \left(\dfrac{x+y}{2}\right)^2 \\ (x+z)(y-z) & (x-z)(y+z) \end{bmatrix},$$

(the entries of Z_3 and W_3 become squares of arithmetic and geometric means on putting $p = x + z$, $q = x - z$, $r = y + z$, $s = y - z$);

$$\mathscr{R}_{2t}(\mu,\mu;Z_4;\mu,\mu)=\mathscr{R}_t(\mu+t,\tfrac{1}{2}-t;W_4;\mu,\mu),$$

(2.4)
$$Z_4=\begin{bmatrix} x & y \\ z & w \end{bmatrix},\quad x+w=y+z,\quad W_4=\begin{bmatrix} \left(\dfrac{x+w}{2}\right)^2 & \left(\dfrac{y+z}{2}\right)^2 \\ xw & yz \end{bmatrix};$$

$$\mathscr{R}_{2t}(\mu,\mu;Z_5;-\mu-t,-\mu-t)=\mathscr{R}_t(\mu+t,\tfrac{1}{2}-t;W_5;-\mu,\tfrac{1}{2}-t),$$

(2.5)
$$Z_5=\begin{bmatrix} x^2 & y^2 \\ z^2 & w^2 \end{bmatrix},\quad x^2+w^2=y^2+z^2,\quad W_5=\begin{bmatrix} \left(\dfrac{x^2+w^2}{2}\right)^2 & \left(\dfrac{xy+zw}{2}\right)^2 \\ \left(\dfrac{xz+yw}{2}\right)^2 & xyzw \end{bmatrix};$$

$$\mathscr{R}_{2t}(\mu,\tfrac{1}{2}-\mu-t;Z_6;-t,-t)=\mathscr{R}_{2t}(\mu,\tfrac{1}{2}-\mu-t;W_6;\mu,\tfrac{1}{2}-\mu-t),$$

(2.6)
$$Z_6=\begin{bmatrix} x^2 & y^2 \\ x^2+z^2 & y^2+w^2 \end{bmatrix},$$

$$W_6=\begin{bmatrix} xy & \left(\dfrac{x+y}{2}\right)^2+\left(\dfrac{z+w}{2}\right)^2 \\ \left(\dfrac{x+y}{2}\right)^2+\left(\dfrac{z-w}{2}\right)^2 & x^2+z^2 \end{bmatrix},$$

$$x^2+z^2=y^2+w^2;$$

$$\mathscr{R}_{2t}(\mu+t,\tfrac{1}{2}-t;Z_7;-t,-t)=\mathscr{R}_{2t}(\mu+t,\tfrac{1}{2}-t;W_7;\mu+t,\tfrac{1}{2}-\mu-2t),$$

(2.7)
$$Z_7=\begin{bmatrix} x^2 & y^2 \\ z^2 & w^2 \end{bmatrix},\quad W_7=\begin{bmatrix} (\lambda-x)(\lambda-y) & (\lambda-x)(\lambda-y) \\ (\lambda-z)(\lambda-w) & zw \end{bmatrix},$$

$$x^2+w^2=y^2+z^2,\quad 2\lambda=x+y+z+w;$$

$$\mathcal{R}_{2t}(\mu, \mu; Z_8; -t, -t) = \mathcal{R}_t(\mu + t, \tfrac{1}{2} - t; W_8; \mu, \tfrac{1}{2} - \mu - t),$$

(2.8)
$$Z_8 = \begin{bmatrix} x^2 & y^2 \\ z^2 & w^2 \end{bmatrix}, \quad x^2 + w^2 = y^2 + z^2, \quad W_8 = \begin{bmatrix} \left(\dfrac{xy + zw}{2}\right)^2 & \left(\dfrac{xy + zw}{2}\right)^2 \\ \left(\dfrac{xw + yz}{2}\right)^2 & xyzw \end{bmatrix}.$$

Most of these transformations contain a quadratic transformation of ${}_2F_1$ as a special case. In particular, (1.10) is obtained by putting $z = x$ and $w = y$ in (2.4), or $y = x$ and $w = z$ in (2.5) or (2.8). Similarly, (1.11) is obtained by putting $z = x$ and $w = y$ in (2.1), or $z = 0$ in (2.3).

Equations (2.5) and (2.6) transform F_2 into F_2 or F_1. (Transformations of F_1 into itself will be discussed in § 4.) In (2.5) we put $t = -\alpha$ and

(2.9)
$$x + w = \cos\frac{\theta + \varphi}{2}, \quad x - w = \sin\frac{\theta + \varphi}{2},$$
$$y + z = \cos\frac{\theta - \varphi}{2}, \quad y - z = \sin\frac{\theta - \varphi}{2}$$

to get the only nonlinear transformation of F_2 into itself,

(2.10)
$$F_2(\alpha, \alpha + \tfrac{1}{2}, \alpha + \tfrac{1}{2}; \mu + \tfrac{1}{2}, \alpha - \mu + \tfrac{1}{2}; \sin^2\theta, \sin^2\varphi)$$
$$= [1 + \sin(\theta + \varphi)]^{-2\alpha} F_2\left[2\alpha, \mu, \alpha - \mu; 2\mu, 2\alpha - 2\mu; \frac{2\sin\theta\cos\varphi}{1 + \sin(\theta + \varphi)}, \frac{2\cos\theta\sin\varphi}{1 + \sin(\theta + \varphi)}\right].$$

In (2.6) we put $t = -\alpha$ and

(2.11) $x = \cos(\theta + \varphi), \quad y = \cos(\theta - \varphi), \quad z = \sin(\theta + \varphi), \quad w = \sin(\theta - \varphi)$

to get

(2.12)
$$F_2(2\alpha, \mu, \mu; \alpha + \tfrac{1}{2}, \alpha + \tfrac{1}{2}; \sin^2\theta, \sin^2\varphi)$$
$$= F_1[\mu, \alpha, \alpha; \alpha + \tfrac{1}{2}; \sin^2(\theta + \varphi), \sin^2(\theta - \varphi)].$$

All eight transformations are proved in essentially the same way by comparing two reduction formulas for F_4. (One can of course bypass the $\mathcal{R}$-function notation to get (2.10) and (2.12).) We shall give details for (2.1) and bare essentials for the other seven. From [7, (4.1), (3.6)] we find

$$(2.13)\quad \begin{aligned} &F_4\left[-t, 2v+t; v+\tfrac{1}{2}, v+\tfrac{1}{2}; \left(\frac{\sin\theta+\sin\varphi}{2}\right)^2, \left(\frac{\sin\theta-\sin\varphi}{2}\right)^2\right] \\ &= R_t\left[2v+t, \tfrac{1}{2}-v-t; \cos^2\left(\frac{\theta+\varphi}{2}\right), 1\right] \\ &\quad \cdot R_t\left[2v+t, \tfrac{1}{2}-v-t; \cos^2\left(\frac{\theta-\varphi}{2}\right), 1\right] \\ &= R_t(v, v; e^{i\theta+i\varphi}, e^{-i\theta-i\varphi})R_t(v, v; e^{i\theta-i\varphi}, e^{-i\theta+i\varphi}) \\ &= \mathscr{R}_t(v, v; Z; v, v), \qquad Z = \begin{bmatrix} e^{i2\theta} & e^{i2\varphi} \\ e^{-i2\varphi} & e^{-i2\theta} \end{bmatrix}. \end{aligned}$$

In the last two steps we have used (1.11) and [4, (6.5)]. From [7, (4.6)] we find

$$(2.14)\quad \begin{aligned} &F_4\left[-t, 2v+t; v+\tfrac{1}{2}, v+\tfrac{1}{2}; \left(\frac{\sin\theta+\sin\varphi}{2}\right)^2, \left(\frac{\sin\theta-\sin\varphi}{2}\right)^2\right] \\ &= \mathscr{R}_t(2v+t, \tfrac{1}{2}-v-t; W; v, v), \qquad W = \begin{bmatrix} \cos^2\theta & \cos^2\varphi \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

Eliminating F_4 between (2.13) and (2.14) gives

$$(2.15)\quad \mathscr{R}_t(v, v; Z; v, v) = \mathscr{R}_t(2v+t, \tfrac{1}{2}-v-t; W; v, v).$$

Multiplying (2.15) by λ^{2t}, we use the homogeneity of $\mathscr{R}$ and put

$$(2.16)\quad x = \lambda e^{i\theta}, \quad y = \lambda e^{i\varphi}, \quad z = \lambda e^{-i\varphi}, \quad w = \lambda e^{-i\theta}$$

to obtain (2.1).

Transformation (2.2) can be proved either by eliminating $F_4(\alpha, \beta; 2\beta, \alpha-\beta+1)$ between [7, (4.1)] and [7, (4.5)] or by eliminating $F_4(\alpha, \alpha+\tfrac{1}{2}; \gamma, 2\alpha-\gamma+\tfrac{3}{2})$ between [7, (4.1)] and (7, (4.4)]. We eliminate $F_4(\alpha, 1+\alpha-\gamma; \gamma, \gamma)$ between [7, (4.5)] and [7, (4.6)] to get (2.3), $F_4(\alpha, \alpha+\tfrac{1}{2}; \gamma, \gamma)$ between [7, (4.4)] and [7, (4.6)] to get (2.4), $F_4(\alpha, \alpha+\tfrac{1}{2}; \gamma, 1+\alpha-\gamma)$ between [7, (4.3)] and [7, (4.4)] to get (2.5), $F_4(2\gamma-1, \beta; \gamma, \gamma)$ between [7, (4.3)] and [7, (4.6)] to get (2.6), $F_4(2\delta-1, \delta; \gamma, \delta)$ between [7, (4.2)] and [7, (4.5)] to get (2.7), and $F_4(\alpha, \alpha+\tfrac{1}{2}; \gamma, \alpha+\tfrac{1}{2})$ between [7, (4.2)] and [7, (4.4)] to get (2.8). Other similar eliminations lead to trivialities or to (1.10) (or (1.11).

An alternative proof of (2.4) begins by multiplying both members by $[(x+w)/2]^{-2t}$, using homogeneity, and expanding both sides in series by [4, (6.3)]. On the left side we then use [5, (5.6), (5.7)] and (3.2) below, and on the right side [7, (2.6)]. Both sides then have the same form, and all steps are reversible.

3. Applications to Gegenbauer polynomials. The transformation of a ${}_2F_1$ polynomial from argument z to $1-z$ is equivalent to

$$(3.1)\quad (\mu+v)_n R_n(\mu, v; x, y) = (\mu)_n R_n(1-\mu-v-n, v; x, x-y),$$

where n is a nonnegative integer. Applied to the right side of (1.11) with t replaced by n, this leads to

$$(2\mu)_n R_n(\mu, \mu; x^2, y^2) = (\mu)_n R_n[\tfrac{1}{2} - \mu - n, \tfrac{1}{2} - \mu - n; (x + y)^2, (x - y)^2]. \tag{3.2}$$

Among all the transformations of R_n into itself (3.2) is the only one with equal parameters on both sides. It relates the Gegenbauer polynomials [defined by the generating function $(1 - 2t\cos\theta + t^2)^{-\nu}$] to the Jacobi polynomials with equal indices [3, (2.8)]:

$$\begin{aligned} \frac{C_n^\nu(\cos\theta)}{C_n^\nu(1)} &= R_n(\nu, \nu; e^{i\theta}, e^{-i\theta}) \\ &= \frac{2^n(\nu)_n}{(2\nu)_n} R_n(\tfrac{1}{2} - \nu - n, \tfrac{1}{2} - \nu - n; \cos\theta + 1, \cos\theta - 1). \end{aligned} \tag{3.3}$$

The normalizing constant is $C_n^\nu(1) = (2\nu)_n/n!$.

Equations (3.1), (3.2), (3.3) have generalizations in terms of $\mathscr{R}$. From (3.1) and (1.4) we find the linear transformation

$$\begin{gathered} (\mu + \nu)_n \mathscr{R}_n(\mu, \nu; Z; \rho, \sigma) = (\mu)_n \mathscr{R}_n(1 - \mu - \nu - n, \nu; W; \rho, \sigma), \\ Z = \begin{bmatrix} x & y \\ z & w \end{bmatrix}, \quad W = \begin{bmatrix} x & y \\ x - z & y - w \end{bmatrix}. \end{gathered} \tag{3.4}$$

Applying this to the right side of (2.1) with t replaced by n, we get the quadratic transformation

$$\begin{gathered} (2\mu)_n \mathscr{R}_n(\mu, \mu; Z_9; \mu, \mu) = (\mu)_n \mathscr{R}_n(\tfrac{1}{2} - \mu - n, \tfrac{1}{2} - \mu - n; W_9; \mu, \mu), \\ Z_9 = \begin{bmatrix} x^2 & y^2 \\ z^2 & w^2 \end{bmatrix}, \quad xw = yz, \quad W_9 = \begin{bmatrix} (x + w)^2 & (y + z)^2 \\ (x - w)^2 & (y - z)^2 \end{bmatrix}. \end{gathered} \tag{3.5}$$

If $y = x$ and $w = z$, (3.4) and (3.5) reduce respectively to (3.1) and (3.2). In general, the left side of (3.5) can be written as a product of two Gegenbauer polynomials by (1.6). Thus we find an important generalization of (3.3),

$$\begin{gathered} \frac{C_n^\nu(\cos\theta) C_n^\nu(\cos\varphi)}{C_n^\nu(1) C_n^\nu(1)} = \mathscr{R}_n(\nu, \nu; X; \nu, \nu) \\ = \frac{2^n(\nu)_n}{(2\nu)_n} \mathscr{R}_n(\tfrac{1}{2} - \nu - n, \tfrac{1}{2} - \nu - n; Y; \nu, \nu), \\ X = \begin{bmatrix} e^{i\theta + i\varphi} & e^{i\theta - i\varphi} \\ e^{-i\theta + i\varphi} & e^{-i\theta - i\varphi} \end{bmatrix}, \quad Y = \begin{bmatrix} \cos(\theta + \varphi) + 1 & \cos(\theta - \varphi) + 1 \\ \cos(\theta + \varphi) - 1 & \cos(\theta - \varphi) - 1 \end{bmatrix}. \end{gathered} \tag{3.6}$$

Gegenbauer's product formula [8, 3.15(20)] follows from (3.6) by (3.3) and (1.4).

We shall now use (3.6) to deduce the addition theorem for Gegenbauer polynomials in a very direct way without assuming prior knowledge of the details of the theorem. We expand a Gegenbauer polynomial with argument $A + BX$, where A and B are constants, in a series of Jacobi polynomials with argument X.

The coefficients can be factored by (3.6) for a particular choice of the indices of the Jacobi polynomials. For other proofs see [3] and the references therein.

The Jacobi series of an entire function f is

$$f(X) = \sum_{m=0}^{\infty} \frac{1}{m!} F^{(m)}(1 + \alpha + m, 1 + \beta + m; -1, 1) \cdot R_m(-\alpha - m, -\beta - m; X + 1, X - 1), \tag{3.7}$$

where $F^{(m)}$ is the Dirichlet average of the derivative $f^{(m)}$ and α and β are the indices of the Jacobi polynomial R_m [3], [6]. If we choose

$$f(X) = C_n^\nu(A + BX) = \frac{2^n(\nu)_n}{n!} R_n(\tfrac{1}{2} - \nu - n, \tfrac{1}{2} - \nu - n; A + BX + 1, A + BX - 1), \tag{3.8}$$

then by [2, (2.6)],

$$f^{(m)}(X) = \frac{2^n(\nu)_n}{(n - m)!} B^m \cdot R_{n-m}(\tfrac{1}{2} - \nu - n, \tfrac{1}{2} - \nu - n; A + BX + 1, A + BX - 1). \tag{3.9}$$

Since this is zero if $m > n$, the series (3.7) terminates as expected. Putting $X = v(-1) + (1 - v)(+1)$ and using (1.4), we find

$$F^{(m)}(1 + \alpha + m, 1 + \beta + m; -1, 1) = \frac{2^n(\nu)_n}{(n - m)!} B^m \cdot \mathscr{R}_{n-m}(\tfrac{1}{2} - \nu - n, \tfrac{1}{2} - \nu - n; W; 1 + \alpha + m, 1 + \beta + m), \tag{3.10}$$

$$W = \begin{bmatrix} A - B + 1 & A + B + 1 \\ A - B - 1 & A + B - 1 \end{bmatrix}.$$

In order to apply (3.6), we choose $\alpha = \beta = \nu - 1$, $A = \cos\theta\cos\varphi$, $B = \sin\theta \sin\varphi$. Then for $0 \leqq m \leqq n$,

$$\begin{aligned} F^{(m)}&(\nu + m, \nu + m; -1, 1) \\ &= \frac{2^n(\nu)_n}{(n - m)!} \sin^m\theta \sin^m\varphi \cdot \mathscr{R}_{n-m}(\tfrac{1}{2} - \nu - n, \tfrac{1}{2} - \nu - n; Y; \nu + m, \nu + m) \\ &= \frac{2^m(\nu)_m}{C_{n-m}^{\nu+m}(1)} \sin^m\theta \sin^m\varphi\, C_{n-m}^{\nu+m}(\cos\theta)\, C_{n-m}^{\nu+m}(\cos\varphi). \end{aligned} \tag{3.11}$$

Substituting in (3.7), we have the addition theorem,

$$C_n^\nu(\cos\theta\cos\varphi + X\sin\theta\sin\varphi) = \sum_{m=0}^{n} \frac{(\nu)_m}{(\nu - \frac{1}{2})_m C_{n-m}^{\nu+m}(1)} \sin^m\theta \sin^m\varphi\, C_{n-m}^{\nu+m}(\cos\theta) \cdot C_{n-m}^{\nu+m}(\cos\varphi)\, C_m^{\nu-(1/2)}(X). \tag{3.12}$$

Another use of (3.6) is in passing from a series of Gegenbauer polynomials to a series of products of Gegenbauer polynomials. Let f be analytic on the inner region Ω of an ellipse with foci -1 and 1. Then f can be expanded in a series of Gegenbauer polynomials which converges uniformly on compact subsets of Ω. By (3.3),

$$f(\cos\psi) = \sum_{n=0}^{\infty} a_n \frac{C_n^\nu(\cos\psi)}{C_n^\nu(1)} \tag{3.13}$$
$$= \sum_{n=0}^{\infty} a_n \frac{2^n(\nu)_n}{(2\nu)_n} R_n(\tfrac{1}{2} - \nu - n, \tfrac{1}{2} - \nu - n; \cos\psi + 1, \cos\psi - 1),$$

where $\cos\psi \in \Omega$. By [6], ν may be any complex number provided $2\nu \neq -1, -2, -3, \cdots$. By (3.7), the coefficients are

$$a_n \frac{2^n(\nu)_n}{(2\nu)_n} = \frac{1}{n!} F^{(n)}(\tfrac{1}{2} + \nu + n, \tfrac{1}{2} + \nu + n; -1, 1). \tag{3.14}$$

Putting $\cos\psi = v\cos(\theta + \varphi) + (1 - v)\cos(\theta - \varphi)$, where $\cos(\theta \pm \varphi) \in \Omega$, we integrate term by term with respect to $dm_{(\nu,\nu)}(v)$. As in (1.1) this requires $\operatorname{Re}\nu > 0$, but this condition can later be removed by analytic continuation. Using (1.4) we find

$$F(\nu, \nu; \cos(\theta + \varphi), \cos(\theta - \varphi)] \tag{3.15}$$
$$= \sum_{n=0}^{\infty} a_n \frac{2^n(\nu)_n}{(2\nu)_n} \mathscr{R}_n(\tfrac{1}{2} - \nu - n, \tfrac{1}{2} - \nu - n; Y; \nu, \nu),$$

where F is the integral average [2] of f and Y is defined in (3.6). Hence

$$F[\nu, \nu; \cos(\theta + \varphi), \cos(\theta - \varphi)] = \sum_{n=0}^{\infty} a_n \frac{C_n^\nu(\cos\theta)\, C_n^\nu(\cos\varphi)}{C_n^\nu(1) C_n^\nu(1)}. \tag{3.16}$$

A case of particular interest is $f(x) = (1 - 2tx + t^2)^{-\lambda}$, for which (3.14) and (3.16) yield

$$R_{-\lambda}[\nu, \nu; 1 - 2t\cos(\theta + \varphi) + t^2, 1 - 2t\cos(\theta - \varphi) + t^2]$$
$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\nu)_n} t^n R_{-\lambda-n}[\tfrac{1}{2} + \nu + n, \tfrac{1}{2} + \nu + n; (1 + t)^2, (1 - t)^2] \tag{3.17}$$
$$\cdot \frac{C_n^\nu(\cos\theta)\, C_n^\nu(\cos\varphi)}{C_n^\nu(1)}, \qquad |t| < \exp(-|\operatorname{Im}\theta| - |\operatorname{Im}\varphi|).$$

This formula was given in different notation by Henrici [10, (65)]. The R-function on the right side reduces by (1.11) and [2, (3.21)] to 1 if $\lambda = \nu$ and to $(1 - t^2)^{-1}$ if $\lambda = \nu + 1$, leading respectively to Ossicini's formula [3, (3.2)] and the well-known Poisson kernel for Gegenbauer series. The latter case provides an integral representation of the solution of certain boundary value problems, for the first of the following equations implies the second [as one sees by substituting (3.17) in the integrand and using the orthogonality of Gegenbauer polynomials]:

$$\sum_{n=0}^{\infty} \alpha_n C_n^\nu(\cos\theta) = f(\cos\theta),$$

$$\sum_{n=0}^{\infty} \alpha_n r^n C_n^\nu(\cos\theta)$$

$$(3.18) \qquad = (1 - r^2)\int_0^\pi f(\cos\varphi)R_{-\nu-1}[\nu, \nu; 1 - 2r\cos(\theta+\varphi) + r^2, 1 - 2r\cos(\theta-\varphi) + r^2]\, d\mu(\varphi),$$

$$d\mu(\varphi) = (\sin\varphi)^{2\nu}\, d\varphi \left(\int_0^\pi (\sin\psi)^{2\nu}\, d\psi\right)^{-1},$$

$$0 \leqq r < 1, \quad 0 \leqq \theta \leqq \pi, \quad \nu > -\tfrac{1}{2}.$$

A second important case is $f(x) = e^{itx}$, for which (3.16) becomes

$$\exp(it\cos\theta\cos\varphi)S(\nu, \nu; it\sin\theta\sin\varphi, -it\sin\theta\sin\varphi)$$

$$(3.19) \qquad = S[\nu, \nu; it\cos(\theta+\varphi), it\cos(\theta-\varphi)]$$

$$= \sum_{n=0}^{\infty} \frac{(it/2)^n}{(\nu)_n C_n^\nu(1)} S(\tfrac{1}{2} + \nu + n, \tfrac{1}{2} + \nu + n; it, -it)C_n^\nu(\cos\theta)C_n^\nu(\cos\varphi).$$

The symmetric S-function [2] is a Bessel function with the branch point at 0 removed to make it entire. It is normalized to 1 at $x = 0$:

$$(3.20) \qquad S(\tfrac{1}{2} + \nu, \tfrac{1}{2} + \nu; ix, -ix) = \Gamma(1 + \nu)(x/2)^{-\nu}J_\nu(x).$$

Substitution in (3.19) gives a formula due to Gegenbauer [12, § 11.5, (9)] which has many uses in relating Bessel functions of integral order to those of half-odd-integral order. For example, we put $t\cos\varphi = kr$ and $t\sin\varphi = \alpha r$ to expand $\exp(ikr\cos\theta)J_m(\alpha r\sin\theta)$, $m = 0, 1, 2, \cdots$, in a series of spherical Bessel functions. Solutions of the wave equation in cylindrical coordinates are thereby related to solutions in spherical coordinates.

4. Generalized Landen transformations. Equations (2.7) and (2.8) respectively imply

$$(4.1) \qquad F_1(2\alpha, \beta, \beta; \alpha + \beta + \tfrac{1}{2}; \sin^2\theta, \sin^2\varphi) = F_1[\beta, \alpha, \alpha; \alpha + \beta + \tfrac{1}{2}; \sin^2(\theta+\varphi), \sin^2(\theta-\varphi)],$$

$$F_1(\alpha, \beta, \beta; 2\alpha; \sin^2\theta, \sin^2\varphi)$$

$$(4.2) \qquad = \left(\frac{1 + \cos\theta\cos\varphi}{2}\right)^{-2\beta} F_1\left[\beta, \alpha, \beta - \alpha + \tfrac{1}{2}; \alpha + \tfrac{1}{2}; \frac{\sin^2\theta\sin^2\varphi}{(1 + \cos\theta\cos\varphi)^2}, \left(\frac{1 - \cos\theta\cos\varphi}{1 + \cos\theta\cos\varphi}\right)^2\right].$$

These appear to be the only nonlinear transformations of F_1 into itself with two free parameters. As indicated below, the Landen transformations of the first and second incomplete elliptic integrals are contained in (4.1). The Bartky transformation of the third complete elliptic integral is contained in (4.2).

In deducing these formulas, we shall avoid some linear transformations of F_1 by using its symmetric variant,

$$(4.3)\quad \begin{aligned} &R_t(\beta_1, \beta_2, \beta_3; z_1, z_2, z_3) \\ &\quad = z_3^t F_1(-t, \beta_1, \beta_2; \beta_1 + \beta_2 + \beta_3; 1 - z_1/z_3, 1 - z_2/z_3). \end{aligned}$$

The R-function is unchanged by permuting 1, 2, 3. It is a single Dirichlet average of x^t over a triangle in $\mathbb{C}_0$ with vertices z_1, z_2, z_3. From (1.8), (4.3), and the row symmetry of $\mathscr{R}$, we find

$$(4.4)\quad \mathscr{R}_t(\mu, \nu + \nu'; Y; \nu, \nu') = R_t(\mu, \nu, \nu'; x, y, z), \qquad Y = \begin{bmatrix} x & x \\ y & z \end{bmatrix}.$$

From (1.9),

$$(4.5)\quad \begin{aligned} &\mathscr{R}_{-\nu-\nu'}(\mu, \mu'; Y_4; \nu, \nu') = z^{-\nu} w^{-\nu'} R_{-\mu}\left(\nu, \nu', \mu + \mu' - \nu - \nu'; \frac{x}{z}, \frac{y}{w}, 1\right), \\ &\qquad Y_4 = \begin{bmatrix} x & y \\ z & w \end{bmatrix}. \end{aligned}$$

Applying (4.5) to the left side of (2.7) and (4.4) to the right side, we find

$$(4.6)\quad \begin{aligned} &R_{-\mu-t}(-t, -t, \tfrac{1}{2} + \mu + 2t; x^2/z^2, y^2/w^2, 1) \\ &\quad = R_{2t}\left[\mu + t, \mu + t, \tfrac{1}{2} - \mu - 2t; \frac{(\lambda - x)(\lambda - y)}{zw}, \frac{(\lambda - z)(\lambda - w)}{zw}, 1\right] \\ &\qquad x^2 + w^2 = y^2 + z^2, \qquad 2\lambda = x + y + z + w. \end{aligned}$$

The substitutions

$$(4.7)\quad \begin{aligned} &x = A \sin(\theta - \varphi) \cos(\theta + \varphi), \qquad y = A \sin(\theta + \varphi) \cos(\theta - \varphi), \\ &z = A \sin(\theta - \varphi), \qquad w = A \sin(\theta + \varphi), \end{aligned}$$

satisfy the parallelogram condition $z^2 - x^2 = w^2 - y^2$ and imply

$$(4.8)\quad \begin{aligned} (\lambda - x)(\lambda - y) &= A^2 \sin(\theta + \varphi) \sin(\theta - \varphi) \cos^2 \varphi, \\ (\lambda - z)(\lambda - w) &= A^2 \sin(\theta + \varphi) \sin(\theta - \varphi) \cos^2 \theta. \end{aligned}$$

With $t = -\alpha$ and $\mu = \alpha + \beta$, (4.6) becomes

$$(4.9)\quad \begin{aligned} &R_{-\beta}(\alpha, \alpha, \beta - \alpha + \tfrac{1}{2}; \cos^2(\theta + \varphi), \cos^2(\theta - \varphi), 1) \\ &\quad = R_{-2\alpha}(\beta, \beta, \alpha - \beta + \tfrac{1}{2}; \cos^2 \theta, \cos^2 \varphi, 1) \\ &\quad = (\cos\theta \cos\varphi)^{1-2\alpha} R_{\alpha-\beta-(1/2)}(\beta, \beta, \alpha - \beta + \tfrac{1}{2}; \cos^2 \theta, \cos^2 \varphi, \\ &\qquad\qquad \cos^2 \theta \cos^2 \varphi). \end{aligned}$$

In the last step we have used Euler's transformation [2, (4.23)] and homogeneity. The equality of the first and second members of (4.9) implies (4.1) by way of (4.3). The equality of the first and third members includes Landen's transformation of the first elliptic integral as the case $\alpha = \beta = \frac{1}{2}$ and of the second integral as the case $\alpha = -\beta = \frac{1}{2}$. The case in which $\alpha = \frac{1}{2}$ but β remains free is the one-parameter generalization of Landen's transformation first given in [1]. To see this we put

$$\cos^2(\theta + \varphi) = z_1/z_3, \qquad \cos^2(\theta - \varphi) = z_2/z_3, \qquad \cos^2\theta = w_2/z_3, \qquad \cos^2\varphi = w_3/z_3, \tag{4.10}$$

and compare with [1, (5.10), (5.11)].

Applying (4.5) to the left side of (2.8) and (4.4) to the right side, we find

$$\begin{aligned} &R_{-\mu}(-t, -t, 2\mu + 2t; x^2/z^2, y^2/w^2, 1) \\ &= R_t\left[\mu + t, \mu, \tfrac{1}{2} - \mu - t; \left(\frac{xy + zw}{2zw}\right)^2, \left(\frac{xw + yz}{2zw}\right)^2, \frac{xy}{zw}\right], \\ &\qquad x^2 + w^2 = y^2 + z^2. \end{aligned} \tag{4.11}$$

Since the transformation now depends only on the ratios x/z and y/w, we replace y by yw/z. Then the condition $x^2 + w^2 = (yw/z)^2 + z^2$ has no effect except to determine w, which no longer occurs in the transformation. Putting $\mu = \alpha$ and $t = -\beta$, we get

$$\begin{aligned} &R_{-\alpha}(\beta, \beta, 2\alpha - 2\beta; x^2, y^2, z^2) \\ &= z^{2\beta - 2\alpha} R_{-\beta}\left[\alpha, \beta - \alpha + \tfrac{1}{2}, \alpha - \beta; \left(\frac{x + y}{2}\right)^2, xy, \left(\frac{xy + z^2}{2z}\right)^2\right]. \end{aligned} \tag{4.12}$$

With $x = z\cos\theta$ and $y = z\cos\varphi$, this becomes (4.2) by way of (4.3). The cases $\alpha = \beta = \frac{1}{2}$ and $\alpha = \beta = \frac{3}{2}$ imply Landen's transformations of the first and second complete elliptic integrals, but these transformations are already contained in (1.11). If $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, both sides of (4.12) are complete elliptic integrals of the third kind and can be reduced to standard integrals. As the three standard complete integrals we choose the homogeneous functions

$$\begin{aligned} &R_K(x, y) = R_{-1/2}(\tfrac{1}{2}, \tfrac{1}{2}; x, y), \qquad R_E(x, y) = R_{1/2}(\tfrac{1}{2}, \tfrac{1}{2}; x, y), \\ &R_L(x, y, z) = R_{-1/2}(\tfrac{1}{2}, \tfrac{1}{2}, 1; x, y, z), \end{aligned} \tag{4.13}$$

all three being symmetric in x and y [13]. With the help of relations between associated R-functions, (4.12) gives

$$\begin{aligned} R_K(x^2, y^2) &= R_K\left[\left(\frac{x + y}{2}\right)^2, xy\right], \\ R_E(x^2, y^2) &= 2R_E\left[\left(\frac{x + y}{2}\right)^2, xy\right] - xyR_K(x^2, y^2), \\ R_L(x^2, y^2, z^2) &= \frac{1}{2}\frac{z - w}{z + w} R_L\left[\left(\frac{x + y}{2}\right)^2, xy, \left(\frac{z + w}{2}\right)^2\right] + \frac{2w}{z + w} R_K(x^2, y^2), \\ xy &= zw. \end{aligned} \tag{4.14}$$

The first two equations, which are Landen's transformations [1, § 5], will be useful in discussing the third, which is Bartky's transformation of the third complete elliptic integral.

Any complete elliptic integral,

$$bR_K(x^2, y^2) + cR_E(x^2, y^2) + dR_L(x^2, y^2, z^2), \tag{4.15}$$

is transformed by (4.14) into

$$b_1 R_K(x_1^2, y_1^2) + c_1 R_E(x_1^2, y_1^2) + d_1 R_L(x_1^2, y_1^2, z_1^2), \tag{4.16}$$

where c_1 is proportional to c and d_1 to d but both are independent of b. A case with $c = 0$ is Bartky's elliptic integral,

$$\begin{aligned} &\frac{2}{\pi}\int_0^{\pi/2} \frac{ax^2\cos^2\theta + bz^2\sin^2\theta}{x^2\cos^2\theta + z^2\sin^2\theta}\frac{d\theta}{(x^2\cos^2\theta + y^2\sin^2\theta)^{1/2}} \\ &\quad = bR_K(x^2, y^2) + \frac{a-b}{2}R_L(x^2, y^2, z^2). \end{aligned} \tag{4.17}$$

The values of $b_1, a_1 - b_1, x_1, y_1, z_1$ are found by inspection of (4.14). A program for automatic computation of this integral by iterating the transformation is given by R. Bulirsch and J. Stoer in [11, Teil III, p. 416]. The program can be used also to calculate cases of (4.15) with $d = 0$, since the second complete integral can be expressed in terms of the first and third:

$$R_E(x, y) = yR_K(x, y) + \frac{x-y}{2}R_L(x, y, x). \tag{4.18}$$

The Bartky transformation of this restricted R_L reduces to the Landen transformation of R_E, as one finds by putting $z = x$ in the third equation of (4.14).

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NONLINEAR DIFFERENTIAL EQUATIONS EQUIVALENT TO SOLVABLE NONLINEAR EQUATIONS*

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Abstract. This paper shows in a simple and direct way the equivalence of the nonlinear differential equation $y''+r(x)y'+q(x)Z(y)=A(y)y'^2+g(x)z(y)[u(y)]^a$, $Z(y)=z(y)u(y)$, to the linear equation $L_1u=g(x)$, $a=0$, or to the nonlinear equation $L_1u=g(x)u^a$, $a\neq 0$, where $L_1=d^2/dx^2+r(x)\,d/dx+q(x)$. The two differential equations in which $A(y)$ is equal to y^{-1} or to $(1-l)y^{-1}$ serve as particular examples. Some nonlinear equations in u are solvable for certain values of the exponent a. An analogous class of nonlinear partial differential equations is presented. These results generalize the earlier work of Herbst.

1. Introduction. If one can establish an equivalence between a given nonlinear differential equation and a linear one, then one can utilize the considerable known theory for linear equations. A number of such equivalences are listed by Kamke in his compendium [15]. There is also a small literature [3], [6]–[8], [19]–[24] on equivalent differential equations based on a note by Pinney [19] (Berkovič and Rozov [4] note that Pinney's equation was solved by V. P. Ermakov in 1880). Technical applications of the equivalence idea are found in [1], [5], [7], [10], [13], [16]–[17].

Here we develop additional nonlinear differential equations which are equivalent to linear or solvable nonlinear ones. We start with two nonlinear equations which are extensions of forms obtained by Herbst [14, eqs. III, IV] and show directly how these two equations transform into linear forms. Next, we define two linearizing transformations T_1 and T_2 and apply them to certain nonlinear equations of the second order. We also give a transformation T which linearizes a more general equation of Herbst [14, eq. II]. Finally, we show that these transformations can also be applied to a class of nonlinear partial differential equations.

2. Transformations T_1 and T_2. The extended versions of the two equations of Herbst mentioned above have the forms

$$y''+r(x)y'+kq(x)y=(1-l)y^{-1}y'^2+y^{1-l}[kq(x)+g(x)],\qquad (')=d/dx, \tag{2.1}$$

$$y''+r(x)y'+kq(x)y=(1-l)y^{-1}y'^2+y^{1-l}[\beta q(x)+kg(x)], \tag{2.2}$$

where β and l are arbitrary parameters and $kl=1$. We can obtain an integral

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combination by dividing both equations by y^{1-l} and grouping as follows:

$$[y''-(1-l)y^{-1}y'^2]/y^{1-l}+r(x)y'/y^{1-l} = kq(x)(1-y^l)+g(x), \tag{2.3}$$

$$[y''-(1-l)y^{-1}y'^2]/y^{1-l}+r(x)y'/y^{1-l}+kq(x)y^l = \beta q(x)+kg(x). \tag{2.4}$$

It follows easily that (2.3) and (2.4) transform into the linear equations

$$u''+r(x)u'+q(x)u=g(x), \tag{2.5}$$

where $u=(y^l-1)/l$ and $u=y^l-l\beta$ for (2.3) and (2.4), respectively. If in (2.1) we let $l\to 0$, we obtain

$$y''+r(x)y'+q(x)y\log y=y^{-1}y'^2+g(x)y \tag{2.1$'$}$$

which transforms into (2.5) on letting $u=\log y$. (Equations (2.1′) and (2.2) are the same as Herbst's eqs. III and IV except for the additional terms $g(x)y$ and $kg(x)y^{1-l}$, respectively.) We shall refer to the removal of the term $y^{-1}y'^2$ from (2.1′) and related equations as transformation T_1 and to the removal of the term $(1-l)y^{-1}y'^2$ from (2.2) and related equations as transformation T_2.

For second order equations, the general form of the nonlinear differential equation equivalent to a linear nonhomogeneous equation has been obtained by deSpautz and Lerman [9]. Thus (2.1′) and (2.2) are explicit examples of their result.

3. Nonlinear equations equivalent to specific nonlinear equations. The transformations T_1 and T_2 reduce the nonlinear equations

$$y''+r(x)y'+q(x)y\log y=y^{-1}y'^2+g(x)y(\log y)^a, \tag{3.1}$$

$$L_k y=(1-l)y^{-1}y'^2+kg(x)y^{1-l+al}, \qquad \beta=0, \tag{3.2}$$

to the simpler nonlinear equations

$$Y''+r(x)Y'+q(x)Y=g(x)Y^a, \tag{3.3}$$

where a is constant and the linear operator L_k is defined as

$$L_k=D^2+r(x)D+kq(x). \tag{3.4}$$

For T_1, we have $Y=\log y$, while for T_2 we have $Y=y^l$. Equation (3.3) can be solved explicitly for some values of the constant a. For example, if

$$a=-3, \qquad g(x)=c, \tag{3.5}$$

then (3.3) becomes a Pinney-type equation, and if

$$a=1-2m, \qquad g(x)=c[u(x)v(x)]^{m-2}, \tag{3.6}$$

(3.3) reduces for $r(x)=0$ to a solvable form [20] in terms of a particular solution with one arbitrary constant of integration. In these equations, c and m are constants; in (3.6), u and v are linearly independent solutions of the equation $Y''+q(x)Y=0$. The case when $r(x)\neq 0$ is considered in [21].

One should note that the equation

$$L_k y = (1-l)y^{-1}y'^2 - (1/4)k\left(\exp\left[-2\int_{x_0}^{x} r(t)\,dt\right]\right)y^{1-4l}, \tag{3.7}$$

developed by Thomas [24] and also recorded by Herbst [14, eq. I], transforms by means of T_2 into the Pinney-type equation

$$Y'' + r(x)Y' + q(x)Y = -(1/4)\left(\exp\left[-2\int_{x_0}^{x} r(t)\,dt\right]\right)Y^{-3}. \tag{3.8}$$

This transformation explicitly demonstrates the equivalence of (3.7) and (3.8) mentioned by Thomas. In a similar manner, all equations of the type

$$L_k y = (1-l)y^{-1}y'^2 + kQ(x)y^{1-2ml} \tag{3.9}$$

transform into

$$L_1 Y = Q(x)Y^{1-2m}, \qquad Y = y^l. \tag{3.10}$$

In addition to (3.5) and (3.6), other forms of the function $Q(x)$ are noted in [21]–[22].

A physically interesting application can be made on the equation

$$4yy'' - 5y'^2 = cy^3, \tag{3.11}$$

which is the condition [17] that eigenvalues λ_n of an inhomogeneous string be given by the formula

$$\lambda_n = \left\{n\pi\Big/\int_0^1 [y(t)]^{1/2}\,dt\right\}^2 + c_0, \qquad c_0 = \text{const}. \tag{3.12}$$

T_2 transforms (3.11) into the equation $Y'' = -c/(16Y^3)$, the general solution of which follows immediately from [19] and is equivalent to that found by Makai [17]. We note that all equations of the form

$$y'' + q(x)y = ay^{-1}y'^2 + cy^{4a-3} \tag{3.13}$$

reduce to the Pinney type [19]. The value of the exponent a and the coefficients $r(x)$, $q(x)$ and $g(x)$ can be adjusted to match a number of physically important equations.

4. Transformation *T*. The two specific transformations T_1 and T_2, defined previously, suggest that there exists a linearizing transformation of the more general equation of Herbst [14, eq. II], which equation has the form

$$y'' + r(x)y' + q(x)Z(y) = A(y)y'^2. \tag{4.1}$$

If it is assumed that if $A(y)$ is given, then Z is found from the linear equation $dZ/dy - A(y)Z = 1$. Expressing Z in the form

$$Z(y) = z(y)u(y), \tag{4.2}$$

where

$$z(y) = \exp\int^{y} A(u)\,du \tag{4.3}$$

and

$$u(y)=\beta+\int^{y}\exp\left(-\int^{u}A(t)\,dt\right)du, \tag{4.4}$$

we find that (4.1) becomes

$$[y''-A(y)y'^2]/z(y)+r(x)y'/z(y)+q(x)u(y)=0. \tag{4.5}$$

Since $u'=y'/z$ and $u''=[y''-A(y)y'^2]/z$, (4.5) is a homogeneous linear equation for $u\{y(x)\}$. We denote the removal of the term $A(y)y'^2$ from (4.1) and related equations as transformation T.

The solution of (4.1) was expressed by Herbst as $y=F(u)$, where F is the solution of

$$\ddot{F}=A(F)\dot{F}^2, \quad (\cdot)=d/du, \quad L_1u=0. \tag{4.6}$$

Operator L_1 corresponds to (3.4) with $k=1$. The solution of (4.6) follows easily from letting $p=\dot{F}$, $p\,dp/dF=\ddot{F}$; we find

$$\alpha\,du=\left(\exp\int A(F)\,dF\right)^{-1}dF, \tag{4.7}$$

where α is an arbitrary constant. Integration and inversion of this equation provides the solution $F(u)$. Equation (4.7) is clearly equivalent to (4.4).

Generalizations of (4.1) follow readily. The nonlinear equation

$$y''+r(x)y'+q(x)Z(y)=A(y)y'^2+g(x)z(y) \tag{4.8}$$

is transformed by T into the nonhomogeneous linear equation in u

$$L_1u=g(x). \tag{4.9}$$

Thus if we can determine $u(y)$ from (4.9), then we can solve the implicit equation (4.4) for $y(x)$. The nonlinear equation

$$y''+r(x)y'+q(x)Z(y)=A(y)y'^2+g(x)z(y)[u(y)]^a \tag{4.10}$$

in y is equivalent, through T, to the following nonlinear equation in u:

$$L_1u=g(x)u^a, \qquad a\neq 0. \tag{4.11}$$

It should be noted that each of the equations discussed here is a special case of the more general form

$$y''=A(x,y)y'^2+B(x,y)y'+C(x,y). \tag{4.12}$$

Equation (4.12) was studied at the beginning of this century by Painlevé [18] and Gambier [11] for conditions under which the critical points of the equation would be fixed points. The coefficient $A(x,y)$ played a dominant role in their analyses. The linearizing transformations T_1 and T_2, for the special case of (4.12) represented by (4.1), correspond to the Painlevé coefficients y^{-1} and $(1-l)y^{-1}$, respectively. These two coefficients represent twenty-six out of fifty canonical types of equations discovered by Painlevé and Gambier. Aside from the case $A=0$, the transformation T represented by (4.4) leads to elliptic integrals for the remaining Painlevé coefficients.

5. Transformation of nonlinear partial differential equations. We conclude this note by observing that the transformation T applies to nonlinear partial differential equations of the form

$$\sum_{i,j=1}^{n} f_{ij}(\mathbf{x})D_jD_iy+\sum_{i=1}^{n} r_i(\mathbf{x})D_iy+q(\mathbf{x})Z(y) = A(y)\sum_{i,j=1}^{n} f_{ij}(\mathbf{x})D_jyD_iy+g(\mathbf{x})z(y), \qquad D_i=\partial/\partial x_i, \tag{5.1}$$

where $\mathbf{x}$ denotes a set of n independent variables. Since (4.3) and (4.4) still define $Z(y)$, it is not difficult to write the n-dimensional version of (4.5). Hence (5.1) is equivalent to the linear equation

$$\sum_{i,j=1}^{n} f_{ij}(\mathbf{x})D_jD_iu+\sum_{i=1}^{n} r_i(\mathbf{x})D_iu+q(\mathbf{x})u=g(\mathbf{x}). \tag{5.2}$$

The solution $u(\mathbf{x})$ of this equation provides a solution of the nonlinear equation (5.1) through inversion of the form (4.4).

As an example, consider the special case of (5.1) for which $A(y)=(1-l)y^{-1}$ and $Z(y)=k(y-y^{1-l})$. The partial differential equation that results from these choices of $A(y)$ and $Z(y)$ generalizes (2.2). The conditions under which initial value problems can be posed for this equation are discussed in [23]. Some physical applications are also noted in [23]. Babikov [2] considers a boundary value problem in three dimensions for the equations

$$\nabla^2y\pm y=by^{-1}(\nabla y)^2\pm y^a, \tag{5.3}$$

where $a>0$, $-\infty<b<\infty$. Babikov essentially makes use of a transformation of type T_2 to bring (5.3) into the form

$$\nabla^2 Y\pm(1-b)Y=\pm(1-b)Y^{(a-b)/(1-b)}, \qquad Y=y^{1-b}. \tag{5.4}$$

We note that (5.4) is an explicit example of (4.1) in three independent variables.

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SERIES EXPANSIONS AND LINEAR DIFFERENTIAL OPERATORS*

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Abstract. Let L denote the linear differential operator $Ly=a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}y^{(1)}+a_ny$, where each a_j is a complex-valued function of class C^{n-j} on the closed interval $[a, b]$, and $a_0(x)\neq 0$ for $x\in[a, b]$. Let $B_1, B_2, \cdots, B_n$ be linearly independent boundary forms, and suppose that the eigenvalue problem $Ly=\lambda y$, $B_1y=B_2y=\cdots=B_ny=0$ is self-adjoint. With each such eigenvalue problem there is associated a series expansion whose coefficients are boundary values. For the case in which 0 is not an eigenvalue of the problem, the expansion has the form $f(x)=\sum_{k=0}^{\infty}\sum_{j=1}^{n}(B_jL^kf)p_{nk+j}(x)$, which is valid for suitably restricted functions f defined on $[a, b]$. The functions $\{p_k\}_{k=1}^{\infty}$ are developed from solutions of the homogeneous equation $Ly=0$ and the Green's function. If the problem has the eigenvalue 0, the representation takes the similar form $f(x)=q_0(x)+\sum_{k=0}^{\infty}\sum_{j=1}^{n}(U_jL^kf)q_{nk+j}(x)$, where q_0 is an eigenfunction corresponding to $\lambda=0$ and where the boundary forms $U_1, U_2, \cdots, U_n$ are linear combinations of $B_1, B_2, \cdots, B_n$. For each expansion, necessary and sufficient conditions for convergence are given in terms of the magnitude of an eigenvalue nearest the origin.

1. Introduction. *Let* n be a positive integer, and let L be the nth order linear differential operator given by

$$Ly=a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}y^{(1)}+a_ny,$$

where each a_j is a complex-valued function of class C^{n-j} on the closed interval $a\leqq x\leqq b$, and $a_0(x)\neq 0$ for x in $[a, b]$. Let M_{jk} and N_{jk}, $1\leqq j, k\leqq n$, be complex constants, and define the boundary forms B_j by

$$B_jy=\sum_{k=1}^{n}M_{jk}y^{(k-1)}(a)+N_{jk}y^{(k-1)}(b),\qquad 1\leqq j\leqq n. \tag{1.1}$$

We shall suppose that these forms are linearly independent. Denoting the relationships $B_jy=0$, $1\leqq j\leqq n$, by $By=0$, we shall suppose throughout that the eigenvalue problem

$$Ly=\lambda y,\qquad By=0 \tag{1.2}$$

is self-adjoint; i.e., $(Lu, v)=(u, Lv)$ for all u and v in $C^n[a, b]$ satisfying $Bu=Bv=0$, where

$$(f, g)=\int_a^b f(t)\overline{g(t)}\,dt.$$

Then the eigenvalues of (1.2) are real and comprise a countably infinite set with no finite limit point.

Suppose first that 0 is not an eigenvalue of (1.2). Let L^k denote the kth interate of the operator L, and define the sequence of linear functionals $\{l_m\}_{m=1}^{\infty}$ by

$$l_{nk+j}(f)=B_jL^kf,\qquad 1\leqq j\leqq n,\quad 0\leqq k<\infty. \tag{1.3}$$

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With each boundary value problem (1.2) having nonzero eigenvalues, we shall associate in a canonical manner a series expansion of the form

$$f(x)=\sum_{m=0}^{\infty} l_m(f)p_m(x)=\sum_{k=0}^{\infty}\sum_{j=1}^{n}(B_jL^kf)p_{nk+j}(x), \tag{1.4}$$

where the functions $\{p_m(x)\}_{m=1}^{\infty}$ are biorthogonal to the functionals $\{l_m\}_{m=1}^{\infty}$:

$$l_jp_k=\delta_{jk}, \qquad 1\leqq j, \quad k<\infty. \tag{1.5}$$

Here, δ_{jk} is the Kronecker delta. In § 2 and § 3 below, we shall derive the expansion (1.4), investigate its mode of convergence and establish necessary and sufficient conditions under which a function f on $[a, b]$ admits a representation of the form (1.4).

The series in (1.4), which we term on *LB-series*, generalizes a number of well-known series expansions. For example, if $Ly=-y''$ and $B_1y=y(0)$, $B_2y=y(1)$, then (1.4) is the Lidstone series. In [7], D. V. Widder employed properties of the Green's function for the system $y''+\lambda y=0$, $y(0)=y(1)=0$, to obtain the classical representation of completely convex functions by Lidstone series. Using the same methods, Pethe and Sharma [6] obtained similar convergence and representation results on the series associated with the boundary value problem $y''+\lambda y=0$, $y'(0)=y(1)=0$. Recently, this author and J. D. Buckholtz [2] extended most of Widder's results, and those of [6], to the case of any Sturm–Liouville system

$$-(Py')'+Qy=\lambda y, \qquad \alpha y(a)+\alpha' y'(a)=\beta y(b)+\beta' y'(b)=0$$

with *positive* eigenvalues.

In the present paper we obtain, under considerably weaker hypotheses, a substantial portion of the results of [2], [6] and [7] as special cases. In addition to unrestricted order of the operator, we make no assumptions concerning positivity of the eigenvalues, and the boundary conditions $B_jy=0$ may be either mixed or separated.

For systems (1.2) having 0 as an eigenvalue, we introduce a modified version of the series (1.4). The endpoint conditions are replaced by linear combinations of same, and we denote these $U_1, U_2, \cdots, U_n$. The corresponding series expansion takes the form

$$f(x)=q_0(x)+\sum_{k=0}^{\infty}\sum_{j=1}^{n}(U_jL^kf)q_{nk+j}(x), \tag{1.6}$$

where q_0 is an eigenfunction belonging to the eigenvalue 0, and where the functions $\{q_m\}_{m=1}^{\infty}$ are biorthogonal to the sequence of linear functionals $\{\psi_m\}_{m=1}^{\infty}$ defined by $\psi_{nk+j}(f)=(U_jL^kf)$, $1\leqq j\leqq n$, $0\leqq k<\infty$.

The most familiar example of the series (1.6) is the biorthogonal expansion in terms of Bernoulli polynomials [1]. Here, we take $n=1$, $Ly=iy'$ and $B_1y=U_1y=y(1)-y(0)$. The eigenvalues of this system are $0, \pm 2\pi, \pm 4\pi, \cdots$. For each m, q_m is a constant multiple of the mth Bernoulli polynomial, and (1.6) reduces to the Bernoulli expansion.

The series (1.6), together with characterizations of its convergence, is developed in § 5.

2. The Green's function for a system. We shall assume throughout this section, and the next, that 0 is not an eigenvalue of (1.2). The only hypothesis we require pertains to eigenfunctions corresponding to an eigenvalue of smallest magnitude. Let $p_1, p_2, \cdots, p_n$ be the uniquely determined solutions of the homogeneous equation $Ly = 0$ which satisfy

$$B_j p_k = \delta_{jk}, \qquad 1 \leqq j, k \leqq n. \tag{2.1}$$

We shall suppose that

(H) For each j, $1 \leqq j \leqq n$, there exists at least one eigenfunction y, belonging to an eigenvalue of smallest absolute value, such that $(p_j, y) \neq 0$.

While the hypothesis (H) holds in each of the examples mentioned in § 1, it is not true in general. For the problem $Ly = -y'' - \pi^2 y$, $B_1 y = y(1) - y(0)$, $B_2 y = y'(1) - y'(0)$, the solutions of (2.1) are $p_1(x) = -\frac{1}{2}\cos \pi x$ and $p_2(x) = -1/(2\pi) \sin \pi x$. The eigenvalue nearest the origin is $\lambda = -\pi^2$, and the corresponding eigenfunctions are all constants. Thus the condition (H) fails for $j = 1$.

Let $G(x, t)$ be the Green's function for the problem (1.2), and define the operator $\mathcal{G}$ by

$$(\mathcal{G}\phi)(x) = \int_a^b G(x, t)\phi(t)\, dt, \qquad a \leqq x \leqq b,$$

for $\phi \in C[a, b]$. Recall that, as a function of x, $G(x, t)$ satisfies

$$LG = 0, \qquad x \neq t, \tag{2.2}$$

$$BG = 0. \tag{2.3}$$

Furthermore, $\mathcal{G}$ acts as a right inverse for L in the sense that

$$L\mathcal{G}\phi = \phi, \qquad \phi \in C[a, b]. \tag{2.4}$$

We now define the sequence $\{p_m\}_{m=1}^{\infty}$ appearing in (1.4) by (2.1) and

$$p_{nk+j} = \mathcal{G}^k p_j, \qquad 1 \leqq j \leqq n, \quad 1 \leqq k < \infty, \tag{2.5}$$

where $\mathcal{G}^k$ is the kth iterate of $\mathcal{G}$. By (2.4), one has $Lp_{nk+j} = p_{n(k-1)+j}$, $k \geqq 1$, and the biorthogonality relations (1.5) follow from (2.1) and (2.3). Note that the functions p_m can be calculated from the recursion formula

$$Lp_{nk+j} = p_{n(k-1)+j}, \qquad Bp_{nk+j} = 0, \qquad k \geqq 1.$$

Finally, let us denote by $\mathcal{L}$ the collection of all complex-valued functions f on $[a, b]$ such that $(L^k f)(x)$ exists for $a \leqq x \leqq b$ and for $k = 0, 1, 3, \cdots$.

If $f \in \mathcal{L}$, then (2.4) implies

$$L(f - \mathcal{G}Lf) = 0.$$

We therefore have

$$f - \mathcal{G}Lf = c_1 p_1 + c_2 p_2 + \cdots + c_n p_n$$

for suitable constant $c_1, c_2, \cdots, c_n$. Applying (2.1) and (2.3), we find

$$B_j f = B_j(f - \mathcal{G}Lf) = c_j, \qquad 1 \leqq j \leqq n,$$

and therefore,

$$f=(B_1f)p_1+(B_2f)p_2+\cdots+(B_nf)p_n+\mathscr{G}Lf. \tag{2.6}$$

This identity (see, e.g., [4, p. 257]) simply reflects that the unique solution to the boundary value problem

$$Lf=\phi, \qquad B_1f=A_1,\cdots,B_nf=A_n$$

is given by

$$f=A_1p_1+\cdots+A_np_n+\mathscr{G}\phi.$$

If we apply (2.6) to the function Lf, we obtain

$$Lf=\sum_{j=1}^{n}(B_jLf)p_j+\mathscr{G}^2L^2f.$$

Substituting this expression into (2.6) yields

$$f=\sum_{k=0}^{1}\sum_{j=1}^{n}(B_jL^kf)p_{nk+j}+\mathscr{G}^2L^2f.$$

Continuing this procedure, we arrive at the identity

$$f=\sum_{k=0}^{m-1}\sum_{j=1}^{n}(B_jL^kf)p_{nk+j}+\mathscr{G}^mL^mf, \tag{2.7}$$

which is valid for all $f\in\mathscr{L}$ and for each positive integer m.

It is well known that the operator norm of $\mathscr{G}$ coincides with its spectral radius. In our case, if ρ_1 is the distance from 0 to the nearest eigenvalue of (1.2), then $\|\mathscr{G}\|=\rho_1^{-1}$. Also, if $\|\cdot\|_2$ denotes the root mean square norm, then $\|\mathscr{G}^mL^mf\|_2\leqq\rho_1^{-m}\|L^mf\|_2$, $m=0,1,2,\cdots$. From this and (2.7) it follows that the condition $\lim_{m\to\infty}\rho_1^{-m}\|L^mf\|_2=0$ insures at least mean square convergence of (1.4). We have therefore proved the following preliminary result.

THEOREM 2.1. *Suppose that $f\in\mathscr{L}$ and that $\lim_{m\to\infty}\rho_1^{-m}\|L^mf\|_2=0$. Then*

$$\lim_{m\to\infty}\|f-\sum_{k=0}^{m-1}\sum_{j=1}^{n}(B_jL^kf)p_{nk+j}\|_2=0.$$

From this result it is clear that the minimum modulus of the eigenvalues of (1.2) is of fundamental importance in LB-series expansions. There are at most two eigenvalues of smallest absolute value, and there are numerous examples in which there are exactly two. To distinguish eigenvalues of opposite sign and equal magnitude, we adopt the following "symmetric notation."

Let us denote by $\{\rho_k\}_{k=1}^{\infty}$, $0<\rho_1<\rho_2<\cdots$, the increasing sequence of absolute values of eigenvalues of (1.2). For each k, $k=1,2,3,\cdots$, let λ_k be an eigenvalue of (1.2) with $|\lambda_k|=\rho_k$, let n_k be the dimension of the eigenspace $S_k=\{y: Ly=\lambda_ky, By=0\}$, and let $\{y_{k1},y_{k2},\cdots,y_{kn_k}\}$ be an orthonormal basis for S_k. Thus we have

$$Ly_{kj}=\lambda_ky_{kj}, \quad By_{kj}=0, \quad (y_{kj},y_{km})=\delta_{jm}$$

for $1 \leqq j, m \leqq n_k$. If $-\lambda_k$ is also an eigenvalue of (1.2), let m_k be the dimension of the space $\hat{S}_k = \{y: Ly = -\lambda_k y, By = 0\}$ and let $\{\hat{y}_{k1}, \hat{y}_{k2}, \cdots, \hat{y}_{km_k}\}$ be an orthonormal basis for $\hat{S}_k$. If $-\lambda_k$ is *not* an eigenvalue of (1.2), take $m_k = 1$ and $\hat{y}_{k1} \equiv 0$.

Now let π_k and $\hat{\pi}_k$ be the projection mappings defined by

$$\begin{aligned} \pi_k\phi &= (\phi, y_{k1})y_{k1} + \cdots + (\phi, y_{kn_k})y_{kn_k}, \\ \hat{\pi}_k\phi &= (\phi, \hat{y}_{k1})\hat{y}_{k1} + \cdots + (\phi, \hat{y}_{km_k})\hat{y}_{km_k} \end{aligned} \tag{2.8}$$

for $k = 1, 2, 3, \cdots$ and $\phi \in C[a, b]$. The eigenfunction expansion [3] for $\mathcal{G}\phi$ then takes the form

$$\mathcal{G}\phi = \sum_{k=1}^{\infty} \lambda_k^{-1}[(\pi_k\phi) - (\hat{\pi}_k\phi)]. \tag{2.9}$$

This expansion converges uniformly in $[a, b]$ and is valid for all $\phi \in C[a, b]$. Since $\mathcal{G}(\pi_k\phi) = \lambda_k^{-1}\, \pi_k\phi$ and $\mathcal{G}(\hat{\pi}_k\phi) = -\lambda_k^{-1}\hat{\pi}_k\phi$, then there follows

$$\mathcal{G}^m\phi = \sum_{k=1}^{\infty} \lambda_k^{-m}[(\pi_k\phi) + (-1)^m\, (\hat{\pi}_k\phi)], \qquad m = 1, 2, 3, \cdots, \tag{2.10}$$

with uniform convergence in $[a, b]$.

LEMMA 2.1. *For each $\phi \in C[a, b]$ and each positive integer k, we have*

$$\begin{aligned} |(\phi, y_{kj})| &\leqq \|\phi\|_2, \qquad 1 \leqq j \leqq n_k, \\ |(\phi, \hat{y}_{kj})| &\leqq \|\phi\|_2, \qquad 1 \leqq j \leqq m_k. \end{aligned} \tag{2.11}$$

Moreover, there exists a positive constant K such that for $k = 1, 2, 3, \cdots$, $r = 0, 1, 2, \cdots, n$ and $a \leqq x \leqq b$, we have

$$\begin{aligned} |y_{kj}^{(r)}(x)| &\leqq K|\lambda_k|, \qquad 1 \leqq j \leqq n_k, \\ |\hat{y}_{kj}^{(r)}(x)| &\leqq K|\lambda_k|, \qquad 1 \leqq j \leqq m_k, \end{aligned} \tag{2.12}$$

where the superscript (r) denotes ordinary differentiation.

Proof. The estimates (2.11) follow from Schwarz' inequality. To prove (2.12), we note that, for each j and k, the equation

$$\lambda_k^{-1} y_{kj}(x) = \int_a^b G(x, t) y_{kj}(t)\, dt$$

may be differentiated up to n times [5], and this results in

$$\begin{aligned} \lambda_k^{-1} y_{kj}^{(r)} &= \int_a^b \frac{\partial^{(r)} G(x, t)}{\partial x^{(r)}} y_{kj}(t)\, dt, \qquad 0 \leqq r \leqq n-1, \\ \lambda_k^{-1} y_{kj}^{(n)}(x) &= \int_a^b \frac{\partial^{(n)} G(x, t)}{\partial x^{(n)}} y_{kj}(t)\, dt + \frac{1}{a_0(x)} y_{kj}(x). \end{aligned} \tag{2.13}$$

The partials of G up to order n are bounded in the square $a \leqq x, t \leqq b$ and so the first inequality in (2.12) follows from (2.13) and Schwarz' inequality. The proof for the functions $\hat{y}_{kj}$ is similar.

THEOREM 2.2. *Suppose that $f \in \mathscr{L}$ and that the sequence $\{\lambda_1^{-k}(L^k f)(x)\}_{k=0}^{\infty}$ converges uniformly to 0 in $[a, b]$. Then*

$$f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{n} (B_j L^k f) p_{nk+j}(x), \qquad a \leqq x \leqq b, \tag{2.14}$$

with uniform convergence in $[a, b]$.

Proof. Because of the identity (2.7), it is enough to show that

$$\lim_{m \to \infty} (\mathscr{G}^m L^m f)(x) = 0$$

uniformly in $[a, b]$. Now by virtue of (2.10), we have

$$\mathscr{G}^m L^m f = \sum_{k=1}^{\infty} \lambda_k^{-m}[(\pi_k L^m f) + (-1)^m (\hat{\pi}_k L^m f)] \tag{2.15}$$

for $m = 1, 2, 3, \cdots$. The sequence of functions $\{\varepsilon_k(x)\}_{k=0}^{\infty}$ defined by $\varepsilon_k(x) = \lambda_1^{-k}(L^k f)(x)$, $k = 0, 1, 2, \cdots$, satisfies $\lim_{k \to \infty} \|\varepsilon_k\|_2 = 0$ and

$$|(L^m f, y_{kj})| \leqq |\lambda_1|^m \|\varepsilon_m\|_2, \qquad 1 \leqq j \leqq n_k,$$
$$|(L^m f, \hat{y}_{kj})| \leqq |\lambda_1|^m \|\varepsilon_m\|_2, \qquad 1 \leqq j \leqq m_k,$$

for $0 \leqq m < \infty$, $1 \leqq k < \infty$. Inserting these estimates into (2.15) and using Lemma 2.1, we find that

$$|(\mathscr{G}^m L^m f)(x)| \leqq K' \|\varepsilon_m\|_2 \sum_{k=1}^{\infty} \left| \frac{\lambda_1}{\lambda_k} \right|^{m-1}, \qquad a \leqq x \leqq b, \quad m \geqq 3,$$

where $K' > 0$ is a constant. Letting $m \to \infty$, we obtain the desired result.

The functions y_{1j}, $1 \leqq j \leqq n_1$, satisfy $\lambda_1^{-k} L^k y_{1j} = y_{1j}$ and do not admit the expansion (2.14). Therefore, the hypothesis in Theorem 2.2 cannot be weakened to boundedness of the sequence $\{\lambda_1^{-k} L^k f\}$. We shall prove later (Theorem 3.3) that a function f for which $\{\lambda_1^{-k} L^k f\}$ is uniformly bounded in $[a, b]$ possesses an expansion which includes the series in (2.14) as a term.

We now seek asymptotic bounds on the sequence $\{p_m\}_{m=1}^{\infty}$. Note first that (2.5) and (2.10) imply

$$p_{nk+j} = \sum_{m=1}^{\infty} \lambda_m^{-k}[(\pi_m p_j) + (-1)^k (\hat{\pi}_m p_j)] \tag{2.16}$$

for $1 \leqq j \leqq n$, $0 \leqq k < \infty$.

LEMMA 2.2. *There exists a positive constant M such that*

$$|p_{nk+j}^{(r)}(x) - \lambda_1^{-k}[(\pi_1 p_j)^{(r)}(x) + (-1)^k (\hat{\pi}_1 p_j)^{(r)}(x)]| \leqq M |\lambda_2|^{-k} \tag{2.17}$$

for $a \leqq x \leqq b$, $0 \leqq r \leqq n$, $0 \leqq k < \infty$ and $1 \leqq j \leqq n$.

Proof. By square-summability of the eigenvalue reciprocals and Lemma 2.1, each of the differentiated series

$$\sum_{m=1}^{\infty} \lambda_m^{-k}[(\pi_m p_j)^{(r)}(x) + (-1)^k (\hat{\pi}_m p_j)^{(r)}(x)],$$
$$0 \leqq r \leqq n, \quad 1 \leqq j \leqq n,$$

converges uniformly and absolutely in $[a, b]$ for $k \geqq 3$. In view of (2.16), we have

$$p_{nk+1}^{(r)}(x) = \sum_{m=1}^{\infty} \lambda_m^{-k}[(\pi_m p_j)^{(r)}(x) + (-1)^k(\hat{\pi}_m p_j)^{(r)}(x)] \tag{2.18}$$

for $a \leqq x \leqq b$, $1 \leqq j \leqq n$, $0 \leqq r \leqq n$ and $3 \leqq k \leqq <\infty$. From this, and (2.12), one sees that

$$|p_{nk+j}^{(r)}(x) - \lambda_1^{-k}[(\pi_1 p_j)^{(r)}(x) + (-1)^k(\hat{\pi}_1 p_j)^{(r)}(x)]|$$
$$\leqq \sum_{m=2}^{\infty} |\lambda_m^{-k}[(\pi_m p_j)^{(r)}(x) + (-1)^k(\hat{\pi}_m p_j)^{(r)}(x)]|$$
$$\leqq |\lambda_2|^{-k+1} \sup_{m,r,x} \left\{\frac{|(\pi_m p_j)^{(r)}(x)| + |(\hat{\pi}_m p_j)^{(r)}(x)|}{|\lambda_m|}\right\} \sum_{m=2}^{\infty} \left|\frac{\lambda_2}{\lambda_m}\right|^{k-1},$$

and the desired result follows from the convergence of $\sum_{m=2}^{\infty} |\lambda_2/\lambda_m|^{k-1}$ for $k \geqq 3$.

Since $|\lambda_1| < |\lambda_2|$, (2.17) yields immediately the following lemma.

LEMMA 2.3. *For $1 \leqq j \leqq n$ and $0 \leqq r \leqq n$, we have*

$$\lim_{k\to\infty} \{\lambda_1^k p_{nk+j}^{(r)}(x) - [(\pi_1 p_j)^{(r)}(x) + (-1)^k(\hat{\pi}_1 p_j)^{(r)}(x)]\} = 0 \tag{2.19}$$

uniformly in $[a, b]$. In particular, there exists a positive constant M' for which

$$|\lambda_1^k p_{nk+j}^{(r)}(x)| \leqq M' \tag{2.20}$$

for $a \leqq x \leqq b$, $0 \leqq r \leqq n$, $0 \leqq k < \infty$, $1 \leqq j \leqq n$.

3. Absolutely convergent series expansions. Concerning the representation (2.14), the question arises as to when the series in (2.14) can be "ungrouped" and written as

$$f(x) = \sum_{k=1}^{\infty} l_k(f) p_k(x), \tag{3.1}$$

where the functionals $\{l_k\}$ are defined by (1.3). In fact, a function f satisfying

$$\lim_{k\to\infty} \lambda_1^{-k}(L^k f)(x) = 0 \tag{3.2}$$

has the representation (3.1) if and only if

$$\lim_{k\to\infty} (B_j L^k f) p_{nk+j}(x) = 0, \qquad 1 \leqq j \leqq n, \quad a \leqq x \leqq b. \tag{3.3}$$

Noting that

$$(B_j L^k f) p_{nk+j}(x) = (\lambda_1^{-k} B_j L^k f)(\lambda_1^k p_{nk+j}(x)), \tag{3.4}$$

we see that the bound (2.20) shows that (3.3) is guaranteed by

$$\lim (\lambda_1^{-k} B_j L^k f) = 0, \qquad 1 \leqq j \leqq n. \tag{3.5}$$

In some cases (3.5) is a consequence of (3.2), and if

$$\sum_{k=0}^{\infty} \sum_{j=1}^{n} |\lambda_1^{-k} B_j L^k f| < \infty,$$

then *absolute* convergence of (2.14), thus insuring (3.1), follows from (3.4) and (2.20).

The same type of reasoning leads to the following characterization of absolutely convergent LB-series. Recalling the definitions (2.8), we now define

$$\pi_j^{(+)}(x) = (\pi_1 p_j)(x) + (\hat{\pi}_1 p_j)(x),$$

$$\pi_j^{(-)}(x) = (\pi_1 p_j)(x) - (\hat{\pi}_1 p_j)(x), \qquad 1 \leqq j \leqq n, \quad a \leqq x \leqq b.$$

THEOREM 3.1. *Let* $\{h_k\}_{k=1}^{\infty}$ *be a complex sequence. Then the following are equivalent*:

(i) $\sum_{k=0}^{\infty} \sum_{j=1}^{n} h_{nk+j} p_{nk+j}(x)$ *converges absolutely at each of* n *points* $x_1, x_2, \cdots, x_n$ *for which* $\pi_j^{(+)}(x_j) \neq 0$ *and* $\pi^{(-)}(x_j) \neq 0$, $1 \leqq j \leqq n$;

(ii) $\sum_{k=0}^{\infty} \sum_{j=1}^{n} |\lambda_1^{-k} h_{nk+j}| < \infty$;

(iii) $\sum_{k=0}^{\infty} \sum_{j=1}^{n} h_{nk+j} p_{nk+j}(x)$ *converges absolutely and uniformly in* $[a, b]$.

Proof. The proof that (iii)$\Rightarrow$(i) amounts to showing that the points $x_1, x_2, \cdots, x_n$ exist. Let j be fixed, $1 \leqq j \leqq n$. By hypothesis (H) we know $\pi_j^{(+)}(x) \not\equiv 0$. From continuity of $\pi_j^{(+)}$, there must exist a subinterval I of $[a, b]$ on which $\pi_j^{(+)}$ does not vanish. Since $L\pi_j^{(-)} = \lambda_1 \pi_j^{(+)}$, then $\pi_j^{(-)}$ does not vanish identically on I. Thus x_j may be chosen as a point in I such that $\pi_j^{(-)}(x_j) \neq 0$.

We now show that (i)$\Rightarrow$(ii)$\Rightarrow$(iii). If (i) holds, then (2.19) implies that there exists a constant $\gamma > 0$ and an integer $k_0 > 0$ such that

$$|\lambda_1^k p_{nk+j}(x_j)| > \gamma, \qquad 1 \leqq j \leqq n, \quad k \geqq k_0.$$

Then

$$\gamma \sum_{k=k_0}^{\infty} |\lambda_1^{-k} h_{nk+j}| \leqq \sum_{k=k_0}^{\infty} |h_{nk+j} p_{nk+j}(x_j)| < \infty, \qquad 1 \leqq j \leqq n,$$

and we see that (ii) holds.

To show that (ii)$\Rightarrow$(iii), we use (2.20). In fact,

$$\sum_{k=0}^{\infty} \sum_{j=1}^{n} |h_{nk+j} p_{nk+j}(x)| = \sum_{k=0}^{\infty} \sum_{j=1}^{n} |\lambda_1^{-k} h_{nk+j} (\lambda_1^k p_{nk+j}(x))|$$

$$\leqq M' \sum_{k=0}^{\infty} \sum_{j=1}^{r} |\lambda_1^{-k} h_{nk+j}|,$$

and this completes the proof.

The following theorem shows that the coefficients $\{h_k\}$ in the expansion (iii) are uniquely determined.

THEOREM 3.2. *Suppose that*

$$f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{n} h_{nk+j} p_{nk+j}(x), \qquad a \leqq x \leqq b,$$

with absolute convergence in $[a, b]$. *Then* $f \in \mathscr{L}$ *and*

$$(3.6) \qquad (L^m f)^{(r)}(x) = \sum_{k=m}^{\infty} \sum_{j=1}^{n} h_{nk+j} p_{n(k-m)+j}^{(r)}(x), \qquad 0 \leqq m < \infty,$$

with absolute and uniform convergence in $[a, b]$. *Furthermore,*

$$B_jL^kf = h_{nk+j}, \qquad 1 \leqq j \leqq n, \quad 0 \leqq k < \infty. \tag{3.7}$$

Proof. Note that conditions (i)–(iii) of Theorem 3.1 hold. Concerning (3.6), a consequence of (2.20) is that

$$\sum_{k=m}^{\infty} \sum_{j=1}^{n} |h_{nk+j}p^{(r)}_{n(k-m)+j}(x)| = |\lambda_1|^m \sum_{k=m}^{\infty} \sum_{j=1}^{n} |\lambda_1^{-k}h_{nk+j}\lambda_1^{k-m}p^{(r)}_{n(k-m)+j}|$$

$$\leqq |\lambda_1|^m M' \sum_{k=m}^{\infty} \sum_{j=1}^{n} |\lambda_1^{-k}h_{nk+j}| \tag{3.8}$$

for $0 \leqq r \leqq n$, $0 \leqq m < \infty$. Therefore, each of the series

$$\sum_{k=m}^{\infty} \sum_{j=1}^{n} p^{(r)}_{n(k-m)+j}(x) \tag{3.9}$$

converges absolutely and uniformly in $[a, b]$. Applying the operator $\mathscr{G}^m$ to the series in (3.9) with $r = 0$ yields

$$\mathscr{G}^m\left\{\sum_{k=m}^{\infty} \sum_{j=1}^{n} h_{nk+j}p_{n(k-m)+j}\right\} = \sum_{k=m}^{\infty} \sum_{j=1}^{n} h_{nk+j}p_{nk+j}$$

$$= f - \sum_{k=0}^{m-1} h_{nk+j}p_{nk+j}$$

Therefore,

$$L^m\mathscr{G}^m\left\{\sum_{k=m}^{\infty} \sum_{j=1}^{n} h_{nk+j}p_{n(k-m)+j}\right\} = L^m\left\{f - \sum_{k=0}^{m-1} h_{nk+j}p_{nk+j}\right\} = L^mf,$$

and, by (2.4), one sees that

$$L^mf = \sum_{k=m}^{\infty} \sum_{j=1}^{n} h_{nk+j}p_{n(k-m)+j}, \qquad 0 \leqq m < \infty. \tag{3.10}$$

Since (3.9) converges uniformly for $r = 1$, then (3.10) implies that (3.6) holds for $r = 1$. Since (3.6) holds for $r = 1$ and (3.9) converges uniformly for $r = 2$, we see that (3.6) holds for $r = 2$. We continue this procedure for $r = 3, 4, \cdots, n$ to complete the proof of (3.6).

The identities (3.6) show that we can apply the boundary operator B_j termwise to the equation

$$L^mf = \sum_{k=m}^{\infty} \sum_{j=1}^{n} h_{nk+j}p_{n(k-m)+j},$$

and this results in

$$B_jL^mf = \sum_{k=m}^{\infty} \sum_{t=1}^{n} h_{nk+t}B_j[p_{n(k-m)+t}] = h_{nm+j},$$

which completes the proof.

Observe now that the estimate (3.8) implies

$$|(L^mf)(x)| \leqq |\lambda_1|^m \mu_m, \qquad a \leqq x \leqq b, \quad 0 \leqq m < \infty,$$

where $\lim_{m\to\infty}\mu_m=0$. This, combined with Theorem 2.2, Theorem 3.2 and our remarks preceding Theorem 3.1, leads to the following corollary.

COROLLARY 3.1. *Let f be a complex-valued function on* $[a,b]$. *Then for f to have an absolutely and uniformly convergent series representation*

$$f(x)=\sum_{k=0}^{\infty}\sum_{j=1}^{n}(B_jL^kf)p_{nk+j}(x),\qquad a\leqq x\leqq b,$$

it is necessary and sufficient that

$$\lim_{k\to\infty}\lambda_1^{-k}(L^kf)(x)=0\qquad \textit{uniformly in } [a,b] \tag{3.11}$$

and

$$\sum_{k=0}^{\infty}\sum_{j=1}^{n}|\lambda_1^{-k}B_jL^kf|<\infty. \tag{3.12}$$

As pointed out earlier, condition (3.11) in Corollary 3.1 cannot be weakened to boundedness of the sequence $\{\lambda_1^{-k}L^kf\}$. There arises the possibility, however, that the series converges, but does not represent the function f.

THEOREM 3.3. *Let f be defined on* $[a,b]$ *and let N be a positive integer. Suppose that* (3.12) *holds, and suppose there exist constants* T, $|\lambda_1|\leqq T\leqq|\lambda_N|$, *and* $C>0$ *such that*

$$|(L^mf)(x)|\leqq CT^m,\qquad 1\leqq m<\infty,\quad a\leqq x\leqq b. \tag{3.13}$$

Then there exist constants c_{kj} *and* $\hat{c}_{kj}$ *such that*

$$f(x)=\sum_{k=1}^{N}\sum_{j=1}^{n_k}c_{kj}y_{kj}(x)+\sum_{j=1}^{m_k}\hat{c}_{kj}\hat{y}_{kj}(x)+\sum_{k=0}^{\infty}\sum_{j=1}^{n}(B_jL^kf)p_{nk+j}(x) \tag{3.14}$$

with uniform and absolute convergence in $[a,b]$. *Conversely, if f has the representation* (3.14), *then* (3.12) *and* (3.13) *hold.*

Proof. The converse statement of the theorem is an easy consequence of Corollary 3.1.

Suppose that (3.12) and (3.13) hold. By Theorem 3.1, the series

$$S(x)=\sum_{k=0}^{\infty}\sum_{j=1}^{n}(B_jL^kf)p_{nk+j}(x)$$

converges absolutely and uniformly in $[a,b]$. Let

$$R(x)=f(x)-S(x),\qquad a\leqq x\leqq b.$$

Now by Corollary 3.1 there exists a constant $C_1>0$ such that

$$|(L^mS)(x)|\leqq C_1|\lambda_1|^m,\qquad 1\leqq m<\infty,$$

and therefore,

$$|(L^mR)(x)|\leqq CT^m+C_1|\lambda_1|^m\leqq C_2T^m,\qquad 0\leqq m<\infty,$$

for a suitable constant $C_2>0$. By uniqueness of coefficients, we have $B_jL^mS=B_jL^mf$, so that $B_jL^mR=0$ for all j and m.

Now fix j and k, $k > N$ and $1 \leqq j \leqq n_k$. By self-adjointness, we have

$$\begin{aligned} |(R, y_{kj})| = |\lambda_k^{-m}(R, L^m y_{kj})| &= |\lambda_k^{-m}(L^m R, y_{kj})| \\ &\leqq |\lambda_k|^{-m} \int_a^b |(L^m R)(x) y_{kj}(x)|\, dx \\ &\leqq C_3 \left|\frac{T}{\lambda_k}\right|^m, \qquad 0 \leqq m < \infty, \end{aligned}$$

where C_3 is a positive constant. Letting $m \to \infty$ in this estimate and noting $|\lambda_k| > T$, we find that

$$(R, y_{kj}) = 0, \qquad k > N, \quad 1 \leqq j \leqq n_k.$$

Similarly, $(R, \hat{y}_{kj}) = 0$ for $k > N$ and $1 \leqq j \leqq m_k$. By completeness of the system of eigenfunctions, then, we have

$$R = \sum_{k=1}^{N} \left\{ \sum_{j=1}^{n_k} (R, y_{kj}) y_{kj} + \sum_{j=1}^{m_k} (R, \hat{y}_{kj}) \hat{y}_{kj} \right\},$$

and the proof is complete.

Using (3.14) and (2.8), we are led immediately to the following analogy to I. J. Schoenberg's classical result on Lidstone series [1, p. 16].

COROLLARY 3.2. *If* $f \in \mathscr{L}$, $B_j L^k f = 0$ *for all* j *and* k *and*

$$|(L^m f)(x)| \leqq C T^m, \qquad a \leqq x \leqq b, \quad 0 \leqq m < \infty,$$

for constants $C > 0$ *and* $T > 0$, *then*

$$f(x) = \sum_{|\lambda_k| \leqq T} \{(\pi_k f)(x) + (\hat{\pi}_k f)(x)\}.$$

4. Generalized Green's functions. In this section and the next, we shall suppose that 0 is an eigenvalue of (1.2). For simplicity, we shall suppose that 0 is a *simple* eigenvlaue; i.e., the eigenspace corresponding to 0 is one-dimensional. Let y_0 be a normalized eigenfunction corresponding to 0:

$$(4.1) \qquad L y_0 = 0, \quad B y_0 = 0, \quad \|y_0\|_2 = 1.$$

The notation for nonzero eigenvalues and corresponding eigenfunctions will be the same as before. Thus λ_1 is a nonzero eigenvalue nearest the origin and $0 < |\lambda_1| < |\lambda_2| < \cdots$. The complete system of eigenfunctions of (1.2) then consists of y_0, all of the functions y_{kj} and all *nonzero* functions $\hat{y}_{kj}$.

For each complex parameter λ, not an eigenvalue of (1.2), let $G_\lambda(x, t)$ be the green's Function for the problem

$$(L - \lambda) y = 0, \qquad By = 0,$$

and define the integral operator $\mathscr{G}_\lambda$ by

$$(\mathscr{G}_\lambda \phi)(x) = \int_a^b G_\lambda(x, t) \phi(t)\, dt$$

for $\phi \in C[a, b]$. Then we have

$$(4.2) \qquad (L - \lambda) \mathscr{G}_\lambda \phi = \phi, \qquad \phi \in C[a, b].$$

The function G_λ is sometimes termed a *generalized Green's function* for (1.2). We recall [3] that it is meromorphic as a function of λ, with simple poles at the eigenvalues of (1.2). As a function of x, $G_\lambda(x, t)$ satisfies

$$LG_\lambda = \lambda G_\lambda, \qquad x \neq t, \tag{4.3}$$

$$BG_\lambda = 0. \tag{4.4}$$

Furthermore, G_λ has the representation

$$G_\lambda(x, t) = -\frac{y_0(x)\overline{y_0(t)}}{\lambda} + \sum_{k=1}^{\infty}\left\{\sum_{j=1}^{n_k}\frac{y_{kj}(x)\overline{y_{kj}(t)}}{\lambda_k - \lambda} - \sum_{j=1}^{m_k}\frac{\hat{y}_{kj}(x)\overline{\hat{y}_{kj}(t)}}{\lambda_k + \lambda}\right\}, \tag{4.5}$$

and we see that the residue of G_λ at $\lambda = 0$ is $-y_0(x)\overline{y_0(t)}$. Let us define

$$A_\lambda(x, t) = G_\lambda(x, t) + \frac{y_0(x)\overline{y_0(t)}}{\lambda},$$

and observe that, for fixed x and t, $A_\lambda(x, t)$ is analytic as function of λ in a certain deleted neighborhood $0 < |\lambda| < \delta$ of $\lambda = 0$, and has a removable singularity at $\lambda = 0$. Thus we can set

$$G_0(x, t) = \lim_{\lambda \to 0} A_\lambda(x, t), \qquad a \leqq x, \quad t \leqq b.$$

Finally, define the operators $\mathcal{G}_0$ and $\mathcal{A}_\lambda$ by

$$(\mathcal{G}_0\phi)(x) = \int_a^b G_0(x, t)\phi(t)\, dt,$$

$$(\mathcal{A}_\lambda(x) = \int_a^b A_\lambda(x, t)\phi(t)\, dt, \qquad \phi \in C[a, b].$$

We now establish some basic properties of these operators.

LEMMA 4.1. *If* $\phi \in C[a, b]$, *then*

$$L^m \mathcal{G}^m \phi = \phi - (\phi, y_0)y_0, \qquad 1 \leqq m < \infty. \tag{4.6}$$

Proof. The Cauchy integral formula implies that, for $0 < \rho < \delta$,

$$G_0(x, t) = \frac{1}{2\pi i}\int_{|\lambda| = \rho}\frac{A_\lambda(x, t)}{\lambda}\, d\lambda. \tag{4.7}$$

Therefore, for any $\phi \in C[a, b]$, one has

$$(\mathcal{G}_0\phi)(x) = \frac{1}{2\pi i}\int_a^b\int_{|\lambda| = \rho}\frac{A_\lambda(x, t)}{\lambda}\phi(t)\, d\lambda\, dt$$

$$= \frac{1}{2\pi i}\int_{|\lambda| = \rho}\int_a^b\frac{A_\lambda(x, t)}{\lambda}\phi(t)\, dt\, d\lambda.$$

Applying the operator L to this equation, we find

$$
\begin{aligned}
(L\mathcal{G}_0\phi)(x) &= L\left\{\frac{1}{2\pi i}\int_{|\lambda|=\rho}\int_a^b \frac{G_\lambda(x,t)+\lambda^{-1}y_0(x)\overline{y_0(t)}}{\lambda}\phi(t)\,dt\,d\lambda\right\}\\
&= \frac{1}{2\pi i}\int_{|\lambda|=\rho} L\left\{\int_a^b \frac{G_\lambda(x,t)+\lambda^{-1}y_0(x)\overline{y_0(t)}}{\lambda}\phi(t)\,dt\right\}d\lambda\\
&= \frac{1}{2\pi i}\int_{|\lambda|=\rho} \lambda^{-1}\left\{L\int_a^b G_\lambda(x,t)\phi(t)\,dt+\frac{(\phi,y_0)}{\lambda}Ly_0\right\}d\lambda \qquad (4.8)\\
&= \frac{1}{2\pi i}\int_{|\lambda|=\rho} \lambda^{-1}(L\mathcal{G}_\lambda\phi)(x)\,d\lambda.
\end{aligned}
$$

It follows from (4.2) that

$$
\begin{aligned}
(L\mathcal{G}_0\phi)(x) &= \frac{1}{2\pi i}\int_{|\lambda|=\rho} \frac{\phi(x)+\lambda(\mathcal{G}_\lambda\phi)(x)}{\lambda}\,d\lambda\\
&= \phi(x)+\frac{1}{2\pi i}\int_{|\lambda|=\rho}(\mathcal{G}_\lambda\phi)(x)\,d\lambda \qquad (4.9)\\
&= \phi(x)+\int_a^b\left\{\frac{1}{2\pi i}\int_{|\lambda|=\rho} G_\lambda(x,t)\,d\lambda\right\}\phi(t)\,dt.
\end{aligned}
$$

By the theorem on residues, this last expression equals

$$\phi(x)+\int_a^b\left(-y_0(x)\overline{y_0(t)}\phi(t)\right)dt=\phi(x)-(\phi,y_0)y_0(x),$$

so that (4.6) holds for $m=1$. Now for any $m\geqq 1$, we have

$$
\begin{aligned}
L^m\mathcal{G}^m\phi &= L^{m-1}\{L\mathcal{G}_0(\mathcal{G}_0^{m-1}\phi)\}\\
&= L^{m-1}\{\mathcal{G}_0^{m-1}\phi-(\mathcal{G}_0^{m-1}\phi,y_0)y_0\}\\
&= L^{m-1}\mathcal{G}_0^{m-1}\phi,
\end{aligned}
$$

and therefore (4.6) holds for all m.

LEMMA 4.2. *For each* $\phi\in C[a,b]$,

$$B(\mathcal{G}_0\phi)=0. \qquad (4.10)$$

Proof. Differentiation of (4.7) gives

$$\frac{\partial^{(r)}G_0(x,t)}{\partial x^{(r)}}=\frac{1}{2\pi i}\int_{|\lambda|=\rho}\frac{\partial^{(r)}A_\lambda(x,t)}{\partial x^{(r)}}\frac{d\lambda}{\lambda}, \qquad 0\leqq r\leqq n-1.$$

Since $B_j A_\lambda = 0$, $1 \leqq j \leqq n$, this shows that

$$BG_0 = 0; \tag{4.11}$$

in particular, (4.10) holds.

LEMMA 4.3. $\mathscr{G}_0 y_0 = 0$.

Proof. Since the unique solution to the problem $(L-\lambda)y = \phi$, $By = 0$, is given by $y = \mathscr{G}_\lambda \phi$, then $(L-\lambda)y_0 = -\lambda y_0$ implies

$$\mathscr{G}_\lambda y_0 = -\frac{1}{\lambda} y_0. \tag{4.12}$$

Therefore,

$$\begin{aligned}(\mathscr{A}_\lambda y_0)(x) &= \int_a^b A_\lambda(x, t) y_0(t)\, dt \\ &= \int_a^b \left\{ G_\lambda(x, t) + \frac{y_0(x)\overline{y_0(t)}}{\lambda} \right\} y_0(t)\, dt \\ &= \mathscr{G}_\lambda y_0(x) + \frac{1}{\lambda} y_0(x) \|y_0\|_2^2 = 0.\end{aligned}$$

Finally,

$$\begin{aligned}(\mathscr{G}_0 y_0)(x) &= \int_a^b G_0(x, t) y_0(t)\, dt \\ &= \int_a^b \left\{ \lim_{\lambda \to 0} A_\lambda(x, t) y_0(t) \right\} dt \\ &= \lim_{\lambda \to 0} (\mathscr{A}_\lambda y_0)(x) = 0,\end{aligned}$$

which is the desired result.

Since (1.2) is self-adjoint, one has, for real λ,

$$G_\lambda(x, t) = \overline{G_\lambda(t, x)},$$

and on the basis of this, it is easy to show that

$$(\mathscr{G}_0 u, v) = (u, \mathscr{G}_0 v) \tag{4.13}$$

for all u and v in $C[a, b]$. In particular,

$$(\mathscr{G}_0 \phi, y_0) = (\phi, \mathscr{G}_0 y_0) = (\phi, 0) = 0, \qquad \phi \in C[a, b]. \tag{4.14}$$

Arguing as in the preceeding lemma, one can show that

$$\mathscr{G}_\lambda y_{kj} = \frac{1}{\lambda_k - \lambda} y_{kj}, \qquad \mathscr{G}_\lambda \hat{y}_{kj} = \frac{-1}{\lambda_k + \lambda} \hat{y}_{kj}$$

for all k and j. Since the eigenfunctions are orthogonal, we also have

$$\mathscr{A}_\lambda y_{kj} = \frac{1}{\lambda_k - \lambda} y_{kj}, \qquad \mathscr{A}_\lambda \hat{y}_{kj} = \frac{-1}{\lambda_k + \lambda} \hat{y}_{kj}.$$

From these equations, it follows easily that

$$\mathcal{G}_0 y_{kj} = \frac{1}{\lambda_k}, \qquad \mathcal{G}_0 \hat{y}_{kj} = \frac{-1}{\lambda_k}\hat{y}_{kj} \tag{4.15}$$

for all k and j.

For any $\phi \in C[a, b]$, the function $\mathcal{G}_0\phi$ belongs to $C^n[a, b]$ and satisfies $B(\mathcal{G}_0\phi) = 0$. Therefore, $\mathcal{G}_0\phi$ may be developed into a series expansion in terms of the eigenfunctions of (1.2). We have

$$\mathcal{G}_0\phi = \sum_{k=1}^{\infty} [\pi_k(\mathcal{G}_0\phi) + \hat{\pi}_k(\mathcal{G}_0\phi)]$$

with uniform convergence in $[a, b]$. By virtue of (4.13) and (4.15), there follows

$$\mathcal{G}_0^m\phi = \sum_{k=1}^{\infty} \lambda_k^{-m}[(\pi_k\phi) + (-1)^m(\pi_k\phi)], \qquad 1 \leqq m < \infty. \tag{4.16}$$

5. More series expansions. In this section, we develop the expansion (1.6) and show that its properties closely parallel those of the expansion (1.4).

Given a fundamental system of solutions $\phi_1, \phi_2, \cdots, \phi_n$ of the homogeneous equation $Ly = 0$, the rank of the matrix

$$\begin{bmatrix} B_1\phi_1 & \cdots & B_1\phi_n \\ \vdots & & \vdots \\ B_n\phi_1 & \cdots & B_n\phi_n \end{bmatrix}$$

is $n-1$. Since the rank is independent of the choice of the functions ϕ_k, we may take $\phi_1 = y_0$ and suppose that $y_0, \phi_2, \phi_3, \cdots, \phi_n$ are mutually orthogonal and have norm 1. We may also suppose that, after possibly reordering the forms $B_1, B_2, \cdots, B_n$, the last $n-1$ rows of the matrix are linearly independent.

If $n > 1$, we denote by $q_2, q_3, \cdots, q_n$ the uniquely determined linearly independent solutions of $Ly = 0$ which satisfy

$$B_j q_k = \delta_{jk}, \qquad (q_k, y_0) = 0, \qquad 2 \leqq j, k \leqq n.$$

Let u be any particular solution to the nonhomogeneous equation $Ly = y_0$, and set

$$U = u + \sum_{j=2}^{n} c_j\phi_j,$$

where the constants $c_2, \cdots, c_n$ are determined by

$$\sum_{j=2}^{n} c_j(B_k\phi_j) = -B_k u, \qquad 2 \leqq k \leqq n.$$

Then $B_k U = 0, 2 \leqq k \leqq n$, but $B_1 U \neq 0$, as otherwise self-adjointness would imply

$$1 = (y_0, y_0) = (y_0, LU) = (Ly_0, U) = (0, U) = 0.$$

Thus, let $B_1 U = \alpha \neq 0$, and define

$$q_1 = \alpha^{-1}U - (\alpha^{-1}U, y_0)y_0.$$

We then have

(5.1) $$Lq_1 = \alpha^{-1} y_0, \quad B_k q_1 = \delta_{k1}, \quad (q_1, y_0) = 0, \qquad 1 \leqq k \leqq n.$$

It is easy to see that these conditions uniquely determine q_1. Taking into account the definition of $q_2, \cdots, q_n$, we now have

(5.2) $$\begin{aligned} &B_j q_k = \delta_{jk}, \qquad 2 \leqq j \leqq n, \quad 1 \leqq k \leqq n, \\ &B_1 q_1 = 1. \end{aligned}$$

Now define the boundary forms $\{U_j\}_{j=1}^n$ by

$$\begin{aligned} &U_1 = B_1 - [(B_1 q_2) B_2 + (B_1 q_3) B_3 + \cdots + (B_1 q_n) B_n], \\ &U_k = B_k, \qquad 2 \leqq k \leqq n, \end{aligned}$$

and observe that, by (5.2),

(5.3) $$U_j q_k = \delta_{jk}, \qquad 1 \leqq j, k \leqq n.$$

We also have

(5.4) $$(q_k, y_0) = 0, \qquad 1 \leqq k \leqq n.$$

Let $\{q_m\}_{m=1}^\infty$ be the sequence defined by

(5.5) $$q_{nk+j} = \mathscr{G}_0^k q_j, \qquad 1 \leqq j \leqq n, \quad 0 \leqq k < \infty.$$

Then from (4.6) and (5.4) there follows

(5.6) $$Lq_{nk+j} = q_{n(k-1)+j}, \qquad 1 \leqq k < \infty,$$

and this, together with (5.3), provides an algorithm for computing the functions $\{q_m\}$.

We now derive the expansion (1.6). If $g \in \mathscr{L}$, then (4.6) implies

$$\begin{aligned} L[g - \mathscr{G}_0 Lg] &= Lg - [Lg - (Lg, y_0) y_0] \\ &= (Lg, y_0) y_0. \end{aligned}$$

Since $L[\alpha(Lg, y_0) q_1] = (Lg, y_0) y_0$, then we have

(5.7) $$g - \mathscr{G}_0 Lg = \alpha(Lg, y_0) q_1 + c_1 y_0 + \sum_{k=2}^n c_k q_k$$

for certain constants $c_1, \cdots, c_n$. Multiplying this equation by $\overline{y_0(t)}$, integrating from a to b and using (4.14) and (5.4) results in

$$c_1 = (g, y_0).$$

If we apply the functionals $U_1, U_2, \cdots, U_n$ to (5.7) and use (5.3) and (4.10), we obtain

$$\begin{aligned} U_k g &= c_k, \qquad 2 \leqq k \leqq n, \\ U_1 g &= \alpha(Lg, y_0). \end{aligned}$$

Therefore,

$$g=(g, y_0)y_0+\sum_{j=1}^{n}(U_jg)q_j+\mathcal{G}_0Lg, \tag{5.8}$$

and this is valid on all of $[a, b]$ for all $g\in\mathcal{L}$. Replacing g by Lg in (5.8) we find that

$$Lg=(Lg, y_0)y_0+\sum_{j=1}^{n}(U_jLg)q_j+\mathcal{G}_0L^2g.$$

It follows from (5.5) and Lemma 4.3 that

$$\mathcal{G}_0Lg=\sum_{j=1}^{n}(U_jLg)q_{n+j}+\mathcal{G}_0^2L^2g.$$

Substituting this into (5.8), we have the identity

$$g=(g, y_0)y_0+\sum_{k=0}^{1}\sum_{j=1}^{n}(U_jL^kg)q_{nk+j}+\mathcal{G}_0^2L^2g.$$

Continuing as in (2.7), we are led to the identity

$$g=(g, y_0)y_0+\sum_{k=0}^{m-1}\sum_{j=1}^{n}(U_jL^kg)q_{nk+j}+\mathcal{G}_0^mL^mg, \tag{5.9}$$

which holds in $[a, b]$ for every $g\in\mathcal{L}$ and every positive integer m.

THEOREM 5.1. *Suppose that $f\in\mathcal{L}$ and that the sequence $\{\lambda_1^{-k}(L^kf)(x)\}_{k=0}^{\infty}$ converges uniformly to 0 in $[a, b]$. Then*

$$f(x)=(f, y_0)y_0(x)+\sum_{k=0}^{\infty}\sum_{j=1}^{n}(U_jL^kf)q_{nk+j}(x), \qquad a\leqq x\leqq b, \tag{5.10}$$

with uniform convergence in $[a, b]$.

Proof. The proof proceeds along the lines of Theorem 2.2. First we observe that (2.11) and (2.12) hold in the present setting. Equation (2.11) is trivial and (2.12) follows, as in Lemma 2.1, from differentiating the equation

$$y_{kj}=(\lambda_k-\lambda)\int_a^b G_\lambda(x, t)y_{kj}(t)\,dt, \qquad \lambda \text{ fixed},$$

n times and using Schwarz' inequality. The remainder of the proof follows from (5.9) and from substituting (2.11) and (2.12) into (4.16). The details are left to the reader.

By (4.16), (5.5) and (2.8), we have

$$q_{nk+j}=\sum_{m=1}^{\infty}\lambda_m^{-k}[(\pi_mq_j)+(-1)^k(\hat{\pi}_mq_j)] \tag{5.11}$$

for $1\leqq j\leqq n$ and $1\leqq k<\infty$. Using this and following the proof of Lemma 2.2, we find Lemma 5.1.

LEMMA 5.1. *There exists a constant $M_1>0$ such that*

$$|q_{nk+j}^{(r)}(x)-\lambda_1^{-k}[(\pi_1q_j)^{(r)}(x)+(-1)^k(\pi_1q_j)^{(r)}(x)]|\leqq M_1|\lambda_2|^{-k} \tag{5.12}$$

for $a \leqq x \leqq b$, $0 \leqq r \leqq n$, $0 \leqq k < \infty$, $1 \leqq j \leqq n$.

Consequently, we have

$$\lim_{k\to\infty} \{\lambda_1^k q_{nk+j}^{(r)}(x) - [(\pi_1 q_j)^{(r)}(x) + (-1)^k (\hat{\pi}_1 q_j)^{(r)}(x)]\} = 0, \tag{5.13}$$

uniformly in $[a, b]$. *Thus there exists a positive constant* M_1' *such that*

$$|\lambda_1^k q_{nk+j}^{(r)}(x)| \leqq M_1' \tag{5.14}$$

for $a \leqq x \leqq b$, $0 \leqq r \leqq n$, $1 \leqq j \leqq n$, $0 \leqq k < \infty$.

Our last hypothesis corresponds to hypothesis (H), § 2. We suppose that

(H′) for each j, $1 \leqq j \leqq n$, there exists at least one eigenfunction y, belonging to an eigenvalue of smallest absolute value, such that $(q_j, y) \neq 0$.

Thus, if we define

$$\Gamma_j^{(+)}(x) = (\pi_1 q_j)(x) + (\hat{\pi}_1 q_j)(x),$$
$$\Gamma_j^{(-)}(x) = (\pi_1 q_j)(x) - (\pi_1 q_j)(x), \qquad a \leqq x \leqq b, \quad 1 \leqq j \leqq n,$$

then the following is a direct extension of Theorem 3.1.

THEOREM 5.2. *Let* $\{h_k\}_{k=1}^{\infty}$ *be a complex sequence. Then the following are equivalent*:

(i) $\sum_{k=0}^{\infty} \sum_{j=1}^{n} h_{nk+j} q_{nk+j}(x)$ *converges absolutely at each of* n *points* $x_1, x_2, \cdots, x_n$ *for which* $\Gamma_j^{(+)}(x_j) \neq 0$ *and* $\Gamma_j^{(-)}(x_j) \neq 0$, $1 \leqq j \leqq n$;

(ii) $\sum_{k=0}^{\infty} \sum_{j=1}^{n} |\lambda_1^{-k} h_{nk+j}| < \infty$;

(iii) $\sum_{k=0}^{\infty} \sum_{j=1}^{n} h_{nk+j} q_{nk+j}(x)$ *converges absolutely and uniformly in* $[a, b]$.

THEOREM 5.3. *Suppose that*

$$f(x) = h_0 y_0(x) + \sum_{k=0}^{\infty} \sum_{j=1}^{n} h_{nk+j} q_{nk+j}(x), \qquad a \leqq x \leqq b, \tag{5.15}$$

with absolute convergence in $[a, b]$. *Then* $f \in \mathscr{L}$,

$$f^{(r)}(x) = h_0 y_0^{(r)}(x) + \sum_{k=0}^{\infty} \sum_{j=1}^{n} h_{nk+j} q_{nk+j}^{(r)}(x), \qquad 0 \leqq r \leqq n-1, \tag{5.16}$$

$$(L^m f)^{(r)}(x) = \alpha^{-1} h_{n(m-1)+1} y_0^{(r)}(x) + \sum_{k=m}^{\infty} \sum_{j=1}^{n} h_{nk+j} q_{n(k-m)+j}^{(r)}(x) \tag{5.17}$$

for $0 \leqq r \leqq n-1$ *and* $1 \leqq m < \infty$. *Furthermore,*

$$h_0 = (f, y_0)$$

and

$$h_{nk+j} = U_j L^k f, \qquad 1 \leqq j \leqq n, \quad 0 \leqq k < \infty.$$

Proof. The estimates (5.14) imply

$$\begin{aligned} &\sum_{k=m}^{\infty} \sum_{j=1}^{n} |h_{nk+j} q_{n(k-m)+j}(x)| \\ &= |\lambda_1|^m \sum_{k=m}^{\infty} \sum_{j=1}^{n} |\lambda_1^{-k} h_{nk+j} \lambda_1^{k-m} q_{n(k-m)+j}(x)| \\ &\leqq |\lambda_1|^m M_1' \sum_{k=m}^{\infty} \sum_{j=1}^{n} |\lambda_1^{-k} h_{nk+j}|. \end{aligned} \tag{5.18}$$

It follows from Theorem 5.2 that the series

$$\alpha^{-1}h_{n(m-1)+1}y_0(x)+\sum_{k=m}^{\infty}\sum_{j=1}^{n}h_{nk+j}q_{n(k-m)+j}(x) \tag{5.19}$$

converges absolutely and uniformly in $[a, b]$. Applying the integral operator $\mathscr{G}_0^m$ termwise to (5.19) and taking into account (5.5) and Lemma 4.3, we obtain

$$\begin{aligned}\mathscr{G}_0^m\left\{\alpha^{-1}h_{n(m-1)+1}y_0(x)+\sum_{k=m}^{\infty}\sum_{j=1}^{n}h_{nk+j}q_{n(k-m)+j}(x)\right\}\\ =\sum_{k=m}^{\infty}\sum_{j=1}^{n}h_{nk+j}q_{nk+j}(x)\\ =f(x)-h_0y_0(x)-\sum_{k=0}^{m-1}\sum_{j=1}^{n}h_{nk+j}q_{nk+j}(x).\end{aligned}$$

By (5.6) and the definition of $q_1, q_2, \cdots, q_n$, we now have

$$\begin{aligned}L^m\mathscr{G}_0^m\{\alpha^{-1}h_{n(m-1)+1}y_0(x)+\sum_{k=m}^{\infty}\sum_{j=1}^{n}h_{nk+j}q_{n(k-m)+j}(x)\}\\ =(L^mf)(x)-\alpha^{-1}h_{n(m-1)+1}y_0(x).\end{aligned}$$

Since the inner product of the series (5.19) with y_0 is $\alpha^{-1}h_{n(m-1)+1}$, (4.6) yields

$$\sum_{k=m}^{\infty}\sum_{j=1}^{n}h_{nk+j}q_{n(k-m)+j}(x)=(L^mf)(x)-\alpha^{-1}h_{n(m-1)+1}y_0(x),$$

and this is (5.17) with $r=0$.

Following (5.18), it is easy to show that the right side of each of (5.16) and (5.17) converges uniformly and absolutely in $[a, b]$ for $r=0, 1, 2, \cdots, n$. Since the "derived" series converge uniformly, we see that (5.16) and (5.17) hold.

If we set $r=0$ in (5.16), take the inner product of the resulting equation with y_0 and use (5.4), there follows

$$(f, y_0)=h_0.$$

Finally, applying the boundary functionals U_j to (5.16) and (5.17) results in

$$h_{nk+j}=U_jL^kf,$$

and this completes the proof.

THEOREM 5.4. *Let f be a complex-valued function on $[a, b]$. Then for f to have an absolutely and uniformly convergent series representation*

$$f(x)=(f, y_0)y_0(x)+\sum_{k=0}^{\infty}\sum_{j=1}^{n}(U_jL^kf)q_{nk+j}(x), \tag{5.20}$$

it is necessary and sufficient that

$$\lim_{k\to\infty}\lambda_1^{-k}(L^kf)(x)=0 \qquad \textit{uniformly in } [a, b] \tag{5.21}$$

and

$$\sum_{k=0}^{\infty} \sum_{j=1}^{n} |\lambda_1^{-k} U_j L^k f| < \infty. \tag{5.22}$$

Proof. If f satisfies (5.22), then Theorem 5.2 shows that the series in (5.20) converges absolutely and uniformly in $[a, b]$. If (5.21) holds, then it follows from Theorem 5.1 that f has the representation (5.20).

Suppose that (5.20) holds. Condition (5.22) is a consequence of (ii), Theorem 5.2. To prove (5.21), first note that (5.17) and (5.18) imply

$$|(L^m f)(x)| \leqq |\alpha^{-1} h_{n(m-1)+1} y_0(x)| + |\lambda_1|^m \mu_m(x),$$

where $\lim_{x\to\infty} \mu_m(x) = 0$ and $h_{n(m-1)+1} = U_1 L^{m-1} f$. Writing this in the form

$$|(L^m f)(x)| \leqq |\lambda_1|^m \{\mu_m(x) + |\lambda_1^{-1} \alpha^{-1} y_0(x)| |\lambda_1^{-(m-1)} h_{n(m-1)+1}|\},$$

and noting that $\lambda_1^{-(m-1)} h_{n(m-1)+1} \to 0$, we obtain (5.21).

Example. Consider the first order self-adjoint problem

$$Ly = iy' = 0, \qquad B_1 y = y(1) - y(0) = 0.$$

The nonzero eigenvalues of this problem are $\pm 2n\pi$, for $n = 1, 2, 3, \cdots$, with corresponding eigenfunctions $\exp[\mp 2n\pi i x]$. We choose $y_0 = i$ to correspond to the eigenvalue 0. A direct calculation yields

$$i^k q_{k+1} = Q_{k+1}, \qquad k = 0, 1, 2, \cdots,$$

where Q_k is the kth Bernoulli polynomial. Furthermore, we have

$$(L^k f)(x) = i^k f^{(k)}(x),$$

so that

$$U_1 L^k f = B_1 L^k f = i^k [f^{(k)}(1) - f^{(k)}(0)], \qquad 1 \leqq k < \infty.$$

Hence

$$(U_1 L^k f) q_{k+1} = [f^{(k)}(1) - f^{(k)}(0)] Q_{k+1}.$$

Since $(f, y_0) y_0 = \int_0^1 f(x)\, dx$, then series (5.20) becomes

$$f(x) = \int_0^1 f(t)\, dt + \sum_{k=0}^{\infty} [f^{(k)}(1) - f^{(k)}(0)] Q_{k+1}(x), \tag{5.23}$$

which is the Bernoulli polynomial expansion [1, p. 29]. In particular, if f is an entire function of exponential type less than 2π, then (5.21) and (5.22) certainly hold. Thus the representation (5.23) converges absolutely and uniformly in $[0.1]$.

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ON q-ANALOGUES OF THE WATSON AND WHIPPLE SUMMATIONS*

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Abstract. In this paper, q-analogues of the terminating cases of the Watson and Whipple summations are presented and proved.

1. Introduction. As basic hypergeometric functions (or q-hypergeometric functions) become increasingly important in pure and applied mathematics (see [3]), the question of obtaining q-analogues for the important hypergeometric series summations becomes significant. If we examine Appendices III and IV in Slater's book [4], we find that among the ${}_2F_1$ summations (III.3–III.7) the first two have q-analogues given by Slater (IV.1, IV.2), Kummer's theorem (III.5) has a q-analogue due to Bailey and Daum (see [2] for references) and the remaining two results (Gauss's second theorem and Bailey's theorem) have q-analogues given in [2].

There are many more known summations for ${}_3F_2$ functions. Most important are Saalschutz's theorem [4, (III.2), p. 243]

$$(1.1)\qquad {}_3F_2\left[\begin{matrix} a, b, -n; 1 \\ c, d \end{matrix}\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}$$

provided $a+b-n+1=c+d$ and n is a nonnegative integer; Dixon's theorem [4, (III.8), p. 243]

$$(1.2)\qquad {}_3F_2\left[\begin{matrix} a, b, c; 1 \\ 1+a-b, 1+a-c \end{matrix}\right] = \frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{a}{2}-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)\Gamma\left(1+\frac{a}{2}-c\right)\Gamma(1+a-b-c)};$$

Watson's theorem [4, (III.23), p. 245]

$$(1.3)\qquad {}_3F_2\left[\begin{matrix} a, b, \frac{c}{2}; 1 \\ \frac{1}{2}(a+b+1), c \end{matrix}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(1+a+b))\Gamma(\frac{1}{2}(1+c))\Gamma(\frac{1}{2}(1-a-b+c))}{\Gamma(\frac{1}{2}(1+a))\Gamma(\frac{1}{2}(1+b))\Gamma(\frac{1}{2}(1-a+c))\Gamma(\frac{1}{2}(1-b+c))};$$

Whipple's theorem [4. (III.24), p. 245]

$$(1.4)\qquad {}_3F_2\left[\begin{matrix} a, b, \frac{1}{2}c; 1 \\ d, e \end{matrix}\right] = \frac{\pi\Gamma(d)\Gamma(e)}{2^{c-1}\Gamma(\frac{1}{2}(a+e))\Gamma(\frac{1}{2}(a+d))\Gamma\frac{1}{2}(d+e))\Gamma(\frac{1}{2}(b+d))}$$

provided $a+b=1$ and $d+e=1+c$.

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In formulas (1.1)–(1.4) we have used the standard hypergeometric series notation

$$(1.5)\qquad {}_pF_q\left[\begin{matrix} a_1, \cdots, a_p; t \\ b_1, \cdots, b_q \end{matrix}\right] = \sum_{n\geqq 0} \frac{(a_1)_n \cdots (a_p)_n t^n}{n!(b_1)_n \cdots (b_q)_n},$$

where $(a)_n = a(a+1)\cdots(a+n-1)$.

There are known q-analogues for (1.1) and (1.2), namely, Jackson's q-analogue of Saalschutz's theorem [4, (IV.4), p. 247]

$$(1.6)\qquad {}_3\phi_2\left[\begin{matrix} a, b, q^{-n}; q, q \\ d, e \end{matrix}\right] = \frac{(d/a;q)_n(d/b;q)_n}{(d;q)_n(d/ab;q)_n},$$

provided $abq^{-n+1} = de$ and n is a nonnegative integer; and the q-analogue of Dixon's theorem [4, (IV.5), p. 247]

$$(1.7)\qquad {}_4\phi_3\left[\begin{matrix} a, -q\sqrt{a}, b, c; q, \dfrac{q\sqrt{a}}{bc} \\ -\sqrt{a}, \dfrac{aq}{b}, \dfrac{aq}{c} \end{matrix}\right] = \frac{(aq;q)_\infty(a^{1/2}q/b;q)_\infty(a^{1/2}q/c;q)_\infty(aq/bc;q)_\infty}{(aq/b;q)_\infty(aq/c;q)_\infty(a^{1/2}q;q)_\infty(a^{1/2}q/bc;q)_\infty}.$$

Here

$${}_r\phi_s\left[\begin{matrix} a_1, \cdots, a_r; q, t \\ b_1, \cdots, b_s \end{matrix}\right] = \sum_{n\geqq 0} \frac{(a_1;q)_n \cdots (a_r;q)_n t^n}{(q;q)_n(b_1;q)_n \cdots (b_s;q)_n},$$

and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$, $(a;q)_\infty\ \ \lim_{n\to\infty}(a;q)_n$.

Identities (1.3) and (1.4), have no known q-analogues. In § 2 we shall prove the following theorems which provide such analogues.

THEOREM 1 (q-analogue of Watson's theorem (1.3)).

$${}_4\phi_3\left[\begin{matrix} a, b, c^{1/2}, -c^{1/2}; q, q \\ (abq)^{1/2}, -(abq)^{1/2}, c \end{matrix}\right] = \frac{a^{n/2}(aq;q^2)_\infty(bq;q^2)_\infty(cq/a;q^2)_\infty(cq/b;q^2)_\infty}{(q;q^2)_\infty(abq;q^2)_\infty(cq;q^2)_\infty(cq/ab;q^2)_\infty},$$

where $b = q^{-n}$ and n is a nonnegative integer.

THEOREM 2 (q-analogue of Whipple's theorem (1.4)).

$${}_4\phi_3\left[\begin{matrix} a, q/a, c^{1/2}, -c^{1/2}; q, q \\ -q, e, cq/e \end{matrix}\right]$$

$$= \frac{q^{n(n+1)/2}(ea;q^2)_\infty(eq/a;q^2)_\infty(caq/e;q^2)_\infty(cq^2/ae;q^2)_\infty}{(e;q)_\infty(cq/e;q)_\infty}$$

provided $a = q^{-n}$ and n is a nonnegative integer.

We remark that while the functions in Theorem 1 and 2 are not ${}_3\phi_2$ functions, they are nonetheless q-analogues of the corresponding ${}_3F_2$ functions; this is because

$$\lim_{q\to 1} {}_4\phi_3\left[\begin{matrix} q^a, q^b, -q^c, q^{-n}; q, t \\ q^e, -q^f, q^g \end{matrix}\right] = {}_3F_2\left[\begin{matrix} a, b, -n; t \\ e, g \end{matrix}\right].$$

2. The main theorems. Interestingly, each of the theorems we have stated relies on Watson's q-analogue of Whipple's theorem [4, (3.4.1.5), p. 100].

Proof of Theorem 1. We observe that the ${}_4\phi_3$ in Theorem 1 is a terminating Saalschutzian series (i.e., the product of the three lower parameters divided by the product of the four upper parameters is q). Hence by Watson's q-analogue of Whipple's theorem (with $a\to -b$, $c\to (bqa^{-1})^{1/2}$, $d\to -(bqa^{-1})^{1/2}$, $e\to c^{1/2}$, $f\to -c^{1/2}$, $g\to b=q^{-n}$)

$$
{}_4\phi_3\left[\begin{matrix} a, c^{1/2}, -c^{1/2}, b; q, q \\ c, (abq)^{1/2}, -(abq)^{1/2}\end{matrix}\right]
$$

$$
=\frac{(b^2q^2c^{-1};q^2)_\infty(-q;q)_\infty(q/c;q)_\infty}{(-bq;q)_\infty(q^2/c;q^2)_\infty(bq/c;q)_\infty}
$$

$$
\cdot {}_8\phi_7\left[\begin{matrix} -b, q\sqrt{-b}, -q\sqrt{-b}, -(bqa^{-1})^{1/2}, (bqa^{-1})^{1/2}, c^{1/2}, -c^{1/2}, b; q, \dfrac{aq}{c} \\ \sqrt{-b}, -\sqrt{-b}, (bqa)^{1/2}, -(bqa)^{1/2}, bqc^{-1/2}, -bqc^{-1/2}, -q \end{matrix}\right]
$$

$$
=\frac{(b^2q^2c^{-1};q^2)_\infty(-q;q)_\infty(q/c;q)_\infty}{(-bq;q)_\infty(q^2/c;q^2)_\infty(bq/c;q)_\infty}\,{}_4\phi_3\left[\begin{matrix} b^2, -bq^2, bqa^{-1}, c; q^2, \dfrac{aq}{c} \\ -b, abq, b^2q^2c^{-1}\end{matrix}\right]
$$

$$
=\frac{(b^2q^2c^{-1};q^2)_\infty(-q;q)_\infty(q/c;q)_\infty(b^2q^2;q^2)_\infty(aq;q^2)_\infty(bq^2c^{-1};q^2)_\infty(abq/c;q^2)_\infty}{(-bq;q)_\infty(q^2/c;q^2)_\infty(bqc^{-1};q)_\infty(abq;q^2)_\infty(b^2q^2c^{-1};q^2)_\infty(bq^2;q^2)_\infty(aq/c;q^2)_\infty}
$$

(by (1.7))

$$
=\frac{(aq;q^2)_\infty(bq;q^2)_\infty(q/c;q^2)_\infty(abq/c;q^2)_\infty}{(q;q^2)_\infty(abq;q^2)_\infty(bqc^{-1};q^2)_\infty(aq/c;q^2)_\infty}
$$

$$
=\begin{cases} 0 & \text{if } n \text{ is odd,} \\[2ex] \dfrac{(aq;q^2)_\infty(bq;q^2{}_\infty(\left(1-\dfrac{aq^{1-n}}{c}\right)\left(1-\dfrac{aq^{3-n}}{c}\right)\cdots\left(1-\dfrac{aq^{-1}}{c}\right)}{(q;q^2)_\infty(abq;q^2)_\infty\left(1-\dfrac{q^{1-n}}{c}\right)\left(1-\dfrac{q^{3-n}}{c}\right)\cdots\left(1-\dfrac{q^{-1}}{c}\right)} & \text{if } n \text{ is even,}\end{cases}
$$

$$
=\begin{cases} 0 & \text{if } n \text{ is odd,} \\[2ex] \dfrac{a^{n/2}(aq;q^2)_\infty(bq;q^2)_\infty(cq/a;q^2)_{n/2}}{(q;q^2)_\infty(abq;q^2)_\infty(cq;q^2)_{n/2}} & \text{if } n \text{ is even,}\end{cases}
$$

$$
=\frac{a^{n/2}(aq;q^2)_\infty(bq;q^2)_\infty(cq/a;q^2)_\infty(cq/b;q^2)_\infty}{(q;q^2)_\infty(abq;q^2)_\infty(cq;q^2)_\infty(cq/ab;q^2)_\infty}. \qquad \square
$$

We now proceed to the q-analogue of Whipple's theorem.

Proof of Theorem 2. As before, we again have a terminating Saalschutzian ${}_4\phi_3$. Hence by Watson's q-analogue of Whipple's theorem (with $a\to -aeq^{-1}$, $c\to aeq^{-1}$, $d\to -a$, $e\to c^{1/2}$, $f\to -c^{1/2}$, $g\to a=q^{-n}$)

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q/a, c^{1/2}, -c^{1/2}, a; q, q \\ cq/e, -q, e \end{matrix}\right] \\
&= \frac{(a^2e^2/c;q^2)_\infty(-e;q)_\infty(e/c;q)_\infty}{(-ae;q)_\infty(e^2/c;q^2)_\infty(ae/c;q)_\infty} \\
&\quad\cdot {}_8\phi_7\left[\begin{matrix} -\frac{ae}{q}, q\sqrt{-\frac{ae}{q}}, -q\sqrt{-\frac{ae}{q}}, \frac{ae}{q}, -a, c^{1/2}, -c^{1/2}, a; q, \frac{eq}{ac} \\ \sqrt{-\frac{ae}{q}}, -\sqrt{-\frac{ae}{q}}, -q, e, -aec^{-1/2}, aec^{-1/2}, -e \end{matrix}\right] \\
&= \frac{(a^2e^2/c;q^2)_\infty(-e;q)_\infty(e/c;q)_\infty}{(-ae;q)_\infty(e^2/c;q^2)_\infty(ae/c;q)_\infty} \\
&\quad\cdot {}_4\phi_3\left[\begin{matrix} \frac{a^2e^2}{q^2}, -aeq, a^2, c; q^2, \frac{eq}{ac} \\ -\frac{ae}{q}, e^2, \frac{a^2e^2}{c} \end{matrix}\right] \\
&= \frac{(a^2e^2/c;q^2)_\infty(-e;q)_\infty(e/c;q)_\infty}{(-ae;q)_\infty(e^2/c;q^2)_\infty(ae/c;q)_\infty} \\
&\quad\cdot \frac{(a^2e^2;q^2)_\infty(eq/a;q^2)_\infty(aeq/c;q^2)_\infty(e^2/c;q^2)_\infty}{(e^2;q^2)_\infty(a^2e^2/c;q^2)_\infty(aeq;q^2)_\infty(eq/ac;q^2)_\infty} \qquad \text{(by (1.7))} \\
&= \frac{(eq;q^2)_\infty(eq/a;q^2)_\infty(e/c;q)_\infty(aeq/c;q^2)_\infty}{(e;q)_\infty(ae/c;q)_\infty(eq/ac;q^2)_\infty} \\
&= \frac{(ea;q^2)_\infty(eq/a;q^2)_\infty(1-eq^{1-n}/c)(1-eq^{3-n}/c)\cdots(1-eq^{n-1}/c)}{(e;q)_\infty(1-eq^{-n}/c)(1-eq^{1-n}/c)\cdots(1-eq^{-1}/c)} \\
&= \frac{q^{n(n+1)/2}(ea;q^2)_\infty(eq/a;q^2)_\infty(cq^{1-n}/e;q^2)_n}{(e;q)_\infty(cq/e;q)_n} \\
&= \frac{q^{n(n+1)/2}(eq;q^2)_\infty(eq/a;q^2)_\infty(caq/e;q^2)_\infty(cq/ae;q)_\infty}{(e;q)_\infty(cq/ae;q^2)_\infty(cq/e;q)_\infty} \\
&= \frac{q^{n(n+1)/2}(ea;q^2)_\infty(eq/a;q^2)_\infty(caq/e;q^2)_\infty(cq^2/ae;q^2)_\infty}{(e;q)_\infty(cq/e;q)_\infty}. \qquad \square
\end{aligned}
$$

In closing this section, we remark that both Theorems 1 and 2 have the defect that they are terminating series while (1.3) and (1.4) are not. The above proofs fail if we remove the termination restriction since we must then look at Bailey's extension of Watson's theorem to the nonterminating case [4, (3.4.2.5), p. 102]. In the case of Theorem 1 an extra term arises which involves the following Saalschutzian series:

$$ {}_4\phi_3\left[\begin{matrix} qc^{-1/2}, -qc^{-1/2}, \frac{bq}{c}, \frac{aq}{c}; q, q \\ \frac{q^2}{c}, -c^{-1}q^{3/2}a^{1/2}b^{1/2}, c^{-1}q^{3/2}a^{1/2}b^{1/2} \end{matrix}\right], $$

which is of the same form as our original series in Theorem 1. Similar problems arise for the nonterminating case of Theorem 2.

3. Conclusion. We remark that the inelegant formulation of Theorems 1 and 2 is due to the fact that these are really theorems about basic hypergeometric series in which some of the finite products are of the form $(A; q)_n$ and some $(B; q^2)_n$. The following modification of the notation of R. P. Agarwal and A. Verma [1] reflects much better the true character of these theorems.

$$ {}_{A+B}\phi_{C+D}\left[\begin{matrix} a_1, \cdots, a_A : b_1, \cdots, b_B; q, q_1, x \\ c_1, \cdots, c_C : d_1, \cdots, d_D \end{matrix}\right] = \sum_{n \geqq 0} \frac{(a_1; q)_n \cdots (a_A; q)_n (b_1; q_1)_n \cdots (b_B; q_1)_n x^n}{(c_1; q)_n \quad (c_C; q)_n (d_1; q_1)_n \cdots (d_D; q_1)_n}. $$

The series in Theorem 1 becomes

$$ {}_3\phi_3\left[\begin{matrix} a, b : c; q, q^2, q \\ q, c : abq \end{matrix}\right], $$

and the series in Theorem 2 becomes

$$ {}_3\phi_3\left[\begin{matrix} a, q/a : c; q, q^2, q \\ e, cq/a : q^2 \end{matrix}\right]. $$

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ON THE NONEXISTENCE OF ENTIRE SOLUTIONS TO NONLINEAR SECOND ORDER ELLIPTIC EQUATIONS*

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Abstract. In this paper, it is shown that there are no global solutions (in all of R^n) to nonlinear elliptic equations of the form

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = f(x, u, \text{grad u})$$

if $f(x, u, p) \geqq g(u)$, where g is a convex, nonnegative function (nondecreasing if $n>2$) such that $g^{-1/2}$ is integrable near infinity unless $g(u(x)) \equiv 0$.

Introduction. It is the purpose of this paper to extend the nonexistence results of Walter [2] for global solutions (in all of R^n) of $\Delta u = f(u)$, to general second order uniformly elliptic operators or equations of the form

$$\mathscr{L}u \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = f(x, u, \text{grad } u), \tag{I}$$

where the $a_{ij}(x)$ are continuously differentiable in R^n and f is a point function of $2n+1$ arguments. The operator $\mathscr{L}$ is assumed to be uniformly elliptic in all of R^n. That is, for all $x, \xi \in R^n$ and some $\mu > 0$.

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqq \mu \sum_{i=1}^{n} \xi_i^2, \qquad a_{ij} = a_{ji}. \tag{II}$$

Our nonexistence arguments are modeled after those of Walter, rather than those of Keller [1], which do not seem to extend to (I). Our sufficient conditions on f to insure nonexistence of global solutions to (I) are thus similar to those of Walter.

Throughout the remainder of this paper we employ both the summation convention and the notation

$$u_{,i} = \frac{\partial u}{\partial x_i}, \qquad u_{,ij} = \frac{\partial^2 u}{\partial x_i \partial x_j} = u_{,ji}. \tag{III}$$

For $x, y \in R^n$, we let $h(x, y)$ denote a fundamental solution to $\mathscr{L}u = 0$, namely,

$$\mathscr{L}_x h(x, y) = -\delta(x-y) \tag{IVa}$$

in the sense of distributions. Of course,

$$\lim_{x\to y} h(x, y) = +\infty. \tag{IVb}$$

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In agreement with what obtains for the Laplace operator, we assume that

$$\text{(Va)} \qquad a_{ij}(x)\frac{\partial h}{\partial x_i}(x, y)\frac{\partial h}{\partial x_j}(x, y) \leqq \phi(h(x, y)),$$

where, for some positive constant $k = k(n)$,

$$\text{(Vb)} \qquad \phi(\alpha) = \begin{cases} k^2\alpha^{2(n-1)(n-2)^{-1}}, & n > 2, \quad 0 < \alpha < \infty, \\ k^2 e^{4\pi\alpha} & n = 2, \quad -\infty < \alpha < \infty. \end{cases}$$

Of course, when $\mathscr{L} = \Delta$ one takes $h(x, y) = (2\pi)^{-1} \ln (|x-y|^{-1})$ for $n = 2$ and $h(x, y) = [\omega_n(n-2)]^{-1}|x-y|^{2-n}$ for $n > 2$, where ω_n is the surface area of the unit sphere in R_n. Equality holds in (Va) in these cases by appropriate choices of k.

Insofar as the nonlinearity, f, is concerned, we assume

$$\text{(VI)} \qquad f(x, u, p) \geqq g(u), \qquad p \in R^n,$$

where g has the following properties:

G-1. g is nonnegative and convex on R^1.

G-2. If $\mathscr{G}(s)$ is any integral of g ($\mathscr{G}' = g$), then for every pair of numbers ε, δ,

$$\int_\delta^\infty [\mathscr{G}(s) - \mathscr{G}(\delta) + \varepsilon^2]^{-1/2}\, ds < \infty.$$

If $n > 2$, we suppose in addition that:

G-3. g is nondecreasing.

1. The nonexistence theorem. Here we prove our basic nonexistence theorem. We are motivated by the fact that for $\mathscr{L} = \Delta$, the level surfaces of the fundamental solution at a fixed point (say $y = 0$) are spheres and the spherical means of the solutions to (I) are precisely integrals of these solutions over these level surfaces. Here we shall obtain similar nonexistence results by using analogous means of the solutions to (I), provided the fundamental solution has the behavior and regularity indicated by (Va) and (Vb), and provided the level surfaces are closed, bounded, continuously differentiable $(n-1)$-dimensional surfaces in R^n with finite nonzero $(n-1)$-dimensional Lebesque measure, with $h(x, y) \to -\infty$ as $|x-y| \to \infty (n = 2)$ and $h(x, y) \to 0$ as $|x-y| \to \infty (n \geqq 3)$. It is clear that for $n \geqq 3$, $h(x, y)$ is just the Green's function for all of space.

THEOREM. *If the fundamental solution $h(x, y)$, the nonlinearity f and the lower bound g satisfy the preceding hypotheses, then no classical solution to* (I) *exists in all of R^n except possibly for solutions which satisfy*

$$\text{(1.1)} \qquad u(x) \in \{s \mid g(s) = 0\}$$

for all $x \in R^n$.

In particular, if $g > 0$ on R^1 no global (entire) solution to (I) *is possible.*

Proof. Fix $y = 0$ as the origin such that $g(u(0)) > 0$, and let $h(x) = h(x, 0)$ denote the fundamental solution. Let

$$D_\alpha = \{x \in R^n \mid h(x) > \alpha\},$$

where

$$S_\alpha = \partial D_\alpha = \{x \in R^n \mid h(x) = \alpha\}.$$

The sets S_α, D_α denote, respectively, the level surfaces of h and the regions interior thereto of the fundamental solution with singularity at the origin. Because of the singular nature of the fundamental solution, the range of α is $(-\infty, \infty)$ if $n=2$ and $(0, \infty)$ if $n>2$. We also have, in view of the maximum principle,

$$D_{\alpha_1} \subsetneqq D_{\alpha_2} \quad \text{if } \alpha_1 > \alpha_2,$$

$$\bigcup_{\alpha>0} D_\alpha = R^n, \qquad n>2,$$

and

$$\bigcup_{\alpha \text{ real}} D_\alpha = R^2, \qquad n=2.$$

Let

$$F(\alpha) = \oint_{S_\alpha} (a_{ij}h_{,i}h_{,j}u/|\text{grad } h|)\, ds \tag{1.1.1}$$

$$= -\oint_{S_\alpha} a_{ij}h_{,i}n_j u\, dS, \tag{1.1.2}$$

where $n_i = -h_{,1}/|\text{grad } h|$ denotes the ith component of the outward normal to D. From Green's theorem and (IVa) we have

$$F(\alpha) = u(0) - \int_{D_\alpha} a_{ij}h_{,i}u_{,j}\, dx \tag{1.2.1}$$

or

$$F(\alpha) = u(0) - \int_\alpha^\infty \left(\oint_{S_\eta} (a_{ij}h_{,i}u_{,j}/|\text{grad } h|)\, ds_\eta \right) d\eta \tag{1.2.2}$$

since

$$dx = dS\, dn = -dh\, dS/|\text{grad } h|.$$

Therefore,

$$F'(\alpha) = \oint_{S_\alpha} (a_{ij}h_{,i}u_{,j}/|\text{grad } h|)\, dS_\eta \tag{1.3.1}$$

$$= -\oint_{S_\alpha} a_{ij}u_{,i}n_j\, dS \tag{1.3.2}$$

$$= -\int_{D_\alpha} \mathscr{L}u\, dx \tag{1.3.3}$$

$$\leqq -\int_{D_\alpha} g(u)\, dx. \tag{1.3.4}$$

Thus since $g(u(0))>0$, $F'(\alpha)$ is negative near $\alpha=+\infty (D_\infty=\{0\})$. Moreover,

$$F'(\alpha)=-\int_\alpha^\infty \left(\oint_{S_\eta} (\mathscr{L}u/|\text{grad } h|)\, dS_\eta\right) d\eta \tag{1.4}$$

so that

$$F''(\alpha)=\oint_{S_\alpha} (\mathscr{L}u/|\text{grad } h|)\, dS \tag{1.5.1}$$

$$\geqq \oint_{S_\alpha} (g(u)/|\text{grad } h|)\, dS \tag{1.5.2}$$

$$\geqq [\phi(\alpha)]^{-1} \oint_{S_\alpha} g(u)(a_{ij}h_{,i}h_{,j}/|\text{grad } h|)\, dS. \tag{1.5.3}$$

Since g is convex, since

$$\int_{S_\alpha} (a_{ij}h_{,i}h_{,j}/|\text{grad } h|)\, dS=-\int_{S_\alpha} a_{ij}h_{,i}n_j\, dS=-\int_{D_\alpha} \mathscr{L}h\, dx=1, \tag{1.6}$$

and since $a_{ij}h_{,i}h_{,j}/|\text{grad } h|>0$, we may apply Jensen's inequality to (1.5.3) to obtain

$$F''(\alpha)\geqq[\phi(\alpha)]^{-1}g(F(\alpha)). \tag{1.7}$$

Inequality (1.7) is the basic inequality in the proof of our theorem. We consider the cases $n=2$ and $n>2$ separately.

Case 1. $n=2$. We have for $a\leqq\alpha_0$,

$$F''(\alpha)\geqq k^{-2}\, e^{-4\pi\alpha}g(F(a)) \tag{1.8.1}$$

$$\geqq k^{-2}\, e^{-4\pi\alpha_0}g(F(\alpha)), \tag{1.8.2}$$

where α_0 is chosen so large that $F'(\alpha_0)<0$. But by assumption G-1 on $g(u)$ it follows from (1.3.4) that $F'(\alpha)\leqq 0$ for all α. In particular then $F'(\alpha)\leqq 0$ for $a\in(\alpha_1,\alpha_0]$, where α_1 is either negative infinity or the limit of the existence interval. For $\alpha<\alpha_0$, we may, therefore, multiply (1.8.2) by $F'(\alpha)$ and integrate to obtain

$$\tfrac{1}{2}[F'(\alpha_0)]^2-\tfrac{1}{2}[F'(\alpha)]^2\leqq(k\, e^{2\pi\alpha_0})^{-2}[\mathscr{G}(F(\alpha_0))-\mathscr{G}(F(\alpha))].$$

We assume that $F(\alpha)$ remains bounded on $(-\infty,\alpha_0]$ and integrate again to obtain

$$I(\alpha)\equiv\int_{F(\alpha_0)}^{F(\alpha)} \{\beta^2[F'(\alpha_0)]^2+\mathscr{G}(s)-\mathscr{G}(F(\alpha_0))\}^{-1/2}\, ds\geqq\beta^{-1}(\alpha_0-\alpha),$$

where $\beta \equiv k\, e^{2\pi\alpha_0}/\sqrt{2}$. By assumption, it follows then that

$$(1.9.1)\quad \infty > \int_{F(\alpha_0)}^{\infty} \{[\beta F'(\alpha_0)]^2 + \mathcal{G}(s) - \mathcal{G}(F(\alpha_0))\}^{-1/2}\, ds \geqq I(\alpha) \geqq \beta(\alpha_0 - \alpha).$$

Letting $\alpha \to -\infty$ on the extreme right of (1.9), we see that we have a contradiction if u exists on R^2. In fact, for some $\alpha_1 > -\infty$,

$$(1.9.2)\qquad \lim_{\alpha \to \alpha_1^+} F(\alpha) = +\infty.$$

Case 2. $n > 2$. Here the preceding arguments must be modified somewhat. Our arguments here closely follow those of Walter [2].

We have from (1.7)

$$(1.10)\qquad F''(\alpha) \geqq k^{-2} \alpha^{-2(n-1)/(n-2)} g(F(\alpha)).$$

Now let

$$(1.11)\qquad t = -(n-2)^{-1} \ln \alpha.$$

We see from (1.4) that $\lim_{\alpha \to +\infty} F'(\alpha) = 0$ so that

$$(1.12.1)\qquad -F'(\alpha) = \int_{\alpha}^{\infty} F''(\eta)\, d\eta$$

$$(1.12.2)\qquad \geqq k^{-2} \int_{\alpha}^{\infty} \eta^{-2(n-1)/(n-2)} g(F(\eta))\, d\eta.$$

Introducing (1.11) into (1.12.2), we find that

$$(1.13)\qquad \frac{dF_1}{dt} \geqq k^{-2}(n-2)^2\, e^{-(n-2)t} \int_{-\infty}^{t} e^{n\sigma} g(F_1(\sigma))\, d\sigma,$$

where we have set $F_1(\sigma) = F(\exp[(2-n)\sigma])$. Now define

$$(1.14)\qquad N'(t) = k^{-2}(n-2)^2\, e^{-(n-2)t} \int_{-\infty}^{t} e^{n\sigma} g(F_1(\sigma))\, d\sigma$$

for $t \geqq t_0$. Since $F_1'(t) \geqq 0$ and g is nondecreasing, we observe from (1.14) that $g(F_1(\sigma)) \leqq g(F_1(t))$ for $\sigma \leqq t$ and hence that

$$(1.14.1)\qquad N'(t) \leqq n^{-1} k^{-2} (n-2)^2\, e^{2t} g(F_1(t))$$

for $t \geqq t_0$. Now let

$$N(t) = F_1(t_0) + \int_{t_0}^{t} N'(\sigma)\, d\sigma.$$

Clearly, for $t \geqq t_0$,

$$F_1'(t) \geqq N'(t) \geqq 0$$

and

$$F_1(t_0) = N(t_0)$$

so that

$$F_1(t) \geqq N(t), \qquad t \geqq t_0.$$

Since g is nondecreasing, for $t \geqq t_0$, we have

$$\begin{aligned} N''(t) &= k^{-2}(n-2)^2 e^{2t} g(F_1(t)) - k^{-2} \\ &\quad \cdot (n-2)^3 e^{-(n-2)t} \int_{-\infty}^{t} e^{n\sigma} g(F_1(\sigma))\, d\sigma \end{aligned} \tag{1.15.1}$$

$$\geqq 2n^{-1}k^{-2}(n-2)^2 e^{2t} g(F_1(t)) \tag{1.15.2}$$

$$\geqq 2n^{-1}k^{-2}(n-2)^2 e^{2t_0} g(N(t)). \tag{1.15.3}$$

Since, as $t \to -\infty$, $F_1(t) \to u(0)$ so that $g(F_1(t)) \to g(u(0)) > 0$, it follows from (1.14) that $N'(t_0) > 0$ for t_0 sufficiently negative, so that $N'(t) > 0$ for all t. Multiplying by $N'(t)$, integrating from t_0 to t and rearranging terms, we obtain, after a second quadrature,

$$\infty > \int_{N(t_0)}^{\infty} \{\gamma^2[N'(t_0)]^2 + \mathcal{G}(s) - \mathcal{G}(N(t_0))\}^{-1/2}\, ds \geqq \gamma^{-1}(t - t_0),$$

where $\gamma = \sqrt{n}\, k e^{-t_0}/2(n-2)$. Letting $t \to +\infty$, we see that u cannot exist on all of R^n. In fact, there exists $T < \infty$ such that

$$\lim_{t \to T^-} N(t) = +\infty$$

and since $N \leqq F$ on $[t_0, \infty)$,

$$\lim_{t \to T^-} F_1(t) = +\infty, \tag{1.16.1}$$

or letting $\alpha_1 = \exp\{-(n-2)T\}$,

$$\lim_{\alpha \to \alpha_1^+} F(\alpha) = +\infty, \tag{1.16.2}$$

and the proof of the theorem is complete.

Remark. Our proof shows that u becomes unbounded in some ball. Indeed from Hölder's inequality

$$F(\alpha) \leqq \left[\int_{S_\alpha} (a_{ij} h_{,i} n_j)^q\, ds\right]^{1/q} \left[\int_{S_\alpha} |u|^p\, dS\right]^{1/p}, \tag{1.17}$$

where $p, q \geqq 1$ and $(1/p) + (1/q) = 1$. Thus from (1.9.2), (1.16.2) and (1.17) we see, since the measures of the S_α are finite,

$$\int_{S_\alpha} |u|^p\, dS \to +\infty$$

as α tends to α_1 from above. Consequently, u becomes unbounded in some ball. This is true in case $n = 2$ or $n > 2$.

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ZERO-FREE PARABOLIC REGIONS FOR SEQUENCES OF POLYNOMIALS*

Dedicated to Nicholas C. Metropolis on the Occasion of his 60th Birthday, June 11, 1975

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Abstract. In this paper, we show that certain sequences of polynomials $\{p_k(z)\}_{k=0}^n$, generated from three-term recurrence relations, have no zeros in parabolic regions in the complex plane of the form $y^2 \leqq 4\alpha(x+\alpha)$, $x > -\alpha$. As a special case of this, no partial sum $s_n(z) = \sum_{k=0}^n z^k/k!$ of e^z has a zero in $y^2 \leqq 4(x+1)$, $x > -1$, for any $n \geqq 1$. Such zero-free parabolic regions are obtained for Padé approximants of certain meromorphic functions, as well as for the partial sums of certain hypergeometric functions.

1. Introduction. In his thesis [11] and in [12], the second author obtained results concerning the existence of unbounded zero-free regions in the complex plane $\mathbb{C}$ for the partial sums of special entire functions $f(z) = \sum_{k=0}^\infty a_k z^k$, with $a_k \geqq 0$ for all k. In particular, it was shown in [12] that the partial sums $s_n(z)$ of the exponential function $f(z) = e^z$, i.e.,

$$s_n(z) := \sum_{k=0}^{n} z^k/k! \tag{1.1}$$

have no zeros in the infinite half-strip $|\mathrm{Im}\, z| \leqq \sqrt{6}$, $\mathrm{Re}\, z \geqq 0$, for any $n = 1, 2, \cdots$. More recently, Newman and Rivlin [8] stated that the parabolic-like domain

$$|y| \leqq -\frac{\pi}{2} + \tau\left(x + \frac{\pi^2}{4\tau^2}\right)^{1/2}, \qquad x \geqq 0, \qquad \tau \doteqdot 1.637\,017, \tag{1.2}$$

is free of zeros of the $s_n(z)$ in (1.1) for all n sufficiently large. However, in [9] this result of (1.2) was retracted, and, using different methods, Newman and Rivlin proved that the smaller region

$$y^2 \leqq dx, \qquad x \geqq 0, \quad d \doteqdot 0.745\,407, \tag{1.3}$$

is zero-free for every $s_n(z)$.

The purpose of the present paper is to establish the existence of zero-free parabolic regions for certain *general* sequences of polynomials. As a special case of our main result, we deduce that the parabolic region

$$y^2 \leqq 4(x+1), \qquad x > -1, \tag{1.4}$$

is zero-free for all the partial sums of the exponential function. As the unbounded set of (1.4) contains the region of (1.3) (and in fact the region of (1.2) as well), we thereby improve upon Newman and Rivlin's results. Furthermore, we obtain zero-free parabolic regions for Padé approximants of certain meromorphic

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functions, as well as for the partial sums of certain ${}_1F_1$ hypergeometric functions. Also, we improve upon the result of Dočev [3], concerning the location of the zeros of generalized Bessel polynomials.

The essential technique of proof utilizes continued fraction expansions, in the spirit of Wall [13].

2. A parabola theorem. Our main result is the following theorem.

THEOREM 2.1. *Let $\{p_k(z)\}_{k=0}^n$ be a sequence of polynomials of respective degrees k which satisfy the three-term recurrence relation*

$$p_k(z)=\left(\frac{z}{b_k}+1\right)p_{k-1}(z)-\frac{z}{c_k}p_{k-2}(z), \qquad k=1,2,\cdots,n, \tag{2.1}$$

where the b_k's and c_k's are positive real numbers for all $1\leqq k\leqq n$, and where $p_{-1}(z):=0$, $p_0(z):=p_0\neq 0$. Set

$$\alpha:=\min\{b_k(1-b_{k-1}c_k^{-1}): k=1,2,\cdots,n\}, \qquad b_0:=0. \tag{2.2}$$

Then, if $\alpha>0$, the parabolic region

$$\mathcal{P}_\alpha:=\{z=x+iy\in\mathbb{C}: y^2\leqq 4\alpha(x+\alpha),\ x>-\alpha\} \tag{2.3}$$

contains no zeros of $p_1(z), p_2(z),\cdots,p_n(z)$.

Proof. Let $z\in\mathcal{P}_\alpha$ be any fixed point which is not a zero of any $p_k(z)$, $1\leqq k\leqq n$, and define

$$\mu_k=\mu_k(z):=\frac{zp_{k-1}(z)}{b_kp_k(z)} \quad \text{for } k=1,2,\cdots,n. \tag{2.4}$$

We shall show inductively that

$$\operatorname{Re}\mu_k\leqq 1 \quad \text{for } k=1,2,\cdots,n. \tag{2.5}$$

This is certainly true for $k=1$; indeed, from (2.4), (2.1) and the fact that $p_0(z):=p_0\neq 0$, we have that

$$\mu_1=\frac{zp_0(z)}{b_1p_1(z)}=\frac{zp_0(z)}{b_1(z/b_1+1)p_0(z)}=\frac{z}{z+b_1},$$

from which it follows that $\operatorname{Re}\mu_1\leqq 1$ if and only if $\operatorname{Re} z\geqq -b_1$. But as $z\in\mathcal{P}_\alpha$ and $b_1\geqq\alpha$ from (2.2), this last condition holds; i.e., $\operatorname{Re} z>-\alpha\geqq -b_1$.

Now, assume inductively that $\operatorname{Re}\mu_{k-1}\leqq 1$ for some k satisfying $2\leqq k\leqq n$. We can express μ_k from (2.4) and (2.1) as

$$\mu_k=\frac{zp_{k-1}(z)}{b_kp_k(z)}=\frac{zp_{k-1}(z)}{(z+b_k)p_{k-1}(z)-b_kc_k^{-1}zp_{k-2}(z)}$$
$$=\frac{z}{z+b_k-b_kc_k^{-1}b_{k-1}\mu_{k-1}}.$$

In other words,

$$\mu_k=T_k(\mu_{k-1}), \tag{2.6}$$

where $T_k(w)$ is the bilinear transformation defined by

$$\xi = T_k(w) := \frac{z}{z + b_k - b_k c_k^{-1} b_{k-1} w}. \tag{2.7}$$

Hence, since Re $\mu_{k-1} \leqq 1$ by hypothesis, then μ_k lies in the image of the half-plane Re $w \leqq 1$ under the transformation T_k. Now, T_k has its pole at

$$w_k := \frac{z + b_k}{b_k c_k^{-1} b_{k-1}},$$

and since Re $z > -\alpha \geqq -(b_k - b_k c_k^{-1} b_{k-1})$ from (2.2), it follows that Re $w_k > 1$. Therefore, T_k maps Re $w \leqq 1$ onto a closed disk D_k in the ξ-plane. The center ξ_k of this disk is the image, under T_k, of the point in the w-plane symmetric to the pole w_k with respect to the line Re $w = 1$, i.e.,

$$\xi_k = T_k(2 - \bar{w}_k) = T_k\left(2 - \frac{\bar{z} + b_k}{b_k c_k^{-1} b_{k-1}}\right) = \frac{z}{2 \operatorname{Re} z + 2b_k(1 - b_{k-1} c_k^{-1})}.$$

Furthermore, since $T_k(\infty) = 0$ lies on the boundary of D_k, the radius r_k of this disk is given by

$$r_k = |\xi_k| = \frac{|z|}{2 \operatorname{Re} z + 2b_k(1 - b_{k-1} c_k^{-1})}.$$

Consequently, the real part of any point in D_k does not exceed the sum

$$\operatorname{Re} \xi_k + r_k = \frac{\operatorname{Re} z + |z|}{2 \operatorname{Re} z + 2b_k(1 - b_{k-1} c_k^{-1})}.$$

Again from (2.2), an upper bound for this last quantity is

$$\frac{\operatorname{Re} z + |z|}{2 \operatorname{Re} z + 2\alpha},$$

which one can directly verify is at most unity because $z \in \mathcal{P}_\alpha$. In particular, since $\mu_k \in D_k$, we have Re $\mu_k \leqq 1$. This completes the induction for establishing (2.5).

Next, we observe that $p_k(0) \neq 0$ for all $k = 0, 1, \cdots, n$; indeed, from (2.1) we have

$$0 \neq p_0(0) = p_1(0) = \cdots = p_k(0).$$

Furthermore, if $p_k(z_0) = p_{k-1}(z_0) = 0$ for some $k \geqq 1$, then evidently $z_0 \neq 0$, so that from (2.1), we deduce that $p_{k-j}(z_0) = 0$ for all $0 \leqq j \leqq k$. In particular, this would imply that $p_0(z_0) = 0$, which is a contradiction. Hence, $p_k(z)$ and $p_{k-1}(z)$ have no zeros in common for each k, $1 \leqq k \leqq n$.

Finally, suppose on the contrary that $p_k(z_0) = 0$ for some $z_0 \in \mathcal{P}_\alpha$, and some k with $1 \leqq k \leqq n$.Clearly, since $p_1(z) = (p_0/b_1)(z + b_1)$ from (2.1), then p_1 has its sole zero at $-b_1$. But as $-b_1 \leqq -\alpha$ from (2.2), this zero by definition (cf. (2.3)) is not in $\mathcal{P}_\alpha$. Thus, $2 \leqq k \leqq n$. Next, $p_k(z_0) = 0$ implies from (2.1) that $(z_0/b_k + 1)p_{k-1}(z_0) = (z_0/c_k)p_{k-2}(z_0)$, and as $p_k(z)$ and $p_{k-1}(z)$ have no common zeros, then on dividing,

$$\frac{c_k}{b_{k-1} b_k}(z_0 + b_k) = \frac{z_0 p_{k-2}(z_0)}{b_{k-1} p_{k-1}(z_0)} = \mu_{k-1}(z_0). \tag{2.8}$$

Now, $z_0 \in \mathscr{P}_\alpha$ implies from (2.5) and continuity considerations that Re $\mu_{k-1}(z_0) \leqq 1$. Thus, taking real parts in (2.8), we obtain

$$\text{Re } z_0 \leqq -b_k(1-b_{k-1}c_k^{-1}) \leqq -\alpha,$$

the last inequality following from (2.2). On the other hand, $z_0 \in \mathscr{P}_\alpha$ implies from (2.3) that Re $z_0 > -\alpha$, which contradicts the above inequality. Thus, $p_k(z)$ has no zeros in $\mathscr{P}_\alpha$ for each k, $1 \leqq k \leqq n$. □

Note, using (2.6), that

$$\mu_k = T_k(\mu_{k-1}) = T_k(T_{k-1}(\mu_{k-2})) = \cdots = T_k T_{k-1} \cdots T_2(\mu_1), \quad 2 \leqq k \leqq n.$$

Hence, the above technique of proof of Theorem 2.1 essentially depends on the finiteness of a continued fraction expansion of μ_k, namely, from (2.7),

$$\mu_k(z) = \cfrac{z}{z + b_k - \cfrac{b_k c_k^{-1} b_{k-1} z}{z + b_{k-1} - \cfrac{b_{k-1} c_{k-1}^{-1} b_{k-2} z}{z + b_{k-2} - \ddots}}}.$$

There is in fact a well-known "parabola theorem" due to Wall [13, p. 57] for continued fractions, but it does not appear to the authors that the finiteness of the above continued fraction expansion for $\mu_k(z)$ with $z \in \mathscr{P}_\alpha$ follows from Wall's parabola theorem.

We remark that, in a certain sense, the result of Theorem 2.1 is sharp. For, consider any three-term recurrence relation (2.1) for which

$$\alpha = b_1.$$

Then, as $p_1(z) = (p_0/b_1)(z + b_1)$, it has its sole zero at $-b_1 = -\alpha$. Therefore, since the parabola $y^2 = 4\alpha(x + \alpha)$ has its vertex at $x = -\alpha$, the parabolic region of (2.3) cannot be enlarged to include the boundary point $z = -\alpha$ of $\mathscr{P}_\alpha$ and still exclude the zeros of $p_1(z), \cdots, p_n(z)$.

We remark further that Theorem 2.1 has an obvious extension to an *infinite* sequence of polynomials $\{p_k(z)\}_{k=0}^{\infty}$ which satisfy (2.1). In such a case, we define

$$\alpha := \inf \{b_k(1 - b_{k-1}c_k^{-1}) \colon k = 1, 2, \cdots\}.$$

Then, the conclusion that the region $\mathscr{P}_\alpha$ of (2.3) is zero-free for *every* $p_k(z)$, $k = 1, 2, \cdots$, remains valid provided that $\alpha > 0$. If, in addition, such an infinite sequence $p_k(z)$ converges uniformly on all compact subsets of $\mathscr{P}_\alpha$ to an analytic function $f(z)$ which is not identically zero, then by the theorem of Hurwitz, $f(z)$ must also be zero-free in the interior of $\mathscr{P}_\alpha$.

Some concrete applications of the parabola theorem will be given in the next sections. For the remainder of the present section, we consider sufficient conditions under which the hypotheses of Theorem 2.1 are fulfilled. We deal first with the partial sums of a power series expansion.

COROLLARY 2.2. *Let* $s_k(z) := \sum_{j=0}^{k} a_j z^j$, $k = 0, 1, \cdots, n$, *and assume that* $a_j > 0$ *for all* $j = 0, 1, \cdots, n$, *and that*

$$\alpha := \min \left\{ \left(\frac{a_{k-1}}{a_k} - \frac{a_{k-2}}{a_{k-1}} \right) \colon k = 1, 2, \cdots, n \right\} > 0, \tag{2.9}$$

where $a_{-1}/a_0 := 0$. *Then, the polynomials* $s_k(z)$, $k = 1, 2, \cdots, n$, *have no zeros in the parabolic region* $\mathcal{P}_\alpha$ *defined in* (2.3).

Proof. One easily verifies that the partial sums $s_k(z)$ satisfy the three-term recurrence relation

$$s_k(z) = \left(\frac{z}{b_k} + 1\right) s_{k-1}(z) - \frac{z}{b_k} s_{k-2}(z), \qquad k = 1, 2, \cdots, n, \tag{2.10}$$

where $s_{-1} := 0$, and where

$$b_k := a_{k-1}/a_k, \qquad k = 0, 1, \cdots, n. \tag{2.11}$$

Consequently, (2.1) holds with $c_k = b_k$, and (2.2) becomes

$$\alpha = \min\{(b_k - b_{k-1}) : k = 1, \cdots, n\},$$

which from (2.11) is the same as (2.9). Applying Theorem 2.1 then establishes the corollary. □

The partial sums of a formal power series

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \qquad a_0 \neq 0, \tag{2.12}$$

can be regarded as special cases of the so-called Padé approximants to $f(z)$ (see Perron [10], or Baker [2]). More precisely, given any pair of nonnegative integers (n, ν), the Padé approximant of type (n, ν) is that rational function $R_{n,\nu}(z)$ of the form

$$R_{n,\nu}(z) = P_{n,\nu}(z)/Q_{n,\nu}(z) \tag{2.13}$$

for which the following conditions are satisfied:

(i) $P_{n,\nu}(z)$ is a polynomial of degree $\leqq n$;

(ii) $Q_{n,\nu}(z)$ is a polynomial of degree $\leqq \nu$ with $Q_{n,\nu}(z) \not\equiv 0$;

(iii) The power series development of $f(z)Q_{n,\nu}(z) - P_{n,\nu}(z)$ about $z = 0$ begins with the $(n + \nu + 1)$st power of z.

In particular, for $\nu = 0$, these conditions are satisfied by

$$P_{n,0}(z) = \sum_{j=0}^{n} a_j z^j, \qquad Q_{n,0}(z) = 1, \qquad n = 0, 1, \cdots.$$

Corresponding to the power series (2.12), we define the Hankel determinants

$$A_n^{(0)} := 1,\ n \geqq 0,\ A_n^{(\nu)} := \det \begin{bmatrix} a_n & a_{n-1} & \cdots & a_{n-\nu+1} \\ a_{n+1} & a_n & \cdots & a_{n-\nu+2} \\ \vdots & \vdots & & \vdots \\ a_{n+\nu-1} & a_{n+\nu-2} & \cdots & a_n \end{bmatrix}, \qquad n \geqq 0, \quad \nu \geqq 1, \tag{2.14}$$

with the convention that

$$a_{-j} := 0 \quad \text{for } j = 1, 2, \cdots.$$

These determinants play an important role in the study of Padé approximants. Indeed, if

$$A_n^{(\nu)} \neq 0, \tag{2.15}$$

then the conditions (i), (ii) and (iii) above are satisfied by the polynomials

$$(2.16)\qquad P_{n,\nu}(z)=\frac{1}{A_n^{(\nu)}}\sum_{j=0}^{n}\det\begin{bmatrix} a_j & a_{j-1} & \cdots & a_{j-\nu} \\ a_{n+1} & a_n & \cdots & a_{n-\nu+1} \\ a_{n+2} & a_{n+1} & \cdots & a_{n-\nu+2} \\ \vdots & \vdots & & \vdots \\ a_{n+\nu} & a_{n+\nu-1} & \cdots & a_n \end{bmatrix}\cdot z^j,$$

$$(2.17)\qquad Q_{n,\nu}(z)=\frac{1}{A_n^{(\nu)}}\det\begin{bmatrix} 1 & z & z^2 & \cdots & z^\nu \\ a_{n+1} & a_n & a_{n-1} & \cdots & a_{n-\nu+1} \\ a_{n+2} & a_{n+1} & a_n & \cdots & a_{n-\nu+2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n+\nu} & a_{n+\nu-1} & a_{n+\nu-2} & \cdots & a_n \end{bmatrix}.$$

In such a case, we refer to the polynomial $P_{n,\nu}(z)$ in (2.16) as the *Padé numerator* of type (n, ν), and to $Q_{n,\nu}(z)$ in (2.17) as the *Padé denominator* of type (n, ν).

We now prove a generalization of Corollary 2.2 for the Padé numerators.

COROLLARY 2.3. *Let $f(z)=\sum_{j=0}^{\infty} a_j z^j$ be a formal power series, and assume that, for a fixed $\nu \geqq 0$, the corresponding Hankel determinants defined in* (2.14) *satisfy*

$$(2.18)\qquad \begin{aligned} &A_k^{(\nu)}>0,\ A_k^{(\nu+1)}>0, \quad \text{for } k=0,1,\cdots,n, \\ &A_k^{(\nu+2)}>0 \quad \text{for } k=0,1,\cdots,n-1. \end{aligned}$$

Then, defining the positive constant α by

$$(2.19)\qquad \alpha := \min\left\{\frac{A_k^{(\nu)}A_{k-1}^{(\nu+2)}}{A_{k-1}^{(\nu+1)}A_k^{(\nu+1)}} : k=1,2,\cdots,n\right\},$$

we find that the Padé numerators $P_{1,\nu}(z), P_{2,\nu}(z), \cdots, P_{n,\nu}(z)$ for $f(z)$ have no zeros in the parabolic region $\mathcal{P}_\alpha$ defined in (2.3).

Proof. A classical identity of Frobenius [5] asserts that

$$(2.20)\qquad P_{k,\nu}(z)=\left(\frac{z}{b_{k,\nu}}+1\right)P_{k-1,\nu}(z)-\frac{z}{c_{k,\nu}}P_{k-2,\nu}(z),$$

where

$$(2.21)\qquad \begin{aligned} b_{k,\nu} &:= \frac{A_k^{(\nu)}A_{k-1}^{(\nu+1)}}{A_{k-1}^{(\nu)}A_k^{(\nu+1)}}, \qquad k \geqq 1, \\ c_{k,\nu} &:= \frac{A_{k-1}^{(\nu)}A_{k-1}^{(\nu+1)}}{A_{k-2}^{(\nu)}A_k^{(\nu+1)}}, \qquad k \geqq 2. \end{aligned}$$

(For notational convenience we set $c_{1,\nu} := 1$.) By assumption (2.18), the $b_{k,\nu}$'s and $c_{k,\nu}$'s are positive real numbers for k $= 1, 2, \cdots, n$. Consequently, the recurrence relation (2.1) holds with $b_k = b_{k,\nu}$, $c_k = c_{k,\nu}$, and (2.2) of Theorem 2.1 becomes

$$\alpha = \min\left\{\frac{A_k^{(\nu)}A_{k-1}^{(\nu+1)}}{A_{k-1}^{(\nu)}A_k^{(\nu+1)}}\left(1-\frac{A_{k-2}^{(\nu+1)}A_k^{(\nu+1)}}{[A_{k-1}^{(\nu+1)}]^2}\right): k=1,\cdots,n\right\}. \tag{2.22}$$

However, using the known identity

$$[A_{k-1}^{(\nu+1)}]^2 - A_{k-2}^{(\nu+1)}A_k^{(\nu+1)} = A_{k-1}^{(\nu+2)}A_{k-1}^{(\nu)} \tag{2.23}$$

in (2.22), we obtain

$$\alpha = \min\left\{\frac{A_k^{(\nu)}A_{k-1}^{(\nu+1)}A_{k-1}^{(\nu+2)}A_{k-1}^{(\nu)}}{A_{k-1}^{(\nu)}A_k^{(\nu+1)}[A_{k-1}^{(\nu+1)}]^2}: k=1,2,\cdots,n\right\},$$

which is the same as the defining formula (2.19). □

In a similar manner, we deduce the following result for the Padé denominators.

COROLLARY 2.4. *Suppose that, for fixed $n \geqq 0$, the Hankel determinants corresponding to the formal power series $f(z)=\sum_{j=0}^{\infty} a_j z^j$ satisfy*

$$\begin{aligned} &A_n^{(k)}>0, \quad A_{n+1}^{(k)}>0, \quad \textit{for } k=1,2,\cdots,\nu, \\ &A_{n+2}^{(k)}>0 \quad \textit{for } k=1,2,\cdots,\nu-1. \end{aligned} \tag{2.24}$$

Then, defining the positive constant α by

$$\alpha := \min\left\{\frac{A_n^{(k)}A_{n+2}^{(k-1)}}{A_{n+1}^{(k-1)}A_{n+1}^{(k)}}: k=1,2,\cdots,\nu\right\}, \tag{2.25}$$

the Padé denominators $Q_{n,1}(z), Q_{n,2}(z),\cdots,Q_{n,\nu}(z)$ for $f(z)$ have no zeros in the parabolic region

$$\hat{\mathcal{P}}_\alpha := \{z=x+iy\in\mathbb{C}: y^2 \leqq 4\alpha(\alpha-x), \alpha>x\}. \tag{2.26}$$

The proof of Corollary 2.4 follows in an analogous fashion from the Frobenius identity

$$Q_{n,k}(-z) = \left(1+\frac{z}{\hat{b}_{n,k}}\right)Q_{n,k-1}(-z) - \frac{z}{\hat{c}_{n,k}}Q_{n,k-2}(-z), \tag{2.27}$$

where

$$\hat{b}_{n,k} := \frac{A_n^{(k)}A_{n+1}^{(k-1)}}{A_{n+1}^{(k)}A_n^{(k-1)}}, \qquad \hat{c}_{n,k} := \frac{A_n^{(k-1)}A_{n+1}^{(k-1)}}{A_{n+1}^{(k)}A_n^{(k-2)}}. \tag{2.28}$$

In concluding this section we remark that the hypotheses (2.18) and (2.24) of the preceding corollaries will be satisfied for all n and ν if $f(z)$ is a meromorphic function of the form

$$f(z) = a_0 e^{\gamma z}\cdot\frac{\prod_{j=1}^{\infty}(1+\lambda_j z)}{\prod_{j=1}^{\infty}(1-\beta_j z)}, \tag{2.29}$$

where $a_0>0$, $\gamma\geqq 0$, $\lambda_j \geqq 0$, $\beta_j\geqq 0$ and $\sum_j(\lambda_j+\beta_j)<\infty$. The convergence properties of the Padé approximants of such functions were studied by Arms and Edrei in [1].

3. Partial sums and Padé approximants of e^z. As a concrete application of the results in § 2, we now obtain zero-free regions for the Padé numerators $P_{n,\nu}(z)$ and denominators $Q_{n,\nu}(z)$ for e^z. Explicitly, these polynomials are given by (cf. Perron [10, p. 433])

$$P_{n,\nu}(z)=\sum_{j=0}^{n}\frac{(n+\nu-j)!n!z^j}{(n+\nu)!j!(n-j)!}, \tag{3.1}$$

$$Q_{n,\nu}(z)=\sum_{j=0}^{\nu}\frac{(n+\nu-j)!\nu!}{(n+\nu)!j!(\nu-j)!}(-z)^j. \tag{3.2}$$

COROLLARY 3.1. *For any $\nu\geqq 0$, each element of the sequence of Padé numerators $\{P_{n,\nu}(z)\}_{n=1}^{\infty}$ for e^z has no zeros in the region*

$$\mathcal{P}_{\nu+1}=\{z=x+iy\in\mathbb{C}\colon y^2\leqq 4(\nu+1)(x+\nu+1),\ x>-(\nu+1)\}. \tag{3.3}$$

Furthermore, for any $n\geqq 0$, each element of the sequence of Padé denominators $\{Q_{n,\nu}(z)\}_{\nu=1}^{\infty}$ has no zeros in the region

$$\hat{\mathcal{P}}_{n+1}=\{z=x+iy\in\mathbb{C}\colon y^2\leqq 4(n+1)(n+1-x),\ x<(n+1)\}. \tag{3.4}$$

Proof. The Hankel determinants $A_m^{(s)}$ for $s\geqq 1$ for e^z are given (cf. [1]) by

$$A_m^{(s)}=\prod_{j=1}^{s}\frac{1}{j(j+1)\cdots(j+m-1)}. \tag{3.5}$$

Thus, for any $n\geqq 0$, the constant α defined in (2.19) is easily verified to be

$$\alpha=\min\{(\nu+1)\colon k=1,2,\cdots,n\}=\nu+1, \tag{3.6}$$

and so, by Corollary 2.3, the region $\mathcal{P}_{\nu+1}$ is zero-free for every $P_{n,\nu}(z)$, $n=1,2,\cdots$.

Similarly, for any $\nu\geqq 0$, the constant α defined in (2.25) equals $n+1$, so that by Corollary 2.4, the region $\hat{\mathcal{P}}_{n+1}$ is zero-free for every $Q_{n,\nu}(z)$, $\nu=1,2,\cdots$. □

In particular, for $\nu=0$, we obtain Corollary 3.2.

COROLLARY 3.2. *No partial sum $P_{n,0}(z)=\sum_{j=0}^{n}z^j/j!$ of e^z, for any $n\geqq 1$, has a zero in the parabolic region*

$$\mathcal{P}_1=\{z=x+iy\in\mathbb{C}\colon y^2\leqq 4(x+1),\ x>-1\}.$$

This result is sharp at $x=-1$, and, as discussed in the introduction, it improves upon an analogous result due to Newman and Rivlin [9].

In Figs. 1 and 2 we plot, respectively, the zeros (shown as asterisks) in the upper half-plane of the Padé polynomials $\{P_{n,0}(z)\}_{n=1}^{40}$ and of $\{P_{n,6}(z)\}_{n=1}^{40}$ for e^z, together with the corresponding bounding parabolas for $\mathcal{P}_1$ and $\mathcal{P}_7$. The computations and the ones mentioned below were carried out by A. Price and P. Comadoll on an IBM 360/65 using a modified version of SUBROUTINE POLRT from the IBM Scientific Subroutine Package. The plots were done on a Calcomp Model 563 plotter.

We remark that the largest parabolic region of the form $y^2<\lambda(x+1)$ which omits the zeros of the Padé polynomials $\{P_{n,0}(z)\}_{n=1}^{40}$ for e^z is approximately given by

$$y^2<7.1940(x+1),\qquad x>-1.$$

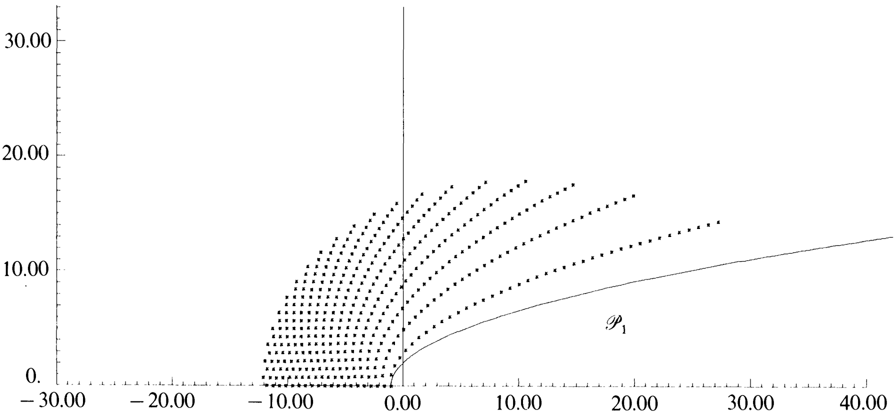

FIG. 1. *Zeros of the polynomials* $P_{n,0}(z)$, $n = 1, 2, \cdots, 40$

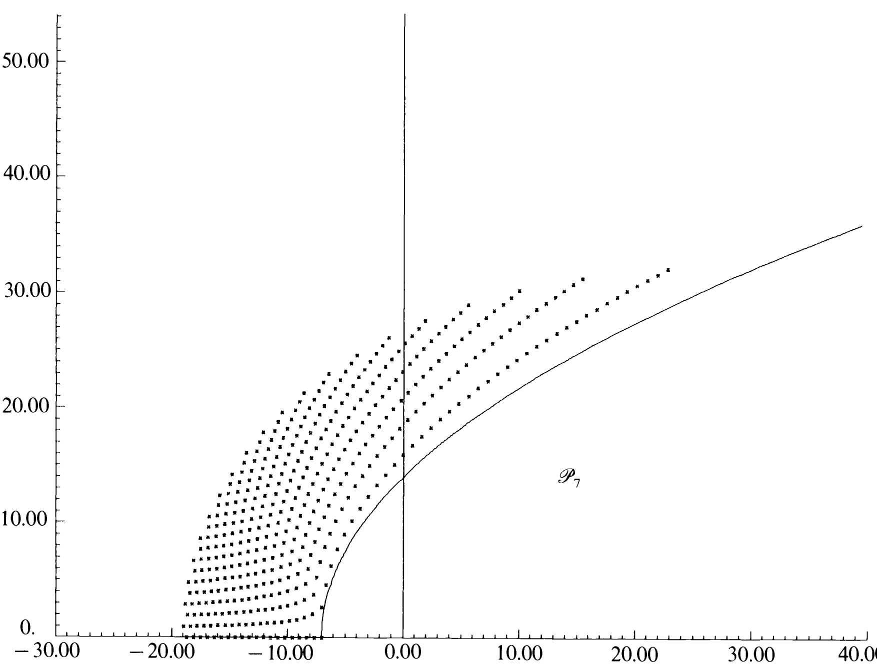

FIG. 2. *Zeros of the polynomials* $P_{n,6}(z)$, $n = 1, 2, \cdots, 40$

On the other hand, Newman and Rivlin [8] have (correctly) established that

$$\left\{ \frac{P_{n,0}(n + \sqrt{2n} \cdot w)}{\exp(n + \sqrt{2n} \cdot w)} \right\}_{n=1}^{\infty}$$

converges uniformly to

$$\frac{1}{\sqrt{\pi}} \int_w^{\infty} e^{-t^2} \, dt := \tfrac{1}{2} \operatorname{erfc}(w),$$

on any compact set in $\operatorname{Im} w \geqq 0$. If t_1 denotes the zero of $\operatorname{erfc}(w)$, having real part negative and smallest (positive) imaginary part, then t_1 is given approximately (cf. Fettis et al. [4]) by

$$t_1 \doteqdot -1.354\,810 + i(1.991\,467).$$

Becase of the uniform convergence above, it then follows from Hurwitz's theorem that, for all n sufficiently large, $P_{n,0}$ has a zero of the form

$$n + \sqrt{2n}\, w_n := x_n + iy_n \quad \text{with } \lim_{n\to\infty} w_n = t_1.$$

From this, we easily deduce that for each fixed β,

$$\lim_{n\to\infty} \frac{y_n^2}{(x_n+\beta)} = 2(\operatorname{Im} t_1)^2 \doteqdot 7.931\,880.$$

In other words, any parabola of the form

$$y^2 < K(x+\beta), \qquad x > -\beta,$$

which is devoid of zeros of $P_{n,0}(z)$ of e^z, for all n sufficiently large, must evidently satisfy

$$K \leqq 2(\operatorname{Im} t_1)^2 \doteqdot 7.931\,880.$$

4. ${}_1F_1$ hypergeometric functions. Using the notation

$$(a)_j := a(a+1)\cdots(a+j-1), \qquad j \geqq 1, \quad (a)_0 := 1, \tag{4.1}$$

for any complex number a, the hypergeometric function ${}_1F_1(c; d; z)$ is defined by

$$ {}_1F_1(c; d; z) := \sum_{j=0}^{\infty} \frac{(c)_j}{(d)_j} \cdot \frac{z^j}{j!}, \tag{4.2}$$

and is an entire function of z, for any choice of c and d with $d \neq 0, -1, -2, \cdots$. For example,

$$e^z = {}_1F_1(c; c; z), \qquad c \neq 0, -1, -2, \cdots, \tag{4.3}$$

and

$$e^z - \sum_{k=0}^{n-1} z^k/k! = \frac{z^n}{n!}\, {}_1F_1(1; n+1; z), \qquad n = 1, 2, \cdots. \tag{4.4}$$

Concerning zero-free regions for certain ${}_1F_1$'s and their partial sums, we prove the next corollary.

COROLLARY 4.1. *With the notation* (2.3) *for the parabolic region* $\mathcal{P}_\alpha$, *all the partial sums*

$$s_n(z) = \sum_{j=0}^{n} \frac{(c)_j}{(d)_j} \frac{z^j}{j!}, \qquad n \geqq 1, \tag{4.5}$$

of ${}_1F_1(c; d; z)$ *have no zeros in the region*

(i) $\mathcal{P}_{d/c}$, *if* $0 < d \leqq c$,

(ii) $\mathcal{P}_1$, *if* $1 \leqq c \leqq d$,

(iii) $\mathcal{P}_\alpha$, $\alpha = (2c - d + cd)/(c^2 + c)$, *if* $0 < c < 1$ *and* $c \leqq d < 2c/(1-c)$.

Consequently, the entire function ${}_1F_1(c; d; z)$ *has no zeros in the corresponding interior region.*

Proof. Putting

$$a_j := \frac{(c)_j}{(d)_j j!}, \qquad a_{-1} := 0, \tag{4.6}$$

we apply Corollary 2.2 with the constant α being defined by

$$\alpha = \inf\left\{\left(\frac{a_{k-1}}{a_k} - \frac{a_{k-2}}{a_{k-1}}\right): k = 1, 2, \cdots\right\}. \tag{4.7}$$

On substituting (4.6) in (4.7), we obtain

$$\alpha = \inf\left\{\frac{d}{c}, g(k): k = 2, 3, \cdots\right\}, \tag{4.8}$$

where

$$g(t) := \frac{t^2 + (2c-3)t + (c-1)(d-2)}{t^2 + (2c-3)t + (c-1)(c-2)} \quad \text{for all } t \geqq 2. \tag{4.9}$$

Next, we observe that

$$g(2) = \frac{2c - d + cd}{c^2 + c}, \qquad \lim_{t \to +\infty} g(t) = 1,$$

and

$$g'(t) = \frac{(2t + 2c - 3)(c-1)(c-d)}{[t^2 + (2c-3)t + (c-1)(c-2)]^2}.$$

From these facts, it follows that the constant α of (4.8) is positive, and is given by d/c, 1, and $(2c - d + cd)/(c^2 + c)$, in the respective cases (i), (ii) and (iii). Applying Corollary 2.2 then proves that all the partial sums have no zeros in the corresponding region $\mathcal{P}_\alpha$, and consequently, the limit function ${}_1F_1$ has no zeros in the interior of $\mathcal{P}_\alpha$ (see the remarks following the proof of Theorem 2.1). □

We remark that when c is not a nonpositive integer and $c - d$ is not a nonnegative integer, then it is known [6] that ${}_1F_1(c; d; z)$ has infinitely many zeros in the complex plane.

In Fig. 3, we plot the zeros in the upper half-plane of the partial sums $\{s_n(z)\}_{n=1}^{40}$ in (4.5) of the hypergeometric function ${}_1F_1(1; 4; z)$, i.e., when $c = 1$, $d = 4$. The corresponding zero-free parabolic region $\mathcal{P}_1$ from (ii) of Corollary 4.1 is also sketched. Two accumulation points of zeros are evident in the figure, and these are necessarily zeros of ${}_1F_1(1; 4; z)$.

COROLLARY 4.2. *For all* $n \geqq 1$, *the remainder*

$$e^z - \sum_{k=0}^{n-1} z^k/k! \tag{4.10}$$

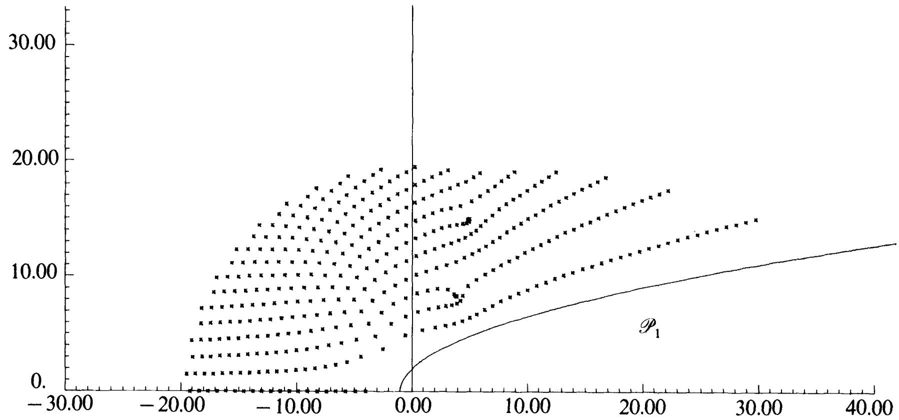

FIG. 3. *Zeros of the partial sums* $s_n(z)$, $n = 1, 2, \cdots, 40$, *for* ${}_1F_1(1; 4; z)$

has no zeros in the region

(4.11) $$\mathscr{P}_1^0 \cup \hat{\mathscr{P}}_1^0 := \{z = x + iy \in \mathbb{C}: y^2 < 4(x+1)\} \cup \{z = x + iy \in \mathbb{C}: y^2 < 4(1-x)\},$$

except at $z = 0$.

Proof. Applying Corollary 4.1 in the case when $c = 1$, $d = n+1$, we deduce from conclusion (ii) that the function ${}_1F_1(1; n+1; z)$ is zero-free in $\mathscr{P}_1^0$, the interior of $\mathscr{P}_1$. Furthermore, the identity (cf. [6])

$$ {}_1F_1(1; n+1; -z) = e^{-z}{}_1F_1(n; n+1, z),$$

together with Corollary 4.1, imply that ${}_1F_1(1; n+1; z)$ is zero-free in $\hat{\mathscr{P}}_1^0$, the interior of $\hat{\mathscr{P}}_1$, defined in (2.26). Hence, by virtue of the representation (4.4), the remainder (4.10) is zero-free in $(\mathscr{P}_1^0 \cup \hat{\mathscr{P}}_1^0)\backslash\{0\}$. □

5. Generalized Bessel polynomials. In this section, we consider the generalized Bessel polynomials

(5.1) $$Y_n^{(\delta)}(z) := \sum_{j=0}^{n} \binom{n}{j} (n+\delta+1)_j \left(\frac{-z}{2}\right)^j,$$

where $(n+\delta+1)_j$ is defined as in (4.1). These polynomials were first introduced by Krall and Frink [7], and in their notation,

$$Y_n^{(\delta)}(z) = y_n(-z, \delta+2, 2).$$

Several authors have investigated the location of the zeros of the polynomials (5.1); among them, Dočev [3] appears to have obtained the strongest result. We state his theorem for real δ as follows.

THEOREM 5.1. *If* $n+\delta+1 > 0$, $\delta \neq -2, -3, -4, \cdots$, *then all the zeros of* $Y_n^{(\delta)}(z)$ *lie in the closed disk*

(5.2) $$D_{n+\delta+1} := \left\{ z \in \mathbb{C}: |z| \leq \frac{2}{n+\delta+1} \right\}.$$

Using Theorem 2.1, we now improve upon Dočev's result.

THEOREM 5.2. *If* $n+\delta+1>0$, *then all the zeros of* $Y_n^{(\delta)}(z)$ *lie in the cardioidal region*

$$(5.3)\quad C_{n+\delta+1}:=\left\{z=re^{i\theta}\in\mathbb{C}: 0<r<\frac{1+\cos\theta}{n+\delta+1}, -\pi<\theta<\pi\right\}\cup\left\{\frac{2}{n+\delta+1}\right\}.$$

Notice that $C_{n+\delta+1}\subset D_{n+\delta+1}$, and this containment is proper, except for $z=2/(n+\delta+1)$.

Proof of Theorem 5.2. It is convenient to introduce the polynomials

$$(5.4)\quad P_m^{(\tau)}(z):=\frac{\Gamma(m+\tau+1)}{\Gamma(2m+\tau+1)}z^m Y_m^{(\tau)}\left(\frac{-2}{z}\right)$$
$$=\frac{\Gamma(m+\tau+1)}{\Gamma(2m+\tau+1)}\sum_{j=0}^{m}\binom{m}{j}(m+\tau+1)_j z^{m-j},\qquad m+\tau+1>0.$$

As can be directly verified, for fixed n and δ, the polynomials $\{P_k^{(n+\delta-k)}(z)\}_{k=0}^n$ satisfy the recurrence relation

$$(5.5)\qquad P_k^{(n+\delta-k)}(z)=\left(\frac{z}{b_k}+1\right)P_{k-1}^{(n+\delta-k+1)}(z)-\frac{z}{c_k}P_{k-2}^{(n+\delta-k+2)}(z),\qquad k\geqq 1,$$

where $P_{-1}^{(n+\delta+1)}(z):=0$, and

$$(5.6)\quad b_k=n+\delta+k,\quad k\geqq 1;\quad c_k=\frac{(n+\delta+k)(n+\delta+k-1)}{(k-1)},\quad k\geqq 2;\quad c_1:=1.$$

Since, by hypothesis, $n+\delta+1>0$, the constants b_k and c_k in (5.6) are positive for all $k\geqq 1$. Furthermore, a simple computation shows that

$$b_k(1-b_{k-1}c_k^{-1})=n+\delta+1\quad\text{for all } k=1,2,\cdots,n,\quad b_0:=0.$$

Hence, the constant α defined in (2.2) is given by

$$\alpha=n+\delta+1,$$

and so from Theorem 2.1, we deduce that all the polynomials $\{P_k^{(n+\delta-k)}(z)\}_{k=1}^n$ are zero-free in the region

$$\mathcal{P}_{n+\delta+1}=\{z=x+iy\in\mathbb{C}: y^2\leqq 4(n+\delta+1)(x+n+\delta+1), x>-(n+\delta+1)\}$$
$$=\{z\in\mathbb{C}: |z|\leqq \operatorname{Re}(z)+2(n+\delta+1), \operatorname{Re} z>-(n+\delta+1)\}.$$

In particular, taking $k=n$, we have that $P_n^{(\delta)}(z)$ is zero-free in $\mathcal{P}_{n+\delta+1}$.

Finally, from (5.4) (with $m=n$, $\tau=\delta$) it follows that no zero of $Y_n^{(\delta)}(w)$ is of the form $w=-2/z$ with $z\in\mathcal{P}_{n+\delta+1}$. In other words, all the zeros of $Y_n^{(\delta)}(w)$ must lie in the region

$$\left\{w\in\mathbb{C}: \left|\frac{2}{w}\right|>\operatorname{Re}\left(\frac{-2}{w}\right)+2(n+\delta+1)\right\}\cup\left\{\frac{2}{n+\delta+1}\right\},$$

which is the same as the region $C_{n+\delta+1}$ in (5.3). □

We remark that the Padé polynomials in (3.1) are related to the polynomials in (5.4) by the formula

$$P_{n,\nu}(z)=P_n^{(\nu-n)}(z).$$

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PRESERVATION OF GEOMETRIC PROPERTIES UNDER AN INTEGRAL TRANSFORMATION*

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Abstract. We consider the transformation $g(\lambda)=\int\phi(\lambda, x)f(x)\,d\mu(x)$, where f and ϕ are nonnegative Borel-measurable functions of nonnegative arguments and μ denotes the Lebesgue measure on $[0, \infty)$ or the counting measure on $\{0, 1, \cdots\}$. We show that under appropriate assumptions on ϕ, various geometric properties of f (such as star-shapedness and superadditivity) are inherited by g.

1. Introduction. Consider the transformation

$$g(\lambda)=\int \phi(\lambda, x)f(x)\,d\mu(x), \tag{1.1}$$

where f and ϕ are nonnegative Borel-measurable functions of nonnegative arguments, μ denotes the Lebesgue measure on $[0, \infty)$ or the counting measure on $\{0, 1, \cdots\}$, and the integral is assumed to exist. We show that various geometric properties possessed by f are inherited by g under appropriate assumptions on ϕ.

Preservations results of a similar kind have been obtained by Schoenberg (1950) for polynomials of a fixed degree, by Karlin and Proschan (1960) and Karlin (1968, p. 130) for totally positive functions, by Karlin (1968, p. 285) for monotonic, convex and generalized convex functions and by Proschan and Sethuraman (1974) for Schur-convex functions.

2. Preliminaries. We shall consider four geometric properties.

DEFINITION 2.1. Let f be a nonnegative function of a nonnegative argument. We say that

(a) f is *star-shaped* if $f(\alpha x)\leqq \alpha f(x)$ for each $x\geqq 0$ and $0\leqq\alpha\leqq 1$,

(b) f is *superadditive* if $f(x+y)\geqq f(x)+f(y)$ for each $x\geqq 0$, $y\geqq 0$,

(c) f is *root-increasing* if $f^{1/x}(x)$ is nondecreasing in $x>0$,

(d) f is *supermultiplicative* if $f(x+y)\geqq f(x)f(y)$ for each $x\geqq 0$, $y\geqq 0$.

Property 2.2. Relationships among geometric properties. The following elementary relationships among the geometric properties are readily verified:

f star-shaped $\Leftrightarrow e^{f(x)}$ root increasing

$\Rightarrow f$ superadditive $\Leftrightarrow e^{f(x)}$ supermultiplicative.

The two equivalences follow directly from the definitions, while the implication is shown in Bruckner and Ostrow (1962). That the reverse implication does not hold is seen by choosing $f(x)=[x]$, a superadditive function which is not star-shaped.

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DEFINITION 2.1′. *Dual geometric properties* may be defined by reversing the direction of the inequality in 2.1(a), (b) and (d), and by replacing "nondecreasing" by "nonincreasing" in (c). The dual geometric properties are called, respectively: (a′) antistar-shaped, (b′) subadditive, (c′), root-decreasing and (d′) submultiplicative. The relationships among (a′), (b′), (c′) and (d′) are analogous to those among (a), (b), (c) and (d) in Property 2.2.

Properties of star-shaped, superadditive functions and their duals are discussed in Bruckner and Ostrow (1962), Bruckner (1962), Rosenbaum (1960), Hille and Phillips (1957) and Johnson (1972). Applications in probability and reliability theory of functions with properties defined in 2.1 and 2.1′ may be found in Esary, Marshall and Proschan (1970), (1973), Marshall and Proschan (1972) and A.-Hameed and Proschan (1973), and applications in statistics may be found in Barlow and Proschan (1966a), (1966b), (1967). Inequalities for star-shaped functions are given in Barlow, Marshall and Proschan (1969).

We next define completely monotonic functions.

DEFINITION 2.3. A nonnegative function f of a nonnegative argument is *completely monotonic* if it has derivatives of all orders and $(-1)^k f^{(k)}(x) \geqq 0$, $x \geqq 0$ and $k = 1, 2, \cdots$.

A comprehensive treatment of completely monotonic functions is given in Chap. 5 of Widder (1946). Their applications in statistics and probability theory can be found in Feller (1971, pp. 439–442). It is easily seen that completely monotonic functions are antistar-shaped.

We shall need to define two more notions in order to state and prove our main results.

DEFINITION 2.4. Let $\phi(\lambda, x)$ be defined on $\Lambda \times X$, where Λ and X are ordered sets. Then $\phi(\lambda, x)$ is said to be *totally positive of order* 2 (TP_2) if $\phi(\lambda, x) \geqq 0$ for $\lambda \in \Lambda$, $x \in X$, and

$$\begin{vmatrix} \phi(\lambda_1, x_1) & \phi(\lambda_1, x_2) \\ \phi(\lambda_2, x_1) & \phi(\lambda_2, x_2) \end{vmatrix} \geqq 0$$

for $\lambda_1 < \lambda_2$, $x_1 < x_2$, with λ_1, λ_2 in Λ and x_1, x_2 in X.

Totally positive functions of order 2 and of higher orders play an important role in analysis, statistics, inventory theory, reliability theory and many other theoretical and applied fields. A comprehensive treatment of the subject is given in Karlin (1968).

DEFINITION 2.5. A function $\phi(\lambda, x)$ is said to obey the *semigroup property* if

$$\phi(\lambda_1 + \lambda_2, x) = \int \phi(\lambda_1, x - y)\phi(\lambda_2, y)\, d\mu(y),$$

where μ denotes the Lebesgue measure on $[0, \infty)$ or the counting measure on $\{0, 1, 2, \cdots\}$, $\lambda \in [0, \infty)$ or alternatively, $\lambda \in \{0, 1, 2, \cdots\}$ and $x \in [0, \infty)$.

3. Preservation of geometric properties. We present the main theorem concerning preservation of the geometric properties of Definition 2.1 under the integral transformation (1.1); preliminary assumptions concerning f, g, ϕ and μ are as stated in the first paragraph of § 1.

THEOREM 3.1. (a) *Let $\phi(\lambda, x)$ be TP_2, and for some $a>0$,*

$$\int x\phi(\lambda, x)\, d\mu(x) = a\lambda \quad \text{for all } \lambda > 0. \tag{3.1}$$

Then f is star-shaped implies that g is star-shaped.

(b) *Let $\phi(\lambda, x)$ satisfy the semigroup property, and $\int \phi(\lambda, x)\, d\mu(x) \equiv 1$. Then f is superadditive implies that g is superadditive.*

(c) *Let $\phi(\lambda, x)$ satisfy the semigroup property. Then f is supermultiplicative implies that g is supermultiplicative.*

(d) *$\phi(\lambda, x)$ satisfies the semigroup property if and only if f is exponential implies that g is exponential.*

(e) *Let $\phi(\lambda, x)$ be TP_2 and satisfy the semigroup property. Then*

(e-1) *f is root-increasing and μ is the Lebesgue measure imply that g is root-increasing.*

(e-2) *f is root-increasing, μ is the counting measure and for all $\lambda > 0$,*

$$\lim_{\xi \downarrow \xi_0} \int \phi(\lambda, x)\xi^x\, d\mu(x) \equiv 0 \quad \text{for some } \xi_0 \in [-\infty, \infty)$$

imply that g is root-increasing.

(f) *$\phi(\lambda, x)$ satisfy the semigroup property. Then f is completely monotonic implies that g is completely monotonic.*

Remark 3.2. Examples of kernels $\phi(\lambda, x)$ satisfying (e−2) are:

(i) the Poisson kernel $\phi(\lambda, x) = e^{-\lambda}(\lambda^x/x!)$, $\lambda \geqq 0$ and $x = 0, 1, \cdots$, and

(ii) the binomial coefficients $\binom{\lambda}{x}$, $\lambda, x = 0, 1, \cdots$.

Proof of Theorem 3.1. (a) For each $c > 0$,

$$g(\lambda) - c\lambda = \int \phi(\lambda, x)\left[f(x) - \frac{c}{a}x\right] d\mu(x).$$

Since f is star-shaped, then $f(x) - (c/a)x$ changes sign at most once in $x \geqq 0$, and if once, from − to +. By the variation diminishing property of the TP_2 function $\phi(\lambda, x)$ (Karlin (1968, p. 21)), it follows that $g(\lambda) - c\lambda$ changes sign at most once, and if once, from − to +. Hence g must be star-shaped.

(b) Write

$$\begin{aligned} g(\lambda_1+\lambda_2) &= \int \phi(\lambda_1+\lambda_2, x) f(x)\, d\mu(x) \\ &= \iint \phi(\lambda_1, x-y)\phi(\lambda_2, y) f(x)\, d\mu(y)\, d\mu(x) \\ &\qquad\qquad [\text{by the semigroup property}] \\ &= \iint \phi(\lambda_1, z)\phi(\lambda_2, y) f(y+z)\, d\mu(y)\, d\mu(z) \\ &\geqq \iint \phi(\lambda_1, z)\phi(\lambda_2, y)[f(y)+f(z)]\, d\mu(y)\, d\mu(z) \\ &\qquad\qquad [\text{since } f \text{ is superadditive}] \end{aligned}$$

$$= g(\lambda_1) + g(\lambda_2)$$

$$\left[\text{since} \int \phi(\lambda, z)\, d\mu(z) \equiv 1\right].$$

Thus g is superadditive.

(c) Using (1.1), the semigroup property and the supermultiplicativity of f, we obtain

$$g(\lambda_1+\lambda_2) \geqq \int\int \phi(\lambda_1, z)\phi(\lambda_2, y) f(z) f(y)\, d\mu(z)\, d\mu(y)$$
$$= g(\lambda_1) g(\lambda_2).$$

Thus g is supermultiplicative.

(d) Let $\phi(\lambda, x)$ satisfy the semigroup property and f be exponential. Then, by the same kind of argument as in (c), we obtain $g(\lambda_1+\lambda_2) = g(\lambda_1)g(\lambda_2)$ for all $\lambda_1 \geqq 0$ and $\lambda_2 \geqq 0$. Since ϕ and f are measurable, it follows from Tonelli's theorem that g is measurable (Royden (1968, p. 270)), and thus $g(\lambda)$ must be exponential, as pointed out by Breiman (1968, p. 305).

Suppose now that (1.1) maps exponential functions into exponential functions. Take $f(x) = e^{-sx}$, $s \geqq 0$. For each fixed λ consider the measure ν_λ defined for every Borel set A explicitly by the relation

$$\nu_\lambda(A) = \int_A \phi(\lambda, x)\, d\mu(x).$$

Define the Laplace transform of ν_λ by

$$g_\lambda(s) = \int e^{-sx}\, d\nu_\lambda(x).$$

Then, by the well-known property of the Laplace transform, for every $\lambda_1 \geqq 0$ and $\lambda_2 \geqq 0$, we have that $g_{\lambda_1}(s)\ g_{\lambda_2}(s)$ is the Laplace transform of the convolution measure $\nu_{\lambda_1} * \nu_{\lambda_2}$; i.e.,

$$g_{\lambda_1}(s) g_{\lambda_2}(s) = \int e^{-sx}\left[\int \phi(\lambda_1, x-y)\phi(\lambda_2, y)\, d\mu(y)\right] d\mu(x). \tag{3.2}$$

Since g is exponential in λ by assumption, then $g_{\lambda_1}(s)g_{\lambda_2}(s) = g_{\lambda_1+\lambda_2}(s)$; i.e.,

$$g_{\lambda_1}(s) g_{\lambda_2}(s) = \int \phi(\lambda_1+\lambda_2, x)\, e^{-sx}\, d\mu(x). \tag{3.3}$$

By the uniqueness of the Laplace transform (see Thm. 1a. of Feller (1971, p. 432)) it follows that

$$\phi(\lambda_1+\lambda_2, x) = \int \phi(\lambda_1, x-y)\phi(\lambda_2, y)\, d\mu(y);$$

i.e., ϕ satisfies the semigroup property.

(e-1) We can assume that for each λ, $\phi(\lambda, x)$ is strictly positive on a set of positive Lebesgue measure; otherwise $g(\lambda)$ would be zero and we would have nothing to prove. Since f is root increasing, it follows that for each fixed a, $0 \leqq a < \infty$, $f(x) - a^x$ changes sign at most once in $x \geqq 0$, and if once, from $-$ to $+$.

Also, by result (d) above, $\int \phi(\lambda, x)a^x\, d\mu(x) = b^\lambda$ for some b depending on a. Moreover, as a increases from 0 to ∞, b increases continuously from 0 to ∞. By the variation diminishing property, it follows that for each nonnegative b,

$$g(\lambda) - b^\lambda = \int \phi(\lambda, x)[f(x) - a^x]\, d\mu(x)$$

changes sign at most once in $\lambda \geqq 0$, and if once, from $-$ to $+$. We conclude that $g(\lambda)$ is root increasing.

(e-2) The proof of (e-2) is similar to that of (e-1), with obvious modifications.

(f) Result (f) follows from Theorem 12a of Widder (1946, p. 160) and (d) above.

Remark 3.3. A dual theorem exists in which each geometric property is replaced by its dual property. Thus the geometric properties of 2.1′ are also preserved under the integral transformation (1.1).

4. Applications.

Moment properties. Karlin, Proschan and Barlow (1961) show that total positivity properties possessed by a probability density $f(x)$ on $[0, \infty)$ are inherited by the sth normalized moment,

$$\lambda_s = \int f(x) \frac{x^s}{\Gamma(s+1)}\, dx, \tag{4.1}$$

wherever finite. We show that, in a similar fashion, certain of the geometric properties described in Definition 2.1 are inherited by the normalized moment λ_s.

THEOREM 4.1. *Let $f(x)$, the density of a nonnegative random variable, possess one of the following properties*: (i) *submultiplicativity or* (ii) *root-decreasing. Then λ_s defined in* (4.1) *possesses the same property.*

Proof. Note that $\phi(s, x) = x^s/\Gamma(s+1)$ is TP_2 and satisfies the semigroup property. See Karlin (1968, p. 140). It follows by Theorem 3.1(c), (e-1) and Remark 3.3. that λ_s inherits the property possessed by the density $f(x)$.

An example of a root-decreasing density is the truncated normal

$$f(x) = \frac{2}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} \quad \text{for } x \geqq 0.$$

Remark 4.2. Alternatively, we may choose $\phi_1(n, x) = \binom{n}{x}$ or $\phi_2(x, n) = \binom{n}{x}$; both ϕ_1 and ϕ_2 are TP_2 and obey the semigroup property. See Karlin (1968, p. 142). A discrete version of Theorem 4.1 then follows.

THEOREM 4.3. *Let $f(x)$, the frequency function of a nonnegative discrete random variable, possess one of the following properties*: (i) *submultiplicativity*, (ii) *root-decreasing. Then the binomial-type moments*

$$B_n^{(1)} = \sum_{x=0}^{\infty} f(x)\binom{n}{x}, \tag{4.2a}$$

$$B_x^{(2)} = \sum_{n=0}^{\infty} f(n)\binom{n}{x} \tag{4.2b}$$

also possess the same property.

Shock models. In Esary, Marshall and Proschan (1973), shock models and wear processes are studied. Under the basic model, $\bar{P}_k$ represents the probability that a device survives k shocks. The shocks occur randomly in time according to a Poisson process with shock rate λ. Then the probability that the device survives until time t without failure is given by

$$\bar{H}(t)=\sum_{k=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\bar{P}_k. \tag{4.3}$$

Some of the results of the Esary, Marshall and Proschan (1973) paper may be obtained as special cases of Theorem 3.1 and Remark 3.3; (b) and (c) of Corollary 4.4 below are contained in Theorems 3.1 and 3.2 of Esary, Marshall and Proschan (1973).

COROLLARY 4.4. (A) *If* $\bar{P}_k$ *is antistar-shaped in* $k=0, 1, 2, \cdots$, *then* $\bar{H}(t)$ *is antistar-shaped in* $t\geqq 0$. (b) *If* $\bar{P}_k$ *is root-increasing* (*root-decreasing*) *in* $k=0, 1, \cdots$, *then* $\bar{H}(t)$ *is root-increasing* (*root-decreasing*) *in* $t\geqq 0$. (c) *If* $\bar{P}_k$ *is supermultiplicative* (*submultiplicative*) *in* $k=0, 1, \cdots$, *then* $\bar{H}(t)$ *is supermultiplicative* (*submultiplicative*) *in* $t\geqq 0$.

Proof. It is easily verified that for fixed $\lambda>0$, $\phi(t, k)=e^{-\lambda t}(\lambda t)^k/k!$ is TP_2 in $t\geqq 0$, $k=0, 1, \cdots$, and satisfies the semigroup property. The conclusions follow from Theorem 3.1 and Remark 3.3.

Bernstein polynomial approximations. Let

$$B_n(x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)x^k(1-x)^{n-k}$$

represent the nth Bernstein polynomial approximation to a continuous function f defined on the unit interval. Karlin (1968, p. 287) shows that for fixed $n=1, 2, \cdots$, $B_n(x)$ is convex whenever f is convex. Using Theorem 3.1 and Remark 3.3, we may show the following corollary.

COROLLARY 4.5. *Let f be star-shaped* (*antistar-shaped*) *on* [0, 1]. *Then for fixed* $n=1, 2, \cdots$, $b_n(x)$ *is star-shaped* (*antistar-shaped*) *on* [0, 1].

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POLYGAMMA FUNCTIONS OF ARBITRARY ORDER*

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Abstract. Functions $\psi^{(\nu)}(x)$ are defined for every complex ν. They are entire functions of ν and generalize the well-known polygamma functions. Some of their properties are derived and their relation to another possible generalization is described.

1. Introduction. The polygamma functions $\psi=\psi^{(0)}$, $\psi^{(1)}$, $\psi^{(2)}, \cdots$ are defined by $\psi^{(n)}(x)=(d/dx)^{n+1}\log\Gamma(x)$. They can be represented by the infinite series [1, Chap. 6]

$$\psi(x)=-\gamma+\sum_{k=1}^{\infty}\frac{x-1}{k(k+x-1)}$$

and

$$\psi^{(n)}(x)=(-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(k+x)^{n+1}}$$

for $n=1,2,3,\cdots$ and $x\neq 0,-1,-2,\cdots$. The polygamma functions have various applications. For example [1, Chap. 6], the sum $\sum_{k=0}^{\infty}R(k)$ can be expressed in terms of values of polygamma functions when R is a rational function defined at the nonnegative integers and of degree $\leqq -2$.

Professor Bertram Ross [3] suggested that sums with functions more general than rational functions might be summed by a similar technique if a suitable generalization of the polygamma function to nonintegral index could be found. The generalization he proposed was quite natural, especially since he was familiar with the properties of fractional integration and differentiation. He suggested the definition $\psi^{(\nu)}(x)=(d/dx)^{\nu+1}\log\Gamma(x)$, where $(d/dx)^{\nu+1}$ is fractional differentiation, the operation inverse to Liouville's fractional integration operator [4], and he proposed (in essence) the study of these functions in a problem presented in the *American Mathematical Monthly* [2].

As we will show here, the functions $\psi^{(\nu)}(x)$ for arbitrary ν are *not* the obvious extension and generalization of the series representation obtained simply by replacing n with ν, $n!$ with $\Gamma(\nu+1)$ and $(-1)^{n+1}$ by $\cos\pi(\nu+1)$ or $e^{i\pi(\nu+1)}$ or the like. The series so obtained are well-known in classical analysis. They are expressible in terms of the Hurwitz zeta function

$$\zeta(s,a)=\sum_{k=0}^{\infty}\frac{1}{(k+a)^{s}}$$

which is represented by this series for Re $s>1$ and $a\neq 0,-1,-2,\cdots$, and is analytically continuable to the whole s-plane as a meromorphic function. Neglecting the sign factor, we have

$$\Gamma(\nu+1)\sum_{k=0}^{\infty}\frac{1}{(k+x)^{\nu+1}}=\Gamma(\nu+1)\zeta(\nu+1,x).$$

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It was Professor Ross' hope that easily calculable representations could be found for the sums of these series. That is a very difficult problem. For example, the case $\nu = 2$ and $x = 1$ would require a convenient expression for $\zeta(3)$, the Riemann zeta function; this old and tantalizing problem is still unsolved.

2. Development. We turn our attention back to the polygamma functions. As $\psi^{(n)}(x)$ is defined only for nonnegative integers n, we are free to extend the definition to functions of arbitrary order in any way we please; but there is a certain structural feature of the whole family of polygammas of integral order that is clearly worth preserving: $(d/dx)^m \psi^{(n)}(x) = \psi^{(m+n)}(x)$ for all nonnegative integers m and n. We will insist that our generalization must satisfy $(d/dx)^\mu \psi^{(\nu)}(x) = \psi^{(\mu+\nu)}(x)$ for all complex numbers μ and ν.

We start off with the Liouville fractional integral

$$I^p \log \Gamma(x) = \frac{1}{\Gamma(p)} \int_0^x (x-t)^{p-1} \log \Gamma(t)\, dt.$$

For a fixed $x > 0$, this integral converges and defines an analytic function of p for Re $p > 0$. It enjoys the property $I^p I^q \log \Gamma(x) = I^{p+q} \log \Gamma(x)$ for Re $p > 0$ and Re $q > 0$. In a formal sense, we are done if we define

$$\psi^{(\nu)}(x) = I^{-\nu-1} \log \Gamma(x).$$

Because of convergence restrictions, this definition is rigorous only if Re $\nu < -1$. It is necessary to obtain an extension of $I^p \log \Gamma(x)$ to all p. This is the result:

$$\begin{aligned} I^p \log \Gamma(x) &= \frac{x^p}{\Gamma(p+1)} \left\{ \log \frac{1}{x} + \gamma + \frac{\Gamma'(p+1)}{\Gamma(p+1)} - \frac{\gamma x}{p+1} \right\} \\ &\quad - \frac{x^p}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} x^s \frac{\Gamma(s)\zeta(s)}{\Gamma(p+1+s)} \frac{\pi}{\sin \pi s}\, ds, \end{aligned} \tag{1}$$

where $1 < \lambda < 2$ and integration is along a vertical line. It is easy to verify that the function $p \to I^p \log \Gamma(x)$ given by this formula is an entire function in the p-plane for each x in the plane cut along the negative real axis (so that $\log 1 = 0$ and $x^s = e^{s \log x}$). We will show that this entire function coincides with the integral expression defining $I^p \log \Gamma(x)$ for Re $p > 0$. It will then follow by the principle of permanence of analytic relations that $I^p I^q \log \Gamma(x) = I^{p+q} \log \Gamma(x)$ for all complex p and q. (We use that principle in the context of operator-valued analytic functions.)

At first glance, the integral $\int_0^x (x-t)^{p-1} \log \Gamma(t)\, dt$ looks like an L^1-convolution, which suggests use of the one-sided Laplace transform; but that leads to asymptotic relationships instead of equalities. It turns out best to use the Mellin transform. The following gives an outline of the calculation.

We use the notation

$$\mathcal{M}[f](s) = \int_0^\infty t^{s-1} f(t)\, dt$$

for the Mellin transform. There is a convolution product

$$(f*_{\mathcal{M}}g)(t)=\int_0^\infty f(v)g\left(\frac{t}{v}\right)\frac{dv}{v}$$

and an isomorphism

$$\mathcal{M}[f*_{\mathcal{M}}g]=\mathcal{M}[f]\cdot\mathcal{M}[g].$$

We also sometimes write

$$\varphi(t)=\frac{\Gamma'(t+1)}{\Gamma(t+1)}+\gamma.$$

Writing $\log\Gamma(t)=\log\Gamma(t+1)-\log t$, we can integrate by parts to get

$$\int_0^x (x-t)^{p-1}\log\Gamma(t)\,dt=\frac{1}{p}\int_0^x (x-t)^p\varphi(t)\,dt$$
$$-\frac{\gamma x^{p-1}}{p(p+1)}+\frac{x^p}{p}\left\{\log\frac{1}{x}+\gamma+\frac{\Gamma'(p+1)}{\Gamma(p+1)}\right\};$$

but

$$\int_0^x (x-t)^p\varphi(t)\,dt=x^{p+1}\left[\frac{1}{v}\left(1-\frac{1}{v}\right)^p H(v-1)\right]*_{\mathcal{M}}[\varphi(v)],$$

where H is the Heaviside unit step function. Since

$$\mathcal{M}\left[\frac{1}{v}\left(1-\frac{1}{v}\right)^p H(v-1)\right](s)=B(1-s,p+1)$$

for Re $s<1$ and $\mathcal{M}[\varphi](s)=-\pi\zeta(1-s)/\sin\pi s$ for $-1<\text{Re } s<0$, we find that

$$\mathcal{M}\left[x^{-p-1}\int_0^x (x-t)^p\varphi(t)\,dt\right](s)=-\frac{\Gamma(p+1)\Gamma(1-s)}{\Gamma(p+2-s)}\frac{\pi}{\sin\pi s}\zeta(1-s)$$

for $-1<\text{Re } s<0$. After division by p, we find that application of the Mellin inversion formula gives

$$\frac{x^{-p-1}}{p}\int_0^x (x-t)^p\varphi(t)\,dt=-\frac{\Gamma(p)}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s}\frac{\Gamma(1-s)\zeta(1-s)}{\Gamma(p+2-s)}\frac{\pi}{\sin\pi s}\,dx,$$

where $-1<\sigma<0$. The formula (1) follows when s is replaced by $1-s$ in this integral.

Two interesting expansions result from (1) when the contour is deformed in one direction or the other to lie along the real axis. In the first case, suppose $0<x<1$. Deform the line contour into one which comes from $+\infty$ on the real axis, loops around $+2$ in the positive sense and returns to $+\infty$. The deformation is justified by Stirling's formula and the hypothesis $0<x<1$. There is a sign change because of the change in orientation in the contour integration. Evaluation of the residues at the simple poles at $s=2, 3, 4, \cdots$ is straightforward, and we get the expansion

$$I^p \log\Gamma(x)=\frac{x^p}{\Gamma(p+1)}\left\{\log\frac{1}{x}+\gamma+\frac{\Gamma'(p+1)}{\Gamma(p+1)}-\frac{\gamma x}{p+1}+\sum_{k=2}^{\infty}(-1)^k\frac{\zeta(k)B(p+1,k)}{k+p}x^k\right\},$$

where the summation in the braces converges for $|x|<1$ and $x=1$.

Now let $x>1$. The contour is deformed in the other direction so that it comes from $-\infty$ on the real axis, loops around $+1$ in the positive sense and returns to $-\infty$. Justification can be supplied by using the functional equation of the Riemann zeta function followed by estimation on a sequence of semicircles whose radii tend to ∞ in a suitable way. The contour encloses simple poles at $s=0, -2, -4, \cdots$ and double poles at $s=1, -1, -3, \cdots$, that at $s=1$ being anomalous. Since calculation of the residues is not quite so straightforward as in the first case, we give some details.

At $s=-2k$ $(k=0, 1, 2, \cdots)$. Using the functional equation of the Riemann zeta function, we get

$$\lim_{s\to-2k}\Gamma(s)\zeta(s)=(2\pi)^{-2k}\frac{B_k}{4k}.$$

The residue at $s=-2k$ is (after use of the reflection formula for the Γ-function)

$$-\frac{B_k}{2k}\Gamma(2k-p)(\sin\pi p)(2\pi x)^{-2k-1}.$$

At $s=1-2k$ $(k=1, 2, 3, \cdots)$ Near $s=-r$, there is a Laurent expansion

$$\Gamma(s)=\frac{(-1)^r}{r!}\frac{1}{s+r}+c_r+O(s+r).$$

We will show subsequently that

$$c_r=\frac{(-1)^r}{r!}\left(\sum_{k=1}^{r}\frac{1}{k}-\gamma\right)=(-1)^r\frac{\psi(r+1)}{\Gamma(r+1)}.$$

By the reflection formula,

$$\frac{\pi}{\sin\pi s}\Gamma(s)=\Gamma(1-s)\Gamma(s)^2,$$

so that the residue at $s=1-2k$ is

$$\frac{(-1)^k B_k c_{2k-1}}{2\pi k}(\sin\pi p)\Gamma(2k-p-1)x^{-2k}.$$

At $s=1$. Since $\zeta(s)=(s-1)^{-1}+\gamma+O(s-1)$ near $s=1$, the residue is $-\gamma$.

The result is an asymptotic expansion, valid in sectors $|\arg x|\leqq\pi-\delta$ as $|x|\to\infty$:

$$I^p\log\Gamma(x)\sim\frac{x^p}{\Gamma(p+1)}\left\{\log\frac{1}{x}+\gamma+\frac{\Gamma'(p+1)}{\Gamma(p+1)}\right\}$$
$$+x^p\sin\pi p\left\{\sum_{k=1}^{\infty}(-1)^k\frac{B_k c_{2k-1}}{2k}\frac{\Gamma(2k-p-1)}{x^{2k}}+\sum_{k=0}^{\infty}\frac{B_k}{2k}\frac{\Gamma(2k-p)}{(2\pi x)^{2k+1}}\right\}.$$

We insert here the evaluation of c_r. Recall that

$$\Gamma(s)=\frac{(-1)^r}{r!}\frac{1}{s+r}+c_r+O(s+r)$$

near the pole at $s=-r$. From the reflection formula,

$$\Gamma(s)=\frac{\pi}{\Gamma(1-s)\sin\pi s}.$$

There are expansions $\pi/\sin\pi s=(-1)^r(s+r)^{-1}+O(1)$ and

$$\frac{1}{\Gamma(1-s)}=\frac{1}{r!}\left[1+\frac{\Gamma'(r+1)}{\Gamma(r+1)}(s+r)+O(s+r)^2\right]$$

near $s=-r$. Thus,

$$c_r=\frac{(-1)^r\Gamma'(r+1)}{r!\ \ \Gamma(r+1)}.$$

To calculate $\Gamma'(r+1)/\Gamma(r+1)$, differentiate the functional equation to get $\Gamma'(t+1)=t\Gamma'(t)+\Gamma(t)$. Set $\Gamma'(n+1)=(n!)(b_n-\gamma)$ with $b_0=0$. Then $b_n=b_{n-1}+1/n$, so that $b_n=\sum_{k=0}^{n}(1/k)$. Therefore,

$$\frac{\Gamma'(r+1)}{\Gamma(r+1)}=\sum_{k=1}^{r}\frac{1}{k}-\gamma,$$

and the formula for c_r follows.

Now we are going to compare our extended polygamma functions to the extensions obtained by using the Hurwitz zeta functions. For reference, we write the full formula

$$\begin{aligned}\psi^{(\nu)}(x)=&\frac{x^{-\nu-1}}{\Gamma(-\nu)}\left\{\log\frac{1}{x}+\gamma+\frac{\Gamma'(-\nu)}{\Gamma(-\nu)}\right\}-\frac{\gamma x^{-\nu}}{\Gamma(1-\nu)}\\&-\frac{x^{-\nu-1}}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}x^s\frac{\Gamma(s)\zeta(s)}{\Gamma(s-\nu)}\frac{\pi}{\sin\pi s}\,ds.\end{aligned}$$

An asymptotic expansion for $\psi^{(\nu)}$ as $|x|\to\infty$ and a power series when $|x|<1$ can be obtained from the corresponding formulas for $I^p\log\Gamma(x)$. For convenience, set

$$\begin{aligned}\Psi^{(\nu)}(x)&=x^{\nu+1}\psi^{(\nu)}(x)-\frac{1}{\Gamma(-\nu)}\left\{\log\frac{1}{x}+\gamma+\frac{\Gamma'(-\nu)}{\Gamma(-\nu)}\right\}+\frac{\gamma x}{\Gamma(1-\nu)}\\&=-\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}x^s\frac{\Gamma(s)\zeta(s)}{\Gamma(s-\nu)}\frac{\pi}{\sin\pi s}\,ds.\end{aligned}$$

Instead of $\zeta(\nu+1,x)$, it is more convenient to treat

$$Z(\nu+1,x)=\Gamma(\nu+1)x^{\nu+1}\zeta(\nu+1,x).$$

We take the Mellin transform of Z (with a modification of the transform variable) and integrate term by term, getting

$$\int_0^\infty x^{-s-1}Z(\nu+1,x)\,dx=\frac{\Gamma(s)\zeta(s)}{\Gamma(s-\nu)}\frac{\pi}{\sin\pi(s-\nu)},$$

which is valid in the strip $1 < \text{Re } s < 1 + \text{Re } \nu$, provided of course that $\text{Re } \nu > 0$. In that strip the transform function is analytic. By the Mellin inversion formula,

$$Z(\nu+1, x) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} x^s \frac{\Gamma(s)\zeta(s)}{\Gamma(s-\nu)} \frac{\pi}{\sin \pi(s-\nu)}\, ds,$$

where $1 < \lambda < 1 + \text{Re } \nu$. Because the integrand is entire analytic in ν, we can fix λ (say, $1 < \lambda < 2$) and then use the integral to continue $Z(\nu+1, x)$ for all ν and all x not on the negative real axis.

It is now clear how intimate is the relation between $\Psi^{(\nu)}(x)$ and $Z(\nu+1, x)$, although they are clearly not interchangeable. Since $1/\Gamma(s)$ has a zero at $s = -n$ and $\sin \pi(s-n) = (-1)^n \sin \pi s$, it follows that

$$Z(n+1, x) = (-1)^{n+1} \Psi^{(n)}(x).$$

Therefore,

$$\psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(k+x)^{n+1}}$$

for $n = 1, 2, 3, \cdots$.

Two of the most important identities satisfied by the polygamma functions of integral order are

$$\psi^{(n)}(x+1) = \psi^{(n)}(x) + (-1)^n n! x^{-n-1}$$

and

$$\psi^{(n)}(1-x) = (-1)^n \psi^{(n)}(x) + (-1)^n \pi \left(\frac{d}{dx}\right)^n \cot \pi x.$$

(These are the functional equation and the reflection formula.) We have not found direct generalizations of these formulas to polygamma functions of arbitrary order, although we do have a formula by means of which $\psi^{(\nu)}(x+1)$ and $\psi^{(\nu)}(1-x)$ can be expressed in terms of polygamma functions whose argument is x. For convenience, let $\vartheta^{(\nu)}(x) = x^{-\nu-1} \Psi^{(\nu)}(x)$. Suppose that $|y| < |x|$. Using the binomial theorem and term by term integration,

$$\vartheta^{(\nu)}(x+y) = -\frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} x^{s-\nu-1} \left(1 + \frac{y}{x}\right)^{s-\nu-1} \frac{\Gamma(s)\zeta(s)}{\Gamma(s-\nu)} \frac{\pi}{\sin \pi s}\, ds$$

$$= \sum_{k=0}^{\infty} \vartheta^{(\nu+k)}(x) \frac{y^k}{k!}.$$

We note in closing that series and asymptotic developments for $Z(\nu+1, x)$ and $\zeta(\nu+1, x)$ can be obtained from the contour integral representation for Z in the same way as was done for $\Psi^{(\nu)}$. When ν is a nonnegative integer, these expansions are well known [1] as properties of the polygamma functions of integral order. The following series are worth giving explicitly. If $|y| < |x|$,

$$\zeta(\nu+1, x+y) = \sum_{k=0}^{\infty} (-1)^k \zeta(\nu+1+k, x) \frac{y^k}{k!}.$$

In particular, if $|y|<1$,

$$\zeta(\nu+1, 1+y)=\sum_{k=0}^{\infty}(-1)^k\zeta(\nu+1+k)\frac{y^k}{k!}.$$

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A SECOND ORDER NONLINEAR DEGENERATE PARABOLIC EQUATION WITH NONLINEAR BOUNDARY CONDITIONS*

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Abstract. Consider the nonlinear degenerate parabolic operator

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x,t,u,\nabla_x u)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x,t,u,\nabla_x u)u_{x_i} - c(x,t,u,\nabla_x u)u_t + d(x,t,u,\nabla_x u)u$$

where u, a^{ij}, b^i, c, d are bounded real-valued functions defined on a domain $D = \Omega \times [0, T] \subset R^{n+1}$. L is degenerate in the sense that $c(x, t, u, \nabla_x u) \geqq 0$ on D. Sufficient conditions are given for the existence of a classical solution to a first initial-boundary value problem with nonlinear boundary conditions: $Lu = f(x, t, u, \nabla_x u)$, $u = \psi(x, t, u, \nabla_x u)$ on the normal boundary of D. The proof of existence is an application of a fixed-point theorem due to Schauder.

1. Introduction. We consider the first initial-boundary value problem for the second order nonuniformly parabolic operator

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x,t,u,\nabla_x u)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x,t,u,\nabla_x u)u_{x_i} - c(x,t,u,\nabla_x u)u_t + d(x,t,u,\nabla_x u)u, \tag{1.1}$$

where u and all coefficients of L are real-valued functions defined for $(x, t) = (x_1, x_2, \cdots, x_n, t)$ in an $(n + 1)$-dimensional, bounded domain D. Subscripts will be used to denote differentiation; $\nabla_x u$ represents the spatial gradient of u.

The operator L is assumed to be parabolic; that is,

$$\sum_{i,j=1}^{n} a^{ij}\xi_i\xi_j \geqq \gamma|\xi|^2 > 0 \tag{1.2}$$

for some positive constant γ and for each real vector $\xi \neq 0$. In general, L will be assumed to be nonuniformly parabolic or, more specifically, time degenerate. That is, $c(x, t, u, \nabla_x u) \geqq 0$ but is not necessarily bounded away from zero on D.

Relying on existence of a unique solution to the first initial-boundary value problem for the linear equation

$$Mu = \sum_{i,j=1}^{n} a^{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} - c(x,t)u_t + d(x,t)u = f(x,t) \tag{1.3}$$

satisfying a "linear" boundary condition on the normal boundary of D, we use a priori estimates of the Schauder type together with a fixed-point theorem to prove existence of a solution to the nonlinear first initial-boundary value problem with nonlinear boundary conditions.

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2. Preliminaries. The notation is similar to that used by A. Friedman in [1]. D is bounded, $(n+1)$-dimensional domain in R^{n+1}. The boundary of $D, \partial D = \bar{B} + B_T + S$, where B is a domain in $R^n \times \{0\}$, B_T $(T > 0)$ is a domain in $R^n \times \{T\}$, and S is a manifold, not necessarily connected, in $R^n \times (0, T]$. $\bar{B} + S$ denotes the parabolic, or normal, boundary of D.

Let $D_T = D \cap (R^n \times (0, T))$, $B_\tau = D \cap (R^n \times \{\tau\})$, and $S_\tau = S \cap (R^n \times (0, \tau])$. Assume B_τ is a domain for each fixed $\tau \in (0, T)$. For every (x, τ) in $D, 0 < \tau < T$, if $S(x, \tau) = D_\tau + B_\tau$, then $\overline{S(x, \tau)} - S(x, \tau) = \bar{B} + S_T$. Assume there is a simple continuous curve α in D connecting B to B_T along which the t-coordinate is nondecreasing.

Hölder continuity of a function f is defined with respect to the metric d where $d(P, Q)$ is given by

$$d(P, Q) = [|x - \bar{x}|^2 + |t - \bar{t}|]^{1/2} \quad \text{for } P = (x, t), \quad Q = (\bar{x}, \bar{t}),$$

and

$$|x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}.$$

For $\alpha \in (0, 1)$, we define $|\bar{u}|_\alpha^D$. As in [1], let

$$|u|_0^D = \sup_D |u|,$$

$$\bar{H}_\alpha^D(u) = \sup_{P, Q \in D} \frac{|u(P) - u(Q)|}{d(P, Q)^\alpha}, \qquad |\bar{u}|_\alpha^D = |u|_0^D + \bar{H}_\alpha^D(u).$$

$\bar{H}_\alpha^D(u)$ is the Hölder coefficient of u. Now $\bar{C}_\alpha(D) = \{u | u : \bar{D} \to R, \ |\bar{u}|_\alpha^D < \infty\}$ is a Banach space with norm $\overline{|\cdot|}_\alpha^D$.

Denote by D_x^m any partial derivative of order m with respect to the variables $x_1, \cdots, x_n$ and let $D_t = \partial/\partial t$. If $D_x u$, $D_x^2 u$, $D_t u$ exist in D, then we define $|\bar{u}|_{2+\alpha}^D$ by $|\bar{u}|_{2+\alpha}^D = |\bar{u}|_\alpha^D + \sum |\overline{D_x u}|_\alpha^D + \sum |\overline{D_x^2 u}|_\alpha^D + |\overline{D_t u}|_\alpha^D$ where the sums are taken over all partial derivatives of the indicated order. Let

$$\bar{C}_{2+\alpha}(D) = \{u | u : \bar{D} \to R, |\bar{u}|_{2+\alpha}^D < \infty\}.$$

$\bar{C}_{2+\alpha}(D)$ is a Banach space with norm $|\cdot|_{2+\alpha}^D$. The bars which appear in the symbols for the above Banach spaces and their corresponding norms, as used in [1], indicate that the estimates are boundary rather than interior estimates. Since we are concerned with boundary estimates only, we shall drop the bar. When it is clear that the domain is D, we shall omit D also.

Definition 2.1. D is said *to have property* $(\bar{\mathrm{E}})$ if for each point Q of S, there exists an $(n+1)$-dimensional neighborhood V such that $V \cap \bar{S}$ can be represented, for some i $(1 \leqq i \leqq n)$, in the form

$$x_i = h(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, t),$$

where h, $D_x h$, $D_x^2 h$, $D_t h$ are Hölder continuous of exponent α. If D has property $(\bar{\mathrm{E}})$ and if the functions $D_x D_t h$, $D_t^2 h$ of the local representations of $\bar{S}$ exist and are continuous functions, then we say that D *has property* $(\bar{\mathrm{E}}')$.

Definition 2.2. A function ψ defined on $\bar{B} + S$ is said *to belong to* $\bar{C}_{2+\alpha}(D)$ if there exist functions Ψ in $\bar{C}_{2+\alpha}(D)$ such that $\Psi = \psi$ on $\bar{B} + S$. Then $|\psi|_{2+\alpha}^D$ is defined by

$$|\psi|_{2+\alpha}^{D} = \inf_{\Psi} |\Psi|_{2+\alpha}^{D},$$

where the infimum is taken over all $\Psi \in \bar{C}_{2+\alpha}(D)$ which coincide with ψ on $\bar{B} + S$.

We introduce the following notation for convenience. If u is a bounded real-valued function on a subset A of R^{n+1}, define

$$M(u; A) = \sup \{ux, t)|(x, t) \in A\}$$

and

$$m(u; A) = \inf \{u(x, t)|(x, t) \in A\}.$$

We shall need to use existence of a unique solution to the linear first initial-boundary value problem

$$\begin{aligned} Nu &= \sum_{i,j=1}^{n} a^{ij}(x, t)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x, t)u_{x_i} - c(x, t)u_t + d(x, t)u \\ &= f(x, t) \quad \text{on } D + B_T, \\ u &= \psi(x, t) \quad \text{on } \bar{B} + S \end{aligned} \tag{2.1}$$

to obtain existence of a solution to the analogous nonlinear one.

We shall consider two cases. For the degenerate case in which $c(x, t)$ is assumed to be nonnegative only, we use the existence-uniqueness theorem which is proved by the author in [2]. For this case, we must assume that a^{ij} is constant.

If $c(x, t)$ is bounded away from zero, then we can use the existence-uniqueness theorem of Friedman [1] and it is not necessary to assume a^{ij} constant. Of course, when there is one space variable we can easily divide a $a(x, t)$ and so it is not necessary to assume $c(x, t)$ attains a positive minimum on D. We now state the two theorems precisely.

THEOREM 2.3 (see [2]). *Assume that a^{ij} is constant for each i, j, that all coefficients of N, defined in* (2.1), *are of class $C^{1,1}(\bar{D})$, $u \in C(\bar{D})$, $u \in C^3(D)$, and $a^{11}\lambda^2 + b^1\lambda \geqq 1$ for some constant $\lambda > 0$. Suppose, further, that the coefficients of N are uniformly Hölder continuous* (*exponent* α) *in D, $|a^{ij}|_\alpha \leqq K_1$, $|b^i|_\alpha \leqq K_1$, $|c|_\alpha \leqq K_1$, $|f_t|_\alpha \leqq K_1$, that $m(c; \bar{B} + S) \geqq \mu > 0$ while $m(c; D) \geqq 0$, that $|f|_\alpha < \infty$, and that* (1.2) *holds. If D has property* ($\bar{\text{E}}'$), *$\psi \in C_{2+\alpha}$ and $N\psi = f$ on ∂B, then there exists a unique solution u of the first initial-boundary value problem* (2.1) *and, furthermore, $u \in C_{2+\alpha}$.*

THEOREM 2.4 (see [1]). *Assume that all coefficients of N, defined in* (2.1), *are of class $C^1(\bar{D})$, $u \in C(\bar{D})$, $u \in C^3(D)$, that the coefficients of N are uniformly Hölder continuous* (*exponent* α) *in D, that there exists a constant K_1 such that $|a^{ij}|_\alpha \leqq K_1$, $|b^i|_\alpha \leqq K_1$, $|d|_\alpha \leqq K_1$, and that $c(x, t) \equiv 1$. Assume also that $|f|_\alpha < \infty$ and that* (1.2) *holds. If D has property* ($\bar{\text{E}}'$), *$\psi \in C_{2+\alpha}$ and $N\psi = f$ on ∂B, then there exists a unique solution u of the first initial-boundary value problem* (2.1) *and, furthermore, $u \in C_{2+\alpha}$.*

It is clear that if $c(x, t)$ is bounded away from zero, then Theorem 2.4 applies.

3. A priori estimates. We shall need Schauder-type estimates for solutions u to the linear problems discussed in Theorems 2.3 and 2.4. We first consider the degenerate problem.

The technique which was used in obtaining the solution in Theorem 2.3 involved perturbing the coefficient c by $1/K$ and considering the problem

$$N^k u = Nu - \frac{1}{k} u_t = f \quad \text{on } D + B_T, \qquad u = \psi \quad \text{on } \bar{B} + S.$$

By Friedman's work, we were guaranteed [2] a unique solution $u_k \in C_{2+\alpha}(D)$. We showed that the sequence $\{u_k\}$ obtained in this manner is Cauchy in the Banach space $C_{2+\alpha}(D)$ and converges in $|\cdot|_{2+\alpha}$ to the unique solution u of the first initial-boundary value problem (2.1). These u_k satisfy $|u_k|_{2+\alpha} \leqq k^{1/2} M(|\psi|_{2+\alpha} + |f|_\alpha)$, where M is independent of $|\psi|_{2+\alpha} + |f|_\alpha$. If $\varepsilon > 0$, there is some $u_{K(\varepsilon)}$ such that $|u - u_{K(\varepsilon)}| < \varepsilon$. It follows that

$$\begin{aligned} |u|_{2+\alpha} &\leqq |u - u_{K(\varepsilon)} + u_{K(\varepsilon)}|_{2+\alpha} \leqq |u - u_{K(\varepsilon)}|_{2+\alpha} + |u_{K(\varepsilon)}|_{2+\alpha} \\ &\leqq \varepsilon + [K(\varepsilon)]^{1/2} M(|\psi|_{2+\alpha} + |f|_\alpha). \end{aligned}$$

We have therefore proved the following theorem.

THEOREM 3.1. *Assume all the conditions of Theorem 2.3. Then if u is a solution to the first initial-boundary value problem (2.1) and $\varepsilon > 0$, there exists a constant $K(\varepsilon)$ and a constant M such that*

$$|u|_{2+\alpha} \leqq \varepsilon + [K(\varepsilon)]^{1/2} M(|\psi|_{2+\alpha} + |f|_\alpha), \tag{3.1}$$

where M is independent of ε, $m(c; D)$ and $|\psi|_{2+\alpha} + |f|_\alpha$.

The approriate theorem for the nondegenerate case is proved in [1].

THEOREM 3.2. *Assume all the conditions of Theorem 2.4. Then a solution u of the first initial-boundary value problem (2.1) satisfies*

$$|u|_{2+\alpha} \leqq M(|\psi|_{2+\alpha} + |f|_\alpha) \tag{3.2}$$

for some constant M depending only on K_1, γ, α and D.

4. The nonlinear problem. We are interested in solving the first initial-boundary value problem

$$\begin{aligned} Lu &= f(x, t, u, \nabla_x u) \quad \text{on } D + B_T, \\ u &= \psi(x, t, u, \nabla_x u) \quad \text{on } \bar{B} + S. \end{aligned} \tag{4.1}$$

For v a fixed element of $C_{2+\alpha}$, consider the linear problem

$$\begin{aligned} L_v u = \sum_{i,j=1}^{n} a^{ij}(x, t, v, \nabla_x v) u_{x_i x_j} &+ \sum_{i=1}^{n} b^i(x, t, v, \nabla_x v) u_{x_i} \\ &- c(x, t, v, \nabla_x v) u_t + d(x, t, v, \nabla_x v) u \\ &= f(x, t, v, \nabla_x v) \quad \text{on } D + B_T, \\ u &= \psi(x, t, v, \nabla_x v) \quad \text{on } \bar{B} + S. \end{aligned} \tag{4.2}$$

Assuming that as a function of x and t, the coefficients satisfy the hypotheses of either Theorem 2.3 or 2.4, we obtain a unique solution $u \in C_{2+\alpha}$ to the initial-boundary value problem (4.2).

Define $\phi : C_{2+\alpha} \to C_{2+\alpha}$ by letting $u = \phi(v)$ be the unique solution to the initial-boundary value problem (4.2).

Under appropriate conditions on the coefficients, we shall show that:

(4.3) $\phi : A \to A$ where A is some closed convex subset of the Banach space $C_{2+\alpha}$;

$$\phi \text{ is continuous in } |\cdot|_{2+\alpha} \text{ on } A. \tag{4.4}$$

$$\phi(A) \text{ is precompact.} \tag{4.5}$$

We shall then apply the Schauder fixed-point theorem of [1, p. 189] to obtain an element $u \in C_{2+\alpha}$ such that $\phi(u) = u$. Then u will be a solution to (4.1).

Assume that the coefficients of L satisfy a Lipschitz condition in u and $\nabla_x u$ in the $|\cdot|_\alpha$-norm. That is, a coefficient $y(x, t, u, \nabla_x u)$ satisfies

$$|y(x, t, u_1, \nabla_x u_1)' - y(x, t, u_2, \nabla_x u_2)|_\alpha \leqq \eta_y |u_1 - u_2|_\alpha + \xi_y |\nabla_x u_1 + \nabla_x u_2|_\alpha \tag{4.6}$$

for some positive constants η_y and ξ_y.

Let $A = \{u \mid |u|_{2+\alpha} \leqq 2M(2M + 2K_1 M + 1)(|\psi|_{2+\alpha} + |f|_\alpha\}$. Analogously, for the degenerate case we shall use for fixed $\varepsilon_0 > 0$,

$$\begin{aligned} A = \{u \mid |u|_{2+\alpha} \leqq 2[\varepsilon_0 &+ [K(\varepsilon_0)]^{1/2} M \varepsilon_0 + [K(\varepsilon_0)]^{1/2} M K_1 \varepsilon_0 \\ &+ [K(\varepsilon_0)]^{1/2} M (2[K(\varepsilon_0)]^{1/2} M \\ &+ 2K_1 [K(\varepsilon_0)]^{1/2} M + 1)(|\psi|_{2+\alpha} + |f|_\alpha)]\}. \end{aligned}$$

A is clearly a closed convex subset of $C_{2+\alpha}$. We now show $\phi : A \to A$.

Let $v \in A$ and $h \in [0, 1]$. $\phi(hv)$ satisfies

$$\begin{aligned} L_{hv}\phi(hv) &= f(x, t, hv, \nabla_x hv) \quad \text{on } D + B_T, \\ \phi(hv) &= \psi(x, t, hv, \nabla_x hv) \quad \text{on } \bar{B} + S. \end{aligned} \tag{4.7}$$

We observe that (4.7) is equivalent to

$$\begin{aligned} L_v \phi(v) &= L_v \phi(v) - L_{hv}\phi(hv) + f \equiv F(v) \quad \text{on } D + B_T, \\ \phi(v) &= \phi(v) - \phi(hv) + \psi(x, t, hv, \nabla_x hv) \equiv \chi(v) \quad \text{on } \bar{B} + S. \end{aligned} \tag{4.8}$$

It follows that

$$
\begin{aligned}
|F(v)|_\alpha &\leqq |L_v\phi(v) - L_{hv}\phi(hv)|_\alpha + |f|_\alpha \\
&\leqq K_1|\phi(v) - \phi(hv)|_{2+\alpha} + |\textstyle\sum\sum [a^{ij}(x,t,v,\nabla_x v) - a^{ij}(x,t,hv,\nabla_x hv)]D_x^2\phi(hv)|_\alpha \\
&\quad + |\textstyle\sum [b^i(x,t,v,\nabla_x v) - b^i(x,t,hv,\nabla_x hv)]D_x\phi(hv)|_\alpha \\
&\quad + |c(x,t,hv,\nabla_x hv) - c(x,t,v,\nabla_x v)|_\alpha |D_t\phi(hv)|_\alpha \\
&\quad + |d(x,t,v,\nabla_x v) - d(x,t,hv,\nabla_x hv)|_\alpha |\phi(hv)|_\alpha + |f|_\alpha \\
&\leqq K_1|\phi(v) - \phi(hv)|_{2+\alpha} + \textstyle\sum\sum (\eta_{a^{ij}}|v - hv|_\alpha + \xi_{a^{ij}}|\nabla_x v - \nabla_x hv|_\alpha)|D_x^2\phi(hv)|_\alpha \\
&\quad + \textstyle\sum (\eta_{b^i}|v - hv|_\alpha + \xi_{b^i}|\nabla_x v - \nabla_x hv|_\alpha)|D_x\phi(hv)|_\alpha \\
&\quad + (\eta_c|v - hv|_\alpha + \xi_c|\nabla_x v - \nabla_x hv|_\alpha)|D_t\phi(hv)|_\alpha \\
&\quad + (\eta_d|v - hv|_\alpha + \xi_d|\nabla_x v - \nabla_x hv|_\alpha)|\phi(hv|_\alpha + |f|_\alpha \\
&\leqq K_1|\phi(v) - \phi(hv)|_{2+\alpha} + 2 \max_{1 \leqq i,j \leqq n} \{\eta_{a^ij}, \xi_{a^ij}\} |v - hv|_{2+\alpha} \textstyle\sum\sum |D_x^2\phi(hv)|_\alpha \\
&\quad + 2 \max_{1 \leqq i \leqq n} \{\eta_{b^i}, \xi_{b^i}\} |v - hv|_{2+\alpha} \textstyle\sum |D_x\phi(hv)|_\alpha \\
&\quad + 2 \max \{\eta_c, \xi_c\} |v - hv|_{2+\alpha} |D_t\phi(hv)|_\alpha \\
&\quad + 2 \max \{\eta_d, \xi_d\} |v - hv|_{2+\alpha} |\phi(hv)|_\alpha + |f|_\alpha \\
&\leqq K_1|\phi(v) - \phi(hv)|_{2+\alpha} + 2|1 - h|\,|v|_{2+\alpha} K_2 |\phi(hv)|_{2+\alpha} + |f|_\alpha,
\end{aligned}
$$

where

$$
K_2 = \max_{1 \leqq i,j \leqq n} \{\eta_{a^{ij}}, \eta_{b^i}, \eta_c, \eta_d, \xi_{a^{ij}}, \xi_{b^i}, \xi_c, \xi_d\}.
$$

Furthermore, the inequality

$$
(4.9) \qquad |F(v)|_\alpha \leqq K_1|\phi(v) - \phi(hv)|_{2+\alpha} + 2|1 - h|\,|v|_{2+\alpha} K_2 |\phi(hv)|_{2+\alpha} + |f|_\alpha
$$

holds in case a^{ij} is constant. If a^{ij} is constant, the term containing $a^{ij}(x, t, v, \nabla_x v) - a^{ij}(x, t, hv, \nabla_x hv)$ vanishes and K_2 is the maximum of the remaining Lipschitz constants.

We shall now be concerned with estimating $|\phi(v) - \phi(hv)|_{2+\alpha}$. $u = \phi(v) - \phi(hv)$ satisfies

$$
(4.10) \qquad
\begin{aligned}
L_v u &= \textstyle\sum\sum [a^{ij}(x,t,hv,\nabla_x hv) - a^{ij}(x,t,v,\nabla_x v)]D_x^2\phi(hv) \\
&\quad + \textstyle\sum [b^i(x,t,hv,\nabla_x hv) - b^i(x,t,v,\nabla_x v)]D_x\phi(hv) \\
&\quad + [c(x,t,v,\nabla_x v) - c(x,t,hv,\nabla_x hv)]\phi(hv) \\
&\quad + f(x,t,v,\nabla_x v) - f(x,t,hv,\nabla_x hv) \quad \text{in } D + B_T, \\
u &= \psi(x,t,v,\nabla_x v) - \psi(x,t,hv,\nabla_x hv) \quad \text{in } \bar{B} + S.
\end{aligned}
$$

Applying Theorem 3.1 or 3.2, whichever is appropriate, to the system (4.10), we have the estimate

$$
(4.11) \quad |\phi(v) - \phi(hv)|_{2+\alpha} \leqq M[|\psi(x,t,v,\nabla_x v) - \psi(x,t,hv,\nabla_x hv)|_{2+\alpha} + |g|_\alpha]
$$

or, analogously, for the degenerate case

$$(4.11') \quad |\phi(v) - \phi(hv)|_{2+\alpha} \leqq \varepsilon_0 + [K(\varepsilon_0)]^{1/2}M[|\psi(x,t,v,\nabla_x v) - \psi(x,t,hv,\nabla_x hv)|_{2+\alpha} + |g|_\alpha],$$

where $g(v)$ is the forcing function appearing in (4.10). It is clear that in a manner similar to that used in obtaining the inequality (4.9), we can estimate $|g|_\alpha$ by

$$2|1-h|\,|v|_{2+\alpha}K_2|\phi(hv)|_{2+\alpha} + 2|f|_\alpha.$$

Hence, substituting (4.11) into (4.9), we see that

$$\begin{aligned}
|F(v)|_\alpha &\leqq K_1 M\{|\psi(x,t,v,\nabla_x v) - \psi(x,t,hv,\nabla_x hv)|_{2+\alpha} \\
&\quad + 2|1-h|\,|v|_{2+\alpha}K_2|\phi(hv)|_{2+\alpha} + 2|f|_\alpha\} \\
&\quad + 2|1-h|\,|v|_{2+\alpha}K_2|\phi(hv)|_{2+\alpha} + |f|_\alpha \\
&\leqq 2K_1 M\{|\psi|_{2+\alpha} + |f|_\alpha + |1-h|\,|v|_{2+\alpha}K_2|\phi(hv)|_{2+\alpha}\} \\
&\quad + 2|1-h|\,|v|_{2+\alpha}K_2|\phi(hv)|_{2+\alpha} + |f|_\alpha \\
&\leqq (2K_1M+1)(|\psi|_{2+\alpha} + |f|_\alpha) + (2K_1M+2)|1-h|K_2|\phi(hv)|_{2+\alpha}|v|_{2+\alpha}.
\end{aligned}$$

The above chain of inequalities yields

$$(4.12) \quad \begin{aligned} |F(v)|_\alpha &\leqq (2K_1M+1)(|\psi|_{2+\alpha} + |f|_\alpha) \\ &\quad + (2K_1M+2)|1-h|K_2|\phi(hv)|_{2+\alpha}|v|_{2+\alpha}. \end{aligned}$$

For the degenerate case, we substitute (4.11′) into (4.9) and obtain

$$(4.12') \quad \begin{aligned} |F(v)|_\alpha &\leqq K_1\varepsilon_0 + (2K_1[K(\varepsilon_0)]^{1/2}M+1)(|\psi|_{2+\alpha} + |f|_\alpha) \\ &\quad + (2K_1[K(\varepsilon_0)]^{1/2}M+2)|1-h|K_2|\phi(hv)|_{2+\alpha}|v|_{2+\alpha}. \end{aligned}$$

We now estimate $|\chi(v)|_{2+\alpha}$. From (4.8) and (4.11) we see that

$$\begin{aligned}
|\chi(v)|_{2+\alpha} &\leqq |\phi(v) - \phi(hv)|_{2+\alpha} + |\psi|_{2+\alpha} \\
&\leqq M[2|\psi|_{2+\alpha} + 2|1-h|\,|v|_{2+\alpha}K_2|\phi(hv)|_{2+\alpha} + 2|f|_\alpha] + |\psi|_{2+\alpha} \\
&\leqq (2M+1)(|\psi|_{2+\alpha} + |f|_\alpha) + 2M|1-h|K_2|\phi(hv)|_{2+\alpha}|v|_{2+\alpha}.
\end{aligned}$$

Thus

$$(4.13) \quad |\chi|_{2+\alpha} \leqq (2M+1)(|\psi|_{2+\alpha} + |f|_\alpha) + 2M|1-h|K_2|\phi(hv)|_{2+\alpha}|v|_{2+\alpha}.$$

Similarly, from (4.8) and (4.11′) we obtain

$$(4.13') \quad \begin{aligned} |\chi|_{2+\alpha} &\leqq \varepsilon_0 + 2[K(\varepsilon_0)]^{1/2}M(|\psi|_{2+\alpha} + |f|_\alpha) \\ &\quad + 2[K(\varepsilon_0)]^{1/2}M|1-h|K_2|\phi(hv)|_{2+\alpha}|v|_{2+\alpha}. \end{aligned}$$

Since $\phi(v)$ satisfies the system (4.8), we may apply Theorem 3.1 or Theorem 3.2 to obtain

$$(4.14) \quad |\phi(v)|_{2+\alpha} \leqq M(|\chi|_{2+\alpha} + |f|_\alpha),$$

$$(4.14') \quad |\phi(v)|_{2+\alpha} \leqq \varepsilon_0 + [K(\varepsilon_0)]^{1/2}M(|\chi|_{2+\alpha} + |f|_\alpha).$$

Substituting (4.12) and (4.13) into (4.14), we see that

$$\begin{aligned}|\phi(v)|_{2+\alpha} &\leqq M[2M|\psi|_{2+\alpha} + |f|_\alpha + 2M|1 - h|K_2|\phi(hv)|_{2+\alpha}|v|_{2+\alpha} \\ &\quad + (2K_1M + 1)(|\psi|_{2+\alpha} + |f|_\alpha) \\ &\quad + (2K_1M + 2)|1 - h|K_2|\phi(hv)|_{2+\alpha}|v|_{2+\alpha}] \\ &= M(2M + 2K_1M + 1)(|\psi|_{2+\alpha} + |f|_\alpha) \\ &\quad + M(2M + 2K_1M + 2)|1 - h|K_2|\phi(hv)|_{2+\alpha}|v|_{2+\alpha}.\end{aligned}$$

Now $|\phi(hv)|_{2+\alpha}$ is uniformly bounded independent of h and v since $\phi(hv)$ satisfies (4.7) and the estimate of Theorem 3.2 holds. Furthermore, $v \in A$ implies that $|v|_{2+\alpha} \leqq 2M(2M + 2K_1M + 1)(|\psi|_{2+\alpha} + |f|_\alpha)$. Choose h sufficiently near 1 that $2M(2M + 2K_1M + 2)|1 - h|K_2|\phi(hv)|_{2+\alpha} < 1$. Therefore, $|\phi(v)|_{2+\alpha} \leqq 2M(2M + 2K_1M + 1)(|\psi|_{2+\alpha} + |f|_\alpha)$ and we conclude that $\phi(v) \in A$.

Proceeding with the degenerate case, we substitute (4.12′) and (4.13′) into (4.14′); then

$$\begin{aligned}|\phi(v)|_{2+\alpha} &\leqq \varepsilon_0 + [K(\varepsilon_0)]^{1/2}M\{\varepsilon_0 + 2[K(\varepsilon_0)]^{1/2}M(|\psi|_{2+\alpha} + |f|_\alpha) \\ &\quad + 2[K(\varepsilon_0)]^{1/2}M|1 - h|K_2|\phi(hv)|_{2+\alpha}|v|_{2+\alpha} \\ &\quad + K_1\varepsilon_0 + (2K_1[K(\varepsilon_0)]^{1/2}M + 2)|1 - h|K_2\phi(hv)|_{2+\alpha}|v|_{2+\alpha}\} \\ &\leqq \varepsilon_0 + [K(\varepsilon_0)]^{1/2}M\varepsilon_0 + [K(\varepsilon_0)]^{1/2}MK_1\varepsilon_0 \\ &\quad + [K(\varepsilon_0)]^{1/2}M\{2[K(\varepsilon_0)]^{1/2}M(|\psi|_{2+\alpha} + |f|_\alpha) \\ &\quad + 2[K(\varepsilon_0)]^{1/2}|1 - h|K_2|\phi(hv)|_{2+\alpha}|v|_{2+\alpha} \\ &\quad + (2K_1[K(\varepsilon_0)]^{1/2}M + 1)(|\psi|_{2+\alpha} + |f|_\alpha) \\ &\quad + (2K_1[K(\varepsilon_0)]^{1/2}M + 2)|1 - h|K_2|\phi(hv)|_{2+\alpha}|v|_{2+\alpha}\} \\ &= \varepsilon_0 + [K(\varepsilon_0)]^{1/2}M\varepsilon_0 + [K(\varepsilon_0)]^{1/2}MK_1\varepsilon_0 \\ &\quad + [K(\varepsilon_0)]^{1/2}M(2[K(\varepsilon_0)]^{1/2}M + 2K_1[K(\varepsilon_0)]^{1/2}M + 1)(|\psi|_{2+\alpha} + |f|_\alpha) \\ &\quad + [K(\varepsilon_0)]^{1/2}M(2[K(\varepsilon_0)]^{1/2}M + 2K_1[K(\varepsilon_0)]^{1/2}M + 2)|1 - h|K_2 \\ &\quad \cdot |\phi(hv)|_{2+\alpha}|v|_{2+\alpha}.\end{aligned}$$

$|\phi(hv|_{2+\alpha}$ is uniformly bounded independent of h and v since $\phi(hv)$ satisfies (4.7) and the estimate of Theorem 3.1 applies. Furthermore, $v \in A$ implies that

$$\begin{aligned}|v|_{2+\alpha} \leqq 2\{\varepsilon_0 + [K(\varepsilon_0)]^{1/2}M\varepsilon_0 + [K(\varepsilon_0)]^{1/2}MK_1\varepsilon_0 + [K(\varepsilon_0)]^{1/2}M(2[K(\varepsilon_0)]^{1/2}M \\ + 2K_1[K(\varepsilon_0)]^{1/2}M + 1)(|\psi|_{2+\alpha} + |f|_\alpha)\}.\end{aligned}$$

Choose h sufficiently near 1 that

$$2[K(\varepsilon_0)]^{1/2}M(2[K(\varepsilon_0)]^{1/2}M + 2K_1[K(\varepsilon_0)]^{1/2}M + 2)K_2|\phi(hv)|_{2+\alpha} \cdot |1 - h| < 1.$$

Then it follows that

$$|\phi(v)|_{2+\alpha} \leqq 2\{\varepsilon_0 + [K(\varepsilon_0)]^{1/2}M\varepsilon_0 + [K(\varepsilon_0)]^{1/2}MK_1\varepsilon_0 + [K(\varepsilon_0)]^{1/2}M(2[K(\varepsilon_0)]^{1/2}M + 2K_1[K(\varepsilon_0)]^{1/2}M + 1)(|\psi|_{2+\alpha} + |f|_\alpha)\}.$$

Hence, $\phi(v) \in A$.

We shall now prove that (4.4) holds: ϕ is continuous in $|\cdot|_{2+\alpha}$ on A. Using (4.10) and (4.11) with v_0 replacing hv, we obtain

$$\begin{aligned} |\phi(v) - \phi(v_0)|_{2+\alpha} \leqq M[&|\psi(x,t,v,\nabla_x v) - \psi(x,t,v_0,\nabla_x v_0)|_{2+\alpha} \\ &+ |f(x,t,v,\nabla_x v) - f(x,t,v_0,\nabla_x v_0)|_\alpha \\ &+ 2|v - v_0|_{2+\alpha}K_2|\phi(v_0)|_{2+\alpha}]. \end{aligned} \tag{4.15}$$

Using uniform boundedness of $|\phi(v_0)|_{2+\alpha}$ and continuity with respect to u in $|\cdot|_{2+\alpha}$ of $\psi(x,t,u,\nabla_x u)$ and $f(x,t,u,\nabla_x u)$, we see that (4.15) immediately implies continuity of ϕ on A in $|\cdot|_{2+\alpha}$. This accounts for the nondegenerate case.

To show continuity of ϕ in case the equation is degenerate, we note that (4.10) and (4.11) imply for arbitrary $\varepsilon > 0$,

$$\begin{aligned} |\phi(v) - \phi(v_0)|_{2+\alpha} \leqq \frac{\varepsilon}{2} + \left[K\left(\frac{\varepsilon}{2}\right)\right]^{1/2} M[&|\psi(c,t,v,\nabla_x v) - \psi(x,t,v_0\nabla_x v_0)|_{2+\alpha} \\ &+ |f(x,t,v,\nabla_x v) - f(x,t,v_0,\nabla_x v_0)|_\alpha \\ &+ 2|v - v_0|_{2+\alpha}K_2|\phi(v_0)|_{2+\alpha}], \end{aligned} \tag{4.15'}$$

where ε, $K(\varepsilon)$, M are independent of

$$|\psi(x,t,v,\nabla_x v) - \psi(x,t,v_0,\nabla_x v_0)|_{2+\alpha} + |f(x,t,v,\nabla_x v) - f(x,t,v_0,\nabla_x v_0)|_\alpha + 2|v - v_0|_{2+\alpha}K_2|\phi(v_0)|_{2+\alpha}.$$

$|\phi(v_0)|_{2+\alpha}$ is uniformly bounded and ψ and f are assumed to be continuous in $|\cdot|_{2+\alpha}$. Hence, there exists $\delta > 0$ such that

$$\begin{aligned} \left[K\left(\frac{\varepsilon}{2}\right)\right]^{1/2} M[&|\psi(x,t,v,\nabla_x v) - \psi(x,t,v_0,\nabla_x v_0)|_{2+\alpha} \\ &+ |f(x,t,v,\nabla_x v) - f(x,t,v_0,\nabla_x v_0)|_\alpha \\ &+ 2|v - v_0|_{2+\alpha}K_2|\phi(v_0)|_{2+\alpha}] < \frac{\varepsilon}{2} \end{aligned}$$

ϕ is therefore continuous on A in $|\cdot|_{2+\alpha}$.

We must verify that $\phi(A)$ is precompact. That is, if $\{\phi(v_m)\}$ is any sequence in $\phi(A)$, there exists a subsequence $\{\phi(v'_m)\}$ which is convergent (in $|\cdot|_{2+\alpha}$) to some element v of A. v need not be in the range of ϕ.

We shall need a theorem which appears in [1, pp. 71–75] and which we state without proof.

THEOREM 4.1. *Let L be a parabolic operator in $\bar{D}$, D satisfying the property* (E′), *and assume that, for some $p \geqq 0$, the functions $D_x^m D_t^k a^{ij}$, $D_x^m D_t^k b^i$, $D_x^m D_t^k c$, $D_x^m D_t^k d$, $D_x^m D_t^k f$ ($0 \leqq m + k \leqq p$) are uniformly Hölder continuous* (*exponent* α) *in every domain whose closure is contained in $D + B + S + B_T$. Assume further that the functions h which appear in the local representations of S are such that $D_x^{m+2} D_t^k h$, $D_x^m D_t^{k+1} h$ ($m \geqq -2$, $k \geqq -1$, $m + k \leqq p$) are Hölder continuous* (*exponent* α). *Assume finally that $\psi \in C_{2+\alpha}$, that $L\psi = f$ on ∂B, and that, as a function of the local parameters of S, ψ is a function satisfying the condition that $D_x^{m+2} D_t^k \psi$, $D_x^m D_t^{k+1} \psi$($m \geqq -2$, $k \geqq -1$, $m + k \leqq p$) are Hölder continuous* (*exponent* α), *whereas on B, $D_x^{m+2}\psi$($-2 \leqq m \leqq p$) are Hölder continuous* (*exponent* α). *If u is the solution of* (2.1), *then in every domain whose closure is contained in $D + B + S + B_t$, the functions $D_x^{m+2} D_t^k u$, $D_x^m D_t^{k+1} u$ ($m \geqq -2$, $k \geqq -1$, $m + 2k \leqq p$) are uniformly Hölder continuous* (*exponent* α).

Remark 4.2. The above theorem also holds for the degenerate case. Since we cannot "solve" for u_t, however, we use the estimates for $|u_t|_\alpha$ obtained in Theorem 3.1 and proceed in a similar fashion as in [1, p. 74]. Furthermore, its proof guarantees a uniform bound for $|D_x^3\phi(v_m)|_\alpha$ and $|D_x D_t \phi(v_m)|_\alpha$ which is independent of m.

Suppose $|D_x^3(v_m)|_\alpha \leqq K$. Then $M(|D_x^3\phi(v_m)|; D) \leqq K$ and $H_\alpha(D_x^3\phi(v_m)) \leqq K$. Therefore, $\{D_x^3\phi(v_m)\}$ is a uniformly bounded and equicontinuous family in the bounded domain D. By the theorem of Ascoli–Arzela, there exists a subsequence $\phi(v_m')$ such that $\{D_x^3\phi(v_m')\}$ is uniformly convergent in D. By successive applications of the Ascoli–Arzela theorem, we obtain a subsequence $\{\phi(v_m'')\}$ of $\{\phi(v_m')\}$ which is uniformly convergent in D together with its first three x-derivatives, its xt-derivative, its txx-derivative, and its first t-derivative. That is,

$$
\begin{aligned}
&\phi(v_m'') \to v, \quad D_x\phi(v_m'') \to D_x v, \quad D_x^2\phi(v_m'') \to D_x^2 v, \\
&D_x^3\phi(v_m'') \to D_x^3 v, \quad D_t D_x^2\phi(v_m'') \to D_t D_x^2 v, \\
&D_x D_t\phi(v_m'') \to D_x D_t(v), \quad D_t\phi(v_m'') \to D_t(v).
\end{aligned} \tag{4.16}
$$

For convenience, let $m'' = m$.

$v \in C_{2+\alpha}$, for suppose P and Q are arbitrary points in D. Since $H_\alpha(D_x^2\phi(v_m)) \leqq K$,

$$|D_x^2\phi(v_m)(P) - D_x^2\phi(v_m)(Q)| d(P, Q)^{-\alpha} \leqq K. \tag{4.17}$$

Letting $m \to \infty$ and using (4.16), we have

$$|D_x^2 v(P) - D_x^2 v(Q)| d(P, Q)^{-\alpha} \leqq K, \tag{4.18}$$

so that $H_\alpha(D_x^2 v) < \infty$. The rest of the argument that $|v|_{2+\alpha} < \infty$ is similar. To show that $\phi(v_m) \to v$ in $|\cdot|_{2+\alpha}$, we consider $|\phi(v_m) - v|_{2+\alpha}$. It is sufficient to prove that $H_\alpha(D_x^2\phi(v_m) - D_x^2(v)) \to 0$ as $m \to \infty$ since all other terms of the norm are treated similarly.

We shall show that for any $\varepsilon > 0$ and for all $P, Q \in D$ with $P \neq Q$, there exists m_0 such that if $m > m_0$,

$$
\begin{aligned}
|I_m(P, Q)| \equiv |D_x^2\phi(v_m)(P) &- D_x^2 v(P) \\
&- D_x^2\phi(v_m)(Q) + D_x^2 v(Q)| d(P, Q)^{-\alpha} < \varepsilon.
\end{aligned} \tag{4.19}
$$

It is sufficient to assume that $P = (x_1, x_2, \cdots, x_n, t)$ and $Q = (x_1 + h, x_2, \cdots, x_n, t)$ since the domain D has the property that there is a simple continuous curve α in D connecting B to B_T along which the t coordinate is nondecreasing. If P and Q are not related in this manner, we use the triangle inequality.

$D_x^3[\phi(v_m) - v] \to 0$ as $m \to \infty$ uniformly in D. Hence, there exists m_0 and h_0 (independent of x_1) such that if $m > m_0$ and $|h| \leqq d(P, Q)^\alpha < |h_0| \leqq 1$, then $|D_x^2[\phi(v_m) - v](P) - D_x^2[\phi(v_m) - v](Q)|\,|h|^{-1} < \varepsilon$. But $|I_m(P, Q) \leqq |D_x^2[\phi(v_m) - v] \cdot (P) - D_x^2[\phi(v_m) - v](Q)|\,|h|^{-1}$. Hence, $|I_m(P, Q)| < \varepsilon$. On the other hand, if $d(P, Q)^\alpha \geqq h_0$, then, since $D_x^2\phi(v_m) \to D_x^2\phi(v)$ uniformly in D,

$$|I_m(P, Q)| \leqq d(P, Q)^{-\alpha}[|D_x^2\phi(v_m)(P) - D_x^2 v(P)| + |D_x^2\phi(v_m)(Q) - D_x^2 v(Q)|] < \varepsilon \quad \text{if } m \geqq m_0,$$

where m_0 is independent of P, Q.

We have, therefore, shown that $\phi(v_m) \to v$ in $|\cdot|_{2+\alpha}$. This completes the proof that ϕ is a compact mapping.

By the Schauder fixed-point theorem, ϕ has a fixed point $u \in A$. u is a solution to (4.1). We have thus proved the following theorems. Theorem 4.3 is for the degenerate parabolic operator, whereas Theorem 4.4 deals with the uniformly parabolic case.

THEOREM 4.3 (THEOREM 4.4). *Assume that the conditions of Theorem* 3.1 *Theorem* (3.2) *and Theorem* 4.1 *hold, that the coefficients are Lipschitz continuous in* $|\cdot|_\alpha$ (*i.e.*, (4.6) *holds*), *that* $\psi(x, t, u, \nabla_x u)$ *and* $f(x, t, u, \nabla_x u)$ *are continuous with respect to u in* $|\cdot|_{2+\alpha}$. *If D has property* (E′), $\psi \in C_{2+\alpha}$ *and* $L\psi = f$ *on* ∂B, *then there exists a solution u of the nonlinear first initial-boundary value problem* (4.1) *and, furthermore,* $u \in C_{2+\alpha}$.

THEOREM 4.5. *If f and* ψ *are functions of x and t alone, then the solution obtained in Theorem* 4.4 *is unique.*

Proof. See [3].

If f and ψ are functions of u and $\nabla_x u$, the question of uniqueness is more complicated. The usual arguments requiring a maximum principle do not readily apply.

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TIME TO ATTAIN A GIVEN TEMPERATURE AT THE CENTER OF A SPHERE DUE TO RADIAL HEAT FLOW*

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Abstract. An explicit formula for the time required for the center of an initially uniformly heated sphere to attain a given temperature due to thermal diffusion is given in terms of the radius and a temperature parameter.

Let the interior of a sphere of radius R and thermal diffusivity a be initially at the temperature T_0 and let the surface be maintained from $t = 0$ at temperature T_s. Then it is well known [1, p. 233] that the temperature at the center is given by

$$(1) \qquad T_c - T_0 = (T_s - T_0)\left[1 + 2\sum_{n=1}^{\infty}(-1)^n \exp(-an^2\pi^2 t/R^2)\right],$$

For many purposes, however, one wants to know the time t that must elapse before this temperature is attained. Due to the existence of the transformed version of (1)

$$(2) \qquad T_c - T_0 = \left[\frac{R(T_s - T_0)}{(\pi a t)^{1/2}}\right]\sum_{n=0}^{\infty} \exp\left(-\frac{(2n+1)^2R^2}{4at}\right),$$

there is little difficulty in determining t in practice since one or the other of the quantities $a\pi^2t/R^2$ or $R^2/4at$ will be greater than unity and the series in (1) or (2) can be truncated after one or two terms. By dimensional analysis one also infers that t varies as R^2, although the functional dependence on T_c is much more difficult to assess.

The purpose of this note is to point out that relation (1) or (2) can be inverted in a way that makes these features explicit. The resulting formula, although it may not have great computational superiority over the pair of formulas (1) or (2), does not eliminate the need to decide between them, and appears not to have been noted before.

In (1), we introduce the dimensionless quantities

$$\beta = \left[\frac{T_c - T_0}{T_s - T_0}\right]^2, \qquad q = \exp\{\pi i(\pi i a t/R^2)\}$$

and note that we have

$$\beta = \theta_4^2(0, q),$$

But [2, p. 479]

$$\theta_4^2(0, q) = \frac{2}{\pi}(1 - k^2)^{1/2}\mathbf{K}(k),$$

$$q = \exp(-\pi\mathbf{K}(k')/\mathbf{K}(k)),$$

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where $\mathbf{K}(k)$ is the complete elliptic integral of the first kind of modulus k. Hence

$$t = \frac{R^2}{a\pi}\frac{\mathbf{K}(k')}{\mathbf{K}(k)} = \frac{2R^2}{\pi^2 a\beta}k'\mathbf{K}(k'), \tag{3}$$

where $k' = (1 - k^2)^{1/2}$ and

$$k'\mathbf{K}(k) = \pi\beta/2 \quad \text{or} \quad \mathbf{K}(ik/k') = \pi\beta/2. \tag{4}$$

Thus, for given T_c (or β), one need only solve (4) for $0 \leqq k \leqq 1$ and insert the resulting value into (3). We see, as expected, that t varies as R^2 and depends only on the square of the ratio $(T_c - T_0)/(T_s - T_0)$. The solution of (4) is facilitated by the series expansion for $\mathbf{K}$ and by the existence of tabulations of $\mathbf{K}$ for imaginary argument [3].

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A FIVE-PARAMETER FAMILY OF POSITIVE KERNELS FROM JACOBI POLYNOMIALS*

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Abstract. The Jacobi polynomials ${}_2F_1(-n, n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1; \alpha_1 + \beta_2; x/E)$ over the range $[0, E]$, $E > 0$, are used to construct a generally nonsymmetric kernel with five parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ whose real parts are taken to be positive. Under further conditions on $\beta_1, \beta_2, \beta_3$ this kernel is shown to be square-integrable, and even continuous over a wide region of the parameter-space. For real values of the parameters the kernel is shown to be essentially positive. Special limiting kernels are obtained by considering various limiting cases: $\alpha_1 \to 0, \alpha_2 \to 0, \alpha_2 = E \to \infty, \beta_3 = E \to \infty$. Some bilinear sums are obtained involving the Jacobi and Laguerre polynomials.

1. Introduction. The motivation of this work can be briefly described as a generalization of our previous work on Jacobi polynomials (Rahman [9]). The kernels we produced in that paper contained four parameters and appeared to be general enough, but in trying to derive some known bilinear sums (Erdélyi [6], Popov [8]) we found out that four parameters did not allow enough freedom to extend the results beyond a certain class. In particular, the family of kernels we obtained in [9] were easily symmetrizable and therefore the bilinear sums we obtained therefrom were characteristic of symmetric square-integrable kernels while Erdélyi's and Popov's bilinear sums had an essential asymmetry. Our search for such an asymmetric kernel resulted in this paper.

The method employed in this work is identical to the one used in [9], but we actually started out with six different parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4$, all with positive real parts, to construct a kernel $K(x, y)$ which, when multiplied by the Jacobi polynomial ${}_2F_1(-n, n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1; \alpha_1 + \beta_2; y)$ and integrated over y from 0 to 1, produces another Jacobi polynomial. We found that this happens if and only if $\beta_4 = \beta_2 + \beta_3 - \beta_1$. Of course, this relation is a reflection of the manner in which the parameters are introduced into the problem. It turns out that in the special case $\beta_1 = \beta_2$ the kernel reduces to the one derived in [9] and obtained previously from stochastic considerations by Cooper [5]. (The references related to this problem will be found in [9].) We therefore assume in this paper that β_1 and β_2 are different. We have found that $K(x, y)$ is essentially nonsymmetric when $\beta_1 \neq \beta_2$ and that its effect on ${}_2F_1(-n, n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1; \alpha_1 + \beta_2; y)$ is a multiple of ${}_2F_1(-n, n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1; \alpha_1 + \beta_1; x)$. This is probably the most basic result of this paper, and, to arrive at this result, we had to make use of some beautiful transformation properties of Saalschützian ${}_3F_2(1)$ and ${}_4F_3(1)$ series described in Bailey [3] and Slater [10]. We have given a detailed derivation of this result in the Appendix even though we proved almost an identical theorem in [9].

By introducing a sixth parameter $E(>0)$ through the transformations $x \to x/E$ and $y \to y/E$, and then passing to the limits $\alpha_2 = E \to \infty$ or $\beta_3 = E \to \infty$ we managed to obtain corresponding results for Laguerre polynomials. We also

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succeeded in deriving some other kernels by considering other limits like $\alpha_1 \to 0$, $\alpha_2 \to 0$.

In § 4 we give a rather detailed analysis of the square-integrability and continuity of the kernels in order that the range of validity of the bilinear sums we write down in § 5 can be justified.

2. Construction of the basic kernel. We start the process of constructing the basic kernel by multiplying the Jacobi polynomial ${}_2F_1(-n, n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1; \alpha_1 + \beta_2; y)$ by $(y - z_1)^{\beta_2 - 1}(z_2 - y)^{\beta_3 - 1}, 0 \leqq z_1 < z_2 \leqq 1$, and integrating the product with respect to y from z_1 to z_2. An elementary integration following a series expansion of the hypergeometric function, a change of variable $u = z_1(1 - y) + yz_2$ and a binomial expansion of u^r yields

$$\int_{z_1}^{z_2} dy(y - z_1)^{\beta_2 - 1}(z_2 - y)^{\beta_3 - 1}\,{}_2F_1(-n, n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1; \alpha_1 + \beta_2; y)$$

$$(2.1) \qquad = (z_2 - z_1)^{\beta_2 + \beta_3 - 1} \sum_{r=0}^{n} \frac{(-n)_r(n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1)_r}{(\alpha_1 + \beta_2)_r r!} \cdot \sum_{m=0}^{r} B(m + \beta_2, r - m + \beta_3)\binom{r}{m} z_1^{r-m} z_2^m,$$

where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ is the beta function.

We now multiply (2.1) by $(z_2 - z_1)^{1 - \beta_2 - \beta_3}(1 - z_2)^{\alpha_2 - 1}(z_2 - x)^{\beta_4 - 1} z_1^{\alpha_1 - 1} \cdot (x - z_1)^{\beta_1 - 1}$, $\operatorname{Re} \beta_4 > 0$, integrate over z_1 from 0 to x and over z_2 from x to 1. Using the transformations

$$(2.2) \qquad z_2 = v + x(1 - v), \qquad z_1 = xw,$$

followed by another binomial expansion of z_2^m, we obtain

$$(2.3) \qquad \begin{aligned} &\int_0^x dz_1\, z_1^{\alpha_1 - 1}(x - z_1)^{\beta_1 - 1} \int_x^1 dz_2 \frac{(1 - z_2)^{\alpha_2 - 1}(z_2 - x)^{\beta_4 - 1}}{(z_2 - z_1)^{\beta_2 + \beta_3 - 1}} \\ &\quad \cdot \int_{z_1}^{z_2} dy(y - z_1)^{\beta_2 - 1}(z_2 - y)^{\beta_3 - 1} \\ &\quad \cdot {}_2F_1(-n, n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1; \alpha_1 + \beta_2; y) \\ &= x^{\alpha_1 + \beta_1 - 1}(1 - x)^{\alpha_2 + \beta_4 - 1} \sum_{r=0}^{n} \frac{(-n)_r(n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1)_r}{(\alpha_1 + \beta_2)_r r!} \\ &\quad \cdot \sum_{m=0}^{r} \binom{r}{m} B(m + \beta_2, r - m + \beta_3) \\ &\quad \cdot \sum_{k=0}^{m} \binom{m}{k} B(k + \beta_4, m - k + \alpha_2) B(r - m + \alpha_1, \beta_1) x^{r-k}. \end{aligned}$$

Apart from the factor $x^{\alpha_1 + \beta_1 - 1}(1 - x)^{\alpha_2 + \beta_4 - 1}$ the expression on the right-hand side of (2.3) is obviously a polynomial of degree n in x. What we shall now try to do is to see if under some restrictions on the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4$

this polynomial can be expressed as a multiple of a Jacobi polynomial of degree n. Clearly we can write this polynomial in the form

$$M_n(x) = \sum_{p=0}^{n} a_{n,p} x^p, \tag{2.4}$$

where

$$a_{n,p} = \sum_{r=p}^{n} \frac{(-n)_r(n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1)_r}{(\alpha_1 + \beta_2)^r r!} \sum_{m=r-p}^{r} \binom{r}{m} B(m + \beta_2, r - m + \beta_3)$$

$$\cdot B(r - m + \alpha_1, \beta_1)\binom{m}{r-p} B(r - p + \beta_4, m - r + p + \alpha_2). \tag{2.5}$$

Simplifying this double sum and using some of the well-known properties of the Pochhammer symbols (see, for example, Slater [10, pp. 243]), we obtain

$$\begin{aligned} a_{n,p} &= \frac{(-n)_p(n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1)_p}{(\alpha_1 + \beta_2)_p(1)_p} \cdot \frac{p!}{(\beta_2 + \beta_3)_p} \\ &\quad \cdot B(\alpha_1, \beta_1)B(\beta_2, \beta_3)B(\beta_4, \alpha_2) \\ &\quad \cdot \sum_{l=0}^{n-p} \frac{(-n + p)_l(n + p + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1)_l(\beta_4)_l}{(\alpha_1 + \beta_2 + p)_l(\beta_2 + \beta_3 + p)_l l!} \\ &\quad \cdot \sum_{k=0}^{p} \frac{(\beta_2)_{k+l}(\beta_3)_{p-k}(\alpha_1)_{p-k}(\alpha_2)_k}{(\alpha_1 + \beta_1)_{p-k}(\alpha_2 + \beta_4)_{k+l}(p - k)!k!} \\ &= C\frac{(-n)_p(n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1)_p}{(\alpha_1 + \beta_1)_p p!} S(n; p), \end{aligned} \tag{2.6}$$

where

$$C = B(\alpha_1, \beta_1)B(\beta_2, \beta_3)B(\beta_4, \alpha_2) \tag{2.7}$$

and

$$\begin{aligned} S(n; p) &= \frac{p!(\alpha_1 + \beta_1)_p}{(\alpha_1 + \beta_2)_p(\beta_2 + \beta_3)_p} \\ &\quad \cdot \sum_{l=0}^{n-p} \frac{(-n + p)_l(n + p + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1)_l(\beta_4)_l}{(\alpha_1 + \beta_2 + p)_l(\beta_2 + \beta_3 + p)_l l!} \\ &\quad \cdot \sum_{k=0}^{p} \frac{(\beta_2)_{k+l}(\beta_3)_{p-k}(\alpha_1)_{p-k}(\alpha_2)_k}{(\alpha_1 + \beta_1)_{p-k}(\alpha_2 + \beta_4)_{k+l}(p - k)!k!}. \end{aligned} \tag{2.8}$$

Note that for $p = 0$,

$$\begin{aligned} S(n; 0) &= \sum_{l=0}^{n} \frac{(-n)_l(n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1)_l(\beta_2)_l(\beta_4)_l}{(\alpha_1 + \beta_2)_l(\beta_2 + \beta_3)_l(\alpha_2 + \beta_4)_l l!} \\ &= {}_4F_3\left[\begin{matrix} -n, n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1, \beta_2, \beta_4 \\ \alpha_1 + \beta_2, \beta_2 + \beta_3, \alpha_2 + \beta_4 \end{matrix}; 1\right], \end{aligned} \tag{2.9}$$

where ${}_4F_3$ is a generalized hypergeometric function defined by

$$(2.10)\qquad {}_4F_3\left[\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix} ; x\right] = \sum_{l=0}^{\infty} \frac{(a_1)_l(a_2)_l(a_3)_l(a_4)_l}{(b_1)_l(b_2)_l(b_3)_l l!} x^l.$$

(See, for example, Bailey [3].)

From the formidable look of the double sum in (2.8) it is not at all obvious that $S(n; p)$ is independent of p. In fact, it is not, unless the β-parameters are related by the equation

$$(2.11)\qquad \beta_4 = \beta_2 + \beta_3 - \beta_1.$$

We prove in the Appendix that when (2.11) is satisfied $S(n; p)$ is indeed independent of p and equals

$$(2.12)\qquad \lambda_n = {}_4F_3\left[\begin{matrix} -n, n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1, \beta_2, \beta_2 + \beta_3 - \beta_1 \\ \alpha_1 + \beta_2, \beta_2 + \beta_3, \alpha_2 + \beta_2 + \beta_3 - \beta_1 \end{matrix} ; 1\right].$$

Assuming (2.11) to be true, we then have

$$(2.13)\qquad M_n(x) = C\lambda_n\, {}_2F_1(-n, n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1; \alpha_1 + \beta_1 : x),$$

where in the expression (2.7) for C we have to replace β_4 by $\beta_2 + \beta_3 - \beta_1$.

The integral operations on the left-hand side of (2.3) can be seen as equivalent to the double integral

$$(2.14)\qquad \begin{aligned} &\int_0^1 dz_1 z_1^{\alpha_1 - 1}(x - z_1)^{\beta_1 - 1} \int_0^1 dz_2 \frac{(1 - z_2)^{\alpha_2 - 1}(z_2 - x)^{\beta_2 + \beta_3 - \beta_1 - 1}}{(z_2 - z_1)^{\beta_2 + \beta_3 - 1}} \\ &\qquad \cdot (y - z_1)^{\beta_2 - 1}(z_2 - y)^{\beta_3 - 1} H(x - z_1)H(y - z_1)H(z_2 - x)H(z_2 - y) \\ &= \int_0^{\min(x,y)} dt\, t^{\alpha_1 - 1}(x - t)^{\beta_1 - 1}(y - t)^{\beta_2 - 1} \\ &\qquad \cdot \int_{\max(x,y)}^1 dz \frac{(1 - z)^{\alpha_2 - 1}(z - x)^{\beta_2 + \beta_3 - \beta_1 - 1}(z - y)^{\beta_3 - 1}}{(z - t)^{\beta_2 + \beta_3 - 1}}, \end{aligned}$$

where $H(x)$ is Heaviside unit function.

Hence equation (2.3) can be put in the compact form

$$(2.15)\qquad \begin{aligned} &\int_0^1 K(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2)\, {}_2F_1(-n, n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1; \alpha_1 + \beta_2; y)\, dy \\ &\qquad = \lambda_n\, {}_2F_1(-n, n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1; \alpha_1 + \beta_1 : x), \end{aligned}$$

where

$$(2.16)\qquad \begin{aligned} K(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2) &= C^{-1} x^{-(\alpha_1 + \beta_1 - 1)}(1 - x)^{-(\alpha_2 + \beta_2 + \beta_3 - \beta_1 - 1)} \\ &\qquad \cdot \int_0^{\min(x,y)} dt\, t^{\alpha_1 - 1}(x - t)^{\beta_1 - 1}(y - t)^{\beta_2 - 1} \\ &\qquad \cdot \int_{\max(x,y)}^1 dz \frac{(1 - z^{\alpha_2 - 1}(z - x)^{\beta_2 + \beta_3 - \beta_1 - 1}(z - y)^{\beta_3 - 1}}{(z - t)^{\beta_2 + \beta_3 - 1}} \end{aligned}$$

is the basic kernel we sought for.

We may introduce a sixth parameter E, real and positive, by using the transformation

$$x = u/E, \qquad y = v/E.$$

Equation (2.16) then transforms to

$$\tag{2.17} \begin{aligned} &\int_0^E K_E(u,v;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2) \\ &\qquad \cdot {}_2F_1(-n, n+\alpha_1+\alpha_2+\beta_2+\beta_3-1;\alpha_1+\beta_2;v/E)\,dv \\ &\qquad = \lambda_n\, {}_2F_1(-n, n+\alpha_1+\alpha_2+\beta_2+\beta_3-1;\alpha_1+\beta_1;u/E), \end{aligned}$$

with

$$\tag{2.18} \begin{aligned} K_E(u,v;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2) &= C^{-1}u^{-(\alpha_1+\beta_1-1)}(E-u)^{-(\alpha_2+\beta_2+\beta_3-\beta_1-1)} \\ &\quad \cdot \int_0^{\min(u,v)} dt\, t^{\alpha_1-1}(u-t)^{\beta_1-1}(v-t)^{\beta_2-1} \\ &\quad \cdot \int_{\max(u,v)}^E dz\, \frac{(E-z)^{\alpha_2-1}(z-u)^{\beta_2+\beta_3-\beta_1-1}(z-v)^{\beta_3-1}}{(z-t)^{\beta_2+\beta_3-1}}. \end{aligned}$$

The kernels (2.16) and (2.18) are more general than the ones we obtained in [9]. In fact, the kernels of [9] correspond to the case $\beta_1 = \beta_2$. In this paper, therefore, we shall assume that $\beta_1 \neq \beta_2$.

Let us denote the weight functions

$$\tag{2.19} \begin{aligned} w_1(x) &= x^{\alpha_1+\beta_2-1}(1-x)^{\alpha_2+\beta_3-1}, \\ w_2(x) &= x^{\alpha_1+\beta_1-1}(1-x)^{\alpha_2+\beta_2+\beta_3-\beta_1-1}. \end{aligned}$$

Then the systems of functions

$$\tag{2.20} \begin{aligned} f_n(x) &= N_1\sqrt{w_1(x)}\, {}_2F_1(-n, n+\alpha_1+\alpha_2+\beta_2+\beta_3-1;\alpha_1+\beta_2;x), \\ g_n(x) &= N_2\sqrt{w_2(x)}\, {}_2F_1(-n, n+\alpha_1+\alpha_2+\beta_2+\beta_3-1;\alpha_1+\beta_1;x), \end{aligned}$$

define two complete orthonormal systems on [0, 1], where N_1 and N_2 are the respective normalization constants defined by

$$\tag{2.21} \begin{aligned} N_1^2 &= \frac{(2n+\alpha_1+\alpha_2+\beta_2+\beta_3-1)\Gamma(\alpha_1+\beta_2+n)\Gamma(n+\alpha_1+\alpha_2+\beta_2+\beta_3-1)}{n!\,\Gamma^2(\alpha_1+\beta_2)\Gamma(n+\alpha_2+\beta_3)}, \\ N_2^2 &= \frac{(2n+\alpha_1+\alpha_2+\beta_2+\beta_3-1)\Gamma(\alpha_1+\beta_1+n)\Gamma(n+\alpha_1+\alpha_2+\beta_2+\beta_3-1)}{n!\,\Gamma^2(\alpha_1+\beta_1)\Gamma(n+\alpha_2+\beta_2+\beta_3-\beta_1)}. \end{aligned}$$

It follows from (2.15) and (2.16) that

$$\tag{2.22} \int_0^1 G(x,y;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2)f_n(y)\,dy = \mu_n g_n(x),$$

where

$$(2.23)\quad G(x,y;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2) = \left[\frac{w_2(x)}{w_1(y)}\right]^{1/2} K(x,y;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2),$$

and

$$(2.24)\quad \mu_n = \frac{N_1}{N_2}\lambda_n.$$

It is obvious that G is symmetric in x and y if and only if $\beta_1 = \beta_2$ in which case $f_n(x) = g_n(x)$ become the eigenfunctions of G with eigenvalues λ_n. However, for $\beta_1 \neq \beta_2$, G is not symmetric and μ_n is not its eigenvalue for any $n = 0, 1, 2, \cdots$. In fact, G may not have any eigenvalue or eigenfunction at all.

We shall now see what happens when we multiply $g_n(y)$ by $G(y,x;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2)$ and integrate over y. Interchanging x and y in (2.14) amounts to interchanging β_1 with β_2 and $\beta_2 + \beta_3 - \beta_1$ with β_3. Hence if we carry out these interchanges of parameters in (2.3) we obtain

$$\int_0^1 G(y,x;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2)g_n(y)\,dy$$

$$(2.25)\quad = \frac{B(\alpha_1,\beta_2)B(\beta_1,\beta_2+\beta_3-\beta_1)B(\beta_3,\alpha_2)}{B(\alpha_1,\beta_1)B(\beta_2,\beta_3)B(\beta_2+\beta_3-\beta_1,\alpha_2)} \cdot N_2\lambda_n'\sqrt{w_1(x)}\,{}_2F_1(-n, n+\alpha_1+\alpha_2+\beta_2+\beta_3-1;\alpha_1+\beta_2;x),$$

where

$$(2.26)\quad \lambda_n' = {}_4F_3\left[\begin{matrix}-n, n+\alpha_1+\alpha_2+\beta_2+\beta_3-1, \beta_1, \beta_3 \\ \alpha_1+\beta_1, \beta_2+\beta_3, \alpha_2+\beta_3\end{matrix};1\right].$$

For $\beta_1 \neq \beta_2$, $\lambda_n' \neq \lambda_n$, however, since this ${}_4F_3$ is Saalschützian (see Appendix for details) a transformation exists for this series and we simply have

$$(2.27)\quad \begin{aligned}\lambda_n' &= {}_4F_3\left[\begin{matrix}\beta_1, \beta_3, n+\alpha_1+\alpha_2+\beta_2+\beta_3-1, -n \\ \beta_2+\beta_3, \alpha_1+\beta_1, \alpha_2+\beta_3\end{matrix};1\right] \\ &= \frac{(\alpha_1+\beta_2)_n(\alpha_2+\beta_2+\beta_3-\beta_1)_n}{(\alpha_1+\beta_1)_n(\alpha_2+\beta_3)_n} \cdot {}_4F_3\left[\begin{matrix}\beta_2+\beta_3-\beta_1, \beta_2, n+\alpha_1+\alpha_2+\beta_2+\beta_3-1, -n \\ \alpha_1+\beta_2, \alpha_2+\beta_2+\beta_3-\beta_1, \beta_2+\beta_3\end{matrix};1\right] \\ &= \frac{(\alpha_1+\beta_2)_n(\alpha_2+\beta_2+\beta_3-\beta_1)_n}{(\alpha_1+\beta_1)_n(\alpha_2+\beta_3)_n}\lambda_n.\end{aligned}$$

(See Bailey [3].) Equation (2.25) then simplifies to

$$(2.28)\quad \int_0^1 G(y,x;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2)g_n(y)\,dy = \mu_n f_n(x).$$

Hence, if we define the so-called left-iterated kernel (Tricomi [11])

$$G_L(x, y) = \int_0^1 G(z, x)G(z, y)\, dz, \tag{2.29}$$

then

$$\int_0^1 G_L(x, y) f_n(y)\, dy = \mu_n^2 f_n(x). \tag{2.30}$$

Similarly, for the right-iterated kernel

$$G_R(x, y) = \int_0^1 G(x, z)G(y, z)\, dz, \tag{2.31}$$

we get

$$\int_0^1 G_R(x, y) g_n(y)\, dy = \mu_n^2 g_n(x). \tag{2.32}$$

The orthonormal systems $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(x)\}_{n=0}^{\infty}$ are therefore the eigenfunctions of the kernels $G_L(x, y)$ and $G_R(x, y)$ respectively, which are obviously symmetric kernels, with the same eigenvalue μ_n^2 for each n. Note that we have abbreviated the iterated kernels by dropping the parameters from their arguments.

Before closing this section it may be worth pointing out how the above results can be expressed in terms of the standard Jacobi polynomials $P_n^{(a,b)}(x)$ which are related to the hypergeometric functions we have used here by the following equation:

$$P_n^{(a,b)}(x) = \frac{(a+1)_n}{n!}\, {}_2F_1\left(-n, n+a+b+1; a+1; \frac{1-x}{2}\right), \quad -1 \leqq x \leqq 1. \tag{2.33}$$

In equation (2.22) we make the transformations

$$x = \frac{1-\xi}{2}, \qquad y = \frac{1-\eta}{2}. \tag{2.34}$$

Then, after some manipulations, (2.22) reduces to

$$\int_{-1}^{1} d\eta H(\xi, \eta; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2)\phi_n(\eta) = \mu_n \psi_n(\xi), \tag{2.35}$$

where

$$
\begin{aligned}
&H(\xi,\eta;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2)\\
&= \frac{C^{-1}}{\sqrt{(1-\eta)^{\alpha_1+\beta_2-1}(1+\eta)^{\alpha_2+\beta_3-1}(1-\xi)^{\alpha_1+\beta_1-1}(1+\xi)^{\alpha_2+\beta_2+\beta_3-\beta_1-1}}}\\
&\quad\cdot\int_{-1}^{\min(\xi,\eta)} dt(1+t)^{\alpha_2-1}(\xi-t)^{\beta_2+\beta_3-\beta_1-1}(\eta-t)^{\beta_3-1}\\
&\quad\cdot\int_{\max(\xi,\eta)}^{1} dz\frac{(1-z)^{\alpha_1-1}(z-\xi)^{\beta_1-1}(z-\eta)^{\beta_2-1}}{(z-t)^{\beta_2+\beta_3-1}},
\end{aligned}
\tag{2.36}
$$

and $\{\phi_n(\xi)\}_{n=0}^{\infty}$, $\{\psi_n(\xi)\}_{n=0}^{\infty}$ are two orthonormal systems on $[-1,1]$ defined by

$$
\begin{aligned}
\phi_n(\xi) = &\left[\frac{n!\Gamma(n+\alpha_1+\alpha_2+\beta_2+\beta_3-1)(2n+\alpha_1+\alpha_2+\beta_2+\beta_3-1)}{2^{\alpha_1+\alpha_2+\beta_2+\beta_3-1}\Gamma(n+\alpha_1+\beta_2)\Gamma(n+\alpha_2+\beta_3)}\right.\\
&\left.\cdot(1-\xi)^{\alpha_1+\beta_2-1}(1+\xi)^{\alpha_2+\beta_3-1}\right]^{1/2}\\
&\cdot P_n^{(\alpha_1+\beta_2-1,\,\alpha_2+\beta_3-1)}(\xi),
\end{aligned}
\tag{2.37}
$$

$$
\begin{aligned}
\psi_n(\xi) = &\left[\frac{n!\Gamma(n+\alpha_1+\alpha_2+\beta_2+\beta_3-1)(2n+\alpha_1+\alpha_2+\beta_2+\beta_3-1)}{2^{\alpha_1+\alpha_2+\beta_2+\beta_3-1}\Gamma(n+\alpha_1+\beta_1)\Gamma(n+\alpha_2+\beta_2+\beta_3-\beta_1)}\right.\\
&\left.\cdot(1-\xi)^{\alpha_1+\beta_1-1}(1+\xi)^{\alpha_2+\beta_2+\beta_3-\beta_1-1}\right]^{1/2}\\
&\cdot P_n^{(\alpha_1+\beta_1-1,\,\alpha_2+\beta_2+\beta_3-\beta_1-1)}(\xi).
\end{aligned}
\tag{2.38}
$$

Similarly (2.28) reduces to

$$
\int_{-1}^{1} d\eta H(\eta,\xi;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2)\psi_n(\eta) = \mu_n\phi_n(\xi),
\tag{2.39}
$$

and consequently, we have

$$
\int_{-1}^{1} H_L(\xi,\eta)\phi_n(\eta)\,d\eta = \mu_n^2\phi_n(\xi),
\tag{2.40}
$$

$$
\int_{-1}^{1} H_R(\xi,\eta)\psi_n(\eta)\,d\eta = \mu_n^2\psi_n(\xi),
\tag{2.41}
$$

where

$$
H_L(\xi,\eta) = \int_{-1}^{1} H(z,\xi)H(z,\eta)\,dz,
$$

$$
H_R(\xi,\eta) = \int_{-1}^{1} H(\xi,z)H(\eta,z)\,dz
$$

are the left- and right-iterated symmetric kernels, respectively.

3. The limiting kernels. Apart from many interesting properties of the basic kernels $K(x, y)$, $K_E(x, y)$, $G(x, y)$ and $G_E(x, y)$ which we shall discuss in the next section, they lend themselves to a number of limiting procedures resulting in a variety of limiting or degenerate kernels some of which have been known in the literature. We shall present here a few of these limiting cases.

Case I. $\alpha_1 \to 0$, $\operatorname{Re}[\alpha_2, \beta_1, \beta_2, \beta_3] > 0$ with $\operatorname{Re}(\beta_2 + \beta_3) > \operatorname{Re}\beta_1$.

Since the measure $\alpha_1 t^{\alpha_1 - 1}\,dt$ behaves like a delta function at $t = 0$ as $\alpha_1 \to 0$, we have

$$\lim_{\alpha_1 \to 0} \frac{1}{\Gamma(\alpha_1)} \int_0^{\min(x,y)} dt\, t^{\alpha_1 - 1} F(t) = F(0).$$

It follows from (2.16) that

$$\begin{aligned}
&K(x, y; 0, \beta_1, \beta_2, \beta_3, \alpha_2) \\
&\quad = \lim_{\alpha_1 \to 0} K(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2) \\
&\quad = \frac{x^{-(\beta_1 - 1)}(1 - x)^{-(\alpha_2 + \beta_2 + \beta_3 - \beta_1 - 1)}}{B(\beta_2, \beta_3)B(\beta_2 + \beta_3 - \beta_1, \alpha_2)} x^{\beta_1 - 1} y^{\beta_2 - 1} \\
&\qquad \cdot \int_{\max(x,y)}^1 dz \frac{(1 - z)^{\alpha_2 - 1}(z - x)^{\beta_2 + \beta_3 - \beta_1 - 1}(z - y)^{\beta_3 - 1}}{z^{\beta_2 + \beta_3 - 1}} \\
&\quad = \frac{(1 - x)^{-(\alpha_2 + \beta_2 + \beta_3 - \beta_1 - 1)} y^{\beta_2 - 1}}{B(\beta_2, \beta_3)B(\beta_2 + \beta_3 - \beta_1, \alpha_2)} \\
&\qquad \cdot \int_{\max(x,y)}^1 dz \frac{(1 - z)^{\alpha_2 - 1}(z - x)^{\beta_2 + \beta_3 - \beta_1 - 1}(z - y)^{\beta_3 - 1}}{z^{\beta_2 + \beta_3 - 1}}.
\end{aligned} \tag{3.1}$$

Correspondingly,

$$\begin{aligned}
&G(x, y; 0, \beta_1, \beta_2, \beta_3, \alpha_2) \\
&= \left[\frac{x^{\beta_1 - 1} y^{\beta_2 - 1}}{(1 - x)^{\alpha_2 + \beta_2 + \beta_3 - \beta_1 - 1}(1 - y)^{\alpha_2 + \beta_3 - 1}}\right]^{1/2} [B(\beta_2, \beta_3)B(\beta_2 + \beta_3 - \beta_1, \alpha_2)]^{-1} \\
&\quad \cdot \int_{\max(x,y)}^1 dz\, z^{-(\beta_2 + \beta_3 - 1)}(1 - z)^{\alpha_2 - 1}(z - x)^{\beta_2 + \beta_3 - \beta_1 - 1}(z - y)^{\beta_3 - 1}.
\end{aligned} \tag{3.2}$$

The μ_n's also approach simpler expressions. For,

$$\begin{aligned}
&\lim_{\alpha_1 \to 0} {}_4F_3\left[\begin{matrix} -n, n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1, \beta_2, \beta_2 + \beta_3 - \beta_1 \\ \alpha_1 + \beta_2, \beta_2 + \beta_3, \alpha_2 + \beta_2 + \beta_3 - \beta_1 \end{matrix}; 1\right] \\
&\quad = {}_3F_2\left[\begin{matrix} -n, n + \alpha_2 + \beta_2 + \beta_3 - 1, \beta_2 + \beta_3 - \beta_1 \\ \beta_2 + \beta_3, \alpha_2 + \beta_2 + \beta_3 - \beta_1 \end{matrix}; 1\right] \\
&\quad = \frac{(\beta_1)_n(\alpha_2)_n}{(\beta_2 + \beta_3)_n(\alpha_2 + \beta_2 + \beta_3 - \beta_1)_n},
\end{aligned} \tag{3.3}$$

since the ${}_3F_2(1)$ series is Saalschützian (see Appendix). By using the limiting values of N_1 and N_2 from (2.21) we obtain

$$\lim_{\alpha_1 \to 0} \mu_n = \frac{(\alpha_2)_n}{(\beta_2 + \beta_3)_n} \tag{3.4}$$
$$\cdot \left[\frac{(\beta_1)_n(\beta_2)_n}{(\alpha_2 + \beta_2 + \beta_3 - \beta_1)_n(\alpha_2 + \beta_3)_n} \cdot \frac{\Gamma(\beta_1)\Gamma(\alpha_2 + \beta_2 + \beta_3 - \beta_1)}{\Gamma(\beta_2)\Gamma(\alpha_2 + \beta_3)} \right]^{1/2}.$$

The limits of $f_n(x)$ and $g_n(x)$ obviously exist and are determined easily from (2.19), (2.20) and (2.21).

Case II. $\alpha_2 \to 0$, $\operatorname{Re}[\alpha_1, \beta_1, \beta_2, \beta_3 >] 0$, $\operatorname{Re}(\beta_2 + \beta_3) > \operatorname{Re}\beta_1$.

As in the previous case the measure $\alpha_2(1 - z)^{\alpha_2 - 1}\, dz$ behaves like $\delta(1 - z)\, dz$ as $\alpha_2 \to 0$. Hence

$$\begin{aligned} &K(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, 0) \\ &\quad = \lim_{\alpha_2 \to 0} K(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2) \\ &\quad = \frac{x^{-(\alpha_1+\beta_1-1)}(1 - y)^{\beta_3 - 1}}{B(\alpha_1, \beta_1)B(\beta_2, \beta_3)} \int_0^{\min(x,y)} dt \frac{t^{\alpha_1 - 1}(x - t)^{\beta_1 - 1}(y - t)^{\beta_2 - 1}}{(1 - t)^{\beta_2 + \beta_3 - 1}}. \end{aligned} \tag{3.5}$$

Correspondingly,

$$\begin{aligned} G(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, 0) = &\left[\frac{(1 - x)^{\beta_2 + \beta_3 - \beta_1 - 1}(1 - y)^{\beta_3 - 1}}{x^{\alpha_1 + \beta_1 - 1} y^{\alpha_1 + \beta_2 - 1}} \right]^{1/2} \\ &\cdot [B(\alpha_1, \beta_1)B(\beta_2, \beta_3)]^{-1} \\ &\cdot \int_0^{\min(x,y)} dt \frac{t^{\alpha_1 - 1}(x - t)^{\beta_1 - 1}(y - t)^{\beta_2 - 1}}{(1 - t)^{\beta_2 + \beta_3 - 1}}. \end{aligned} \tag{3.6}$$

In this case

$$\lambda_n \to \frac{(\alpha_1)_n(\beta_3)_n}{(\alpha_1 + \beta_2)_n(\beta_2 + \beta_3)_n} \tag{3.7}$$

and

$$\mu_n \to \frac{(\alpha_1)_n}{(\beta_2 + \beta_3)_n} \left[\frac{(\beta_2 + \beta_3 - \beta_1)_n(\beta_3)_n}{(\alpha_1 + \beta_2)_n(\alpha_1 + \beta_1)_n} \cdot \frac{\Gamma(\alpha_1 + \beta_1)\Gamma(\beta_2 + \beta_3 - \beta_1)}{\Gamma(\beta_3)\Gamma(\alpha_1 + \beta_2)} \right]^{1/2}. \tag{3.8}$$

If we set

$$\beta_2 + \beta_3 = 1, \quad \alpha_1 = \nu, \quad \beta_1 = 1 - \alpha, \quad \beta_2 = \alpha - \nu, \quad \beta_3 = 1 + \nu - \alpha$$

so that

$$0 < \operatorname{Re}\nu < 1, \quad 0 < \operatorname{Re}\alpha < 1, \quad 0 < \operatorname{Re}(1 + \nu - \alpha) < 1,$$

then the kernel K reduces to the Popov kernel

$$\frac{x^{\alpha - \nu}(1 - y)^{\nu - \alpha}}{B(\nu, 1 - \alpha)B(\alpha - \nu, 1 + \nu - \alpha)} \int_0^{\min(x,y)} dt \frac{t^{\nu - 1}}{(x - t)^{\alpha - 1}(y - t)^{\nu - \alpha + 1}}.$$

So Popov's equation (1) [8] is a special case of (2.16) in this limiting situation (note a misprint in (4) of [8]).

Case III. $\alpha_2 = E \to \infty$, $\mathrm{Re}\,[\alpha_1, \beta_1, \beta_2, \beta_3] > 0$, $\mathrm{Re}\,(\beta_2 + \beta_3) > \mathrm{Re}\,\beta_1$. As $E \to \infty$,

$$B(\beta_2 + \beta_3 - \beta_1, E) \sim \frac{\Gamma(\beta_2 + \beta_3 - \beta_1)}{E^{\beta_2 + \beta_3 - \beta_1}} \quad \text{and} \quad (1 - z/E)^E \sim e^{-z}.$$

Using these limits we obtain

$$\begin{aligned} &K_\infty(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \infty) \\ &\quad = [B(\alpha_1, \beta_1)B(\beta_2, \beta_3)\Gamma(\beta_2 + \beta_3 - \beta_1)]^{-1} x^{-(\alpha_1 + \beta_1 - 1)} e^x \\ &\quad \cdot \int_0^{\min(x,y)} dt\, t^{\alpha_1 - 1}(x - t)^{\beta_1 - 1}(y - t)^{\beta_2 - 1} \\ &\quad \cdot \int_{\max(x,y)}^\infty dz \frac{e^{-z}(z - x)^{\beta_2 + \beta_3 - \beta_1 - 1}(z - y)^{\beta_3 - 1}}{(z - t)^{\beta_2 + \beta_3 - 1}}. \end{aligned} \tag{3.9}$$

Also

$$\begin{aligned} &\lim_{\alpha_2 \to 0} {}_4F_3\left[\begin{matrix} -n, n + \alpha_2 + \alpha_2 + \beta_2 + \beta_3 - 1, \beta_2, \beta_2 + \beta_3 - \beta_1 \\ \alpha_1 + \beta_2, \beta_2 + \beta_3, \alpha_2 + \beta_2 + \beta_3 - \beta_1 \end{matrix}; 1\right] \\ &\quad = {}_3F_2\left[\begin{matrix} -n, \beta_2, \beta_2 + \beta_3 - \beta_1 \\ \alpha_1 + \beta_2, \beta_2 + \beta_3 \end{matrix}; 1\right]. \end{aligned} \tag{3.10}$$

This ${}_3F_2(1)$ series is neither Saalschützian nor well-poised and consequently does not seem to be reducible any further.

In the same limit

$$\frac{N_1}{N_2} \sim \left[\frac{(\alpha_1 + \beta_2)_n \Gamma(\alpha_1 + \beta_1)}{(\alpha_1 + \beta_1)_n \Gamma(\alpha_1 + \beta_2)}\right]^{1/2} E^{(\beta_2 - \beta_1)/2} \tag{3.11}$$

and

$$\begin{aligned} &{}_2F_1(-n, n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1; \alpha_1 + \beta_2; x/E) \\ &\quad \to {}_1F_1(-n; \alpha_1 + \beta_2; x) = \frac{n!}{(\alpha_1 + \beta_2)_n} L_n^{(\alpha_1 + \beta_2 - 1)}(x), \end{aligned} \tag{3.12}$$

where $L_n^{(\alpha)}(x)$ is the associated Laguerre polynomial.

Hence the equation that corresponds to (2.22) in this limit reduces to

$$\int_0^\infty G_\infty(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \infty) f_n(y)\, dy = \mu_n g_n(x), \tag{3.13}$$

where

$$
\begin{aligned}
G_\infty(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \infty) \\
&= [B(\alpha_1, \beta_1)B(\beta_2, \beta_3)\Gamma(\beta_2 + \beta_3 - \beta_1)]^{-1}[x^{\alpha_1+\beta_1-1}y^{\alpha_1+\beta_2-1}]^{-1/2}\, e^{(x+y)/2} \\
&\quad \cdot \int_0^{\min(x,y)} dt\, t^{\alpha_1-1}(x-t)^{\beta_1-1}(y-t)^{\beta_2-1} \\
&\quad \cdot \int_{\max(x,y)}^{\infty} dz \frac{e^{-z}(z-x)^{\beta_2+\beta_3-\beta_1-1}(z-y)^{\beta_3-1}}{(z-t)^{\beta_2+\beta_3-1}}
\end{aligned}
\tag{3.14}
$$

and

$$
\begin{aligned}
f_n(y) &= \left[\frac{n!}{\Gamma(\alpha_1+\beta_2+n)}\right]^{1/2} e^{-y/2}\, y^{(\alpha_1+\beta_2-1)/2} L_n^{(\alpha_1+\beta_2-1)}(y), \\
g_n(x) &= \left[\frac{n!}{\Gamma(\alpha_1+\beta_1+n)}\right]^{1/2} e^{-x/2}\, x^{(\alpha_1+\beta_1-1)/2} L_n^{(\alpha_1+\beta_1-1)}(x)
\end{aligned}
\tag{3.15}
$$

are the orthonormal Laguerre systems on $(0, \infty)$, with

$$
\mu_n = \left[\frac{(\alpha_1+\beta_2)_n\Gamma(\alpha_1+\beta_1)}{(\alpha_1+\beta_1)_n\Gamma(\alpha_1+\beta_2)}\right]^{1/2} {}_3F_2\left[\begin{matrix}-n, \beta_2, \beta_2+\beta_3-\beta_1 \\ \alpha_1+\beta_2, \beta_2+\beta_3\end{matrix} : 1\right].
\tag{3.16}
$$

Case IV. $\alpha_2 \to 0$, $\beta_3 = E \to \infty$, $\mathrm{Re}\,[\alpha_1, \beta_1, \beta_2] > 0$.
From Case II one obtains

$$
\begin{aligned}
K_E(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, 0) \\
&= \frac{x^{-(\alpha_1+\beta_1-1)}(E-y)^{\beta_3-1}}{B(\alpha_1, \beta_1)B(\beta_2, \beta_3)} \int_0^{\min(x,y)} dt \frac{t^{\alpha_1-1}(x-t)^{\beta_1-1}(y-t)^{\beta_2-1}}{(E-t)^{\beta_2+\beta_3-1}}.
\end{aligned}
$$

Since

$$
\lim_{E\to\infty} \frac{\Gamma(\beta_2+E)(E-y)^{E-1}}{\Gamma(E)(E-t)^{\beta_2+E-1}} = e^{-y+t},
$$

we find

$$
\begin{aligned}
K_\infty(x, y; \alpha_1, \beta_1, \beta_2, \infty, 0) \\
&= \frac{x^{-(\alpha_1+\beta_1-1)}\, e^{-y}}{B(\alpha_1, \beta_1)\Gamma(\beta_2)} \int_0^{\min(x,y)} dt\, t^{\alpha_1-1} \cdot e^t (x-t)^{\beta_1-1}(y-t)^{\beta_2-1}.
\end{aligned}
\tag{3.17}
$$

Correspondingly,

$$
\begin{aligned}
G_\infty(x, y; \alpha_1, \beta_1, \beta_2, \infty, 0) &= [B(\alpha_1, \beta_1)\Gamma(\beta_2)]^{-1}[x^{\alpha_1+\beta_1-1}y^{\alpha_1+\beta_2-1}]^{-1/2}\, e^{-(x+y)/2} \\
&\quad \cdot \int_0^{\min(x,y)} dt\, t^{\alpha_1-1}\, e^t (x-t)^{\beta_1-1}(y-t)^{\beta_2-1}
\end{aligned}
\tag{3.18}
$$

with

$$\mu_n = \frac{\Gamma(\alpha_1 + \beta_1)(\alpha_1)_n}{\sqrt{\Gamma(\alpha_1 + \beta_1 + n)\Gamma(\alpha_1 + \beta_2 + n)}} \tag{3.19}$$

and $f_n(y)$, $g_n(x)$ given by (3.15).

Case V. $\alpha_1 \to 0$, $\alpha_2 = E \to \infty$, $\mathrm{Re}\,[\beta_1, \beta_2, \beta_3] > 0$, $\mathrm{Re}\,(\beta_2 + \beta_3) > \mathrm{Re}\,\beta_1$.

Using the result of Case I and going to the limit $\alpha_2 = E \to \infty$ one easily obtains

$$\begin{aligned} K_\infty(x, y; 0, \beta_1, \beta_2, \beta_3, \infty) &= \frac{y^{\beta_2 - 1}\, e^x}{B(\beta_2, \beta_3)\Gamma(\beta_2 + \beta_3 - \beta_1)} \\ &\quad \cdot \int_{\max(x,y)}^{\infty} dz \frac{e^{-z}(z - x)^{\beta_2 + \beta_3 - \beta_1 - 1}(z - y)^{\beta_3 - 1}}{z^{\beta_2 + \beta_3 - 1}}. \end{aligned} \tag{3.20}$$

Correspondingly,

$$\begin{aligned} G_\infty(x, y; 0, \beta_1, \beta_2, \beta_3, \infty) &= \frac{[x^{\beta_1 - 1} y^{\beta_2 - 1}]^{1/2}\, e^{(x+y)/2}}{B(\beta_2, \beta_3)\Gamma(\beta_2 + \beta_3 - \beta_1)} \\ &\quad \cdot \int_{\max(x,y)}^{\infty} dz \frac{e^{-z}(z - x)^{\beta_2 + \beta_3 - \beta_1 - 1}(z - y)^{\beta_3 - 1}}{z^{\beta_2 + \beta_3 - 1}}. \end{aligned} \tag{3.21}$$

The μ_n in this case becomes

$$\mu_n = \frac{[\Gamma(\beta_1 + n)\Gamma(\beta_2 + n)]^{1/2}}{\Gamma(\beta_2)(\beta_2 + \beta_3)_n} \tag{3.22}$$

while $f_n(y)$ and $g_n(x)$ are again given by (3.15) with α_1 replaced by 0.

4. Properties of the kernels. Let us assume, for the time being, that α_1, α_2, $\beta_1, \beta_2, \beta_3$ are all real and, of course, positive. Then it follows from (2.16) that

$$K(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2) \geqq 0 \tag{4.1}$$

for $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$.

Also, since $\lambda_0 = 1$, equation (2.15) gives

$$\int_0^1 K(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2)\, dy = 1. \tag{4.2}$$

These two properties would appear to lend a stochastic interpretation to K, but unless $\beta_1 = \beta_2$, K cannot be symmetrized by multiplying by a weight function and hence cannot be interpreted as a transition probability. However, the left- and right-iterates of K, namely K_L and K_R, corresponding to G_L and G_R of (2.29) and (2.32) respectively, and evidently defined by

$$(4.3)\qquad \begin{aligned} K_L &= \frac{1}{w_1(x)} \int_0^1 w_2(z) K(z, x) K(z, y)\, dz, \\ K_R &= \frac{1}{w_2(x)} \int_0^1 w_1(z) K(x, z) K(y, z)\, dz \end{aligned}$$

have the desired "detailed-balance" properties:

$$(4.4)\qquad \begin{aligned} w_1(x) K_L(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2) &= w_1(y) K_L(y, x; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2), \\ w_2(x) K_R(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2) &= w_2(y) K_R(y, x; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2). \end{aligned}$$

Besides, since

$$(4.5)\qquad \begin{aligned} \frac{N_1^2}{N_2^2} = {} & \frac{\Gamma(\alpha_1 + \beta_1)\Gamma(\alpha_2 + \beta_2 + \beta_3 - \beta_1)}{\Gamma(\alpha_1 + \beta_2)\Gamma(\alpha_2 + \beta_3)} \\ & \cdot \frac{(\alpha_1 + \beta_2)_n(\alpha_2 + \beta_2 + \beta_3 - \beta_1)_n}{(\alpha_1 + \beta_1)_n(\alpha_2 + \beta_3)_n}, \end{aligned}$$

equations (2.30) and (2.32) imply that the kernels

$$(4.6)\qquad \begin{aligned} K_L' &= \frac{\Gamma(\alpha_1 + \beta_2)\Gamma(\alpha_2 + \beta_3)}{\Gamma(\alpha_1 + \beta_1)\Gamma(\alpha_2 + \beta_2 + \beta_3 - \beta_1)} \cdot K_L, \\ K_R' &= \frac{\Gamma(\alpha_1 + \beta_2)\Gamma(\alpha_2 + \beta_3)}{\Gamma(\alpha_1 + \beta_1)\Gamma(\alpha_2 + \beta_2 + \beta_3 - \beta_1)} \cdot K_R \end{aligned}$$

both have the properties (4.1), (4.2) as well as (4.4). Under these conditions K_L' and K_R' can both be interpreted as stochastic kernels.

In the rest of this section we shall be looking into the questions of square-integrability and continuity of the kernels $G(x, y)$.

THEOREM 1. *If* $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3 > 0$, $\beta_2 + \beta_3 - \beta_1 > 0$, *then*

$$(4.7)\qquad \sum_{n=0}^{\infty} \mu_n < \infty$$

if and only if

$$(4.8)\qquad \beta_2 + \beta_3 > 1 + |\beta_1 - \beta_3|.$$

Proof. Since $S(n; p)$ in (2.8) is independent of p for $0 \leqq p \leqq n$, we have

$$\lambda_n = \frac{n!(\alpha_1 + \beta_1)_n}{(\alpha_1 + \beta_2)_n(\beta_2 + \beta_3)_n} \sum_{k=0}^{n} \frac{(\alpha_1)_{n-k}(\beta_2)_k(\beta_3)_{n-k}(\alpha_2)_k}{(\alpha_1 + \beta_1)_{n-k}(\alpha_2 + \beta_2 + \beta_3 - \beta_1)_k(n-k)!k!}.$$

Hence

$$\mu_n = \left[\frac{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2+\beta_3-\beta_1)}{\Gamma(\alpha_1+\beta_2)\Gamma(\alpha_2+\beta_3)}\right]^{1/2}$$

$$(4.9)\qquad \cdot\left[\frac{(\alpha_1+\beta_1)_n(\alpha_2+\beta_2+\beta_3-\beta_1)_n}{(\alpha_1+\beta_2)_n(\alpha_2+\beta_3)_n}\right]^{1/2}\frac{n!}{(\beta_2+\beta_3)_n}$$

$$\cdot\sum_{k=0}^{n}\frac{(\alpha_1)_{n-k}(\beta_2)_k(\beta_3)_{n-k}(\alpha_2)_k}{(\alpha_1+\beta_1)_{n-k}(\alpha_2+\beta_2+\beta_3-\beta_1)_k(n-k)!k!}.$$

Since all the terms on the right-hand side of (4.9) are positive, we have

$$(4.10)\qquad \mu_n > 0, \qquad n = 0, 1, \cdots.$$

In order to find the asymptotic behavior of μ_n for large n, let us consider the series $\sum_{n=0}^{\infty} C_n x^n$ where

$$(4.11)\qquad C_n = \sum_{k=0}^{n}\frac{(\alpha_1)_{n-k}(\beta_3)_{n-k}}{(\alpha_1+\beta_1)_{n-k}(n-k)!}\cdot\frac{(\alpha_2)_k(\beta_2)_k}{(\alpha_2+\beta_2+\beta_3-\beta_1)_k k!}.$$

The finite sum on the right-hand side is obviously a Cauchy product, and hence,

$$(4.12)\qquad \sum_{n=0}^{\infty} C_n x^n = \left[\sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\beta_3)_n}{(\alpha_1+\beta_1)_n n!}x^n\right]\left[\sum_{n=0}^{\infty}\frac{(\alpha_2)_n(\beta_2)_n}{(\alpha_2+\beta_2+\beta_3-\beta_1)_n n!}x^n\right]$$

$$= {}_2F_1(\alpha_1,\beta_3;\alpha_1+\beta_1;x)\,{}_2F_1(\alpha_2,\beta_2;\alpha_2+\beta_2+\beta_3-\beta_1:x)$$

whenever the series on both sides converge.

For $|x| < 1$ the hypergeometric functions are both finite but as $x \to 1-$ at least one of them diverges. If $\beta_1 \neq \beta_3$ one of them converges and the other diverges like

$$(1-x)^{-|\beta_1-\beta_3|} = \sum_{n=0}^{\infty}\frac{(|\beta_1-\beta_3|)_n}{n!}x^n.$$

Hence, for $\beta_1 \neq \beta_3$,

$$(4.13)\qquad C_n \sim \frac{(|\beta_1-\beta_3|)_n}{n!} \sim n^{|\beta_1-\beta_3|-1}.$$

If $\beta_1 = \beta_3$, both hypergeometric functions diverge like $\log(1-x)$ as $x \to 1-$. Therefore

$$(4.14)\qquad C_n \sim \sum_{k=1}^{n-1}\frac{1}{k(n-k)} \sim \frac{2\log n}{n}.$$

Now for large n,

$$\left[\frac{(\alpha_1+\beta_1)_n(\alpha_2+\beta_2+\beta_3-\beta_1)_n}{(\alpha_1+\beta_2)_n(\alpha_2+\beta_3)_n}\right]^{1/2}\frac{n!}{(\beta_2+\beta_3)_n} \sim \left[\frac{n^{\alpha_1+\alpha_2+\beta_2+\beta_3}}{n^{\alpha_1+\alpha_2+\beta_2+\beta_3}}\right]^{1/2}\frac{1}{n^{\beta_2+\beta_3-1}}.$$

Therefore,

$$\mu_n \sim \begin{cases} \dfrac{n^{|\beta_1-\beta_3|-1}}{n^{\beta_2+\beta_3-1}} = n^{-[\beta_2+\beta_3-|\beta_1-\beta_3|]}, & \beta_1 \neq \beta_3, \\ \dfrac{\log n}{n \cdot n^{\beta_2+\beta_3-1}} = n^{-(\beta_2+\beta_3)} \log n, & \beta_1 = \beta_3. \end{cases} \tag{4.15}$$

Hence the theorem.

Remark. It is obvious from the asymptotic property (4.15) of μ_n that if we replace the inequality by the condition

$$1/2 < (\beta_2 + \beta_3) - |\beta_1 - \beta_3| \leqq 1, \tag{4.16}$$

then $\sum_{n=0}^{\infty} \mu_n$ diverges but $\sum_{n=0}^{\infty} \mu_n^2$ is convergent.

If G were a symmetric kernel, then by using some well-known theorems of Hilbert–Schmidt theory (see Tricomi [11] or Goursat [7]) we would be able to draw some definite conclusions about its square-integrability and even continuity from Theorem 1. But for $\beta_1 \neq \beta_2$, which is the case we are considering, G is not symmetric. However, it is clear that if G is square-integrable, then $\sum_{n=0}^{\infty} \mu_n^2$ must be finite. Since our final aim is to derive a bilinear sum whose validity depends on the square-integrability of G, it seems desirable that we investigate this particular property of G in some detail. The conclusions of the previous theorem indicate that we may impose one restriction immediately:

$$\beta_2 + \beta_3 > 1/2 + |\beta_1 - \beta_3|. \tag{4.17}$$

In [9] we proved the square-integrability of the symmetric G by assuming that $\beta_2 + \beta_3 > 1$. It seems we can relax that restriction by requiring $\beta_2 + \beta_3 \geqq 1$. It may be possible to show that G may be square-integrable even if $1/2 < \beta_2 + \beta_3 < 1$, but we have not been able to prove it. It can be shown, however, that when $\beta_2 + \beta_3$ lies between 1/2 and 1 the kernel $G(x, y)$ has discontinuities all along the main diagonal $y = x$.

We shall prove two separate theorems to cover the cases $\beta_2 + \beta_3 = 1$ and $\beta_2 + \beta_3 > 1$ separately, because for the former equality we get an exact result which is subsequently used to deal with the second case.

THEOREM 2. *Let* $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3 > 0$ *such that* $\beta_1 < \beta_2 + \beta_3 = 1$. *Then the kernel* $G(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2)$ *is square-integrable on* (0, 1) *if and only if*

$$|\beta_1 - \beta_3| < 1/2. \tag{4.18}$$

Proof. Let us suppose $0 < x < y < 1$. Then, by using (2.16), (2.19) and (2.23), we obtain, through a pair of obvious transformations,

$$\begin{aligned} G(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2) &= C^{-1}\left[\frac{x^{\alpha_1+\beta_1-1}(1-y)^{\alpha_2+\beta_3-1}}{y^{\alpha_1+\beta_2-1}(1-x)^{\alpha_2-\beta_1}}\right]^{1/2} \\ &\quad \cdot \int_0^1 dt\, t^{\alpha_1-1}(1-t)^{\beta_1-1}(y-xt)^{\beta_2-1} \\ &\quad \cdot \int_0^1 dz\, z^{\beta_3-1}(1-z)^{\alpha_2-1}\{(y-x)+(1-y)z\}^{-\beta_1}. \end{aligned} \tag{4.19}$$

It is clear that G is well-behaved everywhere except possibly on the diagonal $y = x$ and on the two lines $x = 0$, $y = 1$. By replacing z by $1 - z$ in the second integral on the right and by using the integral representation of hypergeometric functions [3] we obtain

$$G(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2) = C^{-1}\left[\frac{x^{\alpha_1+\beta_1-1}(1-y)^{\alpha_2+\beta_3-1}}{y^{\alpha_1+\beta_2-1}(1-x)^{\alpha_2-\beta_1}}\right]^{1/2} y^{\beta_2-1}(1-x)^{-\beta_1} \tag{4.20}$$
$$\cdot B(\alpha_1, \beta_1)B(\alpha_2, \beta_3)\,{}_2F_1(1-\beta_2, \alpha_1; \alpha_1+\beta_1; x/y)$$
$$\cdot\, {}_2F_1\left(\beta_1, \alpha_2; \alpha_2+\beta_3; \frac{1-y}{1-x}\right).$$

If $\beta_3 > \beta_1$ the second hypergeometric function converges everywhere, while the first one also converges if $\beta_1 + \beta_2 > 1$. But since $\beta_2 + \beta_3 = 1$ we have, in fact, $\beta_1 + \beta_2 < 1$ and so there is a singularity on $y = x$ as $(y - x)^{-(1-\beta_1-\beta_2)}$. However, because of (4.18) we also have $\beta_1 + \beta_2 > 1/2$ and hence the singularity is square-integrable. Conversely, if $\beta_1 + \beta_2 > 1/2$, then $\beta_1 + (1 - \beta_3) > 1/2$, i.e., $\beta_3 - \beta_1 < 1/2$.

If, on the other hand, $\beta_3 < \beta_1$, then $\beta_1 + \beta_2 > \beta_2 + \beta_3 = 1$ and so ${}_2F_1(1 - \beta_2, \alpha_1; \alpha_1 - \beta_1; x/y)$ converges while ${}_2F_1(\beta_1, \alpha_2; \alpha_2 + \beta_3; (1 - y)/(1 - x)$ diverges like $(y - x)^{-(\beta_1-\beta_3)}$, but, again, this remains square-integrable if and only if $\beta_1 - \beta_3 < 1/2$.

Finally if $\beta_1 = \beta_3$, $\beta_1 + \beta_2 = \beta_3 + \beta_2 = 1$ and so both hypergeometric functions diverge like $\log(y - x)$. This, however, still remains square-integrable.

We need to examine the behavior of G as $x \to 0$, $y \neq 0$ and $y \to 1$, $x \neq 0$. One can see quite clearly that G behaves as $x^{(\alpha_1+\beta_1-1)/2}$ as $x \to 0$ and so it remains square-integrable since $\alpha_1 + \beta_1 > 0$. Similarly the behavior of G as $y \to 1$ is like $(1 - y)^{(\alpha_2+\beta_3-1)/2}$ and is again square-integrable because $\alpha_2 + \beta_3 > 0$.

Obviously same conclusions would follow if we had assumed $x > y$.

THEOREM 3. *Let* $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3 > 0$ *such that* $\beta_2 + \beta_3 > \max(\beta_1, 1)$. *Then*

(i) $G(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2)$ *is square-integrable on the unit square if and only if*

$$\beta_1 + \beta_2 > 1/2 \quad \text{if } \beta_3 \geqq \beta_1, \tag{4.21}$$

$$\beta_1 - \beta_3 < 1/2 \quad \text{if } \beta_1 > \beta_3; \tag{4.22}$$

(ii) G *is continuous everywhere except possibly at* $\min(x, y) = 0$ *and* $\max(x, y) = 1$ *if* $\beta_3 > \beta_1$ *and*

$$\beta_1 + \beta_2 > 1; \tag{4.23}$$

(iii) G *is continuous everywhere on the unit square if, in addition,*

$$\min(\beta_1, \beta_2) \geqq 1 - \alpha_1, \min(\beta_3, \beta_2 + \beta_3 - \beta_1) \geqq 1 - \alpha_2. \tag{4.24}$$

Proof. To fix ideas let us again suppose that $0 < x < y < 1$. Then, similar to

(4.19), we have the relation

$$\begin{aligned}
&G(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2) \\
&\quad = C^{-1}\left[\frac{x^{\alpha_1+\beta_1-1}(1-y)^{\alpha_2+\beta_3-1}}{y^{\alpha_1+\beta_2-1}(1-x)^{\alpha_2+\beta_2+\beta_3-1}}\right]^{1/2} \\
&\quad \cdot \int_0^1 dt\, t^{\alpha_1-1}(1-t)^{\beta_1-1}(y-xt)^{\beta_2-1} \\
&\quad \cdot \int_0^1 dz\, z^{\beta_3-1}(1-z)^{\alpha_2-1}\frac{\{y-x+(1-y)z\}^{\beta_2+\beta_3-\beta_1-1}}{\{y-xt+(1-y)z\}^{\beta_2+\beta_3-1}}.
\end{aligned} \tag{4.25}$$

In the indicated range of variables x, y, z and t we have

$$\left[\frac{y-x+(1-y)z}{y-xt+(1-y)z}\right]^{\beta_2+\beta_3-1} \leqq 1.$$

Hence

$$\begin{aligned}
&0 \leqq G(x, y) \\
&\leqq C^{-1}\left[\frac{x^{\alpha_1+\beta_1-1}(1-y)^{\alpha_2+\beta_3-1}}{y^{\alpha_1+\beta_2-1}(1-x)^{\alpha_2+\beta_2+\beta_3-\beta_1-1}}\right]^{1/2} y^{\beta_2-1}(1-x)^{-\beta_1}B(\alpha_1, \beta_1)B(\alpha_2, \beta_2) \\
&\quad \cdot {}_2F_1(1-\beta_2, \alpha_1; \alpha_1+\beta_1; x/y)\,{}_2F_1\left(\beta_1, \alpha_2; \alpha_2+\beta_3; \frac{1-y}{1-x}\right).
\end{aligned} \tag{4.26}$$

If $\beta_3 > \beta_1$, ${}_2F_1(\beta_1, \alpha_2; \alpha_2 + \beta_3; (1-y)/(1-x))$ converges on $y = x$ but ${}_2F_1(1-\beta_2, \alpha_1; \alpha_1+\beta_1; x/y)$ diverges like $(y-x)^{\beta_1+\beta_2-1}$ if $\beta_1 + \beta_2 < 1$. However this remains square-integrable if $\beta_1 + \beta_2 > 1/2$ which is indeed the case if (4.21) is true. Obviously G has no singularity on $y = x$ if $\beta_1 + \beta_2 > 1$ and $\beta_3 > \beta_1$.

Now suppose $\beta_3 = \beta_1$. Then ${}_2F_1(\beta_1, \alpha_2; \alpha_2 + \beta_3; (1-y)/(1-x))$ diverges like $\log(y-x)$ while the first hypergeometric function on the right of (4.26) converges since $\beta_2 + \beta_3 = \beta_2 + \beta_1 > 1$.

Finally, let $\beta_1 > \beta_3$. Then $\beta_1 + \beta_2 > \beta_3 + \beta_2 > 1$. Hence ${}_2F_1(1-\beta_2, \alpha_1; \alpha_1 + \beta_1; x/y)$ again converges on $y = x$ but ${}_2F_1(\beta_1, \alpha_2; \alpha_2 + \beta_3; (1-y)/(1-x))$ diverges like $(y-x)^{-(\beta_1-\beta_3)}$ which, however, is square-integrable if $\beta_1 - \beta_3 < 1/2$.

In the case $\beta_1 \geqq \beta_3$ the right-hand side of (4.26) is essentially discontinuous on $y = x$ which, of course, does not imply that G will have singularities as well.

Regarding the possible singularities at $x = 0$ and $y = 1$ it is obvious from (4.26) that G has no singularities on $x = 0$ if $\alpha_1 + \beta_1 \geqq 1$ and none on $y = 1$ if $\alpha_2 + \beta_3 \geqq 1$. By considering $x > y$ we can obtain similarly that $G(y, x)$ has no singularities on $y = 0$ if $\alpha_1 + \beta_2 \geqq 1$ and none on $x = 1$ if $\alpha_2 + \beta_2 + \beta_3 - \beta_1 \geqq 1$. Hence the condition (4.24). However, if any of these inequalities fails to be true, the kernel still remains square-integrable because all the parameters are strictly positive.

4.1. The square-integrability and continuity of the limiting kernels. For real parameters Theorem 3 is obviously applicable to both $G(x, y; 0, \beta_1, \beta_2, \beta_3, \alpha_2)$ and $G(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, 0)$ with corresponding modifications in the inequalities (4.24). However, for $G_\infty(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \infty)$ additional care needs to be taken since the domain is now extended to $(0, \infty)$.

Suppose $0 < x < y < \infty$. Then it follows from (3.14) that

$$\begin{aligned} &G_\infty(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \infty) \\ &\quad = [B(\alpha_1, \beta_1)B(\beta_2, \beta_3)\Gamma(\beta_2 + \beta_3 - \beta_1)]^{-1}\left[\frac{x^{\alpha_1+\beta_1-1}}{y^{\alpha_1+\beta_2-1}}\right]^{1/2} e^{(x+y)/2} \\ &\quad \cdot \int_0^1 dt\, t^{\alpha_1-1}(1-t)^{\beta_1-1}(y-xt)^{\beta_2-1} \\ &\quad \cdot \int_y^\infty dz \frac{e^{-z}(z-x)^{\beta_2+\beta_3-\beta_1-1}(z-y)^{\beta_3-1}}{(z-xt)^{\beta_2+\beta_3-1}}. \end{aligned} \tag{4.27}$$

For $\beta_2 + \beta_3 = 1$ this reduces to

$$\begin{aligned} &[B(\alpha_1, \beta_1)B(\beta_2, \beta_3)\Gamma(\beta_2 + \beta_3 - \beta_1)]^{-1}\left[\frac{x^{\alpha_1+\beta_1-1}}{y^{\alpha_1+\beta_2-1}}\right]^{1/2} \\ &\quad \cdot e^{-(y-x)/2} y^{\beta_2-1} B(\alpha_1, \beta_1)\Gamma(\beta_3)\, {}_2F_1(1-\beta_2, \alpha_1; \alpha_1+\beta_1; x/y) \\ &\quad \cdot (y-x)^{\beta_3-\beta_1} U(\beta_3, \beta_3 - \beta_1 - 1; y - x), \end{aligned} \tag{4.28}$$

where

$$U(a, b, z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}\,dt \tag{4.29}$$

is the confluent hypergeometric function of the second kind (see, for example, Abramowitz and Stegun [1, Chap. 13]). By studying the properties of the U function as $|z| \to 0$ (see [1, p. 508]) we see that G has a singularity at $y = x$ but it is square-integrable if and only if $|\beta_1 - \beta_3| < 1/2$; in other words, the same conclusion as in Theorem 2.

For $\beta_2 + \beta_3 > 1$, we have the inequality

$$\begin{aligned} 0 &\leqq G_\infty(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \infty) \\ &\leqq [B(\alpha_1, \beta_1)B(\beta_2, \beta_3)\Gamma(\beta_2 + \beta_3 - \beta_1)]^{-1} \\ &\quad \cdot B(\alpha_1, \beta_1)\Gamma(\beta_3)\left[\frac{x^{\alpha_1+\beta_1-1}}{y^{\alpha_1+\beta_2-1}}\right]^{1/2} y^{\beta_2-1} e^{-(y-x)/2} \\ &\quad \cdot {}_2F_1(1-\beta_2, \alpha_1; \alpha_1+\beta_1; x/y)(y-x)^{\beta_3-\beta_1}U(\beta_3, \beta_3-\beta_1-1; y-x). \end{aligned} \tag{4.30}$$

Since the right-hand side of this inequality is exactly the same as (4.28), the conclusions (i) and (ii) of Theorem 3 apply also to $G_\infty(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \infty)$, while in (iii) we have only the first of the inequalities, namely, $\min(\beta_1, \beta_2) \geqq 1 - \alpha_1$.

If we let $\alpha_1 \to 0$ we obtain $G_\infty(x, y; 0, \beta_1, \beta_2, \beta_3, \infty)$ of (3.21). Obviously for $\beta_2 + \beta_3 = 1$, and $0 < x < y < \infty$,

$$
\begin{aligned}
&G_\infty(x, y; 0, \beta_1, \beta_2, \beta_3, \infty) \\
&\quad = [B(\beta_2, \beta_3)\Gamma(\beta_2 + \beta_3 - \beta_1)]^{-1} x^{(\beta_1 - 1)/2} y^{(\beta_2 - 1)/2} e^{-(y-x)/2} \\
&\quad \cdot \int_0^\infty dz\, e^{-z} z^{\beta_3 - 1}[(y - x) + z]^{-\beta_1} \\
&\quad = [B(\beta_2, \beta_3)\Gamma(\beta_2 + \beta_3 - \beta_1)]^{-1}\Gamma(\beta_3) x^{(\beta_1 - 1)/2} y^{(\beta_2 - 1)/2} \\
&\quad \cdot e^{-(y-x)/2}(y - x)^{\beta_3 - \beta_1} U(\beta_3, \beta_3 - \beta_1 - 1; y - x)
\end{aligned}
\tag{4.31}
$$

which, for positive-valued parameters $\beta_1, \beta_2, \beta_3$, is square-integrable if and only if $|\beta_1 - \beta_3| < 1/2$. For $\beta_2 + \beta_3 > 1$ the right-hand side of (4.31) is a majorant for $G_\infty(x, y; 0, \beta_1, \beta_2, \beta_3, \infty)$ and again (i) and (ii) of Theorem 3 apply with (iii) replaced by $\min(\beta_1, \beta_2) \geqq 1$.

Finally, let us consider the kernel $G_\infty(x, y; \alpha_1, \beta_1, \beta_2, \infty, 0)$ of (3.18).

In this case $\beta_3 > \beta_1$ and $\beta_2 + \beta_3 > 1$ are automatically satisfied. For $0 < x < y < \infty$ one can reduce this kernel to the form

$$
\begin{aligned}
G_\infty(x, y; \alpha_1, \beta_1, \beta_2, \infty, 0) &= [B(\alpha_1, \beta_1)\Gamma(\beta_2)]^{-1} \left[\frac{x^{\alpha_1 + \beta_1 - 1}}{y^{\alpha_1 - \beta_2 - 1}}\right]^{1/2} e^{-(x+y)/2} \\
&\cdot \int_0^1 dt\, e^{xt} t^{\alpha_1 - 1}(1 - t)^{\beta_1 - 1}(y - xt)^{\beta_2 - 1}.
\end{aligned}
\tag{4.32}
$$

Note that

$$
\begin{aligned}
&G_\infty(x, x; \alpha_1, \beta_1, \beta_2, \infty, 0) \\
&\quad = [B(\alpha_1, \beta_1)\Gamma(\beta_1)]^{-1} x^{(\beta_1 + \beta_2)/2 - 1} e^{-x} \int_0^1 dt\, e^{xt} t^{\alpha_1 - 1}(1 - t)^{\beta_1 + \beta_2 - 2}.
\end{aligned}
\tag{4.33}
$$

This is bounded for all x if $\beta_1 + \beta_2 \geqq 2$, unbounded but square-integrable at $(0, 0)$ if $1 < \beta_1 + \beta_2 < 2$. If $\beta_2 \geqq 1$, then $(y - xt)^{\beta_2 - 1} \leqq y^{\beta_2 - 1}$ and we get

$$
\begin{aligned}
0 \leqq G_\infty(x, y) &\leqq [B(\alpha_1, \beta_1)\Gamma(\beta_2)]^{-1}\left[\frac{x^{\alpha_1 + \beta_1 - 1}}{y^{\alpha_1 + \beta_2 - 1}}\right]^{1/2} \\
&\cdot e^{-(x+y)/2} y^{\beta_2 - 1} \int_0^1 dt\, e^{xt} t^{\alpha_1 - 1}(1 - t)^{\beta_1 - 1}.
\end{aligned}
$$

On the other hand, if $\beta_2 < 1$, then $(y - xt)^{\beta_2 - 1} < (y - x)^{\beta_2 - 1}$ and hence,

$$
\begin{aligned}
G_\infty(x, y) &\leqq [B(\alpha_1, \beta_1)\Gamma(\beta_2)]^{-1}\left[\frac{x^{\alpha_1 + \beta_1 - 1}}{y^{\alpha_1 + \beta_2 - 1}}\right]^{1/2} e^{-(x+y)/2}(y - x)^{\beta_2 - 1} \\
&\cdot \int_0^1 dt\, e^{xt} t^{\alpha_1 - 1}(1 - t)^{\beta_1 - 1}
\end{aligned}
\tag{4.34}
$$

which has a singularity at $y = x$ but nevertheless is square-integrable if $\beta_2 > 1/2$. Evidently $G_\infty(x, y)$ is bounded everywhere in $(0, \infty)$ if $\min(\beta_1, \beta_2) \geqq 1$.

5. The bilinear sums. Theorem 3 of the previous section was proved on the assumption that the parameters $\alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2$ are all real. However, it can be shown that the conclusions of that theorem remain valid even if these parameters are complex provided the inequalities are understood as applied to their real parts. Since $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(x)\}_{n=0}^{\infty}$ are two fundamental orthonormal systems of the kernel $G(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2)$ (see Goursat [7, p. 149]) we have the following bilinear sums:

$$G(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2) = \sum_{n=0}^{\infty} \mu_n g_n(x) f_n(y), \tag{5.1}$$

$$G(y, x; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2) = \sum_{n=0}^{\infty} \mu_n f_n(x) g_n(y). \tag{5.2}$$

(See also Tricomi [11, p. 149].) Whenever G is square-integrable but not bounded these sums converge in the mean. When G is continuous everywhere the bilinear sums on the right of (5.1) and (5.2) converge pointwise to their respective kernels.

If we use (2.36), (2.37) and (2.38) as well as (2.7), (2.12), (2.21) and (2.24) we obtain, after some simplifications, the corresponding sums in terms of the standard Jacobi polynomials:

$$\begin{aligned}
&\int_{-1}^{\min(x,y)} dt(1+t)^{\alpha_2-1}(x-t)^{\beta_2+\beta_3-\beta_1-1}(y-t)^{\beta_3-1} \\
&\quad\cdot \int_{\max(x,y)}^{1} dz \frac{(1-z)^{\alpha_1-1}(z-x)^{\beta_1-1}(z-y)^{\beta_2-1}}{(z-t)^{\beta_2+\beta_3-1}} \\
&= \frac{\Gamma(\alpha_1)\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)\Gamma(\beta_2+\beta_3-\beta_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\beta_2)\Gamma(\beta_2+\beta_3)\Gamma(\alpha_2+\beta_2+\beta_3-\beta_1)2^{\alpha_1+\alpha_2+\beta_2+\beta_3-1}} \\
&\quad\cdot (1-x)^{\alpha_1+\beta_1-1}(1+x)^{\alpha_2+\beta_2+\beta_3-\beta_1-1}(1-y)^{\alpha_1+\beta_2-1}(1+y)^{\alpha_2+\beta_3-1} \\
&\quad\cdot \sum_{n=0}^{\infty} \frac{n!\Gamma(n+\alpha_1+\alpha_2+\beta_2+\beta_3-1)(2n+\alpha_1+\alpha_2+\beta_2+\beta_3-1)}{\Gamma(\alpha_1+\beta_1+n)\Gamma(\alpha_2+\beta_3+n)} \\
&\quad\cdot {}_4F_3\left[\begin{matrix}-n, n+\alpha_1+\alpha_2+\beta_2+\beta_3-1, \beta_2, \beta_2+\beta_3-\beta_1 \\ \alpha_1+\beta_2, \beta_2+\beta_3, \alpha_2+\beta_2+\beta_3-\beta_1\end{matrix}; 1\right] \\
&\quad\cdot P_n^{(\alpha_1+\beta_1-1, \alpha_2+\beta_2+\beta_3-\beta_1-1)}(x) P_n^{(\alpha_1+\beta_2-1, \alpha_2+\beta_3-1)}(y).
\end{aligned} \tag{5.3}$$

In view of (2.27), the simultaneous interchanges $x \leftrightarrow y$, $\beta_1 \leftrightarrow \beta_2$ and $\beta_3 \leftrightarrow \beta_2 + \beta_3 - \beta_1$ leave both sides of (5.3) unchanged.

The bilinear sums for left- and right-iterated kernels are

$$G_L(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2) = \sum_{n=0}^{\infty} \mu_n^2 f_n(x) f_n(y), \tag{5.4}$$

$$G_R(x, y; \alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2) = \sum_{n=0}^{\infty} \mu_n^2 g_n(x) g_n(y), \tag{5.5}$$

which converge whenever G is square-integrable.

Now let us consider the bilinear sums for the limiting kernels. Using (3.2), (3.4) and the corresponding limiting forms of $f_n(y)$ and $g_n(x)$ we obtain

$$\left[\frac{x^{\beta_1-1}y^{\beta_2-1}(1-y)^{1-\alpha_2-\beta_3}}{(1-x)^{\alpha_2+\beta_2+\beta_3-\beta_1-1}}\right]^{1/2}[B(\beta_2,\beta_3)B(\beta_2+\beta_3-\beta_1,\alpha_2)]^{-1}$$

$$\cdot\int_{\max(x,y)}^{1} dz\, z^{-(\beta_2+\beta_3-1)}(1-z)^{\alpha_2-1}(z-x)^{\beta_2+\beta_3-\beta_1-1}(z-y)^{\beta_3-1}$$

$$=\frac{\Gamma(\alpha_2+\beta_2+\beta_3-1)}{\Gamma(\beta_2)\Gamma(\alpha_2+\beta_3)}\sqrt{w_1(y)w_2(x)}$$

$$\cdot\sum_{n=0}^{\infty}\frac{(2n+\alpha_2+\beta_2+\beta_3-1)(\alpha_2)_n(\beta_1)_n(\beta_2)_n(\alpha_2+\beta_2+\beta_3-1)_n}{(\alpha_2+\beta_3)_n(\beta_2+\beta_3)_n(\alpha_2+\beta_2+\beta_3-\beta_1)_n n!}$$

$$\cdot{}_2F_1(-n,n+\alpha_2+\beta_2+\beta_3-1;\beta_2;y)\,{}_2F_1(-n,n+\alpha_2+\beta_2+\beta_3-1;\beta_1;x)$$

which simplifies to

$$\int_{\max(x,y)}^{1} dz\, z^{-(\beta_2+\beta_3-1)}(1-z)^{\alpha_2-1}(z-x)^{\beta_2+\beta_3-\beta\alpha-1}(z-y)^{\beta_3-1}$$

$$=\frac{\Gamma(\alpha_2)\Gamma(\beta_3)\Gamma(\alpha_2+\beta_2+\beta_3-1)\Gamma(\beta_2+\beta_3-\beta_1)}{\Gamma(\alpha_2+\beta_3)\Gamma(\beta_2+\beta_3)\Gamma(\alpha_2+\beta_2+\beta_3-\beta_1)}$$

$$\cdot(1-x)^{\alpha_2+\beta_2+\beta_3-\beta_1-1}(1-y)^{\alpha_2+\beta_3-1}$$

(5.6)
$$\cdot\sum_{n=0}^{\infty}\frac{(2n+\alpha_2+\beta_2+\beta_3-1)(\alpha_2)_n(\beta_1)_n(\beta_2)_n(\alpha_2+\beta_2+\beta_3-1)_n}{(\alpha_2+\beta_3)_n(\beta_2+\beta_3)_n(\alpha_2+\beta_2+\beta_3-\beta_1)_n n!}$$

$$\cdot{}_2F_1(-n,n+\alpha_2+\beta_2+\beta_3-1;\beta_2;y)$$

$$\cdot{}_2F_1(-n,n+\alpha_2+\beta_2+\beta_3-1;\beta_1;x).$$

The left-hand side of this equation can be reduced to an Appell function F_1 defined by

$$F_1(a,b,b',c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!}x^m y^n$$

(5.7)
$$=[B(a,c-a)]^{-1}\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux)^{-b}(1-uy)^{-b'}\,du,$$

$$\operatorname{Re} a>0,\quad \operatorname{Re}(c-a)>0.$$

(See Erdelyi et al. [4, pp. 224 and 231].)

But first, let us transform the variables x, y in (5.6) through the relations

$$x=\frac{1-\xi}{2},\qquad y=\frac{1+\eta}{2},\qquad -1\leqq\xi,\eta\leqq 1.$$

Then

$${}_2F_1\left(-n,n+\alpha_2+\beta_2+\beta_3-1;\frac{1-\xi}{2}\right)=\frac{n!}{(\beta_1)_n}P_n^{(\beta_1-1,\alpha_2+\beta_2+\beta_3-\beta_1-1)}(\xi),$$

and

$$ {}_2F_1\left(-n, n+\alpha_2+\beta_2+\beta_3-1; \beta_2; \frac{1+\eta}{2}\right) = \frac{n!}{(\beta_2)_n}(-1)^n P_n^{(\alpha_2+\beta_3-1,\beta_2-1)}(\eta). $$

If $(1-\eta)/2 \leqq (1+\xi)/2$, i.e., $\xi+\eta \geqq 0$, then using the transformation $z = 1-((1-\eta)/2)u$ in the integral of (5.6) and taking advantage of the integral representation of F_1 we obtain

$$ \begin{aligned} F_1\left(\alpha_2, \beta_2+\beta_3-1, 1+\beta_1-\beta_2-\beta_3, \alpha_2+\beta_3; \frac{1-\eta}{2}, \frac{1-\eta}{1+\xi}\right) \\ = \Gamma(\beta_2+\beta_3-\beta_1)\Gamma(\alpha_2+\beta_3)\left(\frac{1+\xi}{2}\right)^{\alpha_2} \\ \cdot \sum_{n=0}^{\infty} \frac{(-1)^n(\alpha_2)_n\Gamma(\alpha_2+\beta_2+\beta_3+n-1)(2n+\alpha_2+\beta_2+\beta_3-1)n!}{\Gamma(\alpha_2+\beta_3+n)\Gamma(\beta_2+\beta_3+n)\Gamma(\alpha_2+\beta_2+\beta_3-\beta_1+n)} \\ \cdot P_n^{(\beta_1-1,\alpha_2+\beta_2+\beta_3-\beta_1-1)}(\xi)P_n^{(\alpha_2+\beta_3-1,\beta_2-1)}(\eta). \end{aligned} \tag{5.8} $$

If on the other hand, $(1-\eta)/2 > (1+\xi)/2$, i.e., $\xi+\eta < 0$, then the transformation $z = 1-((1+\xi)/2)u$ in (5.6) combined with (5.7) yields

$$ \begin{aligned} F_1(\alpha_2, \beta_2+\beta_3-1, 1-\beta_3, \alpha_2+\beta_2+\beta_3-\beta_1; (1+\xi)/2, (1+\xi)/(1-\eta)) \\ = \Gamma(\beta_3)\Gamma(\alpha_2+\beta_2+\beta_3-\beta_1)\left(\frac{1-\eta}{2}\right)^{\alpha_2} \\ \cdot \sum_{n=0}^{\infty} \frac{(-1)^n(\alpha_2)_n\Gamma(\alpha_2+\beta_2+\beta_3+n-1)(2n+\alpha_2+\beta_2+\beta_3-1)n!}{\Gamma(\alpha_2+\beta_3+n)\Gamma(\beta_2+\beta_3+n)\Gamma(\alpha_2+\beta_2+\beta_3-\beta_1+n)} \\ \cdot P_n^{(\beta_1-1,\alpha_2+\beta_2+\beta_3-\beta_1-1)}(\xi)P_n^{(\alpha_2+\beta_3-1,\beta_2-1)}(\eta). \end{aligned} \tag{5.9} $$

At this point one cannot help feeling tempted to take advantage of a well-known reduction formula for F_1, namely,

$$ F_1(a, b, b', b+b'; x, y) = (1-y)^{-a}\,{}_2F_1\left(a, b; b+b'; \frac{x-y}{1-y}\right) \tag{5.10} $$

(see, for example, Erdélyi et al. [4, p. 238]).

We therefore set $\alpha_2 = \beta_1 - \beta_3$. Then, after some simplifications, the formulas (5.8) and (5.9) reduce respectively to

$$ \begin{aligned} \left(\frac{\xi+\eta}{2}\right)^{\beta_3+\beta_1} {}_2F_1\left(\beta_1-\beta_3, \beta_2+\beta_3-1; \beta_1; \frac{1-(1+\xi\eta)/(\xi+\eta)}{2}\right) \\ = \Gamma(\beta_1)\Gamma(\beta_2+\beta_3-\beta_1) \\ \cdot \sum_{n=0}^{\infty} \frac{(-1)^n(\beta_1-\beta_3)_n\Gamma(\beta_1+\beta_2+n-1)(2n+\beta_1+\beta_2-1)n!}{\Gamma(\beta_1+n)\Gamma(\beta_2+\beta_3+n)\Gamma(\beta_2+n)} \\ \cdot P_n^{(\beta_1-1,\beta_2-1)}(\xi)P_n^{(\beta_1-1,\beta_2-1)}(\eta), \quad \xi+\eta \geqq 0, \end{aligned} \tag{5.11} $$

and

$$\left(-\frac{\xi+\eta}{2}\right)^{\beta_3-\beta_1} {}_2F_1\left(\beta_1-\beta_3,\beta_2+\beta_3-1;\beta_2;\frac{1+(1+\xi\eta)/(\xi+\eta)}{2}\right)$$

$$(5.12)\quad \begin{aligned} &= \Gamma(\beta_2)\Gamma(\beta_3) \\ &\quad\cdot \sum_{n=0}^{\infty} \frac{(-1)^n(\beta_1-\beta_3)_n\Gamma(\beta_1+\beta_2+n-1)(2n+\beta_1+\beta_2-1)n!}{\Gamma(\beta_1+n)\Gamma(\beta_2+\beta_3+n)\Gamma(\beta_2+n)} \\ &\quad\cdot P_n^{(\beta_1-1,\beta_2-1)}(\xi)P_n^{(\beta_1-1,\beta_2-1)}(\eta), \qquad \xi+\eta<0. \end{aligned}$$

For real values of the parameters the left-hand sides of (5.8)–(5.12) must all be positive since they were obtained as special cases of a positive kernel. Furthermore, when $\operatorname{Re}\alpha_2 \leqq 0$, even though the integral representation (5.7) is not valid, since $\Gamma(\alpha_2)$ has cancelled out from both sides, the kernels of the above formulas still exist.

Finally, if $\beta_1 - \beta_3 = -m$, m a nonnegative integer, then the sums on the right-hand side of (5.11) and (5.12) reduce to finite series. Thus

$$\left(\frac{\xi+\eta}{2}\right)^{m} {}_2F_1\left(-m,\beta_1+\beta_2+m-1;\beta_1;\frac{1-(1+\xi\eta)/(\xi+\eta)}{2}\right)$$

$$(5.13)\quad \begin{aligned} &= \Gamma(\beta_1)\Gamma(\beta_2+m) \\ &\quad\cdot \sum_{n=0}^{m} \frac{(-1)^n(-m)_n\Gamma(\beta_1+\beta_2+n-1)(2n+\beta_1+\beta_2-1)n!}{\Gamma(\beta_1+n)\Gamma(\beta_1+\beta_2+m+n)\Gamma(\beta_2+n)} \\ &\quad\cdot P_n^{(\beta_1-1,\beta_2-1)}(\xi)P_n^{(\beta_1-1,\beta_2-1)}(\eta) \end{aligned}$$

and a similar reduction for (5.12). However, in view of the relations between hypergeometric functions and the Jacobi polynomials quoted above, the two formulas now reduce to one, namely,

$$(5.14)\quad \begin{aligned} &\left(\frac{\xi+\eta}{2}\right)^{m} \frac{P_m^{(\beta_1-1,\beta_2-1)}((1+\xi\eta)/(\xi+\eta))}{P_m^{(\beta_1-1,\beta_2-1)}(1)} \\ &\qquad = \sum_{n=0}^{m} C_{n,m} P_n^{(\beta_1-1,\beta_2-1)}(\xi)P_n^{(\beta_1-1,\beta_2-1)}(\eta), \end{aligned}$$

where

$$C_{n,m} = \frac{\Gamma(\beta_2+m)\Gamma(m+1)\Gamma(\beta_1+\beta_2+n-1)\Gamma(\beta_1)\Gamma(n+1)(2n+\beta_1+\beta_2-1)}{\Gamma(\beta_1+\beta_2+m+n)\Gamma(\beta_2+n)\Gamma(m-n+1)\Gamma(\beta_1+n)}. \tag{5.15}$$

Formula (5.14) is known in the literature as Bateman's formula [2].

One might say (5.11) and (5.12) are the infinite series extensions of (5.14) while (5.8) and (5.9) are generalizations of (5.11) and (5.12) respectively.

Next, by considering the kernel (3.6) for the case $\alpha_2 \to 0$, we obtain

$$\int_0^{\min(x,y)} dt\, t^{\alpha_1-1}(x-t)^{\beta_1-1}(y-t)^{\beta_2-1}(1-t)^{-(\beta_2+\beta_3-1)}$$

$$= \frac{\Gamma(\alpha_1)\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+\beta_2+\beta_3-1)}{\Gamma(\alpha_1+\beta_1)\Gamma(\beta_2+\beta_3)} x^{\alpha_1+\beta_1-1} y^{\alpha_1+\beta_2-1}$$

(5.16)
$$\cdot \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_1+\beta_2+\beta_3-1)_n(2n+\alpha_1+\beta_2+\beta_3-1)}{(\beta_2+\beta_3)_n n!}$$
$$\cdot {}_2F_1(-n, n+\alpha_1+\beta_2+\beta_3-1; \alpha_1+\beta_2; y)$$
$$\cdot {}_2F_1(-n, n+\alpha_1+\beta_2+\beta_3-1; \alpha_1+\beta_1; x).$$

Note that this formula reduces to equation (28) of Popov [8] if we set $\beta_2+\beta_3 = 1$ and make the transformations $y \to (1-y)/2$, $x \to (1-x)/2$.

Let us now consider the kernels on $L_2(0,\infty)$-space. First of all, using (3.14), (3.15) and (3.16) we have

$$e^{(x+y)} \int_0^{\min(x,y)} dt\, t^{\alpha_1-1}(x-t)^{\beta_1-1}(y-t)^{\beta_2-1}$$
$$\cdot \int_{\max(x,y)}^{\infty} dz \frac{e^{-z}(z-x)^{\beta_2+\beta_3-\beta_1-1}(z-y)^{\beta_3-1}}{(z-t)^{\beta_2+\beta_3-1}}$$

(5.17)
$$= \frac{B(\alpha_1,\beta_1)B(\beta_2,\beta_3)\Gamma(\beta_2+\beta_3-\beta_1)}{\Gamma(\alpha_1+\beta_2)}$$
$$\cdot \sum_{n=0}^{\infty} n!\, {}_3F_2\left[\begin{matrix}-n, \beta_2, \beta_2+\beta_3-\beta_1 \\ \alpha_1+\beta_2, \beta_2+\beta_3\end{matrix}; 1\right]$$
$$\cdot L_n^{(\alpha_1+\beta_2-1)}(y) L_n^{(\alpha_1+\beta_1-1)}(x).$$

Corresponding to (3.18) and (3.19) we have the bilinear sum

$$x^{-(\alpha_1+\beta_1-1)} y^{-(\alpha_1+\beta_2-1)} \int_0^{\min(x,y)} dt\, t^{\alpha_1-1} e^t (x-t)^{\beta_1-1}(y-t)^{\beta_2-1}$$

(5.18)
$$= \frac{\Gamma(\alpha_1)\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_1+\beta_2)} \sum_{n=0}^{\infty} \frac{(\alpha_1)_n n!}{(\alpha_1+\beta_1)_n(\alpha_1+\beta_2)_n}$$
$$\cdot L_n^{(\alpha_1+\beta_2-1)}(y) L_n^{(\alpha_1+\beta_1-1)}(x).$$

This is essentially the same as equation (13) of Erdélyi [6].

Finally, for (3.21) and (3.22) we get

(5.19)
$$e^{x+y} \int_{\max(x,y)}^{\infty} dz \frac{e^{-z}(z-x)^{\beta_2+\beta_3-\beta_1-1}(z-y)^{\beta_3-1}}{z^{\beta_2+\beta_3-1}}$$
$$= \frac{\Gamma(\beta_3)\Gamma(\beta_2+\beta_3-\beta_1)}{\Gamma(\beta_2+\beta_3)} \sum_{n=0}^{\infty} \frac{n!}{(\beta_2+\beta_3)_n} L_n^{(\alpha_1+\beta_2-1)}(y) L_n^{(\alpha_1+\beta_1-1)}(x).$$

The regions of validity of the above bilinear sums are indicated in the detailed square-integrability and continuity analysis of §4.

Appendix.

THEOREM 4. *For* $\operatorname{Re}(\alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2) > 0$, $\operatorname{Re}(\beta_2 + \beta_3 - \beta_1) > 0$, $\beta_1 \neq \beta_2$ *and positive integers* p, n *such that* $0 \leqq p \leqq n$ *the double sum*

$$S(n;p) = \frac{p!(\alpha_1+\beta_1)_p}{(\alpha_1+\beta_2)_p(\beta_2+\beta_3)_p} \cdot \sum_{l=0}^{n-p} \frac{(-n+p)_l(n+p+\alpha_1+\alpha_2+\beta_2+\beta_3-1)_l(\beta_4)_l}{(\alpha_1+\beta_2+p)_l(\beta_2+\beta_3+p)_l l!} \cdot \sum_{k=0}^{p} \frac{(\alpha_1)_{p-k}(\beta_2)_{k+l}(\beta_3)_{p-k}(\alpha_2)_k}{(\alpha_1+\beta_1)_{p-k}(\alpha_2+\beta_4)_{k+l}(p-k)!k!} \tag{2.8}$$

is independent of p if and only if

$$\beta_4 = \beta_2 + \beta_3 - \beta_1 \tag{2.11}$$

and is equal to

$$\lambda_n = S(n;0) = {}_4F_3\left[\begin{matrix} -n, n+\alpha_1+\alpha_2+\beta_2+\beta_3-1, \beta_2, \beta_2+\beta_3-\beta_1 \\ \alpha_1+\beta_2, \beta_2+\beta_3, \alpha_2+\beta_2+\beta_3-\beta_1 \end{matrix}; 1\right].$$

Proof. The proof depends, as in [9], on the two following results for Saalschützian series:

$$ {}_3F_2\left[\begin{matrix} a, b, -m \\ c, 1+a+b-c-m \end{matrix}; 1\right] = \frac{(c-a)_m(c-b)_m}{(c)_m(c-a-b)_m}, \tag{A.1}$$

$$ {}_4F_3\left[\begin{matrix} x, y, z, -m \\ u, v, w \end{matrix}; 1\right] = \frac{(v-z)_m(w-z)_m}{(v)_m(w)_m} \cdot {}_4F_3\left[\begin{matrix} u-x, u-y, z, -m \\ 1-v+z-m, 1-w+z-m, u \end{matrix}; 1\right], \tag{A.2}$$

where m is a positive integer and

$$u + v + w = x + y + z - m + 1 \tag{A.3}$$

(Bailey [3]).

By using the identity

$$(a)_{N-n} = (-1)^n (a)_N/(1-a-N)_n, \qquad N \geqq n, \tag{A.4}$$

we obtain

$$S(n;p) = \sum_{l=0}^{n-p} \frac{(-n+p)_l(n+p+\alpha_1+\alpha_2+\beta_2+\beta_3-1)_l(\beta_2)_l(\beta_4)_l}{(\alpha_1+\beta_2+p)_l(\beta_2+\beta_3+p)_l(\alpha_2+\beta_4)_l l!} A_{p,l}, \tag{A.5}$$

where

$$A_{p,l} = \frac{(\alpha_1)_p(\beta_3)_p}{(\alpha_1 + \beta_2)_p(\beta_2 + \beta_3)_p} \tag{A.6}$$
$$\cdot {}_4F_3\left[\begin{matrix}1 - \alpha_1 - \beta_1 - p, \alpha_2, \beta_2 + l, -p \\ \alpha_2 + \beta_4 + l, \quad 1 - \alpha_1 - p, 1 - \beta_3 - p\end{matrix}; 1\right].$$

The ${}_4F_3$ series on the right of (A.6) is Saalschützian if and only if $\beta_4 = \beta_2 + \beta_3 - \beta_1$. For $\beta_4 \neq \beta_2 + \beta_3 - \beta_1$ we can show that $S(n; p)$ is indeed dependent on p as can be seen by considering the special values $S(1;0)$ and $S(1;1)$, say.

One can easily show that

$$S(1;0) = 1 - \frac{\beta_2\beta_4(\alpha_1 + \alpha_2 + \beta_2 + \beta_3)}{(\alpha_1 + \beta_2)(\beta_2 + \beta_3)(\alpha_2 + \beta_4)}$$

while

$$S(1;1) = 1 - \frac{\beta_2\beta_4(\alpha_1 + \alpha_2 + \beta_2 + \beta_3) - \alpha_2\beta_2(\beta_4 - \beta_2 - \beta_3 + \beta_1)}{(\alpha_1 + \beta_2)(\beta_2 + \beta_3)(\alpha_2 + \beta_4)}.$$

Using (A.2), (A.4) and (2.11) we get

$$A_{p,l} = \frac{(\alpha_1 + \beta_2 + l)_p(\beta_2 + \beta_3 + l)_p}{(\alpha_1 + \beta_2)_p(\beta_2 + \beta_3)_p}$$
$$\cdot {}_4F_3\left[\begin{matrix}-p, l + p + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1, \beta_2 + l, \beta_4 + l \\ \alpha_1 + \beta_2 + l, \beta_2 + \beta_3 + l, \alpha_2 + \beta_4 + l\end{matrix}; 1\right].$$

Hence

$$S(n;p) = \sum_{l=0}^{n-p}\sum_{k=0}^{p} \frac{(-n + p)_l(n + p + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1)_l(\beta_2)_l(\beta_4)_l}{(\alpha_1 + \beta_2)_l(\beta_2 + \beta_3)_l(\alpha_2 + \beta_4)_l l!}$$
$$\cdot \frac{(p + l + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1)_k(\beta_2 + l)_k(\beta_4 + l)_k(-p)_k}{(\alpha_1 + \beta_2 + l)_k(\beta_2 + \beta_3 + l)_k(\alpha_2 + \beta_4 + l)_k k!}$$
$$= \sum_{l=0}^{n-p}\sum_{k=0}^{p} \frac{(-n + p)_l(n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 + p - 1)_l(\beta_2)_{k+l}(\beta_4)_{k+l}}{(\alpha_1 + \beta_2)_{k+l}(\beta_2 + \beta_3)_{k+l}(\alpha_2 + \beta_4)_{k+l} l!k!} \tag{A.7}$$
$$\cdot \frac{(p + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1)_{k+l}}{(p + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1)_l}.$$

Writing $m = k + l$ we obtain

$$S(n;p) = \sum_{m=0}^{n} \frac{(p + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1)_m(\beta_2)_m(\beta_4)_m}{(\alpha_1 + \beta_2)_m(\beta_2 + \beta_3)_m(\alpha_2 + \beta_4)_m} B_{n,m}, \tag{A.8}$$

where

$$B_{n,m} = \sum_{k=0}^{m} \frac{(-n+p)_{m-k}(n+\alpha_1+\alpha_2+\beta_2+\beta_3+p-1)_{m-k}(-p)_k}{(p+\alpha_1+\alpha_2+\beta_2+\beta_3-1)_{m-k}(m-k)!k!}$$

$$= \frac{(-n+p)_m(n+\alpha_1+\alpha_2+\beta_2+\beta_3+p-1)_m}{(p+\alpha_1+\alpha_2+\beta_2+\beta_3-1)_m m!}$$

$$\cdot \sum_{k=0}^{m} \frac{(-p)_k(2-p-\alpha_1-\alpha_2-\beta_2-\beta_3)_k(-m)_k}{(1+n-p-m)_k(2-n-\alpha_1-\alpha_2-\beta_2-\beta_3-p-m)_k k!}$$

$$= \frac{(-n+p)_m(n+\alpha_1+\alpha_2+\beta_2+\beta_3+p-1)_m}{(p+\alpha_1+\alpha_2+\beta_2+\beta_3-1)_m m!}$$

$$\cdot {}_3F_2\left[\begin{matrix} -p, 2-p-\alpha_1-\alpha_2-\beta_2-\beta_3, -m \\ 1+n-p-m, 2-n-\alpha_1-\alpha_2-\beta_2-\beta_3-p-m \end{matrix}; 1\right].$$

Note that the ${}_3F_2$ series is Saalschützian and hence, by (A.1),

$$B_{n,m} = \frac{(-n+p)_m(n+\alpha_1+\alpha_2+\beta_2+\beta_3+p-1)_m}{(p+\alpha_1+\alpha_2+\beta_2+\beta_3-1)_m m!}$$

$$\cdot \frac{(1+n-m)_m(n+\alpha_1+\alpha_2+\beta_2+\beta_3-1)_m}{(1+n-p-m)_m(n+\alpha_1+\alpha_2+\beta_2+\beta_3+p-1)_m}$$

$$= \frac{(-n)_m(n+\alpha_1+\alpha_2+\beta_2+\beta_3-1)_m}{(p+\alpha_1+\alpha_2+\beta_2+\beta_3-1)_m m!}.$$

Therefore

$$S(n;p) = \sum_{m=0}^{n} \frac{(-n)_m(n+\alpha_1+\alpha_2+\beta_2+\beta_3-1)_m(\beta_2)_m(\beta_4)_m}{(\alpha_1+\beta_2)_m(\beta_2+\beta_3)_m(\alpha_2+\beta_4)_m m!}. \tag{A.9}$$

Hence the theorem.

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SOME POSITIVE KERNELS AND BILINEAR SUMS FOR HAHN POLYNOMIALS*

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Abstract. A five-parameter family of kernels $K_N(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2)$ is constructed by using the Hahn polynomials $Q_n(i;\alpha_1+\beta_k-1,\alpha_2+\beta_2+\beta_3-\beta_k-1,N)$, $k=1,2$, under the assumption that the real parts of the parameters $\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2$ are positive. For real values of these parameters this kernel is shown to be positive. Special limiting kernels are obtained by considering various limiting values of the parameters. Some bilinear formulas for the Hahn and Meixner polynomials are also derived.

1. Introduction. In two recent papers [14], [15] we developed a method of constructing integral kernels from Jacobi polynomials with four [14] or five [15] complex parameters with real positive parts such that the kernels take on positive values in the event that the parameters are all real. In [14] we showed that the Jacobi polynomials ${}_2F_1(-n,n+\alpha+\beta+\gamma+\delta-1;\alpha+\beta:x/E)$, $0\leqq x\leqq E$, are the eigenfunctions of a kernel $K_E(x,y;\alpha,\beta,\gamma,\delta)$ which is symmetric apart from a weight factor. Reference [15] extends this result to generally nonsymmetric kernels $K_E(x,y;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2)$ which produces a multiple of the polynomial ${}_2F_1(-n,n+\alpha_1+\alpha_2+\beta_2+\beta_3-1;\alpha_1+\beta_2;x/E)$ after it is allowed to operate on ${}_2F_1(-n,n+\alpha_1+\alpha_2+\beta_2+\beta_3-1;\alpha_1+\beta_1;x/E)$. Since the present work deals essentially with the discrete analogue of the results obtained in the above references it seems appropriate to quote some of those results here. Specifically,

$$K_E(x,y;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2) = C^{-1}x^{-(\alpha_1+\beta_1-1)}(E-x)^{-(\alpha_2+\beta_2+\beta_3-\beta_1-1)}L_E(x,y), \tag{1.1}$$

where

$$C = B(\alpha_1,\beta_1)B(\beta_2,\beta_3)B(\alpha_2,\beta_2+\beta_3-\beta_1), \tag{1.2}$$

$B(a,b)$ being the usual beta function, and

$$\begin{aligned} L_E(x,y) = \int_0^{\min(x,y)} dt\, t^{\alpha_1-1}(x-t)^{\beta_1-1}(y-t)^{\beta_2-1} \\ \cdot \int_{\max(x,y)}^{E} dz \cdot \frac{(E-z)^{\alpha_2-1}(z-x)^{\beta_2+\beta_3-\beta_1-1}(z-y)^{\beta_3-1}}{(z-t)^{\beta_2+\beta_3-1}}. \end{aligned} \tag{1.3}$$

In the special case $\beta_1=\beta_2$, $L_E(x,y)$ becomes a symmetric kernel and $K_E(x,y;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2)$ reduces to the kernel of [14]. We showed that the eigenvalues of these kernels are positive and hence obtained a set of bilinear formulas. We also showed that under some conditions these kernels are bounded and therefore lend themselves to stochastic interpretation. In fact, as we mentioned in [14], it is through the works of Cooper [5] and Hoare [10], [11] on a family of stochastic urn-models that the kernel $K_E(x,y;\alpha,\beta,\gamma,\delta)$ came to be known,

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although in their works the parameters α, β, γ, δ are essentially positive integers denoting the degrees of freedom of certain stochastic systems.

It would be natural to expect that the discrete analogues of the results stated above must exist. Hoare [12], in fact, did write down the discrete analogue of $K_E(x, y; \alpha, \beta, \gamma, \delta)$ although he apparently did not succeed in solving the corresponding eigenvalue problem. Recently, however, Cooper, Hoare and Rahman [7] did manage to solve the eigenvalue problem for a restrictive class of discrete kernels with positive integral parameters and showed that the Hahn polynomials are their eigenfunctions.

Following Erdélyi and Weber [17], Karlin and McGregor [13], Gasper [8] and Askey [2], we define the Hahn polynomials [9] $Q_n(i)$ by the following relation:

$$Q_n(i) = Q_n(i; \alpha, \beta, N)$$

$$(1.4) \qquad = {}_3F_2\left[\begin{matrix} -n, n+\alpha+\beta+1, -i \\ \alpha+1, -N \end{matrix} ; 1\right] = \sum_{r=0}^{n} \frac{(-n)_r(n+\alpha+\beta+1)_r(-i)_r}{(\alpha+1)_r(-N)_r r!},$$

where $i = 0, 1, \cdots, N$; $n \leqq N$ is the degree of the polynomial; and ${}_3F_2(1)$ is a generalized hypergeometric function defined through the series

$$(1.5) \qquad {}_pF_q\left[\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix} ; x\right] = \sum_{r=0}^{\infty} \frac{(a_1)_r \cdots (a_p)_r}{(b_1)_r \cdots (b_q)_r} \frac{x^r}{r!},$$

(see, for example, Bateman [4], or Slater [16]), where $(a)_0 = 1$ and $(a)_k = a(a+1) \cdots (a+k-1)$ is the usual Pochhammer symbol.

It is well known that the Hahn polynomials are the discrete analogue of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and can be obtained from the Hahn polynomials as a limiting case:

$$(1.6) \qquad \frac{P_n^{(\alpha,\beta)}(1-2x)}{P_n^{(\alpha,\beta)}(1)} = \underset{N\to\infty}{Q_n} (Nx; \alpha, \beta, N).$$

The Hahn polynomials satisfy the orthogonality relation

$$(1.7) \qquad \sum_{i=0}^{N} \rho(i) Q_n(i) Q_m(i) = \frac{\delta_{mn}}{\pi_n}, \qquad m, n = 0, \cdots, N,$$

where $\rho(i)$ is the weight factor defined by

$$(1.8) \qquad \begin{aligned} \rho(i) = \rho(i; \alpha, \beta, N) &= \{B(\alpha+1, \beta+1)(N+1)_{\alpha+\beta+1}\}^{-1} \\ &\cdot (i+1)_\alpha (N-i+1)_\beta \end{aligned}$$

and π_n is the normalization constant given by

$$(1.9) \qquad \begin{aligned} \pi_n &= \pi_n(\alpha, \beta, N) \\ &= \frac{(-1)^n(-N)_n(\alpha+1)_n(\alpha+\beta+1)_n}{n!(N+\alpha+\beta+2)_n(\beta+1)_n} \cdot \frac{2n+\alpha+\beta+1}{\alpha+\beta+1}. \end{aligned}$$

The discrete analogue of the power function x^r can be written as a falling factorial $x^{[r]} = x(x-1)\cdots(x-r+1)$ or a Pochhammer function $(x+1)_r = (x+1)\cdots(x+r)$. Following Hoare [12] we shall use the second analogue consistently throughout this paper. The discrete analogue of K_E and L_E of (1.1) and (1.3) can then be written down almost on sight:

$$K_N(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2) = [C(i+1)_{\alpha_1+\beta_1-1}(N-i+1)_{\alpha_2+\beta_2+\beta_3-1}]^{-1}L_N(i,j), \tag{1.10}$$

where

$$L_N(i,j) = L_N(i,j:\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2)$$
$$= \sum_{k_1=0}^{\min(i,j)} (k_1+1)_{\alpha_1-1}(i-k_1+1)_{\beta_1-1}(j-k_1+1)_{\beta_2-1} \cdot \sum_{k_2=\max(i,j)}^{B} \frac{(N-k_2+1)_{\alpha_2-1}(k_2-i+1)_{\beta_2+\beta_3-\beta_1-1}(k_2-j+1)_{\beta_3-1}}{(k_2-k_1+1)_{\beta_2+\beta_3-1}}. \tag{1.11}$$

When α is not a positive integer the Pochhammer function $(x)_\alpha$ is to be understood as $\Gamma(x+\alpha)/\Gamma(x)$. The parameters α_1, β_1, β_2, β_3, α_2 are again assumed to be generally complex with positive real parts such that $\mathrm{Re}(\beta_2+\beta_3-\beta_1)>0$.

It is tempting to assume that the Hahn polynomials will turn out to be the eigenfunctions of K_N. Fortunately the assumption is true but it needs some work to prove that.

2. Construction of the kernel $K_N(i,j)$. Analogous to the manner in which we constructed the kernel $K_F(x,y:\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2)$ in [15] we proceed by multiplying the Hahn polynomial $Q_n(j;\alpha_1+\beta_2-1,\alpha_2+\beta_3-1,N)$ by $(j+1-k_1)_{\beta_2-1}$ $(k_2+1-j)_{\beta_3-1}$ and summing over j from k_1 to k_2 where $0 \leqq k_1 < k_2 \leqq N$. However, for this summation procedure the particular form of Q_n as given in (1.4) is rather inconvenient since $\sum_{r=0}^{n}$ there should indeed be replaced by $\sum_{r=0}^{\min(j,n)}$ and therefore the summation over j must also take this into account. Fortunately there exists a transformation of the terminating ${}_3F_2(1)$ series, namely,

$${}_3F_2\left[\begin{matrix}a,b,-n\\ e,f\end{matrix};1\right] = \frac{(e-a)_n(f-a)_n}{(e)_n(f)_n} \cdot {}_3F_2\left[\begin{matrix}1-e-f+a+b-n,a,-n\\ 1+a-e-n,1+a-f-n\end{matrix};1\right], \tag{2.1}$$

(see Slater [16, (2.5.11), p. 76]). Using this transformation the Hahn polynomial Q_n can also be written as

$$Q_n(j;\alpha_1+\beta_2-1,\alpha_2+\beta_3-1,N) = \frac{(\alpha_2+\beta_3)_n(N+A)_n}{(\alpha_1+\beta_2)_n(-N)_n}P_n(j), \tag{2.2}$$

where we are writing

$$A = \alpha_1+\alpha_2+\beta_2+\beta_3 \tag{2.3}$$

for abbreviation, and

$$P_n(j) = {}_3F_2\left[\begin{matrix}N+\alpha_2+\beta_3-j,n+A-1,-n\\ \alpha_2+\beta_3,N+A\end{matrix};1\right] = \sum_{r=0}^{n}\frac{(-n)_r(n+A-1)_r}{(\alpha_2+\beta_3)_r(N+A)_r r!}(N+\alpha_2+\beta_3-j)_r. \tag{2.4}$$

We now need the discrete analogue of the binomial formula

$$(x + y)^r = \sum_{m=0}^{r} \binom{r}{m} x^m y^{r-m}$$

which is

$$(a + b)_r = \sum_{m=0}^{r} \binom{r}{m} (a)_m (b)_{r-m}. \tag{2.5}$$

Hence

$$(N + \alpha_2 + \beta_3 - j)_r = \sum_{m=0}^{r} \binom{r}{m} (N + \alpha_2 - k_2)_{r-m} (k_2 + \beta_3 - j)_m$$

and

$$\begin{aligned}
&\sum_{j=k_1}^{k_2} (j + 1 - k_1)_{\beta_2 - 1} (k_2 + 1 - j)_{\beta_3 - 1} P_n(j) \\
&\quad = \sum_{r=0}^{n} \frac{(-n)_r (n + A - 1)_r}{(\alpha_2 + \beta_3)_r (N + A)_r r!} \sum_{m=0}^{r} \binom{r}{m} (N + \alpha_2 - k_2)_{r-m} \\
&\qquad \cdot \sum_{j=0}^{k_2 - k_1} (j + 1)_{\beta_2 - 1} (k_2 - k_1 + 1 - j)_{m + \beta_3 - 1} \\
&\quad = \sum_{r=0}^{n} \frac{(-n)_r (n + A - 1)_r}{(\alpha_2 + \beta_3)_r (N + A)_r r!} \sum_{m=0}^{r} \binom{r}{m} (N + \alpha_2 - k_2)_{r-m} \\
&\qquad \cdot B(\beta_2, m + \beta_3)(k_2 - k_1 + 1)_{\beta_2 + \beta_3 + m - 1}.
\end{aligned} \tag{2.6}$$

In deducing (2.6) we have made use of the formula

$$\sum_{j=0}^{M} (j + 1)_{b-1} (M + 1 - j)_{c-1} = B(b, c)(M + 1)_{b+c-1}; \tag{2.7}$$

for $\operatorname{Re} b, c > 0$. This formula can be seen as a direct consequence of Vandermonde's theorem

$${}_2F_1(-M, b; c; 1) = \frac{(c - b)_M}{(c)_M}, \tag{2.8}$$

(Slater [16, p. 243]) after transforming the Pochhammer symbol $(M + 1 - j)_{c-1}$ according to the formula

$$\Gamma(a - n) = \frac{(-1)^n \Gamma(a)}{(1 - a)_n}. \tag{2.9}$$

We now divide (2.6) by $(k_2 - k_1 + 1)_{\beta_2 + \beta_3 - 1}$, multiply by $(N + 1 - k_2)_{\alpha_2 - 1}$ $\cdot (k_2 + 1 - i)_{\beta_2 + \beta_3 - \beta_1 - 1}$ and sum over k_2 from i to N.

Noting that

$$(k_2 - k_1 + \beta_2 + \beta_3)_m = \sum_{l=0}^{m} \binom{m}{l} (k_2 + \beta_2 + \beta_3 - \beta_1 - i)_l (i - k_1 + \beta_1)_{m-l}$$

we get, after using (2.7),

$$\begin{aligned} &\sum_{k_2=i}^{N} \frac{(N+1-k_2)_{\alpha_2-1}(k_2+1-i)_{\beta_2+\beta_3-1}}{(k_2-k_1+1)_{\beta_2+\beta_3-1}} \sum_{j=k_1}^{k_2} (j+1-k_1)_{\beta_2-1} \\ &\quad \cdot (k_2+1-j)_{\beta_3-1} P_n(j) \\ &= (N-i+1)_{\beta_2+\beta_3-\beta_1-1} \sum_{r=0}^{n} \frac{(-n)_r(n+A-1)_r}{(\alpha_2+\beta_3)_r(N+A)_r r!} \sum_{m=0}^{r} \binom{r}{m} B(\beta_2, m+\beta_3) \\ &\quad \cdot \sum_{l=0}^{m} \binom{m}{l} B(\alpha_2+r-m, \beta_2+\beta_3-\beta_1+l)(i-k_1+\beta_1)_{m-l} \\ &\quad \cdot (N-i+\alpha_2+\beta_2+\beta_3-\beta_1)_{r-m+l}. \end{aligned} \tag{2.10}$$

Finally we multiply (2.10) by $(k_1+1)_{\alpha_1-1}(i-k_1+1)_{\beta_1-1}$ and sum over k_1 from 0 to i getting

$$\begin{aligned} &\sum_{k_1=0}^{i} (k_1+1)_{\alpha_1-1}(i-k_1+1)_{\beta_1-1} \sum_{k_2=i}^{N} \frac{(N+1-k_2)_{\alpha_2-1}(k_2+1-i)_{\beta_2+\beta_3-\beta_1-1}}{(k_2-k_1+1)_{\beta_2+\beta_3-1}} \\ &\quad \cdot \sum_{j=k_1}^{k_2} (j+1-k_1)_{\beta_2-1}(k_2+1-j)_{\beta_3-1}P_n(j) \\ &\quad = C(i+1)_{\alpha_1+\beta_1-1}(N-i+1)_{\beta_2+\beta_3-\beta_1-1}M_n(i), \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} M_n(i) = &\sum_{r=0}^{n} \frac{(-n)_r(n+A-1)_r}{(\alpha_2+\beta_3)_r(N+A)_r r!} \sum_{m=0}^{r} \binom{r}{m} \frac{(\beta_3)_m}{(\beta_2+\beta_3)_m} \\ &\cdot \sum_{l=0}^{m} \binom{m}{l} \frac{(\alpha_2)_{r-m}(\beta_2+\beta_3-\beta_1)_{m-l}}{(\alpha_2+\beta_2+\beta_3-\beta_1)_{r-l}} \frac{(\beta_1)_l}{(\alpha_1+\beta_1)_l} \\ &\cdot (i+\alpha_1+\beta_1)_l(N-i+\alpha_2+\beta_2+\beta_3-\beta_1)_{r-l}. \end{aligned} \tag{2.12}$$

Since

$$\begin{aligned} (i+\alpha_1+\beta_1)_l &= (-1)^l(1-\alpha_1-\beta_1-i-l)_l \\ &= (-1)^l([N-i+\alpha_2+\beta_2+\beta_3-\beta_1+r-l] \\ &\qquad - [N+A+r-1])_l, \end{aligned} \tag{2.13}$$

we have

$$\begin{aligned} &(i+\alpha_1+\beta_1)_l(N-i+\alpha_2+\beta_2+\beta_3-\beta_1)_{r-l} \\ &\quad = \sum_{k=0}^{l} \binom{l}{k} (-1)^k(N+A+r-l+k)_{l-k}(N-i+\alpha_2+\beta_2+\beta_3-\beta_1)_{r-l+k}. \end{aligned}$$

The coefficient of $(N-i+\alpha_2+\beta_2+\beta_3-\beta_1)_p$ in $M_n(i)$ is obtained by setting $r - l + k = p$. Noting that

$$(N + A + p)_{r-p}/(N + A)_r = \frac{1}{(N + A)_p},$$

we see that the coefficient of $(N - i + \alpha_2 + \beta_2 + \beta_3 - \beta_1)_p$ in $M_n(i)$ is obtained as

$$\frac{1}{(N + A)_p} \sum_{r=p}^{n} \frac{(-n)_r(n + A - 1)_r}{(\alpha_2 + \beta_3)_r r!} \sum_{m=r-p}^{r} \frac{(\beta_3)_m(\alpha_2)_{r-m}}{(\beta_2 + \beta_3)_m}$$
$$\sum_{l=r-p}^{m} \binom{m}{l} \frac{(\beta_2 + \beta_3 - \beta_1)_{m-l}}{(\alpha_2 + \beta_2 + \beta_3 - \beta_1)_{r-l}} \binom{l}{l - r + p} \frac{(-1)^{l-r+p}(\beta_1)_l}{(\alpha_1 + \beta_1)_l}$$
$$= \frac{(-n)_p(n + A - 1)_p}{(N + A)_p(\alpha_2 + \beta_2 + \beta_3 - \beta_1)_p p!} S(n; p),$$

where

$$S(n; p) = \frac{p!(\alpha_2 + \beta_2 + \beta_3 - \beta_1)_p}{(\alpha_2 + \beta_3)_p} \sum_{r=0}^{n-p} \frac{(-n + p)_r(n + A + p - 1)_r}{(\alpha_2 + \beta_3 + p)_r r!}$$

(2.14)
$$\cdot \sum_{m=0}^{p} \frac{(\beta_3)_{m+r}(\alpha_2)_{p-m}}{(p - m)!(\beta_2 + \beta_3)_{m+r}}$$
$$\cdot \sum_{l=0}^{m} \frac{(-1)^l(\beta_1)_{l+r}(\beta_2 + \beta_3 - \beta_1)_{m-l}}{(m - l)!l!(\alpha_1 + \beta_1)_{l+r}(\alpha_2 + \beta_2 + \beta_3 - \beta_1)_{p-l}}.$$

Unlike the double series (2.8) of [15] this is a treble series and it needs a separate proof to show that this is independent of p and is equal to

(2.15)
$$\lambda_n' = S(n; 0) = {}_4F_3\left[\begin{matrix} -n, n + \alpha_1 + \alpha_2 + \beta_2 + \beta_3 - 1, \beta_1, \beta_3 \\ \alpha_1 + \beta_1, \beta_2 + \beta_3, \alpha_2 + \beta_3 \end{matrix}; 1\right].$$

We carry out this proof in the Appendix.

The summing operations on the left-hand side of (2.11) can be seen as

$$\sum_{k_1=0}^{N} (k_1 + 1)_{\alpha_1 - 1}(i - k_1 + 1)_{\beta_1 - 1}(j + 1 - k_1)_{\beta_2 - 1}$$
$$\cdot \sum_{k_2=0}^{N} \frac{(N + 1 - k_2)_{\alpha_2 - 1}(k_2 + 1 - i)_{\beta_2 + \beta_3 - \beta_1 - 1}(k_2 + 1 - j)_{\beta_3 - 1}}{(k_2 - k_1 + 1)_{\beta_2 + \beta_3 - 1}}$$

(2.16)
$$\cdot H(i + 1 - k_1)H(j + 1 - k_1)H(k_2 + 1 - i)H(k_2 + 1 - j)$$
$$= \sum_{k_1=0}^{\min(i,j)} (k_1 + 1)_{\alpha_1 - 1}(i - k_1 + 1)_{\beta_1 - 1}(j + 1 - k_1)_{\beta_2 - 1}$$
$$\cdot \sum_{k_2=\max(i,j)}^{N} \frac{(N + 1 - k_2)_{\alpha_2 - 1}(k_2 + 1 - i)_{\beta_2 + \beta_3 - \beta_1 - 1}(k_2 + 1 - j)_{\beta_3 - 1}}{(k_2 - k_1 + 1)_{\beta_2 + \beta_3 - 1}},$$

where

(2.17)
$$H(x) = \begin{cases} 0 & \text{if } x < 1, \\ 1 & \text{if } x \geqq 1, \end{cases}$$

is the Heaviside unit function.

Collecting the above results we finally obtain

$$\sum_{j=0}^{N} K_N(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2)Q_n(j;\alpha_1+\beta_2-1,\alpha_2+\beta_3-1,N)$$
$$(2.18)\qquad = \lambda_n Q_n(i;\alpha_1+\beta_1-1,\alpha_2+\beta_2+\beta_3-\beta_1-1,N), \quad i=0,1,\cdots,N,$$

where

$$(2.19)\qquad \lambda_n = \frac{(\alpha_2+\beta_3)_n(\alpha_1+\beta_1)_n}{(\alpha_1+\beta_2)_n(\alpha_2+\beta_2+\beta_3-\beta_1)_n}\lambda_n' = {}_4F_3\left[\begin{matrix}-n,n+\alpha_1+\alpha_2+\beta_2+\beta_3-1,\beta_2,\beta_2+\beta_3-\beta_1 \\ \alpha_1+\beta_2,\beta_2+\beta_3,\alpha_2+\beta_2+\beta_3-\beta_1\end{matrix};1\right],$$

(see [15]), $n=0,1,\cdots,N$; and

$$K_N(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2) = \frac{C^{-1}}{(i+1)_{\alpha_1+\beta_1-1}(N-i+1)_{\alpha_2+\beta_2+\beta_3-\beta_1-1}}$$
$$(2.20)\qquad \cdot\sum_{k_1=0}^{\min(i,j)}(k_1+1)_{\alpha_1-1}(i-k_1+1)_{\beta_1-1}(j+1-k_1)_{\beta_2-1}$$
$$\cdot\sum_{k_2=\max(i,j)}^{N}\frac{(N+1-k_2)_{\alpha_2-1}(k_2+1-i)_{\beta_2+\beta_3-\beta_1-1}(k_2+1-j)_{\beta_3-1}}{(k_2-k_1+1)_{\beta_2+\beta_3-1}}.$$

which is, of course, the same as (1.20).

Note that for $\beta_1=\beta_2$, the Hahn polynomials on the two sides of (2.18) have the same arguments and hence $Q_n(i)$ is an eigenvector of the matrix $K_N(i,j)$ with the eigenvalue $\lambda_n=\lambda_n'$.

For general values of the parameters let us denote

$$(2.21)\qquad \begin{aligned} Q_n^{(1)}(i) &= Q_n(i;\alpha_1+\beta_1-1,\alpha_2+\beta_2+\beta_3-\beta_1-1,N), \\ Q_n^{(2)}(i) &= Q_n(i;\alpha_1+\beta_2-1,\alpha_2+\beta_3-1,N), \end{aligned}$$

and define the corresponding weight functions

$$(2.22)\qquad \begin{aligned} \rho^{(1)}(i) &= [B(\alpha_1+\beta_1,A-\alpha_1-\beta_1)(N+1)_{A-1}]^{-1} \\ &\quad\cdot(i+1)_{\alpha_1+\beta_1-1}(N-i+1)_{\alpha_2+\beta_2+\beta_3-\beta_1-1}, \\ \rho^{(2)}(i) &= [B(\alpha_1+\beta_2,A-\alpha_1-\beta_2)(N+1)_{A-1}]^{-1} \\ &\quad\cdot(i+1)_{\alpha_1+\beta_2-1}(N-i+1)_{\alpha_2+\beta_3-1}, \end{aligned}$$

where we again use the abbreviation A for $\alpha_1+\alpha_2+\beta_2+\beta_3$.

The Hahn polynomials satisfy the following orthogonality relation:

$$(2.23)\qquad \sum_{i=0}^{N}\rho^{(k)}(i)Q_n^{(k)}(i)Q_m^{(k)}(i) = \frac{\delta mn}{\pi_n^{(k)}},$$

where

$$(2.24)\qquad \pi_n^{(k)} = \frac{(-1)^n(-N)_n(\alpha_1+\beta_k)_n(A-1)_n}{n!(A-\alpha_1-\beta_k)_n(N+A)_n}\cdot\frac{2n+A-1}{A-1}, \qquad k=1,2.$$

Let us now introduce the orthonormal systems

$$R_n^{(k)}(i) = [\pi_n^{(k)}\rho^{(k)}(i)]^{1/2}Q_n^{(k)}(i). \tag{2.25}$$

In terms of $R_n^{(k)}(i)$ equation (2.18) then reads

$$\sum_{j=0}^{N} G_N(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2)R_n^{(2)}(j) = \mu_n R_n^{(1)}(i), \qquad i = 0,1,\cdots,N, \tag{2.26}$$

where

$$\mu_n = \lambda_n\left[\frac{\Gamma(\alpha_1+\beta_1)\Gamma(A-\alpha_1-\beta_1)(\alpha_1+\beta_2)_n(A-\alpha_1-\beta_1)_n}{\Gamma(\alpha_1+\beta_2)\Gamma(A-\alpha_1-\beta_2)(\alpha_1+\beta_1)_n(A-\alpha_1-\beta_2)_n}\right]^{1/2} \tag{2.27}$$

and

$$\begin{aligned} &G_N(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2) \\ &\quad = C^{-1}[(i+1)_{\alpha_1+\beta_1-1}(N-i+1)_{A-\alpha_1-\beta_1-1}(j+1)_{\alpha_1+\beta_2-1} \\ &\quad\quad \cdot(N-j+1)_{A-\alpha_1-\beta_2-1}]^{-1/2}L_N(i,j), \end{aligned} \tag{2.28}$$

$L_N(i,j)$ being given by (1.11).

If $\beta_1 = \beta_2$, $G_N(i,j)$ is symmetric. It can be verified by interchanging $\beta_1 \leftrightarrow \beta_2$ and $\beta_3 \leftrightarrow \beta_2 + \beta_3 - \beta_1$ that

$$\sum_{j=0}^{N} G_N(j,i;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2)R_n^{(1)}(j) = \mu_n R_n^{(2)}(i), \qquad i = 0,1,2,\cdots N. \tag{2.29}$$

If we use the symbol G_N to denote the $(N+1)\times(N+1)$ matrix whose element in the ith row and jth column is $G_N(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2)$ and G_N^T to denote its transpose, then it is evident that $G_N G_N^T$ and $G_N^T G_N$ are both symmetric and their normalized eigenvectors are

$$\begin{aligned} \mathbf{R}_n^{(1)} &= [R_n^{(1)}(0), R_n^{(1)}(1),\cdots, R_n^{(1)}(N)]^T, \\ \mathbf{R}_n^{(2)} &= [R_n^{(2)}(0), R_n^{(2)}(1),\cdots, R_n^{(2)}(N)]^T, \end{aligned} \tag{2.30}$$

respectively, with the same eigenvalue μ_n^2, that is,

$$\begin{aligned} G_N G_N^T\mathbf{R}_n^{(1)} &= \mu_n^2\mathbf{R}_n^{(1)}, \\ G_N^T G_N\mathbf{R}_n^{(2)} &= \mu_n^2\mathbf{R}_n^{(2)}. \end{aligned} \tag{2.31}$$

The symmetric matrices $G_N G_N^T$ and $G_N^T G_N$ are the discrete analogues of the right- and left-iterated kernels of [15].

3. The limiting matrices. The matrices $K_N(i,j)$ and $G_N(i,j)$ may appear rather formidable, but the presence of five parameters enables us to consider various limiting forms of these kernels. In this section we shall look at some of these limiting cases with the understanding that a great many others can be worked out in a similar manner.

Case I. $\alpha_1 \to 0$, $\operatorname{Re}[\beta_1, \beta_2, \beta_3, \alpha_2] > 0$, $\operatorname{Re}(\beta_2 + \beta_3 - \beta_1) > 0$.

Obviously

$$\lim_{\alpha_1 \to 0} \frac{(k_1+1)_{\alpha_1 - 1}}{B(\alpha_1, \beta_1)} = \begin{cases} 0 & \text{if } k_1 \neq 0, \\ 1 & \text{if } k_1 = 0. \end{cases}$$

Hence

$$\begin{aligned} &K_N(i,j; 0, \beta_1, \beta_2, \beta_3, \alpha_2) \\ &= [B(\beta_2, \beta_3) B(\alpha_2, \beta_2 + \beta_3 - \beta_1)(N - i + 1)_{\alpha_2 + \beta_2 + \beta_3 - \beta_1 - 1}]^{-1} (j+1)_{\beta_2 - 1} \\ &\quad \cdot \sum_{k_2 = \max(i,j)}^{N} \frac{(N+1-k_2)_{\alpha_2 - 1}(k_2 + 1 - j)_{\beta_3 - 1}(k_2 + 1 - i)_{\beta_2 + \beta_3 - \beta_1 - 1}}{(k_2+1)_{\beta_2 + \beta_3 - 1}}. \end{aligned} \tag{3.1}$$

Also

$$\begin{aligned} G_N(i,j; 0, \beta_1, \beta_2, \beta_3, \alpha_2) &= [B(\beta_2, \beta_3) B(\alpha_2, \beta_2 + \beta_3 - \beta_1)]^{-1} \\ &\quad \cdot \left[\frac{(i+1)_{\beta_1 - 1}(j+1)_{\beta_2 - 1}}{(N-i+1)_{\alpha_2 + \beta_2 + \beta_3 - \beta_1 - 1}(N-j+1)_{\alpha_2 + \beta_3 - 1}} \right]^{1/2} \\ &\quad \cdot \sum_{k_2 = \max(i,j)}^{N} \frac{(N+1-k_2)_{\alpha_2 - 1}(k_2 + 1 - i)_{\beta_2 + \beta_3 - \beta_1 - 1}(k_2 + 1 - j)_{\beta_3 - 1}}{(k_2+1)_{\beta_2 + \beta_3 - 1}}. \end{aligned} \tag{3.2}$$

The eigenvalues μ_n's take particularly simple form

$$\begin{aligned} \lim_{\alpha_1 \to 0} \mu_n = \frac{(\alpha_2)_n}{(\beta_2 + \beta_3)_n} &\left[\frac{(\beta_1)_n (\beta_2)_n}{(\alpha_2 + \beta_3)_n (\alpha_2 + \beta_2 + \beta_3 - \beta_1)_n} \right. \\ &\left. \cdot \frac{\Gamma(\beta_1)\Gamma(\alpha_2 + \beta_2 + \beta_3 - \beta_1)}{\Gamma(\beta_2)\Gamma(\alpha_2 + \beta_3)} \right]^{1/2}. \end{aligned} \tag{3.3}$$

This is obtained, as in [15], by noting that the Saalschützian ${}_4F_3(1)$ of (2.19) reduces to a Saalschützian ${}_3F_2(1)$ in this limit which can be summed by means of Saalschütz's theorem [16].

The limiting forms of $Q_n^{(k)}(i)$ and $R_n^{(k)}(i)$ are self-evident.

Case II. $\alpha_1 \to 0$, $\alpha_2 = \beta_1 - \beta_3$, $\operatorname{Re}\beta_2 > \operatorname{Re}(\beta_1 - \beta_3) > 0$, $\operatorname{Re}\beta_3 > 0$.

This is indeed a special case of Case I, but because it produces a rather interesting bilinear formula, as we shall see later, let us write down the forms of the matrices and the eigenvalues. We have

$$\begin{aligned} &K_N(i,j; 0, \beta_1, \beta_2, \beta_3, \beta_1 - \beta_3) \\ &= [B(\beta_2, \beta_3) B(\beta_1 - \beta_3, \beta_2 + \beta_3 - \beta_1)(N - i + 1)_{\beta_2 - 1}]^{-1} (j+1)_{\beta_2 - 1} \\ &\quad \sum_{k_2 = \max(i,j)}^{N} \frac{(N+1-k_2)_{\beta_1 - \beta_3 - 1}(k_2 + 1 - j)_{\beta_3 - 1}(k_2 + 1 - i)_{\beta_2 + \beta_3 - \beta_1 - 1}}{(k_2+1)_{\beta_2 + \beta_3 - 1}}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} G_N(i,j: 0, \beta_1, \beta_2, \beta_3, \beta_1 - \beta_3) &= [B(\beta_2, \beta_3) B(\beta_1 - \beta_3, \beta_2 + \beta_3 - \beta_1)]^{-1} \\ &\quad \cdot \left[\frac{(i+1)_{\beta_1 - 1}(j+1)_{\beta_2 - 1}}{(N-i+1)_{\beta_2 - 1}(N-j+1)_{\beta_1 - 1}} \right]^{1/2} \\ &\quad \cdot \sum_{k_2 = \max(i,j)}^{N} \frac{(N+1-k_2)_{\beta_1 - \beta_3 - 1}(k_2 + 1 - i)_{\beta_2 + \beta_3 - \beta_1 - 1}(k_2 + 1 - j)_{\beta_3 - 1}}{(k_2+1)_{\beta_2 + \beta_3 - 1}}, \end{aligned} \tag{3.5}$$

and

$$\lim_{\substack{\alpha_1 \to 0 \\ \alpha_2 = \beta_1 - \beta_3}} \mu_n = \frac{(\beta_1 - \beta_3)_n}{(\beta_2 + \beta_3)_n}. \tag{3.6}$$

The kernels above can be expressed in terms of a Saalschützian ${}_4F_3(1)$ series:

$$K_N(i,j;0,\beta_1,\beta_2,\beta_3,\beta_1 - \beta_3)$$
$$= \frac{\Gamma(\beta_2 + \beta_3)}{\Gamma(\beta_3)\Gamma(\beta_2 + \beta_3 - \beta_1)\Gamma(\beta_1 - \beta_3)} \cdot \frac{(j+1)_{\beta_2 - 1}}{(N - i + 1)_{\beta_2 - 1}}$$
$$\cdot \frac{(N - \max(i,j) + 1)_{\beta_1 - \beta_3 - 1}(\max(i,j) - j + 1)_{\beta_3 - 1} \cdot (\max(i,j) - i + 1)_{\beta_2 + \beta_3 - \beta_1 - 1}}{(\max(i,j) + 1)_{\beta_2 + \beta_3 - 1}}$$
$$\cdot {}_4F_3\left[\begin{matrix} \max(i,j) + 1, \max(i,j) - j + \beta_3, \max(i,j) - i \\ \quad + \beta_2 + \beta_3 - \beta_1, -N + \max(i,j) \\ 1 + \max(i,j) - \min(i,j), \max(i,j) + \beta_2 + \beta_3, \\ \quad 1 - N + \max(i,j) + \beta_3 - \beta_1 \end{matrix} ;1\right].$$

$G_N(i,j)$ has, of course, the same ${}_4F_3(1)$ as a factor.

Note that the above ${}_4F_3(1)$ series is Saalschützian, i.e., the sum of the denominator parameters exceeds the sum of the numerator parameters by 1. For such a series we have the following transformation property:

$$ {}_4F_3\left[\begin{matrix} x, y, z, -m \\ u, v, w \end{matrix} ;1\right] = \frac{(v - z)_m (w - z)_m}{(v)_m (w)_m} \cdot {}_4F_3\left[\begin{matrix} u - x, u - y, z, -m \\ 1 - v + z - m, 1 - w + z - m, u \end{matrix} ;1\right]. \tag{3.7}$$

(See, for example, Bailey [3, p. 56].)

Applying this transformation twice and simplifying the Pochhammer symbols we obtain

$$K_N(i,j;0,\beta_1,\beta_2,\beta_3,\beta_1 - \beta_3)$$
$$= \frac{\Gamma(\beta_2 + \beta_3)}{\Gamma(\beta_3)\Gamma(\beta_2 + \beta_3 - \beta_1)\Gamma(N + \beta_2 + \beta_3)} \cdot (j+1)_{\beta_2 - 1}(N - j + 1)_{\beta_1 - 1} \tag{3.8}$$
$$\cdot \frac{\Gamma(\max(i,j) - j + \beta_3)\Gamma(\max(i,j) - i + \beta_2 + \beta_3 - \beta_1)}{\Gamma(i + \beta_1 - \min(i,j))\Gamma(\max(i,j) - i + \beta_2)}$$
$$\cdot {}_4F_3\left[\begin{matrix} \beta_1 - \beta_3, \beta_2 + \beta_3 - 1, -N + \max(i,j), -\min(i,j) \\ -N, i - \min(i,j) + \beta_1, \max(i,j) - i + \beta_2 \end{matrix} :1\right]$$

and

$$
\begin{aligned}
&G_N(i,j;0,\beta_1,\beta_2,\beta_3,\beta_1-\beta_3)\\
&\quad=\frac{\Gamma(\beta_2+\beta_3)}{\Gamma(\beta_3)\Gamma(\beta_2+\beta_3-\beta_1)\Gamma(N+\beta_2+\beta_3)}\\
&\qquad\cdot[(i+1)_{\beta_1-1}(j+1)_{\beta_2-1}(N-i+1)_{\beta_2-1}(N-j+1)_{\beta_1-1}]^{1/2}\\
&\qquad\cdot\frac{\Gamma(\max(i,j)-j+\beta_3)\Gamma(\max(i,j)-i+\beta_2+\beta_3-\beta_1)}{\Gamma(i-\min(i,j)+\beta_1)\Gamma(\max(i,j)-i+\beta_2)}\\
&\qquad\cdot{}_4F_3\left[\begin{matrix}\beta_1-\beta_3,\beta_2+\beta_3-1,-N+\max(i,j), & -\min(i,j)\\ -N,i-\min(i,j)+\beta_1,\max(i,j)-i+\beta_2 & \end{matrix};1\right].
\end{aligned}
\tag{3.9}
$$

In this case

$$
\rho^{(1,2)}(i)=[B(\beta_1,\beta_2)(N+1)_{\beta_1+\beta_2-1}]^{-1}(i+1)_{\beta_{1,2}-1}(N-i+1)_{\beta_{2,1}-1}
\tag{3.10}
$$

and

$$
\pi_n^{(1,2)}=\binom{N}{n}\frac{(\beta_1+\beta_2-1)_n}{(N+\beta_1+\beta_2)_n}\cdot\frac{(\beta_{1,2})_n}{(\beta_{2,1})_n}\cdot\frac{2n+\beta_1+\beta_2-1}{\beta_1+\beta_2-1},
\tag{3.11}
$$

so that

$$
\begin{aligned}
&R_n^{(1)}(i)\\
&\quad=\left[\frac{2n+\beta_1+\beta_2-1}{\beta_1+\beta_2-1}\cdot\frac{\Gamma(\beta_1+\beta_2)(\beta_1+\beta_2-1)_n}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(N+\beta_1+\beta_2+n)n!(N-n)!}\cdot\frac{(\beta_1)_n}{(\beta_2)_n}\right]^{1/2}\\
&\qquad\cdot[(i+1)_{\beta_1-1}(N-i+1)_{\beta_2-1}]^{1/2}{}_3F_2\left[\begin{matrix}-n,n+\beta_1+\beta_2-1,-i\\ \beta_1,-N\end{matrix};1\right]
\end{aligned}
\tag{3.12}
$$

while $R_n^{(2)}(i)$ has the same expression with β_1 and β_2 interchanged.

The interest in this particular case is further heightened by the fact that one can now consider the limit $\beta_1-\beta_3\to-m$ where m is a nonnegative integer. The $G_N(i,j)$ kernel approaches the limit

$$
\begin{aligned}
&G_N(i,j;0,\beta_1,\beta_2,\beta_1+m,-m)\\
&\quad=\frac{\Gamma(\beta_1+\beta_2+m)}{\Gamma(\beta_1+m)\Gamma(\beta_2+m)\Gamma(N+\beta_1+\beta_2+m)}\\
&\qquad\cdot[(i+1)_{\beta_1-1}(j+1)_{\beta_2-1}(N-i+1)_{\beta_2-1}(N-j+1)_{\beta_1-1}]^{1/2}\\
&\qquad\cdot\frac{\Gamma(\max(i,j)-j+\beta_1+m)\Gamma(\max(i,j)-i+\beta_2+m)}{\Gamma(i-\min(i,j)+\beta_1)\Gamma(\max(i,j)-i+\beta_2)}\\
&\qquad\cdot{}_4F_3\left[\begin{matrix}-m,\beta_1+\beta_2+m-1,-N+\max(i,j),-\min(i,j)\\ -N,i-\min(i,j)+\beta_1,\max(i,j)-i+\beta_2\end{matrix}:1\right].
\end{aligned}
\tag{3.13}
$$

Case III. $\alpha_2\to0$, $\mathrm{Re}(\alpha_1,\beta_1,\beta_1,\beta_3)>0$, $\mathrm{Re}(\beta_2+\beta_3-\beta_1)>0$.
In this case

$$
\lim_{\alpha_2\to0}\frac{(N+1-k_2)_{\alpha_2-1}}{B(\alpha_2,\beta_2+\beta_3-\beta_1)}=\begin{cases}0, & \text{if } k_2\neq N,\\ 1, & \text{if } k_2=N.\end{cases}
$$

Hence

$$K_N(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,0) = [B(\alpha_1,\beta_1)B(\beta_2,\beta_3)]^{-1}\frac{(N-j+1)_{\beta_3-1}}{(i+1)_{\alpha_1+\beta_1-1}}$$

$$\cdot\sum_{k_1=0}^{\min(i,j)}\frac{(k_1+1)_{\alpha_1-1}(i-k_1+1)_{\beta_1-1}(j-k_1+1)_{\beta_2-1}}{(N-k_1+1)_{\beta_2+\beta_3-1}} \tag{3.14}$$

and

$$G_N(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,0)$$

$$= [B(\alpha_1,\beta_1)B(\beta_2,\beta_3)]^{-1}\left[\frac{(N-i+1)_{\beta_2+\beta_3-\beta_1-1}(N-j+1)_{\beta_3-1}}{(i+1)_{\alpha_1+\beta_1-1}(j+1)_{\alpha_1+\beta_2-1}}\right]^{1/2}$$

$$\cdot\sum_{k_1=0}^{\min(i,j)}\frac{(k_1+1)_{\alpha_1-1}(i-k_1+1)_{\beta_1-1}(j-k_1+1)_{\beta_2-1}}{(N-k_1+1)_{\beta_2+\beta_3-1}}. \tag{3.15}$$

The limiting values of the other relevant quantities are given by

$$\lambda_n = \frac{(\alpha_1)_n(\beta_3)_n}{(\alpha_1+\beta_2)_n(\beta_2+\beta_3)_n},$$

$$\mu_n = \frac{(\alpha_1)_n}{(\beta_2+\beta_3)_n}\left[\frac{(\beta_3)_n(\beta_2+\beta_3-\beta_1)_n}{(\alpha_1+\beta_2)_n(\alpha_1+\beta_1)_n}\frac{\Gamma(\alpha_1+\beta_1)\Gamma(\beta_2+\beta_3-\beta_1)}{\Gamma(\beta_3)\Gamma(\alpha_1+\beta_2)}\right]^{1/2} \tag{3.16}$$

(see [15]).

Case IV. $\alpha_2 \to \infty$, $N \to \infty$ such that $\alpha_2 = (c^{-1}-1)N$, $0 < c < 1$.

The Hahn polynomials $Q_n^{(k)}(i)$ approach the Meixner polynomials in this limit:

$$M_n^{(k)}(i;\alpha_1+\beta_k,\ c)$$

$$= \lim_{N\to\infty} {}_3F_2\left[\begin{matrix}-i,-n,n+\alpha_1+\beta_1+\beta_3-1+(c^{-1}-1)N \\ \alpha_1+\beta_k,-N\end{matrix};1\right] \tag{3.17}$$

$$= {}_2F_1(-i,-n;\alpha_1+\beta_k;1-c^{-1}), \qquad k=1,2.$$

Also

$$\pi_n^{(k)} \to c^n\frac{(\alpha_1+\beta_k)_n}{n!}, \tag{3.18}$$

and

$$\rho^{(k)}(i) \to (1-c)^{\alpha_1+\beta_k}c^i\frac{(\alpha_1+\beta_k)_i}{i!} \tag{3.19}$$

so that

$$R_n^{(k)}(i) \to \left\{(1-c)^{\alpha_1+\beta_k}\frac{c^{i+n}(\alpha_1+\beta_k)_n(\alpha_1+\beta_k)_i}{n!\,i!}\right\}^{1/2}M_n^{(k)}(i;\alpha_1+\beta_k,c). \tag{3.20}$$

The matrices K_N and G_N approach the following limits:

$$K_\infty(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,\infty)$$
$$= [B(\alpha_1,\beta_1)B(\beta_2,\beta_3)\Gamma(\beta_2+\beta_3-\beta_1)]^{-1}\frac{(1-c)^{\beta_2+\beta_3-\beta_1}c^{-i}}{(i+1)_{\alpha_1+\beta_1-1}}L_\infty(i,j) \tag{3.21}$$

and

$$G_\infty(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,\infty) = [B(\alpha_1,\beta_1)B(\beta_2,\beta_3)\Gamma(\beta_2+\beta_3-\beta_1)]^{-1}$$
$$\cdot\{(i+1)_{\alpha_1+\beta_1-1}(j+1)_{\alpha_1+\beta_2-1}c^{i+j}\}^{-1/2}(1-c)^{\beta_3+(\beta_2-\beta\alpha)/2}L_\infty(i,j), \tag{3.22}$$

where

$$L_\infty(i,j) = \sum_{k_1=0}^{\min(i,j)}(k_1+1)_{\alpha_1-1}(i-k_1+1)_{\beta_1-1}(j-k_1+1)_{\beta_2-1}$$
$$\cdot\sum_{k_2=\max(i,j)}^{\infty}\frac{c^{k_2}(k_2-i+1)_{\beta_2+\beta_3-\beta_1-1}(k_2-j+1)_{\beta_3-1}}{(k_2-k_1+1)_{\beta_2+\beta_3-1}}. \tag{3.23}$$

As for the eigenvalues μ_n we have

$$\mu_n\to\left[\frac{(\alpha_1+\beta_2)_n\Gamma(\alpha_1+\beta_1)}{(\alpha_1+\beta_1)_n\Gamma(\alpha_1+\beta_2)}\right]^{1/2}{}_3F_2\left[\begin{matrix}-n,\beta_2,\beta_2+\beta_3-\beta_1\\ \alpha_1+\beta_2,\beta_2+\beta_3\end{matrix};1\right]. \tag{3.24}$$

It may be remarked that for large N and α_2 the eigenvalue μ_n in (2.27) and the kernel $G_N(i,j)$ in (2.28) both behave like $N^{(\beta_2-\beta_1)/2}$. The expressions (3.23) and (3.24) therefore imply that, in view of (2.29), we have cancelled out this factor and retained the finite parts.

Case V. $\alpha_1\to 0$, $\alpha_2\to\infty$, $N\to\infty$, $\alpha_2=(c^{-1}-1)N$, $0<c<1$.

This is a special case of Case IV. The limits (3.17) through (3.20) remain the same with α_1 replaced by 0. Combining Cases I and IV we get

$$K_\infty(i,j;0,\beta_1,\beta_2,\beta_3,\infty)$$
$$= [B(\beta_2,\beta_3)\Gamma(\beta_2+\beta_3-\beta_1)]^{-1}(1-c)^{\beta_2+\beta_3-\beta_1}c^{-i}(j+1)_{\beta_2-1}$$
$$\cdot\sum_{k_2=\max(i,j)}^{\infty}\frac{c^{k_2}(k_2+1-i)_{\beta_2+\beta_3-\beta_1-1}(k_2+1-j)_{\beta_3-1}}{(k_2+1)_{\beta_2+\beta_3-1}}, \tag{3.25}$$

$$G_\infty(i,j;\beta_1,\beta_2,\beta_3,\infty) = [B(\beta_2,\beta_3)\Gamma(\beta_2+\beta_3-\beta_1)]^{-1}$$
$$\cdot[(i+1)_{\beta_1-1}(j+1)_{\beta_2-1}c^{-i-j}]^{1/2}(1-c)^{\beta_3+(\beta_2-\beta_1)/2}$$
$$\cdot\sum_{k_2=\max(i,j)}^{\infty}\frac{c^{k_2}(k_2+1-i)_{\beta_2+\beta_3-\beta_1-1}(k_2+1-j)_{\beta_3-1}}{(k_2+1)_{\beta_2+\beta_3-1}} \tag{3.26}$$

and the eigenvalues become

$$\mu_n = \left[\frac{(\beta_2)_n\Gamma(\beta_1)}{(\beta_1)_n\Gamma(\beta_2)}\right]^{1/2} {}_2F_1(-n, \beta_2 + \beta_3 - \beta_1; \beta_2 + \beta_3; 1)$$
$$= \left[\frac{(\beta_1)_n(\beta_2)_n\Gamma(\beta_1)}{\Gamma(\beta_2)}\right]^{1/2} \Big/ (\beta_2 + \beta_3)_n. \tag{3.27}$$

Case VI. $\alpha_2 \to 0$, $\beta_3, N \to \infty$ such that $\beta_3 = (c^{-1} - 1)N$, $0 < c < 1$.

As in Case IV the Hahn polynomials $Q_n^{(k)}(i)$ again approach the Meixner polynomials given in (3.18). Also $\pi_n^{(k)}(i)$, $\rho^{(k)}(i)$ and $R_n^{(k)}(i)$ approach the same limits as in (3.18), (3.19) and (3.20) respectively. Taking the above limits in (3.14) we obtain

$$K_\infty(i,j;\alpha_1,\beta_1,\beta_2,\infty,0) = [B(\alpha_1,\beta_1)\Gamma(\beta_2)(i+1)_{\alpha_1+\beta_1-1}]^{-1}(1-c)^{\beta_2}c^j$$
$$\cdot \sum_{k_1=0}^{\min(i,j)} c^{-k_1}(k_1+1)_{\alpha_1-1}(i-k_1+1)_{\beta_1-1}(j-k_1+1)_{\beta_2-1}. \tag{3.28}$$

The eigenvalue μ_n and the kernel $G(i,j)$ are again of the order $N^{(\beta_2-\beta_1)/2}$ for large N. Cancelling out this common factor as before we obtain

$$G_\infty(i,j;\alpha_1,\beta_1,\beta_2,\infty,0)$$
$$= [B(\alpha_1,\beta_1)\Gamma(\beta_2)]^{-1}\left[\frac{c^{i+j}}{(i+1)_{\alpha_1+\beta_1-1}(j+1)_{\alpha_1+\beta_2-1}}\right]^{1/2}(1-c)^{(\beta_1+\beta_2)/2}$$
$$\sum_{k_1=0}^{\min(i,j)} c^{-k_1}(k_1+1)_{\alpha_1-1}(i-k_1+1)_{\beta_1-1}(j-k_1+1)_{\beta_2-1} \tag{3.29}$$

with

$$\mu_n = (\alpha_1)_n\left[\frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1+\beta_2)(\alpha_1+\beta_1)_n(\alpha_1+\beta_2)_n}\right]^{1/2}. \tag{3.30}$$

4. The bilinear sums for Hahn polynomials. Some properties of the kernels K_N and G_N follow from the manner in which they were constructed. First of all, for real positive values of the parameters $\alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2$ such that $\beta_2 + \beta_3 > \beta_1$, we have the positivity of the kernels:

$$K_N(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2) > 0,$$
$$G_N(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2) > 0. \tag{4.1}$$

Also, for finite N,

$$K_N(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2) \leqq 1. \tag{4.2}$$

This follows from the fact that for $n = 0$, $\lambda_n = 1$, $Q_n^{(k)} = 1$, $k = 1, 2$, and, according to (2.18),

$$\sum_{j=0}^{N} K_N(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2) = 1. \tag{4.3}$$

In the special case $\beta_1 = \beta_2$, $G_N(i,j)$ is a symmetric matrix and $K_N(i,j)$ satisfies the detailed-balance property:

$$\rho(i)K_N(i,j) = \rho(j)K_N(j,i). \tag{4.4}$$

Properties (4.1) through (4.4) imply that $K_N(i,j)$ can be interpreted as a stochastic kernel.

As for the eigenvalue μ_n, which reduces to λ_n in the symmetric case $\beta_1 = \beta_2$, we have already proved in [15] that μ_n is positive and bounded, while λ_n is bounded by 1. Also, for large n,

$$\mu_n \sim \begin{cases} n^{-[\beta_2+\beta_3-|\beta_1-\beta_3|]}, & \beta_1 \neq \beta_3, \\ n^{-(\beta_2+\beta_3)}\log n, & \beta_1 = \beta_3. \end{cases} \tag{4.5}$$

Since for finite N the orthonormal systems $\{\mathbf{R}_n^{(k)}\}$, $n = 0, 1, \cdots, N$; $k = 1, 2$ can both be taken as complete bases for the $(N+1)$-dimensional vector space we have the following spectral representation of the matrix G_N:

$$G_N(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,\alpha_2) = \sum_{n=0}^{N} \mu_n R_n^{(1)}(i)R_n^{(2)}(j). \tag{4.6}$$

This, when written out in terms of the double sum on the left and the generalized hypergeometric functions ${}_4F_3(1)$ and ${}_3F_2(1)$ on the right, assumes a rather formidable look. However, the limiting forms for this bilinear sum corresponding to various limiting cases considered in the previous section may appear somewhat more interesting. Passing to the limit $\alpha_1 \to 0$ in (4.6) and using the special forms we obtained in Case I of § 3 we get, after some simplifications,

$$\begin{aligned}
&\sum_{k_2=\max(i,j)}^{N} \frac{(N-k_2+1)_{\alpha_2-1}(k_2-i+1)_{\beta_2+\beta_3-\beta_1-1}(k_2-j+1)_{\beta_3-1}}{(k_2+1)_{\beta_2+\beta_3-1}} \\
&\quad = \frac{B(\alpha_2,\beta_3)B(\beta_1,\beta_2+\beta_3-\beta_1)}{B(\beta_1,\alpha_2+\beta_2+\beta_3-\beta_1)} \cdot \frac{1}{(N+1)_{\alpha_2+\beta_2+\beta_3-1}} \\
&\quad \cdot (N-i+1)_{\alpha_2+\beta_2+\beta_3-\beta_1-1}(N-j+1)_{\alpha_2+\beta_3-1} \\
&\quad \cdot \sum_{n=0}^{N} \binom{N}{n} \frac{(\alpha_2)_n(\beta_1)_n(\beta_2)_n(\alpha_2+\beta_2+\beta_3-1)_n}{(\alpha_2+\beta_3)_n(\beta_2+\beta_3)_n(\alpha_2+\beta_2+\beta_3-\beta_1)_n(\alpha_2+\beta_2+\beta_3+N)_n} \\
&\quad \cdot \frac{2n+\alpha_2+\beta_2+\beta_3-1}{\alpha_2+\beta_2+\beta_3-1} \\
&\quad \cdot Q_n(i;\beta_1-1,\alpha_2+\beta_2+\beta_3-\beta_1-1,N)Q_n(j;\beta_2-1,\alpha_2+\beta_3-1,N).
\end{aligned} \tag{4.7}$$

This may be compared with the formula (5.5) of [15] for the Jacobi polynomials.

If we set $\alpha_2 = \beta_1 - \beta_3$ in (4.7), replace j by $N - j$ and use (3.7) and (3.8) we obtain

$$(N - i - j + \beta_2)_{\beta_3 - \beta_1}\, {}_4F_3\left[\begin{matrix}\beta_1 - \beta_3, \beta_2 + \beta_3 - 1, -i, -j \\ \beta_1, N - i - j + \beta_2, -N\end{matrix}; 1\right]$$

$$(4.8)\quad = \frac{B(\beta_1, \beta_2 + \beta_3 - \beta_1)\Gamma(N + \beta_2 + \beta_3)}{B(\beta_1, \beta_2)} \cdot \sum_{n=0}^{N} \frac{(\beta_1 - \beta_3)_n}{(\beta_2 + \beta_3)_n} \cdot \frac{(2n + \beta_1 + \beta_2 - 1)(\beta_1 + \beta_2 - 1)_n(\beta_1)_n N!(-1)^n}{(\beta_1 + \beta_2 - 1)\Gamma(N + \beta_1 + \beta_2 + n)n!(N - n)!(\beta_2)_n} \cdot Q_n(i; \beta_1 - 1, \beta_2 - 1; N)Q_n(j; \beta_1 - 1, \beta_2 - 1; N).$$

In expressing $Q_n(N - j)$ in terms of $Q_n(j)$ we have made use of the symmetry relation

$$(4.9)\quad Q_n(N - j; \beta_2 - 1, \beta_1 - 1; N) = (-1)^n \frac{(\beta_1)_n}{(\beta_2)_n} Q_n(j; \beta_1 - 1, \beta_2 - 1, N).$$

(See [13, (1.15)]; note a misprint.)

By virtue of the transformation property (3.7) we may also express the formula (4.8) in the form

$$(i + j - N + \beta_1)_{\beta_3 - \beta_1}\, {}_4F_3\left[\begin{matrix}\beta_1 - \beta_3, \beta_2 + \beta_3 - 1, -N + i, -N + j \\ \beta_2, i + j - N + \beta_1, -N\end{matrix}; 1\right]$$

$$(4.8')\quad = \frac{B(\beta_2, \beta_3)\Gamma(N + \beta_2 + \beta_3)}{B(\beta_1, \beta_2)} \sum_{n=0}^{N} \frac{(-1)^n(\beta_1 - \beta_3)_n(2n + \beta_1 + \beta_2 - 1)(\beta_1 + \beta_2 - 1)_n(\beta_1)_n N!}{(\beta_2 + \beta_3)_n(\beta_1 + \beta_2 - 1)\Gamma(N + \beta_1 + \beta_2 + n)n!(N - n)!(\beta_2)_n} \cdot Q_n(i; \beta_1 - 1, \beta_2 - 1; N)Q_n(j; \beta_1 - 1, \beta_2 - 1; N).$$

The kernels on the left of (4.8) and (4.8′) are positive for $\beta_2 + \beta_3 > \beta_1 > 0$, $\beta_2 > 0$ (when they are real) which follows from (3.8). If we now consider the degenerate case $\beta_1 - \beta_3 = -m$, where m is a nonnegative integer, we obtain, after some simplifications,

$$(N - i - j + \beta_2)_m\, {}_4F_3\left[\begin{matrix}-m, \beta_1 + \beta_2 + m - 1, -i, -j \\ -N, \beta_1, N - i - j + \beta_2\end{matrix}; 1\right]$$

$$(4.10)\quad = \sum_{n=0}^{m} c_{n,m} \frac{\Gamma(N + \beta_1 + \beta_2 + m)\Gamma(N + 1)}{\Gamma(N - n + 1)\Gamma(N + \beta_1 + \beta_2 + n)} \cdot \frac{(\beta_1)_n}{n!} \cdot Q_n(i; \beta_1 - 1, \beta_2 - 1, N)\frac{(\beta_1)_n}{n!} Q_n(j; \beta_1 - 1, \beta_2 - 1; N),$$

where

$$c_{n,m} = \frac{\Gamma(\beta_2 + m)\Gamma(m + 1)\Gamma(\beta_1 + \beta_2 + n - 1)\Gamma(\beta_1)\Gamma(n + 1)(2n + \beta_1 + \beta_2 - 1)}{\Gamma(\beta_1 + \beta_2 + m + n)\Gamma(\beta_2 + n)\Gamma(m - n + 1)\Gamma(\beta_1 + n)}.$$

This is the discrete analogue of Bateman's formula [2]:

$$\left(\frac{x+y}{2}\right)^m \frac{P_m^{(\beta_1-1,\beta_2-1)}((1+xy)/(x+y))}{P_m^{(\beta_1-1,\beta_2-1)}(1)} = \sum_{n=0}^{m} c_{n,m} P_n^{(\beta_1-1,\beta_2-1)}(x) P_n^{(\beta_1-1,\beta_2-1)}(y). \tag{4.11}$$

If $N-i-j \geqq 0$, the kernel on the left-hand side of (4.10) is obviously positive for $\beta_1 > 0$, $\beta_2 > 0$. If, however, $N-i-j<0$, we may use the form (4.8′) rather than (4.8) to show that the kernel still remains positive.

If we replace i, j in (4.10) by $N\xi$ and $N\eta$ respectively and proceed to the limit $N \to \infty$ we obtain (4.11) after using the well-known formula

$$\lim_{N\to\infty} Q_n(N\xi;\alpha,\beta,N) = \frac{P_n^{(\alpha,\beta)}(1-2\xi)}{P_n^{(\alpha,\beta)}(1)}, \tag{4.12}$$

and the transformation $\xi = \frac{1}{2}(1-x), \eta = \frac{1}{2}(1-y)$.

In the case $\beta_1 - \beta_3 \neq -m$ but $\beta_2 + \beta_3 > \beta_1 > 0$, $\beta_2 > 0$, if we take the same continuous limit and use the variables x, y we obtain

$$\begin{aligned} &\left(\frac{x+y}{2}\right)^{\beta_3-\beta_1} {}_2F_1\left(\beta_1-\beta_3, \beta_2+\beta_3-1; \beta_1; \tfrac{1}{2}\left(1-\frac{1+xy}{x+y}\right)\right) \\ &= \frac{B(\beta_1+\beta_2, \beta_2+\beta_3-\beta_1)}{B(\beta_2, \beta_2+\beta_3)} \sum_{n=0}^{\infty} (-1)^n \frac{(\beta_1-\beta_3)_n(\beta_1+\beta_2-1)_n n!}{(\beta_2+\beta_3)_n(\beta_1)_n(\beta_2)_n} \\ &\quad \cdot \frac{2n+\beta_1+\beta_2-1}{\beta_1+\beta_2-1} P_n^{(\beta_1-1,\beta_2-1)}(x) P_n^{(\beta_1-1,\beta_2-1)}(y); \qquad x+y \geqq 0. \end{aligned} \tag{4.13}$$

When $x+y<0$, we may use (4.8′) and derive a similar formula [15]. The kernel on the left of (4.13) is positive since that of (4.8) or (4.8′) is.

Let us now consider the bilinear sum corresponding to the case $\alpha_2 \to 0$. The sum on the right of (3.15) can be expressed in terms of a ${}_4F_3(1)$ series. Thus

$$\begin{aligned} &G_N(i,j;\alpha_1,\beta_1,\beta_2,\beta_3,0) \\ &= \frac{\Gamma(\alpha_1+\beta_1)\Gamma(\beta_2+\beta_3)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)(N+1)_{\beta_2+\beta_3-1}} \\ &\quad \cdot \left[\frac{(N-i+1)_{\beta_2+\beta_3-\beta_1-1}(N-j+1)_{\beta_3-1}}{(i+1)_{\alpha_1+\beta_1-1}(j+1)_{\alpha_1+\beta_2-1}}\right]^{1/2} (i+1)_{\beta_1-1}(j+1)_{\beta_2-1} \\ &\quad \cdot {}_4F_3\left[\begin{matrix}\alpha_1, 1-N-\beta_2-\beta_3, -i, -j \\ -N, 1-i-\beta_1, 1-j-\beta_2\end{matrix}; 1\right]. \end{aligned} \tag{4.14}$$

Using (3.16) and the expressions for $R_n^{(1)}(i)$, $R_n^{(2)}(j)$ for $\alpha_2 \to 0$ we see that equation (4.6) becomes

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix}\alpha_1, 1-N-\beta_2-\beta_3, -i, -j\\ -N, 1-i-\beta_1, 1-j-\beta_2\end{matrix};1\right]\\
&\quad=\frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+\beta_2+\beta_3)}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_1+\beta_2)\Gamma(\beta_2+\beta_3)(N+1)_{\alpha_1+\beta_2+\beta_3-1}}\\
&\quad\cdot\frac{(i+1)_{\alpha_1+\beta_1-1}(j+1)_{\alpha_1+\beta_2-1}}{(i+1)_{\beta_1-1}(j+1)_{\beta_2-1}}\\
&\quad\cdot\sum_{n=0}^{N}\binom{N}{n}\frac{(\alpha_1)_n(\alpha_1+\beta_2+\beta_3-1)_n(2n+\alpha_1+\beta_2+\beta_3-1)}{(\beta_2+\beta_3)_n(N+\alpha_1+\beta_2+\beta_3)_n(\alpha_1+\beta_2+\beta_3-1)}\\
&\quad\cdot Q_n(i;\alpha_1+\beta_1-1,\beta_2+\beta_3-\beta_1-1,N)\\
&\quad\cdot Q_n(j;\alpha_1+\beta_2-1,\beta_3-1,N).
\end{aligned}\tag{4.15}
$$

Note that the ${}_4F_3(1)$ series on the left is not Saalschützian unless $\alpha_1 = \beta_3 - \beta_1$. If we then take this special value of α_1 we obtain

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix}\beta_3-\beta_1, 1-N-\beta_2-\beta_3, -i, -j\\ -N, 1-i-\beta_1, 1-j-\beta_2\end{matrix};1\right]\\
&\quad=\frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_2+2\beta_3-\beta_1)}{\Gamma(\beta_3)\Gamma(\beta_2+\beta_3-\beta_1)\Gamma(\beta_2+\beta_3)(N+1)_{\beta_2+2\beta_3-\beta_1-1}}\\
&\quad\cdot\frac{(i+1)_{\beta_3-1}(j+1)_{\beta_2+\beta_3-\beta_1-1}}{(i+1)_{\beta_1-1}(j+1)_{\beta_2-1}}\\
&\quad\cdot\sum_{n=0}^{N}\binom{N}{n}\frac{(\beta_3-\beta_1)_n(\beta_2+2\beta_3-\beta_1-1)_n(2n+\beta_2+2\beta_3-\beta_1-1)}{(\beta_2+\beta_3)_n(N+\beta_2+2\beta_3-\beta_1)_n(\beta_2+2\beta_3-\beta_1-1)}\\
&\quad\cdot Q_n(i;\beta_3-1,\beta_2+\beta_3-\beta_1-1,N)\\
&\quad\cdot Q_n(j;\beta_2+\beta_3-\beta_1-1,\beta_3-1,N).
\end{aligned}\tag{4.16}
$$

5. The bilinear sums for Meixner polynomials. In this section we shall assume that

$$\operatorname{Re}(\beta_2+\beta_3) > |\operatorname{Re}(\beta_1-\beta_3)| \tag{5.1}$$

so that, according to (4.5),

$$\sum_{n=0}^{\infty} \mu_n < \infty. \tag{5.2}$$

Under this condition the bilinear sums with the Meixner polynomials converge for all c, $0 < c < 1$.

The first and most general formula in this class corresponds to Case IV of

§ 3. We obtain, after some simplifications,

$$\sum_{k_1=0}^{\min(i,j)} (k_1+1)_{\alpha_1-1}(i-k_1+1)_{\beta_1-1}(j-k_1+1)_{\beta_2-1}$$

$$\cdot \sum_{k_2=\max(i,j)}^{\infty} c^{k_2}\frac{(k_2-i+1)_{\beta_2+\beta_3-\beta_1-1}(k_2-j+1)_{\beta_3-1}}{(k_2-k_1+1)_{\beta_2+\beta_3-1}}$$

(5.3)
$$= \frac{\Gamma(\alpha_1)\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)\Gamma(\beta_2+\beta_3-\beta_1)}{\Gamma(\alpha_1+\beta_1)\Gamma(\beta_2+\beta_3)\Gamma(\alpha_1+\beta_2)}$$

$$\cdot (1-c)^{\alpha_1+\beta_1-\beta_3}c^{i+j}(i-1)_{\alpha_1+\beta_1-1}(j+1)_{\alpha_1+\beta_2-1}$$

$$\cdot \sum_{n=0}^{\infty} c^n\frac{(\alpha_1+\beta_2)_n}{n!}\,{}_3F_2\left[\begin{matrix}-n,\beta_2,\beta_2+\beta_3-\beta_1\\ \alpha_1+\beta_2,\beta_2+\beta_3\end{matrix};1\right]$$

$$\cdot M_n(i;\alpha_1+\beta_1,c)M_n(j;\alpha_1+\beta_2,c).$$

If we take the limit $\alpha_1 \to 0$ in (5.3) we get the sum corresponding to Case V of § 3:

(5.4)
$$\sum_{k_2=\max(i,j)}^{\infty} c^{k_2}\frac{(k_2-i+1)_{\beta_2+\beta_3-\beta_1-1}(k_2-j+1)_{\beta_3-1}}{(k_2+1)_{\beta_2+\beta_3-1}}$$

$$= \frac{\Gamma(\beta_3)\Gamma(\beta_2+\beta_3-\beta_1)}{\Gamma(\beta_2+\beta_3)}(1-c)^{\beta_1-\beta_3}c^{i+j}\sum_{n=0}^{\infty}\frac{c^n(\beta_1)_n(\beta_2)_n}{(\beta_2+\beta_3)_n n!}$$

$$\cdot M_n(i;\beta_1,c)M_n(j;\beta_2,c).$$

Finally, the bilinear formula corresponding to the Case IV of § 3 can be expressed as

$$\sum_{k_1=0}^{\min(i,j)} c^{-k_1}(k_1+1)_{\alpha_1-1}(i-k_1+1)_{\beta_1-1}(j-k_1+1)_{\beta_2-1}$$

(5.5)
$$= \frac{\Gamma(\alpha_1)\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_1+\beta_2)}(1-c)^{\alpha_1}(i+1)_{\alpha_1+\beta_1-1}(j+1)_{\alpha_1+\beta_2-1}$$

$$\cdot \sum_{n=0}^{\infty} c^n\frac{(\alpha_1)_n}{n!}M_n(i;\alpha_1+\beta_1,c)M_n(j;\alpha_1+\beta_2,c).$$

To close this section it may be remarked that the bilinear formulas (4.7) and (5.4) for the symmetric case $\beta_1=\beta_2$ were recently obtained by Cooper, Hoare and Rahman [7] by a different method. At the time of writing this paper it was brought to our attention that Al-Salam and Ismail [1] have derived what appears to be a different set of bilinear sums for Meixner polynomials.

Appendix.

THEOREM. *For* $\mathrm{Re}(\alpha_1, \beta_1, \beta_2, \beta_3, \alpha_2) > 0$, $\mathrm{Re}(\beta_2 + \beta_3 - \beta_1) > 0$ *and positive integers p, n such that* $0 \leqq p \leqq n$, *the treble sum*

$$(2.14)\qquad \begin{aligned} S(n;p) = {} & \frac{p!(\alpha_2+\beta_2+\beta_3-\beta_1)_p}{(\alpha_2+\beta_3)_p} \sum_{r=0}^{n-p} \frac{(-n+p)_r(n+A+p-1)_r}{(\alpha_2+\beta_3+p)_r r!} \\ & \cdot \sum_{m=0}^{p} \frac{(\beta_3)_{m+r}(\alpha_2)_{p-m}}{(p-m)!(\beta_2+\beta_3)_{m+r}} \\ & \cdot \sum_{l=0}^{m} \frac{(-1)^l(\beta_1)_{l+r}(\beta_2+\beta_3-\beta_1)_{m-l}}{(m-l)!\,l!\,(\alpha_1+\beta_1)_{l+r}(\alpha_2+\beta_2+\beta_3-\beta_1)_{p-l}} \end{aligned}$$

is independent of p and is equal to

$$(2.15)\qquad \lambda_n' = S(n;0) = {}_4F_3\left[\begin{matrix} -n, n+A-1, \beta_1, \beta_3 \\ \alpha_1+\beta_1, \beta_2+\beta_3, \alpha_2+\beta_3 \end{matrix} ;1\right].$$

Proof. We shall first try to reduce $S(n;p)$ to a double sum and then use Theorem 4 of [15] to draw the main conclusion. By manipulating the Pochhammer symbols, $S(n;p)$ can obviously be written as

$$(\mathrm{A}.1)\qquad S(n;p) = \frac{1}{(\alpha_2+\beta_3)_p} \sum_{r=0}^{n-p} \frac{(-n+p)_r(n+A+p-1)_r(\beta_1)_r(\beta_3)_r}{(\alpha_2+\beta_3+p)_r(\beta_2+\beta_3+p)_r(\alpha_1+\beta_1)_r r!} A_{p,r},$$

where

$$(\mathrm{A}.2)\qquad \begin{aligned} A_{p,r} = {} & \frac{(\beta_2+\beta_3+p)_r}{(\beta_2+\beta_3)_r} \sum_{m=0}^{p} \binom{p}{m} \frac{(\beta_3+r)_m(\alpha_2)_{p-m}}{(\beta_2+\beta_3+r)_m} \\ & \cdot \sum_{l=0}^{m} \binom{m}{l} \frac{(\beta_1+r)_l(\beta_2+\beta_3-\beta_1)_{m-l}(1-\alpha_2-\beta_2-\beta_3+\beta_1-p)_l}{(\alpha_1+\beta_1+r)_l}. \end{aligned}$$

Now, by (2.8),

$$\frac{(1-\alpha_2-\beta_2-\beta_3+\beta_1-p)_l}{(\alpha_1+\beta_1+r)_l} = {}_2F_1(-l, A+p+r-1; \alpha_1+\beta_1+r; 1).$$

Also

$$(\mathrm{A}.3)\qquad \begin{aligned} \frac{(\beta_1+r)_l(\beta_2+\beta_3-\beta_1)_{m-l}}{(\beta_2+\beta_3+r)_m} = {} & \frac{\Gamma(\beta_2+\beta_3+r)}{\Gamma(\beta_1+r)\Gamma(\beta_2+\beta_3-\beta_1)} \\ & \cdot \int_0^1 dt\, t^{\beta_2+\beta_3-\beta_1+m-l-1}(1-t)^{\beta_1+r+l-1}. \end{aligned}$$

Hence

$$\sum_{l=0}^{m}\binom{m}{l}\frac{(\beta_1+r)_l(\beta_2+\beta_3-\beta_1)_{m-l}(1-\alpha_2-\beta_2-\beta_3+\beta_1-p)_l}{(\beta_2+\beta_3+r)_m(\alpha_1+\beta_1+r)_l}$$

$$\text{(A.4)}\qquad =\frac{\Gamma(\beta_2+\beta_3+r)}{\Gamma(\beta_1+r)\Gamma(\beta_2+\beta_3-\beta_1)}\int_0^1 dt\, t^{\beta_2+\beta_3-\beta_1+m-1}(1-t)^{\beta_1+r-1}$$

$$\cdot\sum_{l=0}^{m}\binom{m}{l}\left(\frac{1-t}{t}\right)^l {}_2F_1(-l,A+p+r-1;\alpha_1+\beta_1+r;1).$$

Using the formula

$$\text{(A.5)}\qquad \sum_{n=0}^{\infty}\binom{\lambda}{n}s^n\,{}_2F_1(-n,b;c;z)=(1+s)^{\lambda}\,{}_2F_1\left(-\lambda,b;c;\frac{sz}{1+s}\right)$$

(see Bateman [4, p. 85]) we can see (A.4) as

$$\frac{(\beta_2+\beta_3+r)}{\Gamma(\beta_1+r)\Gamma(\beta_2+\beta_3-\beta_1)}\int_0^1 dt\, t^{\beta_2+\beta_3-\beta_1-1}(1-t)^{\beta_1+r-1}$$

$$\cdot\,{}_2F_1(-m,A+p+r-1;\alpha_1+\beta_1+r;1-t)$$

$$=\frac{\Gamma(\beta_2+\beta_3+r)}{\Gamma(\beta_1+r)\Gamma(\beta_2+\beta_3-\beta_1)}\int_0^1 dt\, t^{\beta_1+r-1}(1-t)^{\beta_2+\beta_3-\beta_1-1}$$

$$\cdot\,{}_2F_1(-m,A+p+r-1;\alpha_1+\beta_1+r;t).$$

Using (A.5) and a relation similar to (A.3) once again we obtain

$$\text{(A.6)}\qquad A_{p,r}=\frac{\Gamma(\beta_2+\beta_3)\Gamma(\beta_2+\beta_3+p+r)\Gamma(\alpha_2+\beta_3+p+r)}{\Gamma(\beta_2+\beta_3-\beta_1)\Gamma(\beta_2+\beta_3+p)\Gamma(\beta_1+r)\Gamma(\alpha_2)\Gamma(\beta_3+r)}$$

$$\cdot\int_0^1\int_0^1 dt\,dz\, t^{\beta_1+r-1}(1-t)^{\beta_2+\beta_3-\beta_1-1}z^{\beta_3+r-1}(1-z)^{\alpha_2-1}$$

$$\cdot\,{}_2F_1(-p,A+p+r-1;\alpha_1+\beta_1+r;tz).$$

Now we make use of the general Euler transform

$$\text{(A.7)}\qquad {}_{p+1}F_{q+1}[c,(a);d,(b),u]=\frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)}\int_0^1 dv\, v^{c-1}(1-v)^{d-c-1}\,{}_pF_q[(a);(b);uv].$$

(See Slater [16, p. 108].)

Using (A.7) twice in the double integral of (A.6) we finally obtain

$$\text{(A.8)}\qquad A_{p,r}=\frac{(\alpha_2+\beta_3+r)_p(\beta_2+\beta_3+r)_p}{(\beta_2+\beta_3)_p}$$

$$\cdot\,{}_4F_3\left[\begin{matrix}-p,A+p+r-1,\beta_1+r,\beta_3+r\\ \alpha_1+\beta_1+r,\beta_2+\beta_3+r,\alpha_2+\beta_3+r\end{matrix};1\right].$$

Hence we get

$$S(n;p) = \sum_{r=0}^{n-p} \frac{(-n+p)_r(n+A+p-1)_r(\beta_1)_r(\beta_3)_r}{(\alpha_2+\beta_3)_r(\beta_2+\beta_3)_r(\alpha_1+\beta_1)_r r!} \tag{A.9}$$
$$\cdot {}_4F_3\left[\begin{matrix}-p, A+p+r-1, \beta_1+r, \beta_3+r \\ \alpha_1+\beta_1+r, \beta_2+\beta_3+r, \alpha_2+\beta_3+r\end{matrix}; 1\right].$$

But this is exactly the same as (A.7) of [15] with the interchange of $\beta_1 \leftrightarrow \beta_2$ and $\beta_3 \leftrightarrow \beta_2 + \beta_3 - \beta_1$. By using Theorem 4 of [15], then, our present theorem also follows.

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METRIC CURVATURE, FOLDING, AND UNIQUE BEST APPROXIMATION*

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Abstract. In this paper, the concepts of metric curvature and folding of a C^1-representable manifold in a normed linear space are studied. With certain restrictions on the metric curvature and/or folding, one can obtain a neighborhood of unique best approximation from the manifold, and in some cases, the manifold can be shown to be Chebyshev. Several familiar examples, including some classes of γ-polynomials, are given.

1. Introduction. The purpose of this paper is to study unique best approximation from subsets of normed linear spaces which are C^1-manifolds with boundary. In order to study such problems from a general geometric vantage point, John R. Rice [12] introduced the concepts of folding and metric curvature (originally called curvature) in smooth, rotund, and finite-dimensional spaces. These concepts were generalized to uniformly smooth spaces by two of the authors in [13] (see also [5], [6]). In §§ 3 and 4 of the present work, we closely examine these concepts for C^1-manifolds with boundaries and obtain results which demonstrate several connections between local uniqueness, metric curvature, and folding.

In $C[a, b]$, there are nonlinear sets, for instance, the set of rational functions of degree no greater than (m, n), that are Chebyshev. We show that the fact that Haar embedded manifolds are Chebyshev (proved in a special case by Daniel Wulbert [17] and generalized by D. Braess [2]) follows from general results on the metric curvature of the manifold (see Theorem 6.1). For example, we see in § 7 that the set

$$\left\{\exp\left(\sum_{i=0}^{N} \alpha_i x^i\right) : \alpha_i \geqq 0\right\}$$

is Chebyshev in $C[a, b]$, where $0 < a < b$.

In $L^2([a, b], \mu)$ it is well known [14, p. 368] that a nonconvex boundedly compact subset is not Chebyshev (i.e., there is a point which does not have a unique best approximation from the set). In § 7 we exhibit for the first time some familiar subsets M of $L^2([a, b], \mu)$ each of which is a C^1-manifold with boundary and has a neighborhood of unique best approximation for M. Thus, for points in this neighborhood of M, steepest descent methods may be attempted. Many nonlinear regression problems fall into the above category.

Both of the above results are special cases of Theorem 5.1 in this paper, which basically states that every manifold M with boundary which has finite metric curvature and positive folding has a neighborhood of unique best approximation from M.

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2. Preliminary definitions and results. Throughout this paper, X will denote a normed linear space. Let A be a subset of X and $r>0$. Then we set

$$
\begin{aligned}
&B(x, r) \equiv \{y \in X : \|x-y\| \leqq r\}, \\
&B^0(x, r) \equiv \{y \in X : \|x-y\| < r\}, \\
(2.1)\qquad &\partial B(x, r) \equiv B(x, r) \backslash B^0(x, r), \\
&\operatorname{dist}(x, A) \equiv \inf\{\|x-a\| : a \in A\}, \\
&P_A(x) \equiv \{a \in A : \|x-a\| = \operatorname{dist}(x, A)\}.
\end{aligned}
$$

If $P_A(x)$ consists of a single point for each $x \in X$, then A is called *Chebyshev*. The mapping $x \to P_A(x)$ is called the *metric projection* from X to subsets of A. For each $x \in X$, the elements of $P_A(x)$ are called the *best approximations to x from A*. A point $a \in A$ is a *local best approximation to x from A* if there is a neighborhood U of a such that $a \in P_{U\cap A}(x)$. If a is the only element of $P_{U\cap A}(x)$ for some neighborhood U of a, then a is called a *strict local best approximation*. Throughout the paper, we will use θ to denote the zero element of any linear space. If A is a cone with vertex at the origin, then $S(A) \equiv \partial B(\theta, 1) \cap A$. In various examples we will use $l^p(n)$, $1 \leqq p \leqq \infty$, to denote R^n with the norms $\|(x_1, \cdots, x_n)\|_p \equiv (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $1 \leqq p < \infty$ and $\|(x_1, \cdots, x_n)\|_\infty = \sup_{1\leqq i \leqq n} |x_i|$.

We will be concerned with approximation from subsets M of X which have the following structure (see Braess [2]).

DEFINITION 2.1. A subset M of X is called a *C^1-representable manifold* (with boundary) if for each $m \in M$ there is a relative neighborhood $U \subset M$ of m satisfying the following three properties:

(i) There is a closed convex body $C \subset R^n$, a relatively open subset V of C, and a homeomorphism $g : V \to U$. (If $g^{-1}(m) = \theta$, then g is said to be *centered* for m.)

(ii) The map g is continuously Fréchet differentiable in V. (The Fréchet derivative of g at $a \in R^n$ is denoted by $g'(a)$.)

(iii) Assuming that g is centered for m, there is a continuous map k from U into $g'(\theta) \cdot (\bigcup_{\alpha>0} \alpha C)$ satisfying $k(m) = \theta$ and

$$
(2.2)\qquad \|u-m-k(u)\| = o(\|k(u)\|) \quad \text{as } u \to m.
$$

For each $m \in M$, we define a tangential cone C_mM at m as follows (see [2]).

DEFINITION 2.2. The vector $h \in X$ is called a *tangent vector* to M at m if there is a continuous map $[0, 1] \ni t \to m_t \in M$ satisfying

$$
(2.3)\qquad \|m_t - m - th\| = o(t) \quad \text{as } t \to 0.
$$

The set of all tangent vectors to M at m is called the *tangential cone* at m, denoted by C_mM.

As noted in [2], if M is an n-dimensional C^1-submanifold, then C_mM is an n-dimensional subspace. Further, it is known that C_mM is a subset of the Dubovickii–Milyutin cone [2].

Since we feel strongly that the geometry of the manifold M is more clearly elucidated by visualizing the tangential cone as being fixed to the manifold at m, we set $TC(m) \equiv m + C_mM$ and call $TC(m)$ the *tangent cone with vertex at m*. For

$x, y \in X$, let $r(x, y) = \{\lambda y + (1-\lambda)x : \lambda \geqq 0\}$ be the ray from x through y. Then the normal cone $N(m)$ of M at m may be defined by

$$N(m) = \{y : r(m, y) \perp TC(m)\}, \tag{2.4}$$

where, by definition, $r(m, y) \perp TC(m)$ if and only if $P_{TC(m)}(y)$ contains m.

Rice introduced the concept of metric curvature in [12] (there called curvature). Here we extend this definition to C^1-representable manifolds. The metric curvature is meant to be a measure of how quickly the manifold M bends *relative to the unit ball of the space* X; hence it depends on the norm of the space X into which M is embedded and thus is not an intrinsic property of M. It is for this reason that we have appended the adjective "metric". For a more complete motivation of the definition below, the reader is urged to read [12, pp. 190–198] and [13]. To define the metric radius of curvature at $m \in M$, we first define for $y \in N(m)$, the metric radius of curvature in the direction y as follows.

DEFINITION 2.3. For $z \in M$, let $\rho(m, y, z)$ be the radius of the smallest ball centered on $r(m, y)$ which contains m and z in its boundary. If there is no such ball, then $\rho(m, y, z) \equiv \infty$ and, if X is rotund, it is easy to see that this ball is unique. *The metric radius of curvature of M at m in the direction y*, written $\rho(m, y)$, is

$$\rho(m, y) \equiv \liminf_{z \to m} \{\rho(m, y, z) : z \in M\}. \tag{2.5}$$

The *metric radius of curvature*, $\rho(m)$, of M at m is

$$\rho(m) \equiv \inf_{y \in N(m)} \rho(m, y). \tag{2.6}$$

The *metric curvature* $\sigma(m)$ of M at m is defined by

$$\sigma(m) \equiv 1/\rho(m).$$

A concept somewhat related to metric curvature is that of folding. The *folding* of a set A at $a \in A$, denoted $\operatorname{fld}(a)$, is

$$\operatorname{fld}(a) \equiv \sup\{t_0 \in R^1 : B(a, t) \cap A \text{ is compact and connected for each } t \leqq t_0\}.$$

We will call an element $m \in M$ a *critical point* of $y \in X$ if $y \in N(m)$. The following lemma was proved by Braess in [2].

LEMMA 2.1. *Each local best approximation to y from M (a C^1-representable manifold) is a critical point.*

Finally we state a fundamental result due to D. Braess [2], which generalises Theorem 3.1 of [13].

THEOREM (Nonzero index theorem). *Let M be a C^1-representable manifold and let $y \in X$. Suppose that $A = \{m \in M : \alpha \leqq \|m - y\| \leqq \beta\}$ is compact and $B(y, \beta) \cap M$ is connected. If $m_1 \in A$ is a strict local best approximation to y and $m_2 \in A$ satisfies $\|m_2 - y\| \leqq \|m_1 - y\|$, then there is a critical point $z \in B(y, \beta) \cap M$ which is not a strict local best approximation to y from M.*

This theorem will be used several times throughout the course of this paper.

3. Metric curvature. In this section, we collect various results on metric curvature, most of which will be used in the later sections. We first state and prove two lemmas which clarify the notion of metric curvature.

LEMMA 3.1. *If $m \in M$, $y \in N(m)$, and m is not a strict local best approximation to y, then $\rho(m, y) \leqq \|m-y\|$.*

The proof of this lemma is a simple variation of Lemma 2.1 of [13].

LEMMA 3.2. *If $m \in M$, $y \in N(m)$, and m is a strict local best approximation to y, then $\rho(m, y) \geqq \|m-y\|$.*

Proof. Suppose to the contrary that $\rho(m, y) < \|m-y\|$. Using the definition of $\rho(m, y)$, we see that for every $\varepsilon > 0$, there exist $z_i \in M$, $z_i \to m$, and $y_i \in r(m, y)$ satisfying

$$\|y_i - z_i\| = \|y_i - m\| \leqq \rho(m, y) + \varepsilon. \tag{3.1}$$

Set $\varepsilon = \|m-y\| - \rho(m, y)$. Then

$$\|z_i - y_i\| = \|y_i - m\| \leqq \|m-y\|. \tag{3.2}$$

It follows that z_i is as close to y as m. Since $z_i \to m$, it is clear that m is not a strict local best approximation to y. This contradiction means that we must have $\rho(m, y) \geqq \|m-y\|$.

One of the useful properties of finite curvature is illustrated in the next theorem.

THEOREM 3.1. *Let M be a C^1-representable manifold with $\sigma(m) < \infty$ for all $m \in M$. Then the map $P_M : X \setminus M \to M$ is a surjection.*

Proof. Suppose $m \in M$ satisfies

$$m \notin P_M(X \setminus M). \tag{3.3}$$

Then for any $y \in N(m)$, m is not a (strict) local best approximation to y from M. Lemma 3.1 then implies that $\rho(m, y) = 0$ for any $y \in N(m)$ and hence $\sigma(m) = \infty$. This is the contrapositive of Theorem 3.1 and completes the proof.

It is not hard to see that there are C^1-manifolds C (even in R^3) for which $P_C(X \setminus C) \neq C$. For instance, consider $g : R^2 \to R^3$ defined by

$$g(x, y) = (x, y, -(1-|x|^{3/2})^{2/3} + (1-|y|^{3/2})^{2/3})$$

in a neighborhood of (0, 0). Note that cross sections of the X-Z and Y-Z planes give locally at (0, 0) the bottom (resp. top) of the $l^{3/2}(2)$ unit sphere. It is clear that if R^3 has the $l^2(3)$ norm and $C = g(R^2)$, then no point of $R^3 \setminus (0, 0)$ projects onto (0, 0), since the $l^{3/2}$ ball bends more rapidly than the l^2 ball.

Many results on metric curvature are somewhat uninitiative. In particular, in nonrotund spaces, finite-dimensional linear varieties which are not Chebyshev have infinite metric curvature at each point. A related fact is contained in the following.

PROPOSITION 3.1. *Let $X = L^1[0, 1]$. Then any C^1-representable manifold M which is locally contained in some finite-dimensional subspace has $\sigma(x) = \infty$ at each point. Furthermore, every neighborhood of M contains a point which has more than one best approximation from M.*

Proof. It is known [14, p. 232] that for any finite-dimensional linear manifold $L \subset L^1[0, 1]$ and $y \in L$, there is a $y \in L^1[0, 1]$ so that $P_L(y)$ contains a relatively open subset in L containing x. Let $m \in M$ and L^m be a finite-dimensional subspace containing a relative neighborhood of $m \in M$. Then there is a $y \in L^1[0, 1]$ so that $P_{L^m}(y)$ contains a neighborhood of m relative to L^m. Since $TC(m) \subset L^m$, it is clear that $y \in N(m)$. But since $P_{L^m}(y)$ contains a neighborhood of $m \in M$, then for any $y' \in r(m, y) \setminus m$, $P_{L^m}(y')$ contains a neighborhood of $m \in M$, and hence m is not a strict local best approximation from m. Using Lemma 3.1, one sees that $\rho(m, y) = 0$ and hence $\sigma(m) = \infty$. Further, if y' is chosen close enough to M, then $P_M(y') = P_{L^m}(y') \cap M$ has more than one element.

As the previous proposition shows, the metric curvature is infinite for many manifolds in nonrotund spaces. However, even in $l^2(2)$, "most" C^1-manifolds have infinite metric curvature everywhere. This is made precise in the following proposition. To see this, we should first remark that if $f : R^1 \to R^1$ is a $C^1(R^1)$-function, then the manifold $M \subset R^2$ given by the graph of f is a C^1-manifold.

PROPOSITION 3.2. *The set of all real-valued continuous periodic functions g on R^1 with period $\alpha > 0$, such that each corresponding manifold $M = Mg = \{(x, \int_0^x g) : x \in R^1\}$ in $l^2(2)$ has infinite curvature everywhere, is a set of second category.*

Proof. Following Chui–Smith [5], [6], we compute the metric radius of curvature at $(x, f(x))$, where $f(x) = \int_0^x g$ for some continuous function g. Since $n \equiv (x + f'(x), -1 + f(x)) \in N[(x, f(x))]$, we need to solve for t in the following equality ($\beta \neq 0$):

$$\begin{aligned} &\|(x, f(x)) - [tn + (1-t)(x, f(x))]\| \\ &\quad = \|(x+\beta, f(x+\beta)) - [tn + (1-t)(x, f(x))]\|. \end{aligned} \tag{3.4}$$

Squaring both sides and solving for t, one obtains

$$t(\beta) \equiv t = \frac{-\beta^2 - [f(x+\beta) - f(x)]^2}{2[f(x+\beta) - f(x) - f'(x)\beta]}. \tag{3.5}$$

We will show that lim inf $t(\beta) = 0$ as β tends to zero for a set of functions f whose derivatives $f' = g$ form a second category set in the space of periodic functions with period $\alpha > 0$. This will show, via (3.4), that the metric radius of curvature is zero at each point. Let $\Gamma = \{g \in C[0, \alpha] : g(0) = g(\alpha)\}$ and then extend g periodically to R^1. Set $f(t) \equiv \int_0^t g(\tau)\, d\tau$ so that $f' \equiv g$. Since f is C^1, $[f(x+\beta) - f(x)]^2 = O(\beta^2)$ as β tends to zero, so that we only need to prove that

$$\liminf_{\beta \to 0} \frac{-\beta^2}{2[f(x+\beta) - f(x) - f'(x)\beta]} = 0 \tag{3.6}$$

for some second category set of $g = f'$ in Γ. To this end, set

$$K_n = \left\{ g : \text{there exists } 0 \leqq x \leqq \alpha \text{ so that } \left| \frac{1}{\beta} \int_x^{x+\beta} g(t)\, dt - g(x) \right| / \beta \leqq n \right. \\ \left. \text{for all } \frac{1}{n} > \beta > 0 \right\}. \tag{3.7}$$

Following arguments similar to those in [8, p. 23], it is easy to see that K_n is closed and nowhere dense. Now clearly $\Gamma \setminus \bigcup_{n=1}^{\infty} K_n$ is second category and each g in this set gives rise to an f whose graph has infinite metric curvature everywhere in $l^2(2)$. This completes the proof of Proposition 3.2.

In [5], [6], two of the authors computed bounds on the metric curvature in a Hilbert space for sufficiently smooth manifolds. The following theorem is an immediate consequence of the definitions and the two earlier studies.

THEOREM 3.2. *Let M be a C^1-representable manifold in a Hilbert space. If there is a C^2 centered parameterization g as in Definition* 2.1, *then for $m = g(\theta)$,*

$$\sigma(m) \leqq \|g''(\theta)\| \, \|(g'(\theta))^{-1}\|^2.$$

In particular, it was shown that for C^2-curves in $l^2(2)$, the concepts of metric curvature and the usual calculus definition of curvature agree.

4. Folding. As in the previous section, we introduced the metric curvature as a measure of "how quickly" a manifold "bends", so we introduce the folding of a set A at a point $a \in A$ as a measure of "how much" the set A turns "back toward a". More precisely, recall that

$$\operatorname{fld}(a) = \operatorname{fld}_A(a) = \sup\{t_0 \in R^1 : B(a, t) \cap A \text{ is compact and connected for all } t \leqq t_0\}.$$

Clearly, $0 \leqq \operatorname{fld}(a) \leqq \infty$. If A turns "smoothly" around a, then we will show that $\operatorname{fld}(a) > 0$; we will also be given a condition which will insure that $\operatorname{fld}(a) = \infty$. It is clear from the definitions that there are connections among the metric curvature, the folding, and (local) uniqueness. We will prove in this section that if the metric curvature is identically zero, then the folding is identically infinity. An example will be given to show that if a manifold is not smooth at a point, then it may have zero or infinite metric curvature, depending on the shape of the ball of X.

The first result of this section involves a condition on the parameterization g which will be satisfied whenever $g'(\theta)$ has a trivial kernel. Thus this theorem generalizes [13, Thm. 1.4].

THEOREM 4.1. *Suppose M is a C^1-representable manifold and, for some $m \in M$, there is a parameterization g, centered at m, such that for all $b \in S(\bigcup_{\alpha>0} \alpha C)$, we have*

$$\|g'(\theta) \cdot b\| > 0. \tag{4.1}$$

Then $\operatorname{fld}(m) > 0$.

Proof. Suppose that $\operatorname{fld}(m) = 0$. Since M is C^1-representable, there relative neighborhood U of m which is homeomorphic to a relatively open subset V of a closed convex body C in R^k; thus, there is an $\varepsilon > 0$ so that $M \cap B(m, \varepsilon)$ is compact. Now if $\operatorname{fld}(m) = 0$, there must exist a sequence $\{\varepsilon_n\}$ such that $0 < \varepsilon_n < \varepsilon$, $\varepsilon_n \to 0$ and $B(m, \varepsilon_n) \cap M \equiv A_n$ is disconnected. Choose a connected component C_n of A_n which does not contain m. Since C_n is compact, there is an $x_n \in C_n$ satisfying

$$\|m - x_n\| = \operatorname{dist}(m, C_n).$$

We may apply Lemma 2.1 to obtain

$$\|m - x_n\| = \text{dist}\,(m, TC(x_n)).$$

But we will show, under the hypothesis of this lemma, that

$$\text{dist}\,(m, TC(x_n)) = o(\|m - x_n\|),$$

which yields a contradiction.

First let $b_n \in R^k$ be such that $g(b_n) = x_n$. Thus

$$\begin{aligned} \text{dist}\,(m, TC(x_n)) &\leqq \|g(\theta) - (g(b_n) + g'(b_n)(-b_n))\| \\ &\leqq \|g(\theta) - g(b_n) - g'(\theta)(-b_n)\| + \|g'(\theta)(-b_n) - g'(b_n)(-b_n)\| \qquad (4.2) \\ &= o(\|b_n\|). \end{aligned}$$

We will now show that

$$\frac{\|g(b_n) - g(\theta)\|}{\|b_n\|} \geqq \delta > 0$$

as $b_n \to \theta$, which together with (4.2) will complete the proof. To this end, set

$$\delta_0 \equiv \inf \left\{ \|g'(\theta)b\|;\ b \in S\left(\overline{\bigcup_{\alpha > 0} \alpha C}\right) \right\}.$$

Note that δ_0 is greater than zero by the compactness of $S(\overline{\bigcup_{\alpha>0} \alpha C})$ and the assumption (4.1). Let

$$\delta_n \equiv \max\,\{\|g'(\theta) - g'(\gamma)\| : \gamma \in [\theta, b_n]\},$$

where $[\theta, b_n]$ is the line segment from θ to b_n. Note that $\delta_n \to 0$ as $n \to \infty$. In the step below, we will use a version of the mean value theorem (cf. [1]). For convenience, set

$$\text{sgn}\ b_n = b_n / \|b_n\|.$$

Then

$$\frac{\|g(b_n) - g(\theta)\|}{\|b_n\|} = \lim_{i \to \infty} \left\| \sum_{j=1}^{i} t_{ij}^{(n)} g'(\gamma_{ij}^{(n)})\ \text{sgn}\ b_n \right\|,$$

where $\sum_{j=1}^{i} t_{ij}^{(n)} = 1$, $0 \leqq t_{ij}^{(n)} \leqq 1$, $\gamma_{ij}^{(n)} \in [\theta, b_n]$. Thus

$$\begin{aligned} \frac{\|g(b_n) - g(\theta)\|}{\|b_n\|} &\geqq \|g'(\theta)\ \text{sgn}\ b_n\| - \left\| \left[g'(\theta) - \lim_{i \to \infty} \sum_{j=1}^{i} t_{ij}^{(n)} g'(\gamma_{ij}^{(n)}) \right] \text{sgn}\ b_n \right\| \\ &\geqq \delta_0 - \lim_{i \to \infty} \left\| \sum_{j=1}^{i} t_{ij}^{(n)} (g'(\theta) - g'(\gamma_{ij}^{(n)})) \right\| \\ &\geqq \delta_0 - \delta_n \\ &\geqq \delta_0 / 2 \end{aligned}$$

for large enough n, thus completing the proof.

In Theorem 4.1, it could happen that (4.1) holds for some parameterization but not for others. For example, with

$$C_1 = \{(x_1, x_2) : x_2 \geqq x_1 \geqq 0\} \quad \text{and} \quad g_1(x_1, x_2) = (x_1^2, x_2),$$

we have that $M = g_1(C_1)$ is a C^1-representable manifold satisfying (4.1). But with

$$C_2 = \{(x_1, x_2) : x_2^2 \geqq x_1 \geqq 0\} \quad \text{and} \quad g_2(x_1, x_2) = (x_1^4, x_2),$$

we have the same $M = g_2(C_2)$ but g_2 does not satisfy condition (4.1).

THEOREM 4.2. *Let M be a connected C^1-representable boundedly compact manifold such that $\sigma(m) = 0$ for all $m \in M$. Then* fld $(m) = \infty$ *for all $m \in M$.*

Proof. The proof proceeds by contradiction. Let $m_0 \in M$ with fld $(m_0) = r < \infty$. Then there is a ball $B(m, \lambda)$ with $0 < \lambda < \infty$ such that $B(m, \lambda) \cap M$ is disconnected. Since M is boundedly compact, there is a $y \in M$ such that y is a local best approximation to m_0 and y is not in the same connected component of $B(m_0, \lambda) \cap M$ as is m_0. In addition, M is connected and locally path-connected, which implies that M is path-connected. Thus there is a $\lambda_1 > \lambda$ so that y and m_0 are in the same connected component G of $B^0(m_0, \lambda_1) \cap M$. G is clearly a C^1-representable manifold. If y is not a strict local best approximation to m_0, then by Lemma 3.1, we see that $\rho(y) \leqq \|m_0 - y\|$. If y is a strict local best approximation to m_0, then the nonzero index theorem implies that there is a $z \in G$ which is a critical point with respect to m_0 but which is not a strict local best approximation to m_0. Hence, again by Lemma 3.1, we have $\rho(z) \leqq \|m_0 - z\|$. In either case, we contradict the zero curvature hypothesis.

In some cases, a simple condition on the embedding map g leads to a result on folding.

PROPOSITION 4.1. *Suppose C is a convex subset of R^n and g is a homeomorphism of C onto $M = g(C)$. Let $m = g(\alpha)$ and suppose that*

$$\|g(\alpha + \lambda(\beta - \alpha)) - g(\alpha)\| \leqq \|g(\beta) - g(\alpha)\| \tag{4.3}$$

for all $\lambda \in (0, 1)$ and all $\beta \in C$. Then fld $(m) = \infty$ *provided M is boundedly compact.*

Proof. Suppose there exists a $\rho > 0$ such that $B(g(\alpha), \rho) \cap M$ is disconnected. Let β be chosen so that $g(\beta)$ and $g(\alpha)$ are in different components. Then there exists a $\lambda \in (0, 1)$ such that

$$\|g(\alpha + \lambda(\beta - \alpha)) - g(\alpha)\| > \|g(\beta) - g(\alpha)\|,$$

which contradicts (4.3).

COROLLARY 4.1. *There exists a non-Chebyshev, boundedly compact manifold in $l^2(2)$ so that* fld $(m) = \infty$ *and $\sigma(m) = \infty$ for all $m \in M$.*

Proof. Let g be a positive periodic function such that $f(t) = \int_0^t g(s)\, ds$ satisfies the conditions of Proposition 3.2. By Proposition 4.1, $M = $ graph (f) has the property that fld $(m) = \infty$ and by Proposition 3.2, $\sigma(m) = \infty$ for all $m \in M$. M is not Chebyshev since it is boundedly compact but not convex.

In particular, this corollary shows that the converse of Theorem 5.4 does not hold.

Folding is a concept which depends very much on the smoothness of the manifold and the shape of the ball. For instance, Theorem 4.1 shows that under certain smoothness conditions, fld $(x) > 0$ for each $x \in M$. On the other extreme, consider the non-C^1-manifold $M = \text{graph}\,(2x \sin(1/x))$ in $l^\infty(R^2)$. It is easy to check that $\text{fld}(0,0)=0$. However, by renorming R^2 with $\|(x_1, x_2)\| = \max\{2|x_1|, |x_2|\}$, $\text{fld}\,(0,0)=\infty$ for $(0,0)\in M$.

5. Central results. The first result in this section gives general conditions on M which entail that M has a neighborhood of unique best approximation. The results of the preceding sections will then allow us to obtain more specialized results.

THEOREM 5.1. *Let M be a C^1-representable manifold in X. Suppose that $\sigma(m)$ is bounded on compact subsets of M and* fld $(m)>0$ *for all $m\in M$. Then M has a neighborhood U in X so that each $u\in U$ has a unique best approximation from M.*

Proof. Let $m\in M$. Pick an ε satisfying $0<\varepsilon<\text{fld}\,(m)$, and set $\rho = \inf\{\rho(z) : z\in M\cap B(m,\varepsilon)\}$. By assumption, $\rho>0$. Choose positive $\varepsilon_0 \equiv \varepsilon(m) < \min(\rho/3, \varepsilon/4)$. We will show that if $y\in B(m,\varepsilon_0)$, then y has a unique best approximation from M. Suppose, to the contrary, that $P_M(y)$ contains two distinct points $\{m_1, m_2\}$ for some $y\in B(m,\varepsilon_0)$. Clearly, $\|y-m_i\|\leqq\|y-m\|$ for $i=1,2$. Thus

$$\{m_1, m_2\}\subseteq B(m, 2\varepsilon_0)\cap M\subseteq B(y, 3\varepsilon_0)\cap M\subseteq B(m, 4\varepsilon_0)\cap M.$$

This, together with the definition of folding, implies that $B(y, 3\varepsilon_0)\cap M$ contains a compact connected component C containing $\{m_1, m_2\}$. Since $y\in N(m_i)$ and $\|m_i - y\|<\rho\leqq\rho(m_i)$, both m_1 and m_2 are strict local best approximations to y. By applying the nonzero index theorem to the C^1-representable manifold C, there is a critical point $z\in C$ with respect to y which is not a strict local best approximation to y. Hence $B(y, \|y-z\|)$ contains a sequence in M converging to z. By Lemma 3.1, we have

$$\rho(z)\leqq\|y-z\|\leqq 3\varepsilon_0<\rho.$$

This is a contradiction since $z\in B(m,\varepsilon)\cap M$ implies $\rho(z)\geqq\rho$. To complete the proof, we note that we can take

$$U = \bigcup_{m\in M} B(m, \varepsilon(m)).$$

If M is a C^1-representable manifold for which the kernel of $g'(\theta)$ is trivial, then $\text{fld}\,(g(\theta))\neq 0$ for all $m\in M$. Hence, for such manifolds, this hypothesis may be dropped in Theorem 5.1. More generally, we may use Theorem 4.1 to obtain immediately the following theorem.

THEOREM 5.2. *Suppose M is a C^1-representable manifold such that, for every $m\in M$, there is a centered parameterization g satisfying $g'(\theta)\cdot b\neq 0$ for every $b\in S(\bigcup_{\alpha>0}\alpha C)$. Then, if the curvature is bounded on compact subsets of M, there is a neighborhood U of M such that every element of U has a unique best approximation from M.*

As another consequence of Theorem 5.1, we obtain the following.

THEOREM 5.3. *Let M be a connected boundedly compact C^1-representable manifold with $\sigma(m)=0$ for all $m\in M$. Then M is Chebyshev.*

Proof. By Theorem 4.2, we have $\text{fld}\,(m)=\infty$ for all $m\in M$. Thus, in the notation of Theorem 5.1, $\varepsilon(m)$ may be taken as large as we like. Hence, every $y\in X$ has a unique best approximation from M.

There is a partial converse to Theorem 5.3, namely, the following.

THEOREM 5.4. *If M is a boundedly compact Chebyshev C^1-representable manifold, then for all $m \in M$, we have* $\operatorname{fld}(m) = \infty$ *and either* $\sigma(m) = 0$ *or* $\sigma(m) = \infty$.

Before proving this theorem, we remark that there exist boundedly compact Chebyshev C^1-representable manifolds M for which $\sigma(m) = 0$ for some $m \in M$ and $\sigma(m') = \infty$ for some $m' \in M$. In particular, consider $X = l^1(2)$ and the C^1-manifold given by the mapping

$$R^1 \ni t \to (t, \sin t).$$

Then M is the graph of the sine curve and for every $(x_1, x_2) \in R^2$,

$$P_M[(x_1, x_2)] = (x_1, \sin x_1).$$

Thus M is Chebyshev; it is also easy to see that $\operatorname{fld}[(t, \sin t)] = \infty$ for all $t \in R^1$. Furthermore, $\sigma[(t, \sin t)] = 0$ as long as $\sin'(t) \neq \pm 1$ (i.e., the tangent plane is not parallel to any facet of the ball). If $\sin'(t) = \pm 1$, or $t = k\pi$ for $k = 0, \pm 1, \cdots$, then it is easy to check that $\sigma((t, \sin t)) = \infty$.

We now prove Theorem 5.4. Since M is boundedly compact and Chebyshev, it is easy to see that $P_M(\cdot)$ is continuous. Thus, by a result of Wulbert (cf. Theorem 3 in [15]), the folding must be infinite at each point. Suppose $\sigma(m) \neq \infty$. Then, for every $y \in N(m)$, there is a y' on the line segment between m and y so that $\|x - y'\| < \rho(m) \leqq \rho(m, y')$. Thus m is a strict local best approximation to y'. Since M is Chebyshev, one may apply Theorem 3 in [15] to show that m is the unique best approximation to y'. Since M is a sun (cf. [14, Thm. 3]), every $y \in r(m, y')$ has m as the unique best approximation from M. Hence, from Lemma 3.2, $\rho(m, y) \geqq \|m - y\|$ for every y. Thus $\rho(m) = \infty$.

6. Chebyshev manifolds. We first apply the results above to Haar embedded manifolds in $C(T)$.

DEFINITION 6.1. [2] Let $0 \leqq m \leqq n$ and $h_1, \cdots, h_n \in C(T)$. For any subset J of natural numbers with

$$\{1, 2, \cdots, m\} \subseteq J \subseteq \{1, 2, \cdots, n\},$$

let the functions $\{h_i : i \in J\}$ be a Haar system. Then

$$K = \left\{ h = \sum_{i=1}^{n} a_i h_i : a_i \in R \text{ for } i = 1, 2, \cdots, m,\ a_i \geqq 0 \text{ for } i = m+1, \cdots, n \right\}$$

is called a *Haar cone*. A C^1-representable manifold $M \subseteq C(T)$ is called a *Haar embedded manifold* if all the tangential cones are Haar cones.

Results on the uniqueness of approximation from such manifolds were first considered by Wulbert [16] and proved in the generality to be stated in Theorem 6.1 below by Braess [2] without reference to metric curvature or folding. An essential property used is that of strong uniqueness.

DEFINITION 6.2. Let A be a subset of X. Then $a \in A$ is called a *strong* (*local*) *best approximation to y from A* if there is a $\lambda > 0$ such that, for every $u \in A$ (in $A \cap U$ for some neighborhood U of x),

$$\|y - u\| \geqq \|y - a\| + \lambda \|u - a\|. \tag{6.1}$$

Intuitively, suppose m is an element of a C^1-representable manifold M. Then, by (2.2), M and $TC(m)$ are "separating slowly". If m is a strong local best approximation to y from $TC(m)$, then $TC(m)$ and $\partial B(y, \|m-y\|)$ are "separating linearly". Hence it should be expected that m is a strong local best approximation to y from M. Similarly, if m is a strong local best approximation to y from M. Similarly, if m is a strong local best approximation to y from M, one should expect the curvature at m in the direction y to be zero since, relative to the sphere $\partial B(y, \|y-m\|)$, the manifold is not bending at all. This is made explicit in the following lemma.

LEMMA 6.1. *Suppose M is a C^1-representable manifold. A necessary and sufficient condition for $m \in M$ to be a strong local best approximation to y from M is that m be a strong best approximation to y from $TC(m)$.*

This theorem was first proved for C^1-representable manifolds without edge for which the kernel of $g'(\theta)$ is trivial by Wulbert [16]. The sufficiency part was proved by Braess [2, Thm. 5.2]. The necessity is proved as follows. Since m is a strong local best approximation to y from M, there is a neighborhood U of m in M and a $\lambda > 0$ such that

$$\|y-u\| \geqq \|y-m\| + \lambda\|u-m\| \quad \text{for } u \in U.$$

If m is not a strong best approximation to y out of $TC(m)$, then there is a sequence h_n in C_mM so that

$$\|y-m-h_n\| < \|y-m\| + (1/n)\|h_n\|. \tag{6.2}$$

Since $h_n \in C_mM$, there is a continuous map from $[0, 1]$ into M with $t \to m_n(t)$ satisfying

$$\|m_n(t) - m - th_n\| = o(t) \quad \text{as } t \to 0.$$

Thus

$$\|m_n(t) - m\| \geqq t\|h_n\| - o(t). \tag{6.3}$$

Now for t so small that $m_n(t) \in U$, we have

$$\begin{aligned}
\|y-m\| + \lambda\|m_n(t) - m\| &\leqq \|y - m_n(t)\| \\
&\leqq \|y - m - th_n\| + \|m_n(t) - m - th_n\| \\
&= \|(1-t)(y-m) + t(y-m-h_n)\| + o(t) \\
&\leqq \|y-m\| + t[\|y-m-h_n\| - \|y-m\|] + o(t).
\end{aligned}$$

Cancelling $\|y-m\|$ and using (6.2), one obtains

$$\lambda\|m_n(t) - m\| \leqq t(1/n)\|h_n\| + o(t).$$

By (6.3),

$$\lambda t\|h_n\| - o(t) \leqq t(1/n)\|h_n\| + o(t).$$

Dividing by t and letting $t \to 0$, we obtain

$$\lambda\|h_n\| \leqq (1/n)\|h_n\|,$$

which is a contradiction.

In order to compute the curvature, we need the following lemma, which may be considered in the light of the defining property of suns.

LEMMA 6.2. *Suppose x is a strong best approximation to y from a convex set A. Then every point on $r(x, y)$ has x as a strong best approximation from A.*

Proof. Let $\lambda > 0$ satisfy

$$\|y-z\| \geqq \|y-x\| + \lambda\|x-z\| \quad \text{for all } z \in A.$$

Let $y' = \gamma y + (1-\gamma)x \in r(x, y)$, $\gamma \geqq 0$.

Case 1. $0 \leqq \gamma \leqq 1$. We have the estimates

$$\begin{aligned} \|y'-z\| &\geqq \|y-z\| - \|y-y'\| \\ &\geqq \|y-x\| + \lambda\|x-z\| - \|y-y'\| \\ &= \|y'-x\| + \lambda\|x-z\|. \end{aligned}$$

(Note: convexity not used here.)

Case 2.

$$\begin{aligned} 1 \leqq \gamma : \|y'-x\| &= \|\gamma y + (1-\gamma)x - z\| \\ &= \gamma\left\|y - \left[\left(1-\frac{1}{\gamma}\right)x + \frac{1}{\gamma}z\right]\right\| \\ &\geqq \gamma\left[\|y-x\| + \lambda\left\|x - \left\{\left[\left(1-\frac{1}{\gamma}\right)x + \frac{1}{\gamma}z\right]\right\}\right\|\right] \\ &= \gamma(y-x)\| + \lambda\|x-z\| \\ &= \|y'-x\| + \lambda\|x-z\|. \end{aligned}$$

PROPOSITION 6.1. *Suppose m is a strong best approximation to y in $TC(m)$. Then $\rho(m, y) = \infty$.*

Proof. Lemma 6.2 implies that every point $y' \in r(m, y)$ has x as a strong local best approximation in $TC(m)$. Hence by Lemma 6.1, m is a strong best approximation to y' in M. By Lemma 3.2, $\rho(m, y) \geqq \|m-y'\|$. Thus $\rho(m, y) = \infty$ for every $y \in N(m)$.

Braess has shown that, for Haar embedded manifolds, the best approximation to any $y \in X = C(T)$ from any tangent cone $TC(m)$ is strong. This, together with Proposition 6.1 and Theorem 5.3, immediately imply the following result of Braess (cf. [2]).

THEOREM 6.1. *Every connected, boundedly compact, Haar embedded manifold of $C(T)$ is Chebyshev.*

Wulbert has shown that the best approximation from any Chebyshev subspace of $L^1(\mu)$ is strong. Hence, we obtain the following theorem similarly.

THEOREM 6.2. *If $m \subset L^1(\mu)$ is a C^1-representable manifold such that every tangential cone is a Chebyshev subspace, then M is Chebyshev.*

7. Examples. In this section, we exhibit familiar examples of nonlinear manifolds in either $L^2([a, b], \mu)$ or $C[a, b]$ which are C^1-representable. For manifolds in $L^2([a, b], \mu)$, we will show that there is a neighborhood of uniqueness of best approximation. In $C[a, b]$, we will present explicitly some C^1-representable manifolds which are Chebyshev.

Let T be a closed interval of the real line and let γ be a continuous real-valued mapping on $T\times[a, b]$. We further assume that $\gamma(t, \cdot)$ is continuously Fréchet differentiable in the first variable. The Fréchet derivative with respect to $t \in T$ will be denoted $\gamma'(t, \cdot)$. Let $x_1 < x_2 < \cdots < x_{2N}$ be elements of T and $\varepsilon_1, \cdots, \varepsilon_N$ be positive numbers. Then the subset of γ-polynomials [3], [4], [7], [9]

$$(7.1) \quad M = \left\{ \sum_{i=1}^{N} \alpha_i \gamma(t_i, \cdot) : \alpha_i \geqq \varepsilon_i > 0 \text{ and } x_{2i-1} \leqq t_i \leqq x_{2i},\ i = 1, \cdots, N \right\}$$

may easily be seen to be a C^1-representable manifold when the set of $2N$-functions $\{\gamma(t_i, \cdot), \gamma'(t_i, \cdot)\}_{i=1}^{N}$ are linearly independent for all t_i satisfying the constraints in (7.1).

Suppose, in addition, that $\gamma(t, \cdot)$ is twice continuously Fréchet differentiable. Then viewing M as a subset of $L^2([a, b], \mu)$, μ a positive Borel measure, we can apply Theorem 3.2 to conclude that the curvature of M is bounded on compact sets. Theorem 5.2 then implies that M has a neighborhood of unique best approximations from M.

In particular, we remark that if γ is an extended sign regular kernel of order $2N(ESR_{2N})$ (cf. [10, p. 49]) and if the dimension of $L^2([a, b], \mu)$ is no smaller than $2N$, then M has a neighborhood of unique best approximation from M.

An example of a manifold M as in (7.1) which is not ESR_{2N} but has a neighborhood of unique best approximation is given by the spline kernel $\gamma(t, x) = (x-t)_+^n$, where $n \geqq 3$.

We now turn our attention to manifolds in $C[a, b]$. In [15], Wulbert showed that if $g : R^1 \to R^1$ satisfies $g'(x) \geqq \varepsilon > 0$, then the manifold given by

$$(7.2) \qquad M = \left\{ g\left(\sum_{i=1}^{N} \alpha_i U_i \right) : \alpha_i \in R^1 \text{ and } \{U_i\}_{i=1}^{N} \text{ is a Haar system} \right\}$$

is a Haar embedded manifold (hence Chebyshev by Theorem 6.1).

It is easy to see that certain manifolds with boundary are also Haar embedded. Consider

$$(7.3) \qquad M = \left\{ \exp\left(\sum_{i=0}^{N} \alpha_i x^i \right) : \alpha_i \geqq 0 \right\}.$$

Then if $0 < a < b$, it is easy to see from Theorem 6.1 that M is a C^1-representable Chebyshev manifold in $C[a, b]$ since $\{x^i\}_{i=0}^{N}$ is a Descartes system [10, p. 25] on $[a, b]$ and M is clearly boundedly compact. Note that if there is no restriction on the coefficients in (7.3), then M is not closed ($\theta \notin M$) and hence not Chebyshev.

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SOME A PRIORI AND ISOPERIMETRIC INEQUALITIES ASSOCIATED WITH A CLASS OF BOUNDARY VALUE PROBLEMS FOR DOUBLY CONNECTED DOMAINS*

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Abstract. In this paper isoperimetric inequalities are derived for the eigenvalues which appear as the optimal constants in certain classes of boundary value problems for multiply connected regions. These eigenvalues are related to the fundamental modes of vibrating membranes with rigid inclusions.

1. Introduction. The motivation for this paper stems from the fact that the optimal constants in the types of explicit a priori inequalities which are used in computing bounds for solutions of various problems in partial differential equations are eigenvalues in some associated eigenvalue problems (see, e.g., [2]). It follows then that isoperimetric lower bounds for these various eigenvalues in terms of geometric properties of the region on which the problem is defined will lead to simple, explicit and in a certain sense optimal a priori inequalities.

The eigenvalue problems whose eigenvalues arise in the study of the Dirichlet and Neumann problems for the Laplace equation have been well studied (see [2] and the references cited therein). In this paper we consider another class of problems associated with the Laplace operator, namely, the Poisson equation defined on a doubly connected domain in the plane with the value of the function prescribed on the outer boundary and the value of the tangential derivative prescribed on the inner boundary. Such a problem is not uniquely defined so additional auxiliary conditions must be imposed to insure uniqueness. The simplest example of a problem of this type is the determination of the stress function in the elastic torsion problem for a hollow beam.[1]

2. Some a priori inequalities. Let G be a doubly connected region in the plane bounded internally by a curve γ and externally by a curve Γ. Let H denote the interior of γ. Let A_0 be the total area enclosed by Γ and let A_i be the area of H. Suppose we are interested in obtaining a priori bounds for the energy $D(v) \equiv \int_G |\text{grad } v|^2 \, dA$ for the solution of the following problem for the Poisson equation:

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[1] This problem has been studied by Pólya and Weinstein [1]. We shall subsequently make use of symmetrization arguments described in [1].

$$
\begin{aligned}
&\Delta v = F \text{ in } G,\\
&v = f \quad \text{on } \Gamma,\\
&\partial v/\partial s = g \text{ on } \gamma,\\
&\oint_\gamma \frac{\partial v}{\partial n}\, ds = K.
\end{aligned}
\tag{1}
$$

Here $\partial/\partial s$ and $\partial/\partial n$ denote, respectively, the tangential and normal derivative on γ, the quantities F, f and g are the given data functions and the prescribed constant K eliminates the indeterminacy of v on γ. We assume throughout that Γ, γ and the data are sufficiently smooth so that the indicated operations are valid.

Let us approximate v by a function φ on which for the moment we impose only smoothness requirements. Then set

$$
w = v - \varphi. \tag{2}
$$

Our aim is to compute an explicit a priori bound for $D(w)$ of the following type:

$$
\begin{aligned}
D(w) \leqq C_1\left\{\frac{1}{A_i}\left(K - \oint_\gamma \frac{\partial \varphi}{\partial n}\, ds\right)^2 + \int_G (F-\Delta\varphi)^2\, dA\right\}\\
+ C_2\left\{\oint_\Gamma \left[\frac{\partial}{\partial s}(f-\varphi)\right]^2 ds + \oint_\gamma \left(g - \frac{\partial \varphi}{\partial s}\right)^2 ds\right\}.
\end{aligned}
\tag{3}
$$

The constants C_1 and C_2 will, as we shall see, be related to the lowest eigenvalues of certain associated problems.

To pursue this question we decompose w as follows: Set

$$
w = u + h,
$$

where

$$
\begin{aligned}
&\Delta u = F - \Delta\varphi \quad \text{in } G,\\
&u = 0 \quad \text{on } \Gamma,\\
&\frac{\partial u}{\partial s} = 0 \quad \text{on } \gamma,\\
&\oint_\gamma \frac{\partial u}{\partial n}\, ds = K - \oint_\gamma \frac{\partial \varphi}{\partial n}\, ds,
\end{aligned}
\tag{4}
$$

and

$$
\begin{aligned}
&\Delta h = 0 \quad \text{in } G,\\
&h = f - \varphi \quad \text{on } \Gamma,\\
&\frac{\partial h}{\partial s} = \left(g - \frac{\partial \varphi}{\partial s}\right) \quad \text{on } \gamma,\\
&\oint_\gamma \frac{\partial h}{\partial n}\, ds = 0.
\end{aligned}
\tag{5}
$$

Under the appropriate smoothness hypotheses, the solutions of (4) and (5) will exist, and $D(w)=D(u)+D(h)$; hence the bound for $D(w)$ will be obtained once we have determined bounds for $D(u)$ and $D(h)$. We first consider $D(u)$.

Since $\partial u/\partial s=0$ on γ, it follows that $u=\beta$ on γ. This constant β is unknown a priori. Now

$$D(u)=\beta\oint_\gamma \frac{\partial u}{\partial n}ds-\int_G u\,\Delta u\,dA, \tag{6}$$

and Schwarz's inequality for vectors gives

$$[D(u)]^2\leqq\left[\int_G u^2dA+\beta^2A_i\right]\left[\frac{1}{A_i}\left(\oint_\gamma \frac{\partial u}{\partial n}ds\right)^2+\int_G (\Delta u)^2\,dA\right]. \tag{7}$$

If we now define λ to be the first eigenvalue of the following Rayleigh quotient,

$$\lambda=\inf_{w\in\mathcal{B}}\frac{D(w)}{\int_G w^2\,dA+\beta_1^2A_i}, \tag{8}$$

where $\mathcal{B}$ is the class of Dirichlet integrable functions which satisfy

$$w=0\quad\text{on }\Gamma,$$

$$\partial w/\partial s=0\quad(w=\beta_1)\quad\text{on }\gamma,$$

then we are led to

$$D(u)\leqq\frac{1}{\lambda}\left\{\frac{1}{A_i}\left(K-\oint_\gamma\frac{\partial\varphi}{\partial n}ds\right)^2+\int_G (F-\Delta\varphi)^2\,dA\right\}. \tag{9}$$

In (8) the value of β_1 is not prescribed a priori. Inequality (9) thus gives us the first part of (3) provided we are able to obtain a lower bound for λ. To obtain the bound for $D(h)$ we note that since $\oint_\gamma \partial h/\partial n\,ds=0$, the conjugate function ψ of h is single-valued in G. Thus if we choose the arbitrary constant so that $\oint_{\Gamma\cup\gamma}\psi\,ds=0$, we have

$$\begin{aligned}D(h)=D(\psi)&=-\oint_{\Gamma\cup\gamma}\psi\frac{\partial h}{\partial s}ds\\ &\leqq\left[\oint_{\Gamma\cup\gamma}\psi^2\,ds\right]^{1/2}\left[\oint_{\Gamma\cup\gamma}\left(\frac{\partial h}{\partial s}\right)^2ds\right]^{1/2}\\ &\leqq\left[\frac{1}{p_2}D(\psi)\right]^{1/2}\left[\oint_{\Gamma\cup\gamma}\left(\frac{\partial h}{\partial s}\right)^2ds\right]^{1/2},\end{aligned} \tag{10}$$

where p_2 is the first nonzero eigenvalue in the Stekloff problem for G (see [2]). It follows then that

$$D(h)\leqq\frac{1}{p_2}\left\{\oint_\Gamma\left[\frac{\partial}{\partial s}(f-\varphi)\right]^2ds+\oint_\gamma\left(g-\frac{\partial\varphi}{\partial s}\right)^2ds\right\}. \tag{11}$$

Since the equality sign can hold in (9) if u satisfies the Euler equation of (8), i.e.,

$$\Delta u+\lambda u=0 \quad \text{in } G,$$

(12)
$$u=0 \quad \text{on } \Gamma,$$

$$u=\beta \quad \text{on } \gamma,$$

where the value of β is determined by the condition

(13)
$$\oint_\gamma \frac{\partial u}{\partial n}\, ds=\lambda\beta A_i,$$

it follows that the optimal C_1 in (3) is $1/\lambda$ and the optimal C_2 is $1/p_2$. If we use the triangle inequality

(14)
$$\left(\sqrt{D(v)}-\sqrt{D(\varphi)}\right)^2 \leqq D(v-\varphi),$$

we may state the result as the following theorem:

THEOREM 1. *The energy expression $D(v)$ associated with the solution v of problem* (1) *is bounded by* (14) *where*

$$D(v-\varphi)\leqq \frac{1}{\lambda}\left\{\frac{1}{A_i}\left(K-\oint_\gamma \frac{\partial\varphi}{\partial n}\, ds\right)^2+\int_G (F-\Delta\varphi)^2\, dA\right\}$$
$$+\frac{1}{p_2}\left\{\oint_\gamma \left[\frac{\partial}{\partial s}(f-\varphi)\right]^2 ds+\oint_\gamma \left(g-\frac{\partial\varphi}{\partial s}\right)^2 ds\right\}.$$

Here φ is any sufficiently smooth function.

We see then that explicit bounds for $D(v)$ can be obtained provided lower bounds for λ and p_2 are known. We shall give an isoperimetric lower bound for λ in § 3. There is no isoperimetric lower bound known for p_2 in the case of a multiply connected domain G. It is possible, however, to find a lower bound using techniques as given in [3].

We make brief mention of a second problem which is similar to (1) but which leads to a slightly different eigenvalue problem. Using the same notation as before we now let v be a solution of

$$\Delta v=F \quad \text{in } G,$$

$$v=f \quad \text{on } \Gamma,$$

(15)
$$v=g+\tilde{\beta} \quad \text{on } \gamma,$$

$$\oint_\gamma \frac{\partial v}{\partial n}\, ds+\alpha\tilde{\beta}=0.$$

Here $\tilde{\beta}$ is unknown a priori and α is a given positive constant.

If we now approximate v by a function φ such that

$$w\equiv v-\varphi=0 \quad \text{on } \Gamma, \qquad w=\tilde{\beta}-\beta_1 \quad \text{on } \gamma,$$

we obtain

$$
\begin{aligned}
D(w)+\alpha(\tilde{\beta}-\beta_1)^2 &= (\tilde{\beta}-\beta_1)\left[\oint_\gamma \frac{\partial w}{\partial n}ds+\alpha(\tilde{\beta}-\beta_1)\right]-\int_G w\,\Delta w\,dA \\
&= -(\tilde{\beta}-\beta_1)\left[\oint_\gamma \frac{\partial \varphi}{\partial n}ds+\alpha\beta_1\right]+\int_G w\,\Delta w\,dA.
\end{aligned} \tag{16}
$$

Again Schwarz's inequality gives

$$
\begin{aligned}
D(w)+\alpha(\tilde{\beta}-\beta_1)^2 \leqq &\left[\int_G w^2\,dA+(\tilde{\beta}-\beta_1)^2 A_i\right]^{1/2} \\
&\cdot\left[\frac{1}{A_i}\left(\oint_\gamma \frac{\partial \varphi}{\partial n}ds+\alpha\beta_1\right)^2+\int_G (\Delta w)^2\,dA\right]^{1/2}.
\end{aligned} \tag{17}
$$

As before, if we knew a lower bound for the eigenvalue $\hat{\lambda}(\alpha)$ defined by the following Rayleigh quotient,

$$
\hat{\lambda}(\alpha)=\inf_{\psi\in\mathcal{B}}\frac{D(\psi)+\alpha\beta^2}{\int_G \psi^2\,dA+\beta^2 A_i}, \tag{18}
$$

it would then follow that

$$
D(w)+\alpha(\tilde{\beta}-\beta_1)^2 \leqq \frac{1}{\hat{\lambda}(\alpha)}\left[\frac{1}{A_i}\left(\oint_\gamma \frac{\partial \varphi}{\partial n}ds+\alpha\beta_1\right)^2+\int_G (F-\Delta\varphi)^2\,dA\right], \tag{19}
$$

and we would obtain a simultaneous bound for $D(v)$ and $\tilde{\beta}$. We also derive an isoperimetric lower bound for $\hat{\lambda}(\alpha)$ in § 3.

We remark in passing that the Rayleigh–Ritz technique may be used to make the right-hand side of (3) or (19) small once the explicit bounds for the eigenvalues have been inserted.

3. Isoperimetric inequalities for λ and $\hat{\lambda}(\alpha)$. We consider a membrane spanning the region G with a rigid portion H which is allowed to move in such a way that H stays in a horizontal plane during the motion of the membrane. We are led then to problem (12), (13), and the corresponding Rayleigh quotient is given by (8). We establish the following theorem for λ.

THEOREM 2. *For given values of A_0 and A_i the eigenvalue λ defined by* (8) *is a minimum either* (i) *for the circular annulus with the rigid portion in the center or* (ii) *for the circular membrane of area A_0-A_i (i.e., without rigid portion).*

The proof of Theorem 2 is similar to the one given in Pólya and Weinstein's paper [1] in the case of the torsion problem for multiply connected domains. An important difference, however, is that the solution of the torsion problem in the case of a circle with ring-shaped "holes" is elementary, whereas in our case we are concerned with zeros of complicated equations involving Bessel functions.

The first part of the proof is completely analogous to the situation in [1]; hence we do not carry out all the details.

We apply a Schwarz-symmetrization to the "hill" given by the solution $u(x, y)$ of (12), (13). The symmetrized hill u^* will have either a "plateau" as a top or a ring-shaped plateau. The first situation will appear if $\beta=u_{\max}=u(\gamma)$. It is sufficient to consider the second situation only. We extend u continuously inside

H by setting

(20) $$\tilde{u}(P) \equiv \beta$$

for any point $P \in H$, and analogously

(21) $$\tilde{u}^*(P^*) = \beta$$

for P^* in the ring H^*, the symmetrized portion corresponding to H. The same arguments as in [1] show now that

(22) $$\lambda = \frac{D_{G \cup H}(\tilde{u})}{\int_{G \cup H} \tilde{u}^2 \, dA} \geqq \frac{D_{G^* \cup H^*}(\tilde{u}^*)}{\int_{G^* \cup H^*} (\tilde{u}^*)^2 \, dA} \geqq \lambda^*.$$

Here, $G^* \cup H^*$ denotes the circle with the same area A_0, H^* is the ring with area A_i, $\tilde{u}^*$ is the symmetrized function u (as in [1]), and λ^* is the first eigenvalue corresponding to G^*. In order to establish Theorem 2 we consider now the following situation:

Let R be the outer radius of the membrane and t and $b(t)$ the radii of the ring, where $t < b(t)$ and $A_i = \pi(b^2(t) - t^2)$. We then have for $u = u(r, t)$, the differential equation

(23) $$\Delta u + \lambda u = 0 \quad \text{for } 0 \leqq r \leqq t, \quad b(t) \leqq r \leqq R,$$

together with the boundary conditions

(24) $$u(R, t) = 0 \qquad u(t, t) = u(b(t), t) = \beta(t),$$

and

(25) $$2\pi \left\{ t \frac{\partial u}{\partial r}(t, t) - b(t) \frac{\partial u}{\partial r}(b(t), t) \right\} = \lambda A_i u(b(t), t),$$

which is the side condition corresponding to (13) in the present case.

We wish now to investigate the sign of $\lambda'(t) \equiv d\lambda/dt$. In what follows we use the notation

(26) $$u_t = \frac{\partial u}{\partial t}(r, t), \qquad u_r = \frac{\partial u}{\partial r}(r, t).$$

We start with the equation

(27) $$\lambda \left[\int_G u^2 \, dA + \beta^2(t) A_i \right] \equiv \lambda P = D(u),$$

where $\beta(t) = u(b(t), t)$ and G is the domain consisting of the circle $0 \leqq r \leqq t$ and the ring $b(t) \leqq r \leqq R$. Differentiating (27) we get

(28) $$\begin{aligned} \lambda' P + \lambda \Big\{ 2 \int_G & u u_t \, dA + 2 u u_t A_i + 2\pi (t u^2(t, t) - b b' u^2(b, t)) \\ & + 2 u u_r(b, t) A_i b' \Big\} \\ = {} & 2\pi \{ t u_r^2(t, t) - b b' u_r^2(b, t) \} + 2 D(u, u_t), \end{aligned}$$

$D(u, u_t)$ denoting the expression $\int_G \operatorname{grad} u \cdot \operatorname{grad} u_t \, dA$. An easy calculation shows then that (28) can be rewritten as

$$\lambda' P = 2\pi t((u_r(b, t))^2 - (u_r(t, t))^2), \tag{29}$$

which may be put into the form

$$\lambda' P = Q\left(\frac{\lambda\beta}{2}(b+t) + u_r(t, t)\right), \tag{30}$$

where

$$Q = \frac{2A_i t}{b^2}\left(-u_r(t, t) + \frac{\lambda\beta}{2}(b-t)\right) \geqq 0.$$

In fact, $Q > 0$ unless $A_i = 0$ or $t = 0$. If we write (30) as

$$\lambda' P = Q\lambda\left(\frac{\beta}{2}(b+t) - \frac{1}{t}\int_0^t ru\,dr\right), \tag{31}$$

then it is clear that for sufficiently small t, λ' will be positive while if b is close to R (β sufficiently small) λ' will be negative.

We shall show that there is only one critical point of λ in the interval $0 \leqq t \leqq (A - A_i)^{1/2}\pi^{-1/2}$ and that this is a maximum point. This will show that for the circular membrane with a rigid ring the first eigenvalue will always be greater than the smaller of the two extreme cases, that is,

(i) when the rigid part is a centrally located disk, or

(ii) when the ring is at the outer boundary of the membrane.

To carry out the arguments it is convenient to introduce the actual form of the solution u in $0 \leqq r \leqq t$, i.e.,

$$u(r, t) = B(t)J_0(\sqrt{\lambda(t)}\,r), \tag{32}$$

for some positive function $B(t)$. Then

$$\lambda' p = J_0(x)\left(x + \sqrt{x^2 + \frac{A_i\lambda}{\pi}}\right) - 2J_1(x), \tag{33}$$

where $x = t\sqrt{\lambda(t)}$ and $p = 2PQ^{-1}\sqrt{\lambda(t)}\,B^{-1}$. Clearly

$$x + \sqrt{x^2 + \frac{A_i\lambda}{\pi}} \geqq \max\left(\sqrt{\frac{A_i\lambda}{\pi}}, 2x\right).$$

Thus for each $x \in [0, j_0]$ ($j_0 = 2.4048\cdots$) the curve

$$y = \left(x + \sqrt{x^2 + \frac{A_i\lambda}{\pi}}\right)J_0(x)$$

lies above the curves

$$y_1(x) = \sqrt{\frac{A_i\lambda}{\pi}}\,J_0(x), \qquad y_2(x) = 2xJ_0(x).$$

In particular then the first positive value x_0 at which λ' vanishes satisfies the inequality $x_0 \geqq 1.84 \cdots$, the first zero of

$$xJ_0(x) - J_1(x) = 0. \tag{34}$$

It is easy to see that at a critical value of t, i.e., when $\lambda' = 0$, the sign of λ'' is given by the sign of

$$\frac{d}{dx}\left\{J_0(x)\left(x+\sqrt{x^2+\frac{A_i\lambda}{\pi}}\right)-2J_1(x)\right\} = J_0(x)\left(g^{-1}+g-\frac{x^2}{2}(1+g)^2\right), \tag{35}$$

with

$$g = \sqrt{1+\frac{A_i\lambda}{\pi x^2}}.$$

But for $x \geqq x_0$ we have

$$g^{-1}+g-\frac{x^2}{2}(1+g)^2 \leqq g^{-1}+g-\frac{(1.84)^2}{2}(1+g)^2 \equiv H(g).$$

Since $H(g)$ is decreasing for $g \geqq 1$, we arrive at

$$\frac{d}{dx}\left\{J_0(x)\left(x+\sqrt{x^2+\frac{A_i\lambda}{\pi}}\right)-2J_1(x)\right\} \leqq H(1)J_0(x) = -4.78J_0(x). \tag{36}$$

Since $\lambda'' \leqq 0$ at any critical point it follows that there is a single maximum for $\lambda(t)$ in the interval under consideration. We have thus established Theorem 2.

The subsequent discussion will show that the alternative in Theorem 2 cannot be sharpened, in general, since the eigenvalue in case (i) may be larger or smaller than that of case (ii) depending on the geometry of G. To see this we consider now a circular annulus G with fixed area $A = \pi(b^2 - t^2)$ containing a centrally located hole H of area πt^2. We have then the differential equation

$$\begin{aligned} \Delta u + \lambda u &= 0 \quad \text{for } t \leqq r \leqq b(t), \\ u &= 0 \quad \text{for } r = b(t), \\ u &= \beta \quad \text{for } r = t \end{aligned} \tag{37}$$

with (25) replaced by

$$-2u_r(t,t) = \lambda t \beta. \tag{38}$$

It follows from the boundary condition at $r = b(t)$ that

$$u_t(b,t) + \frac{t}{b}u_r(b,t) = 0. \tag{39}$$

A computation similar to that following (28) leads to

$$\lambda' P = 2\pi t\{u_r^2(t,t) - u_r^2(b,t)\}, \tag{40}$$

with $P=\int_G u^2\,dA+\pi\beta^2t^2$ in this case. We now make use of the identity

$$\int_t^b ru_r\Big(\frac{\partial}{\partial r}(ru_r)+\lambda ru\Big)\,dr=0 \tag{41}$$

to solve (integration by parts) for $u_r(b,t)$, and get

$$\lambda' P=\frac{2\pi t\lambda}{b^2}\Big\{t^2\beta^2\Big(\frac{\lambda A}{4\pi}-1\Big)-2\int_t^b ru^2\,dr\Big\}. \tag{42}$$

Here we have also introduced (38). It is clear from (42) that if t is sufficiently small, λ' will be negative, while an asymptotic analysis of the Bessel functions in the solution shows that $\lambda'>0$ for A sufficiently small. A rather crude use of inequalities shows us in fact that

$$\text{if } A_0/A_i>9, \quad \text{then } \lambda'<0. \tag{43}$$

This establishes the assertion that, in general, Theorem 2 cannot be sharpened by eliminating one of the two alternatives.

In the case of the eigenvalue $\hat\lambda(\alpha)$ defined by (18), the symmetrization arguments used for the proof of Theorem 2 remain valid. By employing arguments similar to those used previously in analyzing the eigenvalue problems for the circular annuli one can establish theorems similar to Theorem 2 for $\hat\lambda(\alpha)$. For instance one can obtain the following result.

THEOREM 3. *Let $\hat\lambda_L$ be a lower bound for the eigenvalue $\hat\lambda_C(\alpha)$ defined by* (18) *in the case of a circular annulus. Let $\delta=1-\alpha/(\hat\lambda_L A_i)>0$, and let x_0 be the first zero of*

$$\delta\{x+(x^2+A_i\hat\lambda_L\pi^{-1})^{1/2}\}J_0(x)-2J_1(x). \tag{44}$$

Then, if $x_0^2\geqq 1/\delta$, the alternative of Theorem 2 holds for $\hat\lambda(\alpha)$.

Remark 1. A convenient lower bound $\hat\lambda_L$ for $\hat\lambda_C(\alpha)$ is for instance $\pi j_0^2/A_0$.

Remark 2. A similar theorem could be stated if $\delta<0$.

Remark 3. A discussion analogous to the one after Theorem 2 can be made here showing that the result of Theorem 3 is again best possible in a certain sense.

4. Additional inequalities.

(a) One could also get a lower bound for λ in the following way (see, e.g., [4]):

For any $w\in\mathcal{B}$ set

$$w=\psi+\beta h, \tag{45}$$

where

$$\Delta h=0 \quad \text{in } G, \qquad h=0 \quad \text{on } \Gamma, \qquad h=1 \quad \text{on } \gamma, \tag{46}$$

$$\Delta\psi=\Delta w \quad \text{in } G, \qquad \psi=0 \quad \text{on } \Gamma\cup\gamma. \tag{47}$$

If we extend h and ψ continuously as constants inside H, we may write, using a bar to denote the extended functions,

$$\frac{1}{\lambda}=\max\ \frac{\int_{G\cup H}(\bar\psi+\beta\bar h)^2\,dA}{D_{G\cup H}(\bar\psi+\beta\bar h)}. \tag{48}$$

Applying Schwartz's inequality we get after some routine calculation

$$\frac{1}{\lambda} \leqq \frac{\int_G h^2\, dA + A_i}{D_G(h)} + \max_{\psi=0 \text{ on } \partial G} \frac{\int_G \psi^2\, dA}{D_G(\psi)} = \frac{B+A_i}{4\pi C} + \frac{1}{\lambda_f}. \tag{49}$$

Here, $B = \int_G h^2\, dA$, C = capacity per unit height of the infinitely long condenser generated by γ and Γ, and λ_f is the first eigenvalue of the fixed membrane on G. Isoperimetric bounds for C and λ_f are known [2].

For $\hat{\lambda}(\alpha)$ we would get analogously

$$\frac{1}{\hat{\lambda}(\alpha)} \leqq \frac{B+A_i}{4\pi C + \alpha} + \frac{1}{\lambda_f}. \tag{50}$$

(b) We can also obtain isoperimetric upper bounds using conformal mapping and the same reasoning as in [5]. The result is then the following theorem.

THEOREM 4. *For given values of the capacity C of G and A_i (and α), the eigenvalue $\lambda(\hat{\lambda}(\alpha))$ is maximal in the case of circular ring with inner area A_i.*

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REGULARIZATION AND APPROXIMATION OF SECOND ORDER EVOLUTION EQUATIONS*

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Abstract. We give a nonstandard method of integrating the equation $Bu''+Cu'+Au=f$ in Hilbert space by reducing it to a first order system in which the differentiated term corresponds to energy. Semigroup theory gives existence for hyperbolic and for parabolic cases. When $C=\varepsilon A$, $\varepsilon \geqq 0$, this method permits the use of Faedo–Galerkin projection techniques analogous to the simple case of a single first order equation; the appropriate error estimates in the energy norm are obtained. We also indicate certain singular perturbations which can be used to approximate the equation by one which is dissipative or by one to which the above projection techniques are applicable. Examples include initial-boundary value problems for vibrations (possibly) with inertia, dynamics of rotating fluids, and viscoelasticity.

1. Introduction. Let A and C be continuous linear operators from a Hilbert space V into its antidual V'. Let W be a Hilbert space, of which V is a dense subspace continuously imbedded, and let B be continuous and linear from W to W'. We naturally identify W' with a subspace of V' and use $\langle \cdot , \cdot \rangle$ to denote the various dualities.

Problem 1. Given $u_1 \in V$, $u_2 \in W$, $f \in C((0,\infty), W')$, find $u \in C([0,\infty), V) \cap C^1((0,\infty), V) \cap C^1([0,\infty), W) \cap C^2((0,\infty), W)$ such that $u(0)=u_1$, $u'(0)=u_2$, and

$$Bu''(t)+Cu'(t)+Au(t)=f(t), \qquad t>0. \tag{1.1}$$

We shall rewrite this as a first order system. Define the Hilbert product spaces $V_l = V\times V$, $V_m = V \times W$ and the operators

$$M(x_1, x_2) \equiv (Ax_1, Bx_2), \qquad L(x_1, x_2) \equiv (-Ax_2, Ax_1 + Cx_2)$$

from V_m to V'_m and V_l to V'_l, respectively. If u is a solution of Problem 1, then $w \equiv (u, u')$ is a solution of the next problem.

Problem 2. Given $(u_1, u_2) \in V_m$, $f \in C((0,\infty), w')$, find $w \in C([0,\infty), V_m) \cap C^1((0,\infty), V_m)$ such that $w(0)=(u_1,u_2)$ and

$$Mw'(t)+Lw(t)=(0,f(t)), \qquad t>0. \tag{1.2}$$

Our plan is as follows. In § 2 we obtain existence and uniqueness results under hypotheses which imply that Problems 1 and 2 are equivalent. Examples of initial-boundary value problems to which our results apply are given in § 3. Approximate solutions are obtained in § 4 from standard Faedo–Galerkin projection techniques. When $C=\varepsilon A$, $\varepsilon \geqq 0$, the L-projection factors into the A-projection onto a subspace of V; then we can give energy norm error estimates for models of finite-element subspaces when A is an elliptic operator of order 2. Finally, in § 5 we examine the error resulting from certain perturbations of (1.1) into more regular models which are parabolic. In certain models these regularizations represent *artificial viscosity* or *artificial inertia.*

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2. Existence and uniqueness. We shall seek hypotheses for which Problem 1 is well-posed. Recall that the operator $A : V \to V'$ is *monotone* if $\operatorname{Re}\langle Ax, x\rangle \geqq 0$, $x \in V$, and *symmetric* if $\langle Ax, y\rangle = \overline{\langle Ay, x\rangle}$, $x, y \in V$. Such an operator induces a seminorm $\|x\|_a = \langle Ax, x\rangle^{1/2}$, $x \in V$, and we have a Cauchy–Schwartz inequality

$$|\langle Ax, y\rangle| \leqq \|x\|_a \|y\|_a, \qquad x, y \in V.$$

Let u be a solution of Problem 1. If M is symmetric, then $w \equiv (u, u')$ satisfies

$$D_t\langle Mw(t), w(t)\rangle + 2\operatorname{Re}\langle Lw(t), w(t)\rangle = 2\operatorname{Re}\langle f(t), u'(t)\rangle, \tag{2.1}$$

so we obtain

$$\langle Mw(t), w(t)\rangle + 2\operatorname{Re}\int_0^t \langle Lw, w\rangle = \langle Mw(0), w(0)\rangle + 2\operatorname{Re}\int_0^t \langle f, u'\rangle. \tag{2.2}$$

This is equivalent to the identity

$$\langle Au(t), u(t)\rangle + \langle Bu'(t), u'(t)\rangle + 2\operatorname{Re}\int_0^t \langle Cu', u'\rangle$$
$$= \langle Au(0), u(0)\rangle + \langle Bu'(0), u'(0)\rangle + 2\operatorname{Re}\int_0^t \langle f, u'\rangle.$$

Suppose B is also monotone, and denote by $\|\cdot\|_{W_b'}$ the norm on the Hilbert space W_b' which is the antidual of W with the seminorm $\|\cdot\|_b$ induced by B. The last term in (2.1) is bounded by

$$2\|f\|_{W_b'}\|u'\|_b \leqq T\|f\|_{W_b'}^2 + T^{-1}\|u'\|_b^2,$$

where $T > 0$ is arbitrary, so (2.1) gives

$$D_t(e^{-t/T}\langle Mw(t), w(t)\rangle) + e^{-t/T} 2\operatorname{Re}\langle Lw(t), w(t)\rangle \leqq Te^{-t/T}\|f\|_{W_b'}^2.$$

Integrating this inequality gives the a priori estimate

$$\langle Mw(t), w(t)\rangle + 2\operatorname{Re}\int_0^t \langle Lw, w\rangle \leqq e\langle Mw(0), w(0)\rangle + Te\int_0^t \|f\|_{W_b'}^2, \qquad 0 \leqq t \leqq T. \tag{2.3}$$

We summarize the above as the following proposition.

PROPOSITION 1. *Let u be a solution of Problem* 1 *on the interval* $[0, T]$ *and assume that A and B are symmetric and monotone. Then we have*

$$\langle Au(t), u(t)\rangle + \langle Bu'(t), u'(t)\rangle + 2\int_0^t \operatorname{Re}\langle Cu', u'\rangle$$
$$\leqq e\langle Au_1, u_1\rangle + e\langle Bu_2, u_2\rangle + Te\int_0^t \|f\|_{W_b'}^2, \qquad 0 \leqq t \leqq T.$$

From the representation $u(t) = u_1 + \int_0^t u'$ by the (strong) integral in W and the fact that $\|\cdot\|_b$ is a continuous seminorm on W, it follows from $\|u'\|_b = 0$ on $[0, T]$ that $\|u\|_b$ is constant on $[0, T]$. This gives the following proposition.

PROPOSITION 2. *Let A and B be symmetric and monotone and let C be monotone. If u is a solution of Problem* 1 *on* $[0, T]$ *with* $u_1 = u_2 = 0$ *and* $f(\cdot) = 0$,

then

$$\|u(t)\|_a = \|u(t)\|_b = 0, \qquad 0 \leqq t \leqq T.$$

Thus, there is at most one solution of Problem 1 *if* $\ker(A) \cap \ker(B) = \{0\}$.

We could continue to permit B to be *degenerate* as in [15]; it would be necessary to modify the definitions above and work in dual spaces but nothing essential is changed. For the remainder of this section we shall assume B is W-*coercive*: there is a $c > 0$ such that

$$\langle Bx, x\rangle \geqq c\|x\|_W^2, \qquad x \in W.$$

This holds, for instance, if $B : V \to V'$ is given symmetric and (strictly) positive and if W is the completion of V with the norm $\|\cdot\|_b$.

We consider the question of existence. In addition to the hypotheses of Proposition 2, assume A is V-coercive and B is W-coercive. Define $D \equiv \{x \in V_l : Lx \in V_m'\}$. Since A and B are isomorphisms, M is also, and we can define an operator $N : D \to V_m$ by $N = M^{-1} \circ L$. Note that $(x, y)_m \equiv \langle Mx, y\rangle$ gives an (equivalent) inner product on V_m for which we have the identity

$$(Nx, y)_m = \langle Lx, y\rangle, \qquad x \in D, \quad y \in V_l.$$

It follows that N is *accretive*:

$$\operatorname{Re}(Nx, x)_m \geqq 0, \qquad x \in D.$$

To show that $-N$ generates a strongly continuous semigroup of contractions on V_m, it suffices to show that $\lambda + N$ is onto V_m for every $\lambda > 0$. But this is equivalent to the following lemma.

LEMMA 1. *$\lambda M + L$ maps D onto V_m' for every $\lambda > 0$.*

Proof. Let $f_1 \in V'$, $f_2 \in W'$. Since A is V-coercive, so also is $A + \lambda C + \lambda^2 B$, and each maps onto V', so there exist $x_1, x_2 \in V$ for which

$$(A + \lambda C + \lambda^2 B)x_2 = \lambda f_2 - f_1,$$
$$\lambda A x_1 = A x_2 + f_1.$$

It follows that $Ax_1 + Cx_2 = -\lambda B x_2 + f_2 \in W'$, hence $(x_1, x_2) \in D$, and that $(\lambda M + L)(x_1, x_2) = (f_1, f_2)$.

Our first existence result follows directly from the preceding discussion and standard results on the generation of semigroups [9].

PROPOSITION 3. *Let A be symmetric and V-coercive, B be symmetric and W-coercive, and C be monotone. If $u_1, u_2 \in V$ with $Au_1 + Cu_2 \in W'$ and if $f \in C^1([0, \infty), W')$ are given, then there is a (unique) solution of Problem* 1. *The equation* (1.1) *is satisfied up to the initial time*: $(u, u') \in C^1([0, \infty), V \times W)$. *From this it follows that* $(u, u') \in C([0, \infty), V \times V)$.

PROPOSITION 4. *In addition to the hypotheses of Proposition* 3, *suppose that $C + \lambda B$ is V-coercive for $\lambda > 0$. If $u_1 \in V$, $u_2 \in W$ and $f : [0, \infty) \to W'$ is Hölder continuous, then there is a (unique) solution of Problem* 1.

Proof. For each $\lambda > 0$ and $x = (x_1, x_2) \in D$ we have

$$\operatorname{Re}((\lambda + N)x, x)_m = \lambda\langle Ax_1, x_1\rangle + \langle(\lambda B + C)x_2, x_2\rangle,$$

so $\lambda + N$ is V_l-coercive and, hence, sectorial. Thus, $-N$ generates an analytic semigroup [9].

In the situation of Proposition 4, either the equation is *irreversible* or N is bounded, i.e., $V = W$ [7]. In particular, Proposition 4 applies to *parabolic* problems while Proposition 3 is appropriate in the *hyperbolic* situation.

3. Examples. We illustrate some of our preceding results with initial-boundary value problems which occur in various applications. These existence-uniqueness results are far from best possible, but will serve as models for our following work.

Let G be a nonempty open set in $\mathbb{R}^n$ lying on one side of its smooth $(n-1)$-dimensional boundary, ∂G. $H^1(G)$ is the Hilbert space of (equivalence classes of) functions in $L^2(G)$, all of whose (distribution) derivatives of first order belong to $L^2(G)$. The inner product is given by

$$(\varphi, \psi)_{H^1} = \sum_{j=0}^{n} (D_j\varphi, D_j\psi)_{L^2(G)},$$

where D_j, $1 \leqq j \leqq n$, denotes a partial derivative and D_0 is the identity. Let Γ_0 be an open subset of ∂G and $\Gamma_1 = \partial G \sim \Gamma_0$. Let V be that subspace of $H^1(G)$ consisting of those functions whose traces vanish on Γ_0. We shall denote the *gradient* $\nabla\varphi = (D_1\varphi, \cdots, D_n\varphi)$ and *Laplacian* $\Delta\varphi = \sum_{j=1}^n D_j^2\varphi$ as indicated. Also, ν will denote the unit outward normal on ∂G, and $D_\nu\varphi = \nabla\varphi \cdot \nu$ is the directional normal derivative. See [12] for details.

Example 1. Define $A : V \to V'$ by

$$\langle A\varphi, \psi\rangle = \int_G \nabla\varphi \cdot \overline{\nabla\psi}, \qquad \varphi, \psi \in V.$$

For each $\varphi \in V$, the restriction of $A\varphi$ to the space $C_0^\infty(G)$ is the distribution $-\Delta\varphi$. Regularity theory for elliptic equations shows that Green's formula

$$\langle A\varphi, \psi\rangle = \int_G (-\Delta\varphi)\bar{\psi} + \int_{\partial G} D_\nu\varphi\bar{\psi}$$

is meaningful whenever $\Delta\varphi \in L^2(G)$. Take $W = L^2(G)$ and $\langle B\varphi, \psi\rangle = (\varphi, \psi)_{L^2(G)}$. Let $R \geqq 0$ and $r \geqq 0$ be given and define

$$\langle C\varphi, \psi\rangle = R\int_G \varphi\bar{\psi} + r\int_{\partial G} \varphi\bar{\psi}, \qquad \varphi, \psi \in V.$$

Finally, let $F(x, t)$ be a real-valued function in $C^1(\bar{G} \times [0, \infty))$ and set $f(t) = F(\cdot, t)$, $t \geqq 0$. Propositions 2 and 3 show that for each pair $u_1, u_2 \in V$ with $\Delta u_1 \in L^2(G)$ and $D_\nu u_1 + ru_2 = 0$ on Γ_1 there is a unique generalized solution $u = u(x, t)$ of

$$D_t^2 u + RD_t u - \Delta u = F(x, t), \qquad x \in G, \quad t \geqq 0,$$

$$u(x, 0) = u_1(x), \qquad D_t u(x, 0) = u_1(x),$$

$$u(x, t) = 0, \qquad x \in \Gamma_0,$$

$$D_\nu u(x, t) + rD_t u(x, t) = 0, \qquad x \in \Gamma_1.$$

This hyperbolic problem is the classical wave equation with weak dissipation distributed through $G(R>0)$ or along $\partial G(r>0)$.

Example 2. Let A and B be as above and set

$$\langle C\varphi, \psi\rangle = \varepsilon \int_G \nabla\varphi \cdot \overline{\nabla\psi}, \qquad \varphi, \psi \in V,$$

where $\varepsilon>0$. Propositions 2 and 4 show that for each pair $u_1 \in V$ and $u_2 \in L^2(G)$ there is a unique generalized solution $u = u(x, t)$ of

$$\begin{aligned} &D_t^2 u - \varepsilon \Delta D_t u - \Delta u = F(x, t), && x \in G, \quad t>0,\\ &u(x, 0) = u_1(x), \qquad D_t u(x, 0) = u_2(x), &&\\ &u(x, t) = 0, && x \in \Gamma_0,\\ &D_\nu(u(x, t) + \varepsilon D_t u(x, t)) = 0, && x \in \Gamma_1. \end{aligned}$$

This is a parabolic problem arising from certain models in classical hydrodynamics or viscoelasticity. Strong dissipation results from the presence of the positive constant ε which represents *viscosity* in the model [8].

Example 3. Take A as above but set $C=0$, $W=V$, and define

$$\langle B\varphi, \psi\rangle = \int_G (\varphi\bar{\psi} + \varepsilon \nabla\varphi \cdot \overline{\nabla\psi}), \qquad \varphi, \psi \in V,$$

where $\varepsilon>0$. Let $G(s, t)$ be a real-valued function in $C^1(\Gamma_1 \times [0, \infty))$ and define $f: [0, \infty) \to V'$ by

$$\langle f(t), \varphi\rangle = \int_G F(\cdot\,, t)\varphi + \int_{\partial G} G(\cdot\,, t)\varphi, \qquad \varphi \in V.$$

Then either of Propositions 3 or 4 shows that for each pair $u_1, u_2 \in V$ there is a generalized solution $u = u(x, t)$ of

$$\begin{aligned} &D_t^2 u - \varepsilon\, \Delta D_t^2 u - \Delta u = F(x, t), && x \in G, \quad t \geqq 0,\\ &u(x, 0) = u_1(x), \qquad D_t u(x, 0) = u_2(x), &&\\ &u(x, t) = 0, && x \in \Gamma_0,\\ &D_\nu(u(x, t) + \varepsilon D_t^2 u(x, t)) = G(x, t), && x \in \Gamma_1. \end{aligned}$$

This problem arises in classical vibration models in which ε represents *inertia* [13, § 278].

Example 4. Here we choose $W=V$ and $C=0$ as before, but define

$$\langle B\varphi, \psi\rangle = \int_G \nabla\varphi \cdot \overline{\nabla\psi}, \qquad \varphi, \psi \in V,$$

$$\langle A\varphi, \psi\rangle = \int_G \left\{ a \sum_{j=1}^{n-1} D_j\varphi D_j\bar{\psi} + b D_n\varphi D_n\bar{\psi} \right\},$$

where $a \geqq 0$ and $b \geqq 0$. Define f as in the preceding example. From either of Propositions 3 or 4 (and possibly after an exponential shift to obtain an equivalent problem with A replaced by the coercive $A+\lambda^2 B$) we obtain for each pair u_1, $u_2 \in V$ the existence of a generalized solution $u=u(x,t)$ of the problem

$$-\Delta D_t^2 u - a \sum_{j=1}^{n-1} D_j^2 u - bD_n^2 u = F(x,t), \qquad x \in G, \quad t \geqq 0,$$

$$u(x,0)=u_1(x), \qquad D_t u(x,0) = u_2(x),$$

$$u(x,t)=0, \qquad x \in \Gamma_0,$$

$$D_\nu(D_t^2 u) + a \sum_{j=1}^{n-1} \nu_j D_j u + b\nu_n D_n u = 0, \qquad x \in \Gamma_1.$$

Such problems arise in models of "fat bodies" of homogeneous incompressible fluid in rotation. These include the internal waves in which the term with $b>0$ results from the rotation while that with $a>0$ is contributed by a vertical temperature gradient [10, § 6]. Similarly, certain models of wave motion in a rotating stratified fluid [17] lead to the equation

$$(D_t + D_1)^2 \Delta u + dD_1^2 u = 0.$$

An elementary change of variable reduces this to the form above.

Various models of diffusion processes lead to problems similar to Examples 3 and 4 but with D_t^2 replaced by D_t and without the initial condition on $D_t u(x,0)$. These are resolved as Problem 2 with $M=B$ and $L=A$ in the respective examples [14].

Many other similar problems arising from models of waves in fluids or solids could be added. If one considers *transverse vibrations* (instead of longitudinal vibrations) of rods, then we obtain problems like Examples 1 and 3 but with Δu replaced by $\Delta^2 u$. Consideration of *shear* forces could add a term $\Delta^2 u$ to Example 3. Finally, we mention the models of coupled heat-sound systems and plate vibrations which lead to *systems* in the form of (1.1) in which the operators are 2×2 matrix-operators. Our results apply to these as well.

4. Approximation by projection. In order to describe the approximation methods we shall discuss, we denote as indicated the following forms:

$$a(x,y)=\langle Ax, y\rangle, \qquad c(x,y)=\langle Cx, y\rangle, \qquad x, y \in V,$$

$$b(x,y)=\langle Bx, y\rangle, \qquad x, y \in W,$$

$$m(x,y)=\langle Mx, y\rangle, \qquad x, y \in V_m = V\times W,$$

$$l(x,y)=\langle Lx, y\rangle, \qquad x, y \in V_l = V\times V.$$

These forms permit a weak formulation of Problem 2.

LEMMA 2. *If $w(\cdot)$ is a solution of Problem 2, then*

$$m(w'(t), v) + l(w(t), v) = \langle (0, f(t)), v\rangle, \qquad v \in V_l, \quad t>0. \tag{4.1}$$

Let S be a closed subspace of V. We shall consider an approximation of $w(\cdot)$ by a function $W:[0,\infty)\to S\times S$ which satisfies

$$m(W'(t), v) + l(W(t), v) = \langle (0, f(t)), v\rangle, \qquad v \in S\times S, \quad t>0, \tag{4.2}$$

and for which $W(0)$ is specified below. We note that if $U:[0,\infty)\to S$ is the corresponding approximation of a solution u of Problem 1, i.e.,

$$b(U''(t),v)+c(U'(t),v)+a(U(t),v)=\langle f(t),v\rangle,\qquad v\in S,\quad t>0, \tag{4.3}$$

then the pair $W=(U,U')$ satisfies (4.2). If ker $(A)=\{0\}$, then (4.2) and (4.3) are equivalent. When S has finite dimension, (4.3) is the expansion method of S (Faedo [5]).

We obtain error estimates in the energy norm $\|x\|_m\equiv m(x,x)^{1/2}$ by comparing each of $w(\cdot)$ and $W(\cdot)$ with the pointwise L-*projection* $W_l(t)$ of $w(t)$ onto $S\times S$: for each $t>0$, $W_l(t)\in S\times S$ is defined by

$$l(W_l(t),v)=l(w(t),v),\qquad v\in S\times S. \tag{4.4}$$

From (4.1), (4.2) and (4.4) we obtain for each $v\in S\times S$,

$$m(w'(t)-W_l'(t),v)=m(W'(t)-W_l'(t),v)+l(W(t)-W_l(t),v).$$

Setting $v=W(t)-W_l(t)$ and using the monotonicity of L give

$$D_t\|W(t)-W_l(t)\|_m^2\leqq 2\|w'(t)-W_l'(t)\|_m\|W(t)-W_l(t)\|_m.$$

Since the function $t\mapsto\|W(t)-W_l(t)\|_m$ is absolutely continuous, hence, differentiable almost everywhere with

$$D_t\|W(t)-W_l(t)\|_m^2=2\|W(t)-W_l(t)\|_m D_t\|W(t)-W_l(t)\|_m,$$

we obtain the estimate

$$D_t(\|W(t)-W_l(t)\|_m)\leqq\|w'(t)-W_l'(t)\|_m \tag{4.5}$$

off of the set of $t>0$ for which $\|W(t)-W_l(t)\|_m=0$. But (4.5) trivially holds at an accumulation point of this set, and there are at most a countable number of isolated points of this set, so (4.5) holds almost everywhere on $(0,\infty)$. Integrating (4.5) yields the following lemma.

LEMMA 3. *Let A and B be symmetric and monotone and let C be monotone. If u is such that $w\in C([0,\infty),V_l)$ (cf. Proposition 3) and if $w', W_l'\in L^1((0,\varepsilon),V_m)$ for some $\varepsilon>0$, then*

$$\|W(t)-W_l(t)\|_m\leqq\|W(0)-W_l(0)\|_m+\int_0^t\|w'-W_l'\|_m,\qquad t\geqq 0.$$

If the initial value $W(0)\in S\times S$ is chosen by M-projection, i.e.,

$$m(W(0),v)=m(w(0),v),\qquad v\in S\times S,$$

then $\|W(0)-w(0)\|_m\leqq\|W_l(0)-w(0)\|_m$, so the triangle inequality yields

$$\|W(0)-W_l(0)\|_m\leqq 2\|w(0)-W_l(0)\|_m.$$

If $W(0)$ is chosen by L-projection (4.4), then $W(0)=W_l(0)$ and the preceding estimate holds trivially. Either way we obtain the following proposition.

PROPOSITION 5. *In the situation of Lemma* 3, *if* $W(0)$ *is chosen by* M-*projection or by* L-*projection, then*

$$\|w(t)-W(t)\|_m \leqq \|w(t)-W_l(t)\|_m + 2\|w(0)-W_l(0)\|_m + \int_0^t \|w'-W_l'\|_m, \quad t \geqq 0. \tag{4.6}$$

Thus, the error in approximating (4.1) by (4.2) is determined by the error in the corresponding stationary Galerkin approximation (4.4).

Hereafter we restrict our attention to the case of $C=\varepsilon A$, $\varepsilon \geqq 0$, for then (4.4) factors into a pair of A-projections of V onto S. That is, denoting the error by $e(t)=w(t)-W_l(t) \in V_l$, we see that (4.4) is equivalent to $(j=1, 2)$

$$a(e_j(t), v)=0, \qquad v \in S, \quad t \geqq 0, \tag{4.7}$$

so $U_l(t)(U_l'(t))$ is the A-projection of $u(t)$ (respectively, $u'(t)$) onto S, where $W_l(t)=(U_l(t), U_l'(t))$, $t \geqq 0$. This gives

$$\|u(t)-U_l(t)\|_a \leqq \inf\{\|u(t)-v\|_a : v \in S\},$$

and similar estimates hold for the various derivatives of the error.

We shall combine the preceding remarks with approximation-theoretic results. Denote by $H^k(G)$ the space of functions φ which with all derivatives $D^\alpha\varphi$ of order $|\alpha|$ at most k belong to $L^2(G)$. Such a space is complete with the norm

$$\|\varphi\|_{H^k}^2 = \sum\{\|D^\alpha\varphi\|_{L^2(G)}^2 : |\alpha| \leqq k\}.$$

For appropriate functions v from an interval $[0, T]$ into a normed space N with norm $\|\cdot\|_N$, we recall the norms

$$\|v\|_{L^p(N)} = \left(\int_0^T \|v(t)\|_N^p\,dt\right)^{1/p}, \qquad 1 \leqq p < \infty,$$

$$\|v\|_{L^\infty(N)} = \text{ess sup}\,\{\|v(t)\|_N : 0 \leqq t \leqq T\}.$$

Our approximation result is based on an approximation assumption that is typical of multivariate spline and finite element spaces [16].

PROPOSITION 6. *Let* V *be a closed subspace of* $H^1(G)$ (*as in* § 3), *and* $\{S_h : 0<h<1\}$ *a collection of finite-dimensional subspaces of* V *which satisfy the following approximation assumption*: *There are a constant* M *and an integer* $k \geqq 1$ *such that*

$$\inf\{\|\varphi-\psi\|_{H^1} : \psi \in S_h\} \leqq Mh^{k-1}\|\varphi\|_{H^k}, \qquad \varphi \in V \cap H^k(G), \quad 0<h<1. \tag{4.8}$$

Let A *and* B *be symmetric and monotone,* A *be* V-*coercive, and set*

$$K_a \equiv \sup\{a(\varphi, \varphi)^{1/2} : \|\varphi\|_{H^1} \leqq 1\},$$

$$K_b \equiv \sup\{b(\varphi, \varphi)^{1/2} : \|\varphi\|_{H^1} \leqq 1\}.$$

Let $u \in C^1([0, T], V)$ *be a solution of Problem* 1 *with* $C=\varepsilon A$ *for some* $\varepsilon \geqq 0$ *and assume that*

$$u, u' \in L^\infty([0, T], H^k(G)), \qquad u'' \in L^1([0, T], H^k(G)).$$

Then the approximate solution U defined by (4.3) *with $S = S_h$ and initial data chosen by M-projection* (*or L-projection*) *satisfies the estimate*

$$(\|u(t)-U(t)\|_a^2+\|u'(t)-U'(t)\|_b^2)^{1/2} \leqq Ch^{k-1}, \qquad 0 \leqq t \leqq T, \tag{4.9}$$

where $C = M\{3K_a\|u\|_{L^\infty(H^k)} + (3K_b + TK_a)\|u'\|_{L^\infty(H^k)} + K_b\|u''\|_{L^1(H^k)}\}$.

Additional Remarks. The coercivity of A implies that (4.9) bounds the H^1-norm of $u(t)-U(t)$. Similar remarks apply to $u'(t)-U'(t)$ when B is coercive.

The preceding proofs give estimates for problems of first order in time in the form of Problem 2.

Proposition 6 applies directly to Examples 2 and 3 of § 3. After an elementary change of variable, Example 1 with $r = 0$ is included. In the following section we indicate how Example 4 can be perturbed into a "nearby" problem to which Proposition 6 applies.

Since B is not required to be coercive, Proposition 6 gives error estimates for problems like the following:

$$\begin{aligned} -\Delta u(x, t) &= F(x, t), && x \in G, \quad t \geqq 0,\\ D_t^2 u(x, t) + D_\nu u(x, t) &= 0, && x \in \Gamma_1,\\ u(x, 0) = u_1(x), \qquad D_t u(x, 0) &= u_2(x), && \\ u(x, t) &= 0, && x \in \Gamma_0, \quad t \geqq 0. \end{aligned}$$

Such problems arise as linear approximations of gravity waves [11], [15].

The preceding techniques lead directly to energy estimates of error in the approximation of equations with higher order elliptic coefficients. Such examples were mentioned at the end of § 3. For related results, see [1], [2], [3], [16], [18].

5. Perturbations. Three methods will be given for perturbing (1.1) into "nearby" equations with desirable properties. We shall assume that A, B and C are all monotone and that A and B are symmetric. None are necessarily coercive, so the functions $\|\cdot\|_a$ and $\|\cdot\|_b$ are continuous seminorms on V and W, respectively; denote the corresponding seminorm spaces by V_a and W_b. The first two methods are appropriate for the most common situation (e.g., Example 1) in which A is strictly stronger than B and C. The first method corresponds to an introduction of artificial viscosity for strong dissipation in the model (cf., Example 2) while the second method is suggestive of an introduction of artificial inertia. The third method is a means of perturbing (1.1) into an equation to which we can apply our approximation results of § 4. It is appropriate for situations (e.g., Example 4 with $a = 0$ or $b = 0$) in which B is an elliptic operator and A is not coercive.

Parabolic regularization. We modify (1.1) by replacing C with $C + \varepsilon A$, $\varepsilon > 0$. If u_ε is the corresponding solution of Problem 1 on $[0, T]$ and $w_\varepsilon = (u_\varepsilon, u_\varepsilon')$, then we have

$$Mw_\varepsilon'(t) + L_\varepsilon w_\varepsilon(t) = (0, f(t)), \qquad 0 \leqq t \leqq T, \tag{5.1}$$

where $L_\varepsilon = L + \varepsilon\begin{pmatrix} 0 & 0 \\ 0 & A\end{pmatrix}$. If u is a solution of Problem 1 and $w = (u, u')$ the corresponding solution of Problem 2, then

$$M(w'(t)-w'_\varepsilon(t))+L_\varepsilon(w(t)-w_\varepsilon(t))=(0,\varepsilon Au'(t)),\qquad 0\leqq t\leqq T, \tag{5.2}$$

and from (2.2) we obtain

$$\|w(t)-w_\varepsilon(t)\|_m^2+2\operatorname{Re}\int_0^t\langle (C+\varepsilon A)(u'-u'_\varepsilon),u'-u'_\varepsilon\rangle$$
$$=2\operatorname{Re}\int_0^t\langle \varepsilon Au',u'-u'_\varepsilon\rangle\leqq\varepsilon\int_0^t(\|u'\|_a^2+\|u'-u'_\varepsilon\|_a^2).$$

Since C is monotone, it follows that

$$\|w(t)-w_\varepsilon(t)\|_m^2+\varepsilon\int_0^t\|u'-u'_\varepsilon\|_a^2\leqq\varepsilon\int_0^t\|u'\|_a^2,\qquad 0\leqq t\leqq T.$$

PROPOSITION 7. *If $u'\in L^2([0,T],V_a)$, then*

$$\|u(t)-u_\varepsilon(t)\|_a^2+\|u'(t)-u'_\varepsilon(t)\|_b^2+\varepsilon\int_0^t\|u'-u'_\varepsilon\|_a^2\leqq\varepsilon\int_0^t\|u'\|_a^2.$$

In particular, $u_\varepsilon\to u$ ($u'_\varepsilon\to u'$) in $L^\infty([0,T],V_a)$ (respectively, $L^\infty([0,T],W_b)$) and u'_ε is bounded in $L^2([0,T],V_a)$.

From (5.1) and (2.3) one shows easily that $\|u_\varepsilon\|_{L^\infty(V_a)}$, $\|u'_\varepsilon\|_{L^\infty(W_b)}$ and $\sqrt{\varepsilon}\|u'_\varepsilon\|_{L^2(V_a)}$ are bounded. The existence of a solution of (1.1) can be deduced from existence for (5.1) and weak*-compactness of closed balls in L^∞ [12, Chap. 3.8].

When $C=0$ and A is coercive, Proposition 6 applies both to (1.1) and (5.1). However, the strongly dissipative parabolic equation (5.1) may be more desirable for numerical work [16, Chap. 7.3].

Sobolev regularization. In this perturbation of (1.1) we replace B with $B+\varepsilon A$, $\varepsilon>0$. Denoting by u_ε a solution of the perturbed problem and letting $w_\varepsilon=(u_\varepsilon,u'_\varepsilon)$ as before, we have

$$M_\varepsilon w'_\varepsilon(t)+Lw_\varepsilon(t)=(0,f(t)),\qquad 0\leqq t\leqq T, \tag{5.3}$$

where $M_\varepsilon=M+\varepsilon\begin{pmatrix}0&0\\0&A\end{pmatrix}$. With u and w as before, we have

$$M_\varepsilon(w'(t)-w'_\varepsilon(t))+L(w(t)-w_\varepsilon(t))=(0,\varepsilon Au''(t)),\qquad 0\leqq t\leqq T, \tag{5.4}$$

so (2.2) gives the estimate

$$\|w(t)-w_\varepsilon(t)\|_m^2+\varepsilon\|u'(t)-u'_\varepsilon(t)\|_a^2\leqq 2\varepsilon\int_0^t\operatorname{Re}\langle Au'',u'-u'_\varepsilon\rangle$$
$$\leqq\varepsilon\int_0^t(\|u''\|_a^2+\|u'-u'_\varepsilon\|_a^2),\quad 0\leqq t\leqq T.$$

Setting $H(t)=\int_0^t\|u''\|_a^2+\|u'-u'_\varepsilon\|_a^2$, we have

$$H'(t)\leqq\|u''(t)\|_a^2+H(t),\qquad 0\leqq t\leqq T,$$

and hence,

$$H(t) \leqq \int_0^t e^{t-\tau} \|u''(\tau)\|_a^2 \, d\tau \leqq e^t \int_0^t \|u''\|_a^2.$$

Our original estimate now gives the following proposition.

PROPOSITION 8. *If* $u'' \in L^2([0, T], V_a)$, *then*

$$\|u(t)-u_\varepsilon(t)\|_a^2 + \|u'(t)-u'_\varepsilon(t)\|_b^2 + \varepsilon \|u'(t)-u'_\varepsilon(t)\|_a^2 \leqq \varepsilon \, e^t \int_0^t \|u''\|_a^2, \qquad 0 \leqq t \leqq T.$$

In particular, $u_\varepsilon \to u$ $(u'_\varepsilon \to u')$ *in* $L^\infty([0, T], V_a)$ *(respectively,* $L^\infty([0, T], W_b)$*) and* u'_ε *is bounded in* $L^\infty([0, T], V_a)$.

From (5.3) and (2.3) it follows that $\|u_\varepsilon\|_{L^\infty(V_a)}$, $\|u'_\varepsilon\|_{L^\infty(W_b)}$ and $\sqrt{\varepsilon}\|u'_\varepsilon\|_{L^\infty(V_a)}$ are bounded. We can obtain existence proofs from such a priori inequalities.

Discrete analogues of this method appear as Laplace-modified Galerkin techniques [1], [2] for equation (1.1) with $B = I$ and first order equations, (1.1) with $B = 0$. In these numerical schemes, ε is chosen as a first or second power of the time increment.

A nonsingular perturbation. For our final method we modify Problem 1 by replacing A with $A + \varepsilon B$. Letting u_ε denote the corresponding solution and $w_\varepsilon = (u_\varepsilon, u'_\varepsilon)$ as before, we have

$$M_\varepsilon w'_\varepsilon(t) + L_\varepsilon w_\varepsilon(t) = (0, f(t)), \qquad 0 \leqq t \leqq T, \tag{5.5}$$

where $M_\varepsilon = M + \varepsilon\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$, $L_\varepsilon = L + \varepsilon\begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix}$. If u and w are respective solutions of Problems 1 and 2, then we have

$$M_\varepsilon(w'(t) - w'_\varepsilon(t)) + L_\varepsilon(w(t) - w_\varepsilon(t)) = (0, \varepsilon B u(t)), \qquad 0 \leqq t \leqq T, \tag{5.6}$$

so Proposition 1 gives us the following.

PROPOSITION 9. *If* u *and* u_ε *are respective solutions of Problem* 1 *and the indicated perturbed problem, then*

$$\|u(t)-u_\varepsilon(t)\|_a^2 + \|u'(t)-u'_\varepsilon(t)\|_b^2 + \varepsilon \|u(t)-u_\varepsilon(t)\|_b^2 \leqq \varepsilon T e \int_0^t \|u\|_b^2, \qquad 0 \leqq t \leqq T.$$

The point of Proposition 9 is to perturb Example 4 into a form to which Proposition 6 can be applied. An attempt to do so by introducing the unknown $v(t) = e^{-\lambda t} u(t)$ leads to Problem 1 for v with A replaced by the coercive $A + \lambda^2 B$ but at the expense of introducing a term $2\lambda A u'(t)$, thus making Proposition 6 nonapplicable.

Similar techniques work for corresponding problems with a first order time derivative. Such a problem arises with the equation

$$D_t(\Delta u(x, t)) + \beta D_1 u(x, t) = 0$$

for divergence-free Rossby waves [10, § 7].

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CAUSALITY OF TIME-VARYING NONLINEAR OPERATORS AND THE EXTENDIBILITY OF THEIR DOMAINS*

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Abstract. When applying operator theory to model the behavior of the electrical system, it is advantageous to initially consider a restrictive domain of infinitely differentiable functions with compact support. It is shown that by the assumption of causality and related postulates, the domain of an operator, continuous from D into D', can be continuously and uniquely extended onto D_R, the space of right-sided testing functions. A converse theorem is established too.

1. Introduction. When applying operator theory to mathematically model the behavior of physical systems, the process usually involves two stages. First, the setting is constructed in such a way that a large class of operators amenable to the analysis is obtained. This is accomplished by choosing a restrictive space with a strong topology to serve as the domain and a broad space equipped with a weak topology to be the range space. In the second stage, certain postulates, motivated by the physical nature of the system, are proposed and imposed on the operator. Their effect is to restrict the class of permissible operators, provide them with characterizations and to extend the initial domain and to restrict the initial range. In this work, the attention is focused in the last subject, namely, the extendibility of the domain and the restrictivity of the range which are obtainable by the postulate of causality and related postulates. The importance of the subject originates from the fact that extending the initially restrictive domain will allow a larger set of admissible input signals to the system. On the other hand, restricting the initially broad range space will furnish information about the expected output signals.

We consider the postulate of causality and related postulates like, finite memory, finite lag and localness. It is shown that they allow us to extend the operator, initially assumed to operate on functions of compact support, to a larger domain which consists of the functions whose supports are limited only from the left. Furthermore, the extended operator enjoys the properties of continuity and causality on the extended domain as well. This is very satisfying from a physical point of view. The operator is initially defined for infinitely differentiable functions of compact support. However, although there is physical sense in "starting" the inputs at a finite time, there is no justification to assume that they should also "end" at a finite time in the future. The natural inputs to a causal system are functions which are not limited with regard to their support or growth on the right-hand side of the time axis. Being able to extend the domain of definition in a unique way and such that the extended operator remains continuous and causal, to include those functions is significant from a physical point of view. As expected, this extendibility can uniquely be tied to the postulate of causality or related postulates.

Newcomb [2] postulates an electrical network to be defined by an operator (or actually by a binary relation) which maps D_R into itself, where D_R is the space

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of right-sided testing functions. Among other things, it is shown in this work that this need not be taken as a basic assumption. Instead, if one takes the usual course and defines the operator on input functions of compact support and adds the assumption of causality, the extendibility of the domain onto D_R is automatically fulfilled.

The physical system under consideration is, of course, the electrical network. The analysis will be carried out for signals which are scalar-valued time functions. Extending the results for the n-dimensional case is straightforward, provided that an extended definition of causality to R^n (Meidan [1]) is adopted.

The analysis will be carried out in the framework of Schwartz's distribution theory. This provides the space D, of infinitely differentiable testing functions of compact support as the initial domain and the space D' of distributions as the initial range. This framework is compatible with other works in the literature (e.g., Zemanian [5]). The domain D will be equipped with the strong testing function topology and the range D' with the weak dual topology. This will meet the abovementioned requirements from the initial domain and range spaces.

At this point, the postulate of continuity will be imposed. In view of the above topologies, this is a fairly weak restriction. However, it implies boundedness. One should remember that, for nonlinear operators, continuity does not, in general, imply boundedness. However, in view of the domain space chosen, this is the case for the operators under consideration.

In the literature, the assumption of linearity is usually imposed at this stage. This proves to be very powerful from the standpoint of the analysis. In particular it allows us to invoke Schwartz's kernel theorem (Schwartz [3]). Since in this work linearity is relinquished, the analysis will be carried out without the help of Schwartz's kernels. We will also see, especially when the converse subject is pursued, that certain properties automatically fulfilled for linear operators, in view of the nonlinearity, will have to be assumed independently.

2. Preliminaries and notations.

Spaces of testing functions. Let R denote the set of real numbers and D the space of complex-valued infinitely differentiable functions on R with compact support. We equip D with the testing functions topology. By this topology, a sequence $\{\psi_n\}$ of functions in D converges to zero if the sequence of functions as well as their derivatives of any order converge uniformly to zero, and if, in addition, all functions have their supports contained in a compact subset of R. D_R denotes the space of right-sided testing functions. It consists of infinitely differentiable functions whose supports are bounded on the left. D_R is equipped with a topology according to which a sequence $\{\psi_n\}$ of functions in D_R converges to zero if the sequences of functions and all their derivatives converge to zero uniformly on all compact subsets of R and if, in addition, all functions have their supports limited on the left by a fixed real number. The space D_L on left-sided testing functions is defined in a similar way. The last space of testing functions to be considered is E. It consists of infinitely differentiable testing functions on R with no restriction on the support. It carries a topology by which a sequence $\{\psi_n\}$ of testing functions in E converges to zero if the functions and all their derivatives converge uniformly to zero on every compact subset of R. The relation between these spaces can be described in terms of the following set inclusion:

$$\begin{array}{ccccc} & & D_L & & \\ & \subset & & \subset & \\ D & & & & E. \\ & \subset & & \subset & \\ & & D_R & & \end{array}$$

This inclusion defines natural injections of one space into the other. In view of the assigned topologies, these injections are continuous.

Let Ω be an open subset of R. We will need spaces of testing functions restricted to Ω. Hence $D(\Omega)$ denotes the space of testing functions defined on Ω whose supports are contained in a compact subset of Ω. The spaces $D_R(\Omega)$, $D_L(\Omega)$, $E(\Omega)$ are defined accordingly.

Spaces of distributions. D' is the dual space of D, namely, the space of the linear and continuous functionals on D, so-called distributions. If $\psi \in D$ and $f \in D'$, we denote by $\langle f,\psi \rangle$ the complex number which f assigns to ψ. $D'(\Omega)$ is the dual of $D(\Omega)$. Let Ω_1 and Ω_2 be two open sets in R such that $\Omega_1 \subset \Omega_2$. Hence $D(\Omega_1) \subset D(\Omega_2)$. Let f be in $D'(\Omega_2)$. We restrict f to the testing functions of $D(\Omega_1)$, and get a member of $D'(\Omega_1)$. It will be called the restriction to Ω_1 of a distribution on Ω_2. A distribution on R is said to vanish on an open set Ω of R if its restriction to Ω is the zero distribution. The support of a distribution is a closed set which is the complement of the largest open set for which the distribution vanishes. D_R' denotes the space of right-sided distributions, i.e., distributions whose supports are bounded on the left. It is not the dual of D_R. In fact, it is the dual of D_L. Similarly D_L' is the dual of D_R. E' denotes the dual of E, and it consists of the distributions with compact support. The topologies assigned to the various spaces of distributions are the weak dual topologies. Let $\{f_n\}$ be a sequence of distributions in one of the above mentioned spaces. The sequence is said to converge to zero if the sequence of complex numbers $\langle f_n,\psi \rangle$ converges to zero for every ψ in the corresponding testing space. Equality in D', between two distributions f_1 and f_2 means that $\langle f_1,\psi \rangle = \langle f_2,\psi \rangle$ for every $\psi \in D$. Two distributions, f_1 and f_2, can be equal on an open subset Ω of R. This means that their restrictions to $D(\Omega)$ are equal.

The operators. Let N denote an operator, nonlinear in general, mapping D into D'. We assume that it is continuous with respect to the testing function topology of D and the weak dual topology of D'. This is a fairly weak assumption in view of the topologies involved. For linear operators, the continuity implies boundedness. This is not the case, in general, for nonlinear operators. However, we will show that, in view of the particular domain space, this holds for the operators at hand.

THEOREM 1. *Let N be an operator mapping D into D'. If N is continuous, it is also bounded.*

Proof. Let A be a bounded set in D. We have to show that $N(A)$ is bounded in D'. Indeed, since D is a Montel space, all its bounded sets are relatively compact. Since N is continuous, $N(A)$ is relatively compact too. But this implies that $N(A)$ is bounded, which completes the proof.

This result allows us to consider only the continuity of the operator and to be assured that its boundedness holds too. We will consider now the question of extending the domain of the definition of the operator.

Let N denote a continuous operator (not necessarily linear) from D into D'. Let A be a space of testing functions which contains D and which has a weaker topology than D. N is said to be extendible onto A if an operator exists which is continuous from A into D' and which coincides with N when restricted to D. We will not distinguish notationwise between the operator and its extension. Since the main subject at hand is extending the domain of definition of operators, we will state some well-known theorems regarding this. The operator is extendible from D onto A, if and only if it is continuous with respect to the relative topology induced on D by A. Since this topology is, by assumption, weaker than the initial topology, this is a stronger continuity requirement. The extension is unique if and only if D is dense in A.

3. Causality and the extendibility of the domain.

DEFINITION. Let N be an operator mapping D into D'. N is said to be *causal* on D if, for every $t_0 \in R$, we have that $N\psi_1 = N\psi_2$ (in the sense of equality of distributions) on the open interval $(-\infty, t_0)$ whenever $\psi_1, \psi_2 \in D$ and $\psi_1 = \psi_2$ on $(-\infty, t_0)$.

By the definition, the causality of the operator implies its being single-valued. Hence we will always assume that the operators are single-valued. Also, no generality is lost if we assume that $N(0) = 0$ where the zero on the left-hand side is the zero testing function in D and on the right-hand side, the zero distribution in D'. Indeed, let N' be an operator such that $N'(0) = f_0 \neq 0$. Then, define a shifted operator $N = N' - f_0$. N has the same continuity property as N', but its reponse to the zero input function is zero. It should be noted that for linear operators, $N(0)$ must be always zero.

Based on this assumption, we get a necessary condition for the causality of the operator. It is only for linear operators that this condition is also sufficient.

LEMMA. *Let N be a causal operator from D into D'* (*such that* $N(0) = 0$). *If* $\psi \in D$ *such that* $\operatorname{supp} \psi \subset [t_0, \infty)$, *then* $\operatorname{supp} N\psi \subset [t_0, \infty)$.

We are now in a position to state the main extendibility theorem.

THEOREM 2. *Let N be a continuous operator from D into D'. If N is causal, it is extendible as a continuous and causal operator mapping D_R into D'_R.*

The rule of extension. Let $\psi(t)$ be a testing function in D_R. We wish to define $N(\psi)$. Choose $t_i \in R$ and a testing function $\lambda_i(t)$ which is equal to unity over the interval $(-\infty, t_i)$ and vanishes on the closed interval $[t_i + \delta, \infty)$, $\delta > 0$. By the partition of unity (e.g., see [4]), such a function can be found. Consider $\lambda_i(t)\psi(t)$. It is a testing function of compact support. Hence, we can apply the operator N to it, and $f^{(i)} = N(\lambda_i\psi)$ is a distribution in D'. Consider its restriction to the open interval $\Omega_i = (-\infty, t_i)$. In view of the causality of N, this restriction is independent of the different possible choices of λ_i. Assume now that t_i traverses R. A family of distributions $\{f^{(i)} | f^{(i)} = N(\psi\lambda_i), i \in R\}$ is obtained. The family is pairwise consistent in the following sense: Let $f^{(i)}$, $f^{(j)}$ be two members of the family with Ω_i and Ω_j their respective open sets, i.e., $\Omega_i = (-\infty, t_i)$, $\Omega_j = (-\infty, t_j)$. Then $f^{(i)}$ and $f^{(j)}$ coincide on $\Omega_i \cap \Omega_j$. This pairwise consistency follows directly from the causality of N. We now invoke a theorem ([4, Thm. 24.1]). It follows from this theorem that, in view of the pairwise consistency of $\{f^{(i)}\}$, there exists a unique distribution f on R, such that f, when restricted to each Ω_i, coincides with the respective $f^{(i)}$. By setting $f = N\psi$, the requested rule for extending the domain is established.

Proof of Theorem 2. That the above rule applies to all members of D_R is obvious. The uniqueness of the extended operator is clear from the construction of the extension rule. It remains to prove the continuity of the extended operator. Let $\{\psi_n\}$ be a sequence converging to ψ in D_R. (It will not be sufficient to assume that $\psi = 0$ in view of the possible nonlinearity of the operator). We have to show that $N\psi_n$ (where N denotes here the extended operator) converges in D' to $N\psi$. Let $t_i \in R$ and $\lambda_i(t)$ be a testing function with properties as above. By hypothesis, $\{\psi_n\}$ converges in D_R. This means that all supports of ψ_n are bounded from the left by, say T, and that for every $t_i \in R$, the sequence and all its derivatives converge uniformly on the compact set $[T, t_i + \delta]$, $\delta > 0$ (and t_i is defined in the rule of extension). Hence the sequence $\{\psi_n(t)\lambda_i(t)\}$ converges in D to $\psi(t)\lambda_i(t)$, and $\{f_n^{(i)}\} = \{N(\psi_n\lambda_i)\}$ converges in D' to $f^{(i)} = N(\psi\lambda_i)$. Again, consider t_i traversing R. A family of sequences $\{f_n^{(i)}\}$ and a family of distributions $\{f^{(i)}\}$, $i \in R$, is obtained. For each n, the family of distributions $\{f_n^{(i)}\}$ is pairwise consistent. Hence, there exists a sequence of distributions $\{f_n\}$ such that their restrictions to $\Omega_i = (-\infty, t_i)$ coincide with $f_n^{(i)}$ and a distribution f whose restriction to Ω_i is equal to $f^{(i)}$. By the extension rule, $f_n = N\psi_n$ and $f = N\psi$, and since $\{f_n^{(i)}\}$ converges to $f^{(i)}$, we have that $\{f_n\}$ converges to f, which establishes the continuity of N on D_R.

Based on the extension rule, it is easy to verify that the extended operator is causal on its extended domain.

Having dealt with the domain, we will now consider the range space. By the causality, it is clear that $N\psi \in D'_R$ for every $\psi \in D_R$. D'_R, which is a subspace of D', carries a topology stronger than the relative topology induced by D'. We claim that N is actually continuous with respect to this topology. Indeed, let $\{\psi_n\}$ be a sequence converging to ψ in D_R. Then $\{N\psi_n\}$ is contained in D'_R and converges to $N\psi$ with respect to each testing function of compact support. We have to show that the convergence holds also for testing functions whose supports are not bounded on the left. Let ψ denote such a testing function in D_L. Since the supports of all ψ_n are bounded by a fixed number, say T, on the left, $N\psi_n$ have their supports bounded by T as well. Choose $\lambda_T(t)$ to be a testing function equal to one over the interval (T, ∞) and vanishing on $(-\infty, T\text{-}\delta)$, $\delta > 0$. Then, $\phi \in D_L$, $\langle N\psi_n, \phi\rangle = \langle N\psi_n, \phi\lambda_T\rangle$ for each n. But since the right-hand side converges, so does the left, which completes the proof.

There are other postulates similar in nature to causality. These are the concepts of finite lag, finite memory, localness and memorylessness.

DEFINITION. Let N be an operator mapping D into D'. N is said to be *of finite lag* if there exists $T \in R$, $T \geqq 0$, such that the composite operator $\sigma_T N$ is causal, where σ_T denotes the shifting operator in D', namely, $\sigma_T f(t) = f(t - T)$. Clearly, a causal operator is a finite lag operator with zero lag time, $T = 0$.

DEFINITION. Let N be an operator mapping D into D'. N is said to have *a finite memory* T on D if, for every $t_0 \in R$, we have that $N\psi_1 = N\psi_2$ on the open interval (t_0, ∞) whenever $\psi_1, \psi_2 \in D$ and $\psi_1 = \psi_2$ on $(t_0 - T, \infty)$.

DEFINITION. Let N be an operator from D into D'. N is said to be a *local operator* if there exist $T_1, T_2 \geqq 0$ such that for every $t_1 < t_2$ in R and $\psi_1(t), \psi_2(t)$ in D

with $\psi_1(t)=\psi_2(t)$ on (t_1-T_1, t_2+T_2), we have that $N\psi_1=N\psi_2$ on (t_1, t_2). If $T_1=T_2=0$, we say that the operator is *memoryless.* Clearly, a local operator is both of finite lag and of finite memory.

The physical motivation of these postulates is clear. There may be applications where a noncausal operation is requested. However, should this be an operation of finite lag, it can be made causal by a finite delay. There may be operators, especially those simulated by a digital computer, which are restricted by their memory. All these postulates may be related via the operation of time reversal. Let γ denote the operation of time reversal, i.e., $\gamma\psi(t)=\gamma(-t)$. Then γ is an isomorphism in all the spaces of testing functions and distributions under consideration. γ also defines an isomorphism between the class of finite lag operators and the class of finite memory operators. In view of this, the extension theorems for these operators are clear.

THEOREM 3. *Let N be an operator continuous from D into D'. Then,*

(i) *if N is of finite lag, it is extendible as a continuous operator of finite lag from D_R into D_R';*

(ii) *if N is of finite memory, it is extendible as a continuous finite memory operator from D_R into D_R';*

(iii) *if N is local, it is extendible as a continuous and local operator from E into D'.*

The rules of extension are similar to the one given for the causal operator, and the proof of the theorem follows along the same lines.

4. Converse theorems. Let N be an operator from D into D' which has a continuous extension onto E. Can one draw a conclusion with regard to the localness of the operator? We will give a positive answer to this question, but not before generalizations of the concept of localness are introduced. Some of the difficulties encountered when dealing with nonlinear operators will come to light in the process. Without loss of generality, only the case of local operators will be pursued. The other cases follow along similar lines.

One of the powers of linear operators stems from the fact that local properties are actually global. This is not the case for nonlinear operators. The following can serve as an example.

DEFINITION. Let f be a continuous functional on a countably normed space X, and $\gamma_n(\psi)$, $\psi\in X$, denote a sequence of nondecreasing seminorms generating the topology of X. f is said to be *locally finite order continuous* of order m if the following holds: For every $\psi_0\in X$, there exists an integer m such that for every $\varepsilon>0$, there exists a $\delta>0$, such that $\gamma_m(\psi-\psi_0)<\delta$ implies $|f(\psi)-f(\psi_0)|<\varepsilon$. f is *(globally) finite order continuous of order m* if a uniform m can be found independent of ψ_0.

It is a standard result of functional analysis that linear operators on countably normed spaces are globally of finite order. The following lemma will establish that for nonlinear operators we do have a finite order continuity, but only in a local sense; i.e., the order can, in general, change as a function of ψ_0 in X.

LEMMA. *Let f be a continuous functional on a countably normed space X. Then f is locally finite order continuous.*

Proof. As before, let $\{\gamma_n\}$ denote a generating sequence of nondecreasing seminorms. The lemma will be proved by contradiction. Suppose that for some ψ_0 such a finite integer does not exist. Then a sequence $\{\psi_n\}$ in X can be found such that $\gamma_n(\psi_n - \psi_0) < 1/n$ and $|f(\psi_n) - f(\psi_0) \geqq \varepsilon$ for some $\varepsilon > 0$. $\{\psi_n\}$ obviously converges to ψ_0 in X, but $\{f(\psi_n)\}$ does not converge to $f(\psi_0)$. This contradicts the assumed continuity of f, hence completing the proof.

The above exhibits the fact that the property of localness of the operator should be considered in a local sense depending both on the time and on the function $\psi_0 \in X$.

DEFINITION. Let N be an operator from E into D'. Let ψ_0 be in E and $t_1 < t_2$ be two real numbers. N is said to be of a *local variable localness* if there exist nonnegative numbers $T_1(\psi_0, t_1, t_2)$ and $T_2(\psi_0, t_1, t_2)$, such that any ψ in E with $\psi = \psi_0$ on $(t_1 - T_1, t_2 + T_2)$ implies $N\psi = N\psi_0$ on (t_1, t_2). If T_1 and T_2 do not depend on ψ_0, we say that N is of a *variable localness*. Similarly, if T_1, T_2 do not depend on t_1, t_2, then N is of a *local localness*. Finally, if T_1, T_2 are fixed, N is *local*.

THEOREM 4. *Let N be extendible as a continuous operator from E into D'. Then N is of a local variable localness. If, in addition, N is finite order continuous, then it is of a variable localness. If N is time invariant, then it is of a local localness. Finally, if N is both finite order continuous and time invariant, then N is local.*

Proof. Let ψ_0 be in E and fix t_1 and t_2, $t_1 < t_2$. Let K denote the compact set $[t_1, t_2]$ and $D(K)$ the set of testing functions with support contained in K. Consider the form $\langle N\psi, \theta\rangle$, $\psi \in E$ and $\theta \in D(K)$. It is a separately continuous functional on the Cartesian product space $E \times D(K)$. It is, in fact, (jointly) continuous on the product space. This follows as a consequence of the Banach–Steinhauss theorem. (We quote Theorem 34.1 of [4] as a direct reference.) Moreover, since both spaces E and $D(K)$ are countably normed spaces, $\langle N\psi, \theta\rangle$ is locally finite order continuous on E, and in view of its linearity on $D(K)$, finite order continuous on $D(K)$. Let γ_m (β_m, respectively) denote the sequence of nondecreasing seminorms on E ($D(K)$, respectively). Then we have the following: For every $\psi_0 \in E$, there exists a compact set K_1 of R and an integer m such that for every $\varepsilon > 0$, a $\delta > 0$ can be found such that

$$\gamma_m(\psi - \psi_0) = \sup_{t \in K_1} \sup_{k \leqq m} |D^k[\psi(t) - \psi_0(t)]| < \delta \tag{1}$$

and

$$\beta_m(\theta) = \sup_{t \in K} \sup_{k \leqq m} |D^k \theta(t)| < \delta \tag{2}$$

imply that

$$|\langle N\psi - N\psi_0, \theta\rangle| < \varepsilon. \tag{3}$$

Now consider the convex hull of $K \cup K_1$. It is a closed interval, say $[t_1 - T_1, t_2 + T_2]$. This defines two positive numbers T_1 and T_2. Let ψ be in E such that $\psi = \psi_0$ on the open interval $(t_1 - T_1, t_2 + T_2)$. By the continuity of ψ and ψ_0, we have that $\psi = \psi_0$ also on the closed interval $[t_1 - T_1, t_2 + T_2]$, consequently, $\gamma_m(\psi - \psi_0)$ of (1) is equal to zero. It follows that for every $\theta \in D(K)$, (3) holds for arbitrary ε. Hence $N\psi_1 = N\psi_2$ on (t_1, t_2). This completes the proof. The other special cases of the theorem are straightforward consequences.

It should be noted that a slight weakening of the assumption of the finite order continuity of N can be considered in the previous theorem. It is sufficient that the finite order continuity holds with respect to the compact sets of the real line on which the supremum is taken. There is no necessity to have a finite order continuity with respect to the order of the derivatives.

Examples. We consider a few examples:

(I) Let $g(t)$ be an integrable function and define

$$N\psi = g(t) \cdot \int_{-\infty}^{\infty} \psi(\tau)\, d\tau, \qquad \psi \in D. \tag{4}$$

N is linear time-varying, but does not meet any of the postulates considered. As is well known, it is extendible onto L^1, but it cannot be extended onto E, whose elements can grow at infinity without restriction.

(II) Consider,

$$N\psi = g(t) \int_{-\infty}^{t} \psi(\tau)\, d\tau, \qquad \psi \in D. \tag{5}$$

It is causal and thus extendible onto D_R. Indeed, $\psi \in D_R$ is integrable over $(-\infty, t]$ for every $t \in R$. The continuity of N on D_R is easily verified.

(III) Let

$$N\psi = g(t) \int_{-a(\psi,t)}^{a(\psi,t)} \psi(\tau)\, d\tau, \qquad \psi \in E, \tag{6}$$

where

$$a(\psi, t) = |t| \max_{|t| \leqq 1} |\psi(t)|.$$

It is a nonlinear time-varying operator. It exhibits a local finite continuity on E. Indeed, fix $\psi_0 \in E$ and $\phi \in D$. The functional $(N\psi, \phi)$ is continuous at ψ_0 with respect to the seminorm,

$$\gamma(\psi) = \sup_{t \in K} |\psi(t)|,$$

where

$$K = \operatorname{supp} \phi.$$

Also, in view of the finite limits of the integration of (6), we have that N exhibits a local and variable localness.

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THE RESOLVENT KERNEL OF AN INTEGRODIFFERENTIAL EQUATION IN HILBERT SPACE*

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Abstract. Let $\mathbf{y}(t, \mathbf{x}, \mathbf{f})$ denote the solution of $\mathbf{y}'(t) + \int_0^t a(t-s)\mathbf{L}\mathbf{y}(s)\,ds = \mathbf{f}(t), \mathbf{y}(0) = \mathbf{x}, t \geqq 0$, where $\mathbf{L}$ is a self-adjoint densely defined operator on a Hilbert space $\mathbf{H}$, with $\mathbf{L} \geqq \mu > 0$. Let $\mathbf{U}(t)\mathbf{x} = \mathbf{y}(t, \mathbf{x}, \mathbf{0})$. It is shown that if $a(t)$ is continuous ($t \geqq 0$) and completely monotonic ($t > 0$), but not constant, then $\mathbf{U}(t)$ is a bounded operator on $\mathbf{H}$ with $\|\mathbf{U}(t)\| \leqq \sqrt{2}$, $\|\mathbf{U}(t)\| \to 0$ $(t \to \infty)$, and $\int_0^\infty \|\mathbf{U}(t)\|\,dt < \infty$. This result is useful when the representation $\mathbf{y}(t, \mathbf{x}, \mathbf{f}) = \mathbf{U}(t)\mathbf{x} + \int_0^t \mathbf{U}(t-s)\mathbf{f}(s)\,ds$ holds. The proof starts with the inequality $\|\mathbf{U}(t)\| \leqq \sup_{\lambda \geqq \mu} |u(t, \lambda)|$, where $u_t(t, \lambda) + \lambda \int_0^t a(t-s)u(s, \lambda)\,ds = 0$, $u(0, \lambda) = 1$.

1. Introduction. Let $\mathbf{H}$ be a Hilbert space and $\mathbf{L}$ a self-adjoint operator with domain $\mathbf{D}$ dense in $\mathbf{H}$ and spectral decomposition $\mathbf{L}\mathbf{x} = \int_\mu^\infty \lambda\, d\mathbf{E}_\lambda \mathbf{x} (\mathbf{x} \in \mathbf{D})$ with $\mu > 0$. We consider the integrodifferential equation

$$\mathbf{y}'(t) + \int_0^t a(t-s)\mathbf{L}\mathbf{y}(s)\,ds = \mathbf{f}(t), \mathbf{y}(0) = \mathbf{y}_0 \qquad \left(' = \frac{d}{dt}\right), \tag{1.1}$$

where $a: R^+ \to R^+$ and $\mathbf{f}: R^+ \to \mathbf{H}$ are continuous ($R^+ = [0, \infty)$) and $\mathbf{y}_0 \in \mathbf{H}$. A solution of (1.1) is a continuously differentiable function $\mathbf{y}: R^+ \to \mathbf{H}$ such that $\mathbf{y}(t) \in \mathbf{D}$, $\mathbf{L}\mathbf{y}(t)$ is continuous, and (1.1) holds on R^+. Generalized solutions, such as solutions of an integrated form of (1.1), may also be of interest.

A solution or generalized solution of (1.1) is often given by the resolvent formula

$$\mathbf{y}(t) = \mathbf{U}(t)\mathbf{y}_0 + \int_0^t \mathbf{U}(t-s)\mathbf{f}(s)\,ds, \tag{1.2}$$

where $\mathbf{U}(t): \mathbf{H} \to \mathbf{H}$ (defined precisely in (1.7) below) is the solution operator for the homogeneous version of (1.1).

THEOREM 1. *Suppose $a(t)$ is continuous on R^+ and completely monotonic on $(0, \infty)$, but not constant. Then*

(i) $\mathbf{U}(t)$ *belongs to the space $\mathscr{B}(\mathbf{H})$ of bounded linear operators on $\mathbf{H}$, and $\|\mathbf{U}(t)\| \leqq 1 (t \in R^+)$; $\mathbf{U}(t)$ maps $\mathbf{D}$ into $\mathbf{D}$; and $t \to \mathbf{U}(t)$ is continuous on $(0, \infty)$ in the strong operator topology.*

(ii) $\|\mathbf{U}(t)\| \to 0 (t \to \infty)$, $\int_0^\infty \|\mathbf{U}(t)\|\,dt < \infty$, *and* $\int_0^\infty \mathbf{U}(t)\mathbf{x}\,dt = \mathbf{L}^{-1}\mathbf{x}/\int_0^\infty a(t) \cdot dt (\mathbf{x} \in \mathbf{H})$.

(In this paper $(\int_0^\infty a(t)\,dt)^{-1}$ is interpreted as zero when $a(t) \notin L^1(0, \infty)$.)

The first two relations of conclusion (ii) constitute our main result. The remaining conclusions of Theorem 1 hold if $a(t)$ is merely assumed to be nonnegative, nonincreasing, convex, and locally integrable, but not piecewise linear

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[12]; indeed, conclusions of the latter ("strong") type hold in some cases when the kernel $a(t)\mathbf{L}$ is replaced by $\mathbf{a}(t, \mathbf{L}) = \int_\mu^\infty a(t, \lambda)\, d\mathbf{E}_\lambda$ (see [13], [14]).

The next theorem gives sufficient conditions for the representation (1.2) and thus exhibits one case where Theorem 1 applies to (1.1).

THEOREM 2. *Under the assumptions of Theorem* 1, *suppose* $\mathbf{y}_0$ *and* $\mathbf{f}(t)$ *belong to* $\mathbf{D}(t \in R^+)$ *and* $t \to \mathbf{Lf}(t)$ *is (strongly) continuous on* R^+. *Then* (1.2) *gives the unique solution of* (1.1).

Our proof of Theorem 2 follows the procedure used by Krein [15, p. 135] for an ordinary differential equation in Banach space. See [19] for more extensive discussion of the validity of (1.2).

With $a(t)$ as above, consider the scalar problem

$$u'(t, \lambda) + \lambda \int_0^t a(t - s)u(s, \lambda)\, ds = 0, \qquad u(0, \lambda) = 1 \tag{1.3}$$

for $\mu \leqq \lambda < \infty$. It is a consequence of J. J. Levin's results [16] (see [10, Thm. 2]), together with the general theory of Volterra equations [18], that (1.3) has a unique solution, differentiable in t on R^+ and continuous in λ, with

$$|u(t, \lambda)| \leqq 1, \qquad \lim_{t \to \infty} u(t, \lambda) = 0. \tag{1.4}$$

A recent result of Shea and Wainger [20] implies, moreover, that

$$\int_0^\infty |u(t, \lambda)|\, dt < \infty, \tag{1.5}$$

and we have shown in [8] that

$$\int_0^\infty u(t, \lambda)\, dt = \left[\lambda \int_0^\infty a(t)\, dt\right]^{-1}. \tag{1.6}$$

We define

$$\mathbf{U}(t) = \int_\mu^\infty u(t, \lambda)\, d\mathbf{E}_\lambda, \qquad 0 \leqq t < \infty. \tag{1.7}$$

(See [5] for the operational calculus of self-adjoint operators on $\mathbf{H}$.) Then $\mathbf{U}(t) \in \mathscr{B}(\mathbf{H})$ with

$$\|\mathbf{U}(t)\| \leqq \sup_{\mu \leqq \lambda < \infty} |u(t, \lambda)|. \tag{1.8}$$

Conclusion (i) of Theorem 1 follows immediately from (1.4) and (1.7), and conclusion (ii) is a direct consequence of (1.6), (1.8) and the following result for (1.3).

THEOREM 3. *Let* $\mu > 0$. *If* $a(t)$ *is continuous on* R^+ *and completely monotonic on* $(0, \infty)$ *but not constant, then*

$$\lim_{t \to \infty} \left[\sup_{\mu \leqq \lambda < \infty} |u(t, \lambda)|\right] = 0, \tag{1.9}$$

$$\int_0^\infty \left[\sup_{\mu \leqq \lambda < \infty} |u(t, \lambda)|\right] dt < \infty. \tag{1.10}$$

The change of variables $r = \sqrt{\lambda}t$ transforms (1.3) into

$$\frac{d}{dr}v(r, \lambda) + \int_0^r b(r - s, \lambda)v(s, \lambda)\, ds = 0$$

with $v(r, \lambda) = u(r/\sqrt{\lambda}, \lambda), b(r, \lambda) = a(r/\sqrt{\lambda})$. As $\lambda \to \infty$, $b(r, \lambda) \to a(0)$ uniformly on compact subsets of $\{0 \leqq r < \infty\}$.

Thus for $T > 0$,

$$\max_{0 \leqq t \leqq T/\sqrt{\lambda}} |u(t, \lambda) - \cos \sqrt{a(0)\lambda}\, t| \to 0, \qquad \lambda \to \infty.$$

It follows easily that $t \to \mathbf{U}(t)$ is not continuous in the operator norm topology at $t = 0$ if $\mathbf{L}$ is unbounded. We have been unable to determine whether $\mathbf{U}(t)$ is continuous in the norm topology on $(0, \infty)$.

The problem

$$y_t(x, t) = \int_0^t a(t - s)y_{xx}(x, s)\, ds + f(x, t)$$

with prescribed data

$$y(x, 0) = y_0(x) \quad (0 \leqq x \leqq \pi), \qquad y(0, t) = y(\pi, t) = 0 \quad (t \geqq 0),$$

provides an example for our results. Here $\mathbf{L} = -\partial^2/\partial x^2$ and (1.2) becomes

$$y(t) = \sum_{n=1}^{\infty} \sin nx[u(t, n^2)y^{(n)} + \int_0^t u(t - s, n^2)f_n(s)\, ds],$$

where $y^{(n)}$ and $f_n(s)$ are the Fourier sine coefficients of y_0 and $f(\cdot, s)$, respectively. If y_0 and f are in C^2 and vanish at $x = 0, \pi$, and if, for example, $\int_0^\pi f^2(x, t)\, dx \to 0$ $(t \to \infty)$, Theorem 1 shows that $\int_0^\pi y^2(x, t)\, dx \to 0$ $(t \to \infty)$. The spectrum of L is discrete in this example, but this property is not required in Theorems 1 and 2.

C. M. Dafermos [1] and J. S. W. Wong [21], motivated by problems in viscoelasticity theory, have recently obtained results on asymptotic behavior for abstract equations similar to (1.1). Generally these results are less restrictive than ours in requirements on the form of (1.1); thus Dafermos discusses operator-valued kernels $\mathbf{G}(t, s)$, while Wong obtains results for certain nonlinear equations. In return for our restrictions on the form of (1.1) and for our requirement of complete monotonicity, we obtain results on the operator norm of the resolvent; this permits less restrictive conditions on the forcing term in the applications. The papers [10], [11], [12] give related results without the requirement of complete monotonicity.

R. K. Miller [19] has begun to investigate questions of existence, uniqueness, and continuous dependence (in the spirit of [15]) for wide classes of integro-differential equations in abstract spaces. The possible applicability of such equations is indicated, for example, by recent work of P. L. Davis [2], [3], [4] on "hyperbolic" integro-partial differential equations arising in electromagnetics and heat conduction problems.

The transform methods we use go back to the work [17] of Levin and Nohel, where some discussion of parameter dependence can also be found. A. Friedman and M. Shinbrot [7] introduced spectral decomposition methods for Volterra equations in Banach space; Friedman [6] uses a decomposition like (1.7) for the self-adjoint case.

2. A transform representation for *u*. Since $a(t)$ is completely monotonic and not constant, there is a nondecreasing function $\alpha : R^+ \to R^+$ such that

$$a(t) = \int_0^\infty e^{-xt}\, d\alpha(x)$$

with $0 = \alpha(0) \leqq \alpha(0+) < \alpha(\infty) = a(0) < \infty$. Denote by $\hat{a}(z)$ the Laplace transform of a:

$$\hat{a}(z) = \int_0^\infty e^{-zt} a(t)\, dt, \qquad \operatorname{Re} z \geqq 0, \quad z \neq 0.$$

Then

$$\hat{a}(i\tau) = A(\tau) \equiv \int_0^\infty \frac{d\alpha(x)}{x + i\tau}, \qquad \tau \neq 0. \tag{2.1}$$

The representation

$$\pi u(t, \lambda) = \operatorname{Re}\left\{ \frac{1}{it\lambda} \int_0^\infty e^{i\tau t} \frac{i/\lambda + A'(\tau)}{[i\tau/\lambda + A(\tau)]^2}\, d\tau \right\}, \qquad t > 0, \tag{2.2}$$

is an immediate consequence of [9, Thm. 1]. Since (2.2) is the key to our proof of Theorem 3, we recall briefly its derivation. Taking Laplace transforms in (1.3) and using the complex inversion formula, one sees that

$$2\pi u(t) = e^{\sigma t} \int_{-\infty}^\infty e^{i\tau t} \frac{d\tau}{\sigma + i\tau + \lambda \hat{a}(\sigma + i\tau)}, \qquad t > 0,$$

where σ is a suitable positive number. A contour shift moves σ to 0, where $\hat{a}(i\tau)$ becomes $A(\tau)$; finally, we factor λ^{-1}, integrate by parts and change the integration variable on $(-\infty, 0)$ to get (2.2).

For convenience of notation we shall assume in §§ 2 and 3 that $\mu = 1$; this is no restriction, since a can be replaced by a/μ.

We next state a sequence of inequalities for $A(\tau)$ and its real and imaginary parts, which we write, respectively, as

$$\varphi(\tau) = \int_0^\infty \frac{x\, d\alpha(x)}{x^2 + \tau^2}, \qquad -\psi(\tau) = -\tau\theta(\tau)$$

with

$$\theta(\tau) = \int_0^\infty \frac{d\alpha(x)}{x^2 + \tau^2}.$$

For $\tau > 0$ we have

(2.3) $$\varphi(\tau) = \frac{1}{\tau}\int_0^\infty \frac{d\alpha(x)}{x/\tau + \tau/x} \leqq \frac{a(0)}{2\tau},$$

(2.4) $$\theta(\tau) = \frac{1}{\tau^2}\int_0^\infty \frac{d\alpha(x)}{(x/\tau)^2 + 1} \leqq \frac{a(0)}{\tau^2},$$

(2.5) $$|\varphi'(\tau)| = \int_0^\infty \frac{2x\tau\, d\alpha(x)}{(x^2 + \tau^2)^2} \leqq \theta(\tau) \leqq \frac{a(0)}{\tau^2},$$

(2.6) $$|\theta'(\tau)| = 2\tau\int_0^\infty \frac{d\alpha(x)}{(x^2 + \tau^2)^2} \leqq \frac{2a(0)}{\tau^3},$$

(2.7) $$|\psi'(\tau)| = |\theta(\tau) + \tau\theta'(\tau)| \leqq 3a(0)/\tau^2.$$

Similarly,

(2.8) $$|\varphi''(\tau)| + |\psi''(\tau)| \leqq 24a(0)/\tau^3, \qquad \tau > 0.$$

In [9, Lem. 2.1] we use the representation (2.1) to show that

(2.9) $$|A'(\tau)| + |\tau A''(\tau)| \leqq 34|\hat{a}'(\tau)| \leqq 34\hat{a}(\tau)/\tau$$

when $\tau > 0$; from the same lemma we conclude that there is a ρ, $0 < \rho \leqq \sqrt{a}(0)$, such that

(2.10) $$|A(\tau) + i\tau/\lambda| \geqq \hat{a}(\tau)/4, \qquad 0 < \tau \leqq \rho, \quad 1 \leqq \lambda \leqq \infty.$$

In fact, the proof of [9, Lem. 2.1] shows that $A(\tau) \geqq \hat{a}(\tau)/2\sqrt{2}$, so such a ρ surely exists. Note also that $\hat{a}'(\tau) \leqq 0$ and

(2.11) $$0 \leqq -\int_0^\rho [\hat{a}'(\tau)/\hat{a}^2(\tau)]\, d\tau \leqq \hat{a}^{-1}(\rho) < \infty.$$

It is a consequence of (2.10) when $\hat{a}(0+) = \infty$ and of continuity when $\hat{a}(0+) < \infty$ that

(2.12) $$A(\tau) \to \hat{a}(0+) = \int_0^\infty a(t)\, dt, \qquad \tau \to 0+.$$

When $\hat{a}(0+) = \infty$, (2.9) shows that

(2.13) $$\tau A'(\tau)/A^2(\tau) \to 0, \qquad \tau \to 0.$$

When $\infty > \hat{a}(0+) = \int_0^\infty d\alpha(x)/x$,

$$\tau A'(\tau) = \int_0^\infty \frac{-2x^2\tau^2 + i[\tau^3 x - \tau x^3]}{(x^2 + \tau^2)^2}\frac{d\alpha(x)}{x} \to 0, \qquad \tau \to 0,$$

so (2.13) holds in all cases.

Finally, note that

(2.14) $$\theta(\tau) = \frac{1}{\tau^2}\int_0^\infty \frac{d\alpha(x)}{(x/\tau)^2 + 1} \geqq \frac{C^2(\rho)}{\tau^2}, \qquad \rho \leqq \tau < \infty.$$

Similarly

$$\varphi(\tau) = \frac{1}{\tau^2}\int_0^\infty \frac{x\,d\alpha(x)}{(x/\tau)^2 + 1} \geqq \frac{M}{\tau^2}, \qquad \rho \leqq \tau < \infty. \tag{2.15}$$

The positive constant

$$C^2 = C^2(\rho) = \int_0^\infty \frac{d\alpha(x)}{[1 + (x/\rho)^2]^2}$$

(for which (2.14) holds) will be held fixed through the remainder of this paper. M will denote generically a positive *a priori* constant, independent of t and λ; its value may change from line to line.

3. Proof of Theorem 3. Write (2.2) as

$$\pi u(t, \lambda) = \operatorname{Re}\left[\frac{1}{it\lambda}\int_0^\rho + \frac{1}{it\lambda}\int_\rho^\infty\right] \tag{3.1}$$

$$= \operatorname{Re}\,[I_1(t, \lambda) + I_2(t, \lambda)].$$

We shall show that there is a function $P(t)$ such that

$$|I_2(t, \lambda)| \leqq Mt^{-2}, \tag{3.2}$$

$$|I_1(t, \lambda) - \lambda^{-1}P(t)| \leqq Mt^{-2} \tag{3.3}$$

when $0 < t < \infty$, $1 \leqq \lambda < \infty$. From (1.4) and (1.5) it follows that $\operatorname{Re} P(t) \to 0$ $(t \to \infty)$ and $\int_1^\infty |\operatorname{Re} P(t)|\,dt < \infty$; since $|u(t, \lambda)| \leqq 1$ (from (1.4)), (1.9) and (1.10) must hold.

(It need not be the case that $u(t, \lambda) = O(t^{-2})$ $(t \to \infty)$. For example, if $a(t) = (1 + t)^{-1}$, [9, Cor. 3.4] shows that $u(t, 1) \sim [t \log^2 (t/d)]^{-1}$ for a certain positive d).

We write

$$\frac{i\tau}{\lambda} + A(\tau) = i\tau\left[\frac{1}{\lambda} - \theta(\tau)\right] + \varphi(\tau). \tag{3.4}$$

Note that $\theta'(\tau) < 0$, so $\theta \downarrow 0$ as $\tau \uparrow \infty$. There is thus at most one point $\omega = \omega(\lambda)$ on $[\rho, \infty)$ such that $\theta(\omega) = \lambda^{-1}$. If no such point exists, we set $\omega(\lambda) = \rho$. Equations (2.4) and (2.14) imply that

$$C\lambda^{1/2} \leqq \omega(\lambda) \leqq [a(0)\lambda]^{1/2}. \tag{3.5}$$

(This is true even when $\omega = \rho$, since we chose $\rho^2 < a(0)$.) The main technical difficulty in the analysis of $I_2(t, \lambda)$ is that the imaginary part of the denominator in the integrand may vanish at $\tau = \omega(\lambda)$. To deal with this, we divide $[\rho, \infty)$ into four subsets depending on λ):

$E_1 = [\rho, \infty) \cap [0, \omega(\lambda) - C\lambda^{1/3}/2]$,

$E_2 = \{[\rho, \infty) \cap [\omega(\lambda) - C\lambda^{1/3}/2, \omega(\lambda) - C/2]\} \cup [\omega(\lambda) + C/2, \omega(\lambda) + C\lambda^{1/3}/2]$,

$E_3 = [\rho, \infty) \cap [\omega(\lambda) - C/2, \omega(\lambda) + C/2]$,

$E_4 = [\omega(\lambda) + C\lambda^{1/3}/2, \infty)$.

Since

$$\frac{d}{d\tau}[\lambda^{-1} - \theta(\tau)] = \frac{2}{\tau^3}\int_0^\infty \frac{d\alpha(x)}{[1 + x^2/\tau^2]^2} \geqq 2C^2/\tau^3, \qquad \tau \geqq \rho,$$

and $\lambda^{-1} - \theta(\tau)$ can vanish only at $\tau = \omega(\lambda)$, integration from $\omega(\lambda)$ to τ reveals that

$$\left|\frac{1}{\lambda} - \theta(\tau)\right| \geqq \frac{C^2|\omega(\lambda) - \tau|[\omega(\lambda) + \tau]}{\tau^2\omega^2(\lambda)}, \qquad \tau \geqq \rho. \tag{3.6}$$

Using (3.4) and (3.5), we find that

$$\left|\frac{i\tau}{\lambda} + A(\tau)\right| \geqq \frac{C^3\lambda^{-1/6}}{2\tau\sqrt{a(0)}}, \qquad \tau \in E_1, \tag{3.7}$$

$$\left|\frac{i\tau}{\lambda} + A(\tau)\right| \geqq C^2|\omega(\lambda) - \tau|/a(0)\lambda, \qquad \tau \in E_2, \tag{3.8}$$

$$\left|\frac{i\tau}{\lambda} + A(\tau)\right| \geqq \frac{C^3}{2a(0)\lambda^{2/3}}, \qquad \tau \in E_4. \tag{3.9}$$

On E_3 we use (2.15) and (3.4) to deduce that

$$\left|\frac{i\tau}{\lambda} + A(\tau)\right| \geqq M\tau^{-2}, \qquad \tau \in E_3. \tag{3.10}$$

Finally, we shall need the estimate

$$\left|\frac{i\tau}{\lambda} + A(\tau)\right| \geqq \frac{\tau}{2\lambda}, \qquad \tau^2 > 2a(0)\lambda, \tag{3.11}$$

a consequence of (2.4) and (3.4).

To obtain (3.2), we integrate I_2 by parts. Using (2.3), (2.4), (2.5), and (2.7), we see easily that

$$\lim_{\tau\to\infty} \frac{i/\lambda + A'(\tau)}{[i\tau/\lambda + A(\tau)]^2} = 0.$$

Thus

$$\begin{aligned} I_2(t, \lambda) = {} & \frac{-e^{i\rho t}}{\lambda t^2}\,\frac{i/\lambda + A'(\rho)}{[i\rho/\lambda + A(\rho)]^2} \\ & + \frac{1}{\lambda t^2}\int_{E_1\cup E_2\cup E_3\cup E_4} e^{i\tau t}\left\{\frac{A''(\tau)}{[i\tau/\lambda + A(\tau)]^2} - \frac{[i/\lambda + A'(\tau)]^2}{[i\tau/\lambda + A(\tau)]^3}\right\} d\tau. \end{aligned} \tag{3.12}$$

The modulus of the integrated term is clearly less than $M/\lambda t^2$. Now (2.8) shows that $|A''(\tau)| \leqq M/\tau^3$, while (2.5) and (2.7) imply $|A'(\tau)| \leqq M/\tau^2$. Using (3.7) through

(3.11) and (3.5), we can estimate (3.12):

$$|I_2(t,\lambda)| \leqq \frac{M}{\lambda t^2} + \frac{M}{\lambda t^2}\int_{E_1} \{\lambda^{1/3}\tau^{-1} + [\lambda^{-2} + \lambda^{-1}\tau^{-2} + \tau^{-4}]\lambda^{1/2}\tau^3\}\,d\tau$$
$$+ \frac{M}{t^2}\int_{E_2}\left\{\frac{\lambda^{-1/2}}{[\omega(\lambda)-\tau]^2} + \frac{1}{|\omega(\lambda)-\tau|^3}\right\}d\tau + \frac{M}{\lambda t^2}\int_{E_3}[\tau + \lambda^{-2}\tau^6]\,d\tau$$
$$+ \frac{M}{\lambda t^2}\int_{E_4}\{\lambda^{4/3}\tau^{-3} + [\lambda^{-1}\tau^{-2} + \tau^{-4}]\lambda^2\}\,d\tau$$
$$+ \frac{M}{\lambda t^2}\left[\int_{C\sqrt{\lambda}}^{[2a(0)\lambda]^{1/2}} d\tau + \int_{[2a(0)\lambda]^{1/2}}^{\infty} \lambda\tau^{-3}\,d\tau\right].$$

Using (3.5) and the definitions of the sets E_j, we estimate these integrals directly and obtain (3.2).

Observe that

$$\frac{1}{i\tau/\lambda + A(\tau)} = \frac{1}{A(\tau)} - \frac{i\tau/\lambda}{[i\tau/\lambda + A(\tau)]A(\tau)}.$$

Therefore,

$$\frac{i/\lambda + A'(\tau)}{[i\tau/\lambda + A(\tau)]^2} = \frac{A'(\tau)}{A^2(\tau)} + \frac{i}{\lambda A^2(\tau)}$$
$$+ \frac{[i/\lambda + A'(\tau)]}{A^2(\tau)}\left[-\frac{2i\tau/\lambda}{[i\tau/\lambda + A(\tau)]} - \frac{\tau^2/\lambda^2}{[i\tau/\lambda + A(\tau)]^2}\right].$$

Set

$$P(t) = \frac{1}{it}\int_0^{\rho} e^{i\tau t}\frac{A'(\tau)}{A^2(\tau)}\,d\tau.$$

Integration by parts (using (2.13)) and direct (but laborious) use of estimates (2.9) and (2.10) show that

$$|I_1(t,\lambda) - \lambda^{-1}P(t)| \leqq \frac{M}{t^2\lambda^2}\int_0^{\rho}\frac{1 + |A'(\tau)| + |\tau A''(\tau)|}{\hat{a}^3(\tau)}\,d\tau.$$

Now (2.9) and (2.11) yield (3.3). This completes the proof of Theorem 3.

4. Proof of Theorem 2. Given $\mathbf{y}_0$ and $\mathbf{f}(t)$ as in the theorem and $\mathbf{U}(t)$ defined by (1.7), note that the function

$$a(s-r)\mathbf{L}\mathbf{U}(r)\mathbf{y}_0 = a(s-r)\mathbf{U}(r)\mathbf{L}\mathbf{y}_0$$

from $\{0 \leqq r \leqq s \leqq t\}$ to $\mathbf{H}$ is continuous ($t \geqq 0$). Set $h(t) = \int_0^t a(s)\,ds$. Then

$$
\begin{aligned}
\mathbf{y}_0 - \int_0^t \int_0^s a(s-r)\mathbf{LU}(r)\mathbf{y}_0\,dr\,ds &= \mathbf{y}_0 - \int_0^t h(t-r)\mathbf{LU}(r)\mathbf{y}_0\,dr \\
&= \mathbf{y}_0 - \int_0^t h(t-r)\int_\mu^\infty \lambda u(r,\lambda)\,d\mathbf{E}_\lambda \mathbf{y}_0\,dr \\
&= \mathbf{y}_0 - \int_\mu^\infty \left[\lambda \int_0^t h(t-r)u(r,\lambda)\,dr\right] d\mathbf{E}_\lambda \mathbf{y}_0 \qquad (4.1)\\
&= \mathbf{y}_0 + \int_\mu^\infty [u(t,\lambda)-1]\,d\mathbf{E}_\lambda \mathbf{y}_0 \\
&= \mathbf{U}(t)\mathbf{y}_0.
\end{aligned}
$$

Differentiating both sides [14, p. 6], we see that

$$
\frac{d}{dt}[\mathbf{U}(t)\mathbf{y}_0] + \int_0^t a(t-s)\mathbf{LU}(s)\mathbf{y}_0\,ds = \mathbf{0}. \qquad (4.2)
$$

(The change of order of integration at line (4.1) uses the fact that the vector integral converges uniformly ($0 \leqq r \leqq t$) in the **H**-norm; this follows from (1.4).)

Since $\mathbf{Lf}(t)$ is continuous, the function $a(t-s)\mathbf{LU}(s-r)\mathbf{f}(r)$ is jointly continuous on $\{0 \leqq r \leqq s \leqq t\}, t \geqq 0$, so

$$
\begin{aligned}
\int_0^t a(t-s)\mathbf{L}\int_0^s \mathbf{U}(s-r)\mathbf{f}(r)\,dr\,ds &= \int_0^t\int_0^s a(t-s)\mathbf{LU}(s-r)\mathbf{f}(r)\,dr\,ds \\
&= \int_0^t\int_0^{t-r} [a(t-r-s)\mathbf{LU}(s)\mathbf{f}(r)]\,ds\,dr \qquad (4.3)\\
&= -\int_0^t \frac{d}{dt}[\mathbf{U}(t-r)\mathbf{f}(r)]\,dr,
\end{aligned}
$$

where the last step uses (4.2) with $\mathbf{y}_0$ replaced by $\mathbf{f}(r)$. This last step also shows that the integrand in (4.3) is jointly continuous, so the formula

$$
\frac{d}{dt}\int_0^t \mathbf{U}(t-r)\mathbf{f}(r)\,dr = \mathbf{f}(t) + \int_0^t \frac{d}{dt}[\mathbf{U}(t-r)\mathbf{f}(r)]\,dr. \qquad (4.4)
$$

is valid. Then (4.4), together with (4.2) and the calculation leading to expression (4.3), shows that the function $\mathbf{y}(t)$ of (1.2) solves the initial value problem (1.1).

For uniqueness, note that the difference $\mathbf{z}(t)$ between two solutions of (1.1) must satisfy the "weak," integrated, homogeneous version of (1.1),

$$
\mathbf{z}(t) + \mathbf{L}\int_0^t h(t-s)\mathbf{z}(s)\,ds = \mathbf{0}.
$$

Then for $\mu \leqq \lambda < \infty$,

$$
\|\mathbf{E}_\lambda \mathbf{z}(t)\| \leqq \lambda \int_0^t h(t-s)\|\mathbf{E}_\lambda \mathbf{z}(s)\|\,ds \leqq \lambda h(t)\int_0^t \|\mathbf{E}_\lambda \mathbf{z}(s)\|\,ds.
$$

Thus $\mathbf{E}_\lambda \mathbf{z}(t) \equiv 0$, so $\mathbf{z}(t) \equiv 0$. (This argument appeared in [11].)

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OSCILLATION AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THIRD ORDER DIFFERENTIAL DELAY EQUATIONS*

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Abstract. In this paper we introduce comparison techniques for studying the oscillatory and asymptotic behavior of solutions of third order differential equations with retarded argument. This allows the use of "Kneser-type", as well as integral, criteria for deciding the behavior of the solutions.

1. Introduction. The primary purpose of this paper is to investigate the oscillatory and asymptotic behavior of the solutions of the linear third order differential equation with a retarded argument of the form

$$y'''(t)+p(t)y(g(t))+q(t)y(t)=0, \tag{1.1}$$

where $p(t)$, $q(t)$ and $g(t)$ are assumed to be continuous on the half-line $[0, +\infty)$ with $g(t) \leqq t$ and $\lim_{t\to\infty} g(t)=+\infty$. We assume that under the initial condition

$$y(t)=\phi(t), \quad t\leqq t_0 \quad \text{and} \quad y^{(k)}(t_0)=y_k, \qquad k=1,2, \tag{1.2}$$

(1.1) has a solution which exists for all $t \geqq t_0$ (i.e., there exists a function $y(t) \in C^3[t_0, \infty)$ which satisfies (1.1) for all $t \geqq t_0$. A complete discussion of the initial value problem for differential equations with retarded argument may be found in [3]. A solution of (1.1) is said to be oscillatory in case for each $t_1 > t_0$, there exists $t_2 > t_1$ with $y(t_2)=0$ and $y(t)\equiv 0$ does not hold on any subinterval $(t_1, \infty) \subset [0, \infty)$. A solution of (1.1) is said to be nonoscillatory if there exists $t_1 > t_0$ such that $y(t) \neq 0$ for $t \geqq t_1$ or if $y(t)\equiv 0$ for $t \geqq t_1$.

There have been numerous recent papers discussing the oscillatory and asymptotic behavior of solutions of higher order linear and nonlinear differential equations with retarded argument and we refer the reader to [8], [9], [10], [12]–[16] and the references therein. Most of the oscillation criteria in the above references have been obtained through appropriate integral conditions. Our technique here, on the other hand, involves comparison theorems using known properties of linear differential equations without delays and allows the use of "Kneser-type" rather than integral criteria for deciding the oscillatory or asymptotic behavior of solutions of (1.1). Section 2 is devoted to equation (1.1) under the assumption that the coefficients $p(t)$, $q(t)$ are nonnegative. In § 3 we discuss the case $p(t) \leqq 0$, $q(t) \leqq 0$, and in § 4 some extensions are considered for third order nonlinear equations with delays. We also discuss examples which are not covered by previous investigations for both the linear and nonlinear case.

2. Nonnegative coefficients. We begin with a preliminary lemma which concerns the relation between solutions of the general third order linear ordinary differential equation

$$L_3y=y'''+a(t)y''+b(t)y'+c(t)y=0 \tag{2.1}$$

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and the corresponding Riccati equation

$$u''+f(t, u, u')=0, \tag{2.2}$$

where $f(t, u, u')=3uu'+a(t)u'+u^3+a(t)u^2+b(t)u+c(t)$. Recall that an nth order linear differential equation is said to be disconjugate on an interval I in case no nontrivial solution has more than $n-1$ zeros on I.

LEMMA 2.1. *Equation* (2.1) *is disconjugate on the interval I iff there exists $\alpha(t)$, $\beta(t)\in C^2(I)$ with $\alpha(t)<\beta(t)$ and*

$$\alpha''(t)+f(t, \alpha(t), \alpha'(t))\geqq 0, \qquad \beta''(t)+f(t, \beta(t), \beta'(t))\leqq 0, \qquad t\in I.$$

A proof of this result may be found, for example in [4] (see [2] for higher order analogues of this lemma). Functions $\alpha(t)$, $\beta(t)$ as in the previous lemma are called lower and upper solutions, respectively, for (2.2) (see [7]).

THEOREM 2.2. *Let $p(t)\geqq 0$, $q(t)\geqq 0$, $p(t)+q(t)>0$, $t\geqq t_0$ and let $y(t)$ be a solution of* (1.1) *which exists on $[t_0, \infty)$ with $y(t)\not\equiv 0$ on any subinterval of $[t_0, \infty)$. Assume there exists a real number λ, $0<\lambda<\frac{1}{2}$, such that the equation*

$$y'''(t)+\left(\lambda\frac{g^2(t)}{t^2}p(t)+q(t)\right)y=0 \tag{2.3}$$

is not disconjugate on any half-line $[t_1, \infty)$ of $[t_0, \infty)$. Then either

(a) *there exists $M>0$ with $|y(t)|<M$ for all $t\geqq t_0$ or*

(b) *$y(t)$ changes sign on $[t_1, \infty)$ for all $t_1>t_0$.*

Proof. We assume that (b) does not hold and will show that $y(t)y'(t)\leqq 0$ for all large t, so that $|y(t)|$ is eventually nonincreasing. To be specific, assume that $y(t)\geqq 0$ and $y(g(t))\geqq 0$ on $[t_1, \infty)$ for some $t>t_0$. Therefore $y'''(t)\leqq 0$ for $t\geqq t_1$ so that $y''(t)\geqq 0$, $t\geqq t_1$. In fact, $y''(t)>0$, $t\geqq t_1$, by our assumptions. Now if (a) does not hold, then there exists $t_2>t_1$ with $y'(t_2)>0$ so that on $[t_2, \infty)$, we have $y(t)>0$, $y'(t)>0$, $y''(t)>0$, $y'''(t)\leqq 0$. We claim now that for any $0<\mu<\frac{1}{2}$ there exists $T_\mu>t_2$ with

$$\frac{y(g(t))}{y(t)}\geqq\mu\left(\frac{g(t)}{t}\right)^2, \qquad t\geqq T_\mu. \tag{2.4}$$

To see this, merely note that we may write

$$\frac{y(g(t))}{y(t)}=\frac{y(g(t))}{y'(g(t))}\cdot\frac{y'(g(t))}{y'(t)}\cdot\frac{y'(t)}{y(t)}. \tag{2.5}$$

By a result of Lazer [11, Lemma 3.2], we have

$$\liminf_{g(t)\to\infty}\frac{y(g(t))}{g(t)y'(g(t))}\geqq\tfrac{1}{2}. \tag{2.6}$$

By Lemma 2.1 of [5], since $V(t)=y'(t)$ satisfies $V(t)>0$, $V'(t)>0$, $V''(t)\leqq 0$, $t\geqq t_2$, given $0<k<1$, there exists $T_k\geqq t_2$ with

$$\frac{y'(g(t))}{y'(t)}\geqq k\frac{g(t)}{t}, \qquad t\geqq T_k. \tag{2.7}$$

The mean value theorem and the fact that $y'(t)$ is increasing for $t \geqq t_2$ imply

$$\liminf_{t\to\infty} \frac{ty'(t)}{y(t)} \geqq 1. \tag{2.8}$$

Hence, (2.4) follows from (2.6), (2.7) and (2.8), so that with $\mu = \lambda$ a calculation shows that $\beta(t) = y'(t)/y(t) > 0$ satisfies

$$\beta'' + 3\beta\beta' + \beta^3 + \lambda p(t)\left(\frac{g(t)}{t}\right)^2 + q(t) \leqq 0. \tag{2.9}$$

With $\alpha(t) \equiv 0$, it now follows from Lemma 2.1 that (2.3) is disconjugate and this contradiction proves the result.

Our next result is similar to the previous theorem and uses a second order comparison equation.

THEOREM 2.3. *Let* $p(t) \geqq 0$, $q(t) \geqq 0$, $p(t)+q(t) > 0$, $t \geqq t_0$ *and let* $y(t)$ *be a solution of* (1.1) *which exists on* $[t_0, \infty)$ *with* $y(t) \not\equiv 0$ *on any subinterval of* $[t_0, \infty)$. *Assume there exists a real number* λ, $0 < \lambda < \frac{1}{2}$, *such that the equation*

$$y'' + \lambda t\left(\left(\frac{g(t)}{t}\right)^2 p(t) + q(t)\right) y = 0 \tag{2.10}$$

is oscillatory on $[t_1, \infty)$ *for some* $t_1 \geqq t_0$. *Then the conclusion of Theorem* 2.2 *holds.*

Proof. We proceed as in Theorem 2.2, but instead of (2.4) we obtain from (2.6) and (2.7) the inequality

$$y(g(t)) \geqq \mu \frac{(g(t))^2}{t} y'(t), \qquad t \geqq T_\mu, \quad 0 < \mu < \tfrac{1}{2}, \tag{2.11}$$

and again from Lemma 3.2 of [11],

$$y(t) \geqq \mu t y'(t), \qquad t \geqq T_\mu. \tag{2.12}$$

Therefore, with $u(t) = y'(t)$, we obtain from (1.1)

$$u'' + \mu t\left(\left(\frac{g(t)}{t}\right)^2 p(t) + q(t)\right) u \leqq 0; \tag{2.13}$$

but with $\mu = \lambda$, (2.13) implies that (2.10) is disconjugate on $[T_\lambda, \infty)$ (see [6, Thm. 7.2, p. 362]). This proves the theorem.

We shall next establish conditions under which bounded solutions of (1.1) are either oscillatory or tend monotonically to zero, along with their derivatives.

THEOREM 2.4. *Let* $p(t) \geqq 0$, $q(t) \geqq 0$, $p(t)+q(t) > 0$, *and let* $y(t)$ *be a solution of* (1.1) *which exists on* $[t_0, \infty)$ *and satisfies* $|y(t)| < M$ *for some* $M > 0$. *Assume*

$$\liminf_{t\to\infty} t^3(p(t)+q(t)) > 0 \tag{2.14}$$

and that the equation

$$y'''(t) + (p(t)+q(t))y = 0 \tag{2.15}$$

is not disconjugate on any half-line $[t_1, \infty)$ *of* $[t_0, \infty)$. *Then either* $y(t)$ *is oscillatory or* $y(t)$ *tends monotonically to zero along with its first two derivatives.*

Proof. We assume, to be specific, that $y(t) \geqq \delta > 0$ and $y(g(t)) \geqq \delta > 0$ for $t \geqq t_1$ and for some $\delta > 0$, and we shall show that this leads to a contradiction. Since $y'''(t) \leqq 0$ it follows that $y''(t) > 0$ on $[t_1, \infty)$, and since $y(t) \leqq M$, we must have $y'(t) < 0$ on $[t_1, \infty)$. Let $u(t) = y'(t)/y(t) < 0$ and let $\varepsilon > 0$ be given. Choose $T \geqq t_1$ so that $y(T) \leqq \delta(1+\varepsilon)$. Then on (T, ∞), the mean value theorem and the fact that $y'(t)$ is increasing imply

$$u(t) > \frac{-\varepsilon}{t-T}. \tag{2.16}$$

Let $\alpha(t) = -\varepsilon/(t-T)$. Then we have

$$\alpha'' + 3\alpha\alpha' + \alpha^3 + p(t) + q(t) = -(t-T)^{-3}(\varepsilon^3 + 3\varepsilon^2 + 2\varepsilon) + p(t) + q(t). \tag{2.17}$$

Now the right-hand side of (2.17) is nonnegative for sufficiently small ε because of (2.14), and hence $\alpha(t)$ is a lower solution of the Riccati equation corresponding to (2.15). Since $y(g(t)) \geqq y(t)$ for $t \geqq T$, it follows that $u(t)$ is an upper solution of the same Riccati equation, and hence Lemma 2.1 shows that (2.15) is disconjugate on $[T, \infty)$. This contradiction shows that $y(t) \to 0$. The fact that $y'(t) \to 0$ and $y''(t) \to 0$ is clear so the theorem is proved.

The next two results give additional conditions under which the conclusion of Theorem 2.4 is valid.

THEOREM 2.5. *Let* $p(t) \geqq 0$, $q(t) \geqq 0$, $p(t) + q(t) > 0$, *and let* $y(t)$ *be a solution of* (1.1) *which exists on* $[t_0, \infty)$ *and satisfies* $|y(t)| \leqq M$ *for some* $M > 0$. *Let there exist* μ, $0 < \mu < \frac{1}{32}$ *such that the equation*

$$y'''(t) + \mu(g(t))^2 p(t) y''(t) + q(t) y(t) = 0 \tag{2.18}$$

is not disconjugate on any half-line $[t_1, \infty)$. *Assume either condition* (a) *or* (b) *below holds*:

(a) $\liminf_{t\to\infty} t^3 q(t) > 0$,

(b) $\liminf_{t\to\infty} \mu t (g(t))^2 p(t) > 2$.

Then the conclusion of Theorem 2.4 *holds.*

THEOREM 2.6. *Let* $p(t) \geqq 0$, $q(t) \geqq 0$, $p(t) + q(t) > 0$, *and let* $y(t)$ *be a solution of* (1.1) *which exists on* $[t_0, \infty)$ *and satisfies* $|y(t)| \leqq M$ *for some* $M > 0$. *Assume* (2.14) *holds and that the equation*

$$y'''(t) + \frac{(g(t)-t)^2}{2} p(t) y''(t) + (g(t)-t) p(t) y'(t) + (p(t) + q(t)) y = 0 \tag{2.19}$$

is not disconjugate on any half-line $[t_1, \infty)$. *Then the conclusion of Theorem* 2.4 *holds.*

Proof of Theorem 2.5. We apply a lemma of Sficas [12, Lemma 3]. If $y(t)$ is as in Theorem 2.4 and $y(t)$ does not tend to zero, then for any $k > 1$,

$$y(g(t)) \geqq \frac{(g(t))^2 y''(t)}{32k} \tag{2.20}$$

for all large t. Hence, $u(t)=y'(t)/y(t)$ is an upper solution of the Riccati equation corresponding to (2.18) with $k=1/32\mu$ and $u(t)>-\varepsilon/t-T$ for $t\geqq T_\varepsilon$, as in Theorem 2.4. Condition (a) or (b) implies that $\alpha(t)=-\varepsilon/(t-T)$ is a lower solution of the corresponding Riccati equation, for sufficiently small $\varepsilon>0$, so by Lemma 2.1, we obtain a contradiction. Therefore, we conclude $y(t)\to 0$.

Proof of Theorem 2.6. The proof is similar to Theorems 2.4 and 2.5. Here we use the mean value theorem to write

$$y(g(t))\geqq y(t)+y'(t)(g(t)-t)+y''(t)\frac{(g(t)-t)^2}{2} \tag{2.21}$$

and the remainder of the proof is the same. Condition (2.14) insures that $\alpha(t)=-\varepsilon/(t-T)$ is a lower solution of the Riccati equation corresponding to (2.19) for small enough $\varepsilon>0$.

Examples. If $p(t)\geqq 0$, $q(t)\geqq 0$, $p(t)+q(t)>0$ and if $\liminf t^3(\lambda(g(t)/t)^2\cdot p(t)+q(t))>2/(3\sqrt{3})$ for some $0<\lambda<\frac{1}{2}$, then (2.3) and (2.15) are oscillatory by comparison with the third order Euler equation (see [18, p. 163] for example). Hence any solution of (1.1) is either oscillatory or tends monotonically to zero along with its derivatives. For example, suppose that $q(t)=\mu_1 t^{-3}$, $(g(t)/t)^2p(t)=\mu_2 t^{-3}$, where μ_1, $\mu_2>0$. If $2/(3\sqrt{3})<\frac{1}{2}\mu_1+\mu_2<\mu_1+\mu_2<\frac{1}{2}$, then Theorems 2.2 and 2.4 (but not Theorem 2.3) show that any nonoscillatory solution tends monotonically to zero. If $\frac{1}{2}\mu_1+\mu_2<2/(3\sqrt{3})<\frac{1}{2}<\mu_1+\mu_2$, then we may use Theorems 2.3 and 2.4 (but not Theorem 2.2). This asymptotic behavior may not be concluded from any of the references. Additional examples may be constructed similarly using other known oscillation criteria for (2.3) or (2.15). It is clear that the previous theorems have analogues for the linear equation with several delays

$$y'''(t)+\sum_{i=1}^{n} p_i(t)y(g_i(t))+q(t)y(t)=0, \tag{2.22}$$

and their formulation is left to the interested reader.

Remark. By a theorem of Švec [17] (see also [19]), if the equation $y'''(t)+r(t)y(t)=0$, $r(t)>0$, has an oscillatory solution, then any nonoscillatory solution tends to zero. Theorems 2.2 and 2.4 may be viewed as an analogue of this result. In fact, if $p(t)\equiv 0$, then they yield an alternate proof of the Švec theorem in all cases except when $\liminf_{t\to\infty} t^3q(t)=0$ and $\limsup_{t\to\infty} t^3q(t)\geqq 2/3\sqrt{3}$.

3. Nonpositive coefficients. In this section, we discuss solutions of (1.1) under the assumption $p(t)\leqq 0$, $q(t)\leqq 0$, $t\leqq t_0$; with these assumptions, (1.1) will always have nonoscillatory solutions satisfying $\operatorname{sgn} y(t)=\operatorname{sgn} y'(t)=\operatorname{sgn} y''(t)\neq 0$ and $\lim_{t\to\infty}|y(t)|=\lim_{t\to\infty}|y'(t)|=+\infty$ (e.g., if $y(t)=\phi(t)\geqq 0$, $t\leqq t_0$, $y'(t_0)\geqq 0$, $y''(t_0)>0$). The following theorem gives a criterion under which this is the only type of nonoscillatory solution possible.

THEOREM 3.1. *Let* $p(t)\leqq 0$, $q(t)\leqq 0$, $t\geqq t_0$ *and let* $y(t)$ *be a nonoscillatory solution of* (1.1). *Let there exist* k, $0<k<1$, *such that the equation*

$$y'''(t)+\left(kp(t)\frac{g(t)}{t}+q(t)\right)y(t)=0 \tag{3.1}$$

is not disconjugate on any half-line $[t_1, \infty)$. *Then there exists* $t_2 \geqq t_0$ *such that*

$$\operatorname{sgn} y(t) = \operatorname{sgn} y'(t) = \operatorname{sgn} y''(t) \neq 0, \qquad t \geqq t_2,$$

and $\lim_{t\to\infty} |y(t)| = \lim_{t\to\infty} |y'(t)| = \infty$.

Proof. To be specific, assume $y(t)$, $y(g(t)) \geqq 0$, $t \geqq T \geqq t_0$, so that $y'''(t) \geqq 0$ for $t \geqq T$. If the result is not true, then $y''(t) < 0$ for $t \geqq T$ and hence $y'(t) > 0$ for $t \geqq T$. Therefore, by Lemma 2.1 of [5], there exists $T_k \geqq T$ with $y(g(t)) \geqq k(g(t)/t)y(t)$ for $t \geqq T_k$ and so with $u(t) = y'(t)/y(t) > 0$ we have

$$u'' + 3uu' + u^3 + q(t) + kp(t)\frac{g(t)}{t} \geqq 0, \qquad t \geqq T_k. \tag{3.2}$$

That is, $u(t)$ is a lower solution of the Riccati equation for (3.1). Since $y'(t)$ is decreasing, the mean value theorem implies that $u(t) < 1/(t-T)$, $t \geqq T_k$. Letting $\beta(t) = 1/(t-T)$, a calculation shows that $\beta(t)$ is an upper solution of the Riccati equation for (3.1). Hence, by Lemma 2.1 equation (3.1) is disconjugate on $[T_k, \infty)$, and this contradiction proves the theorem.

Examples. If $p(t) \leqq 0$, $q(t) \leqq 0$ and if

$$\liminf_{t\to\infty} t^3 \left| \frac{g(t)}{t} p(t) + q(t) \right| > \frac{2}{3\sqrt{3}}, \tag{3.3}$$

then equation (3.1) will be oscillatory for k sufficiently close to 1. We recall that the third order equation

$$y'''(t) + r(t)y(t) = 0, \qquad r(t) > 0, \tag{3.4}$$

is oscillatory iff its adjoint equation

$$y'''(t) - r(t)y(t) = 0 \tag{3.5}$$

is oscillatory [18]. A sufficient condition for the existence of an oscillatory solution of (3.4) is the existence of λ, $0 < \lambda < \frac{1}{2}$, such that the second order equation

$$u'' + \lambda t r(t) u = 0 \tag{3.6}$$

is oscillatory [11, Thm. 3.1]. Now if $\int_{t_0}^{\infty} t^{1+\delta} r(t)\, dt = \infty$ for some $0 < \delta < 1$, then (3.6) is oscillatory [6, p. 368], and hence (3.4) has an oscillatory solution. Thus, if $\int_{t_0}^{\infty} t^{\delta+1}(q(t) + kp(t)(g(t)/t))dt = -\infty$ for some $0 < k < 1$, then any nonoscillatory solution of (1.1) satisfies the conclusion of Theorem 3.1.

4. Nonlinear equations. In this section we shall discuss the application of the previous techniques to some nonlinear equations. For simplicity we shall consider only equations of the form

$$y'''(t) + p(t)y^{\alpha}(g(t)) = 0, \tag{4.1}$$

where $\alpha > 0$ is the quotient of odd positive integers.

THEOREM 4.1. *Let* $p(t) > 0$, $\alpha > 1$ *and assume that the equation*

$$y'''(t) + \lambda p(t)(g(t))^{2\alpha} t^{-\alpha-1} y(t) = 0 \tag{4.2}$$

has an oscillatory solution for all $\lambda > 0$. *Then for any nonoscillatory solution* $y(t)$ *of* (4.1), $|y(t)|$ *is eventually nonincreasing.*

Proof. To be specific, we assume that $y(t)$ is a solution of (4.1) with $y(t) \geqq 0$, $y(g(t)) \geqq 0$ for $t \geqq t_1$. If $y(t)$ is not nonincreasing, then as in Theorem 2.2, we get $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$, $y'''(t) \leqq 0$ for $t \geqq T \geqq t_1$ and for any $0 < \mu < \frac{1}{2}$ we have $y(g(t)) \geqq \mu(g(t)/t)^2 y(t)$ for $t \geqq T_\mu \geqq T$. Therefore, for $t \geqq T_\mu$, $y(t)$ satisfies

$$y'''(t) + \mu^\alpha p(t)\left(\frac{g(t)}{t}\right)^{2\alpha} y^\alpha(t) \leqq 0. \tag{4.3}$$

Now the mean value theorem implies

$$y(t) \geqq mt, \tag{4.4}$$

where $m > 0$ is a suitable constant. From (4.3) and (4.4) we obtain

$$y'''(t) + \lambda_0 p(t)(g(t))^{2\alpha} t^{-\alpha-1} y(t) \leqq 0, \tag{4.5}$$

where $\lambda_0 = m^{\alpha-1}\mu^\alpha$. Therefore, $u = y'/y$ is an upper solution for the Riccati equation corresponding to (4.2) with $\lambda = \lambda_0$, and since $\alpha(t) \equiv 0$ is a lower solution, (4.2) is disconjugate. This proves the theorem.

In the next theorem we use a necessary and sufficient condition in the theory of second order nonlinear oscillations to show that for $\alpha > 1$ any nonoscillatory solution of (4.1) must be bounded if a certain integral diverges.

THEOREM 4.2. *Let* $p(t) \geqq 0$, $\alpha > 1$ *with*

$$\int_{t_0}^{\infty} p(t)(g(t))^{2\alpha} t^{1-\alpha}\, dt = +\infty. \tag{4.6}$$

Then any nonoscillatory solution of (4.1) *is nonincreasing in absolute value.*

Proof. We proceed as in Theorem 4.1 and Theorem 2.3 to obtain the inequality (2.11). Then from (2.11) and (4.1) with $u = y'$ we get

$$u''(t) + \mu^\alpha (g(t))^{2\alpha} t^{-\alpha} p(t)(u(t))^\alpha \leqq 0, \qquad t \geqq T_\mu. \tag{4.7}$$

Therefore $u > 0$ is an upper solution of the equation

$$y''(t) + \mu^\alpha (g(t))^{2\alpha} t^{-\alpha} p(t)(y(t))^\alpha = 0, \tag{4.8}$$

and since $\alpha(t) \equiv 0$ is a lower solution, we conclude (see [7], for example) that there exists a nonoscillatory solution $V(t)$ of (4.8) with $0 < V(t) \leqq u(t)$ on $[T_\mu, \infty)$. But then by a well-known theorem of Atkinson [1], we have

$$\int_{T_\mu}^{\infty} p(t)(g(t))^{2\alpha} t^{1-\alpha}\, dt < +\infty. \tag{4.9}$$

This proves the theorem.

We next establish a result for the sublinear case.

THEOREM 4.3. *Let* $p(t) > 0$ *and* $0 < \alpha < 1$. *Assume*

$$\liminf_{t \to \infty} t^3 p(t) > 0 \tag{4.10}$$

and that the equation

$$y'''(t) + \lambda p(t) y = 0 \tag{4.11}$$

is not disconjugate on any half-line $[t_1, \infty)$ *for all* $\lambda > 0$. *Then any bounded solution* $y(t)$ *of* (4.1) *which exists on* $[t_0, \infty)$ *is either oscillatory or tends to zero monotonically along with its derivatives.*

Proof. If $y(t)$ is a bounded nonoscillatory solution of (4.1) which does not tend to zero, then there exists $\delta > 0$ and $T \geqq t_0$ such that

$$y(g(t)) > y(t) > \delta, \quad y'(t) < 0, \quad y''(t) > 0, \quad y'''(t) \leqq 0 \tag{4.12}$$

for all $t \geqq T$. Hence, for $t \geqq T$, $y(t)$ satisfies

$$y'''(t) + \lambda_0 p(t) y(t) \leqq 0, \tag{4.13}$$

where $\lambda_0 = y(g(T))^{\alpha-1}$. We may now argue as in Theorem 2.4 to show that $u = y'/y$ is an upper solution for the Riccati equation corresponding to (4.11) with $\lambda = \lambda_0$ and that $u(t) > -\varepsilon/(t-T)$ for $t > T = T_\varepsilon$. Condition (4.10) insures that $\alpha(t) = -\varepsilon/(t-T)$ is a lower solution of the Riccati equation corresponding to (4.11) with $\lambda = \lambda_0$ for sufficiently small $\varepsilon > 0$. This contradiction proves the theorem.

Remark. Theorem 4.3 is true, as stated, for $\alpha > 1$ also, but in this case condition (4.10) guarantees that $\int_{t_0}^{\infty} t^2 p(t)\, dt = +\infty$ so that any nonoscillatory solution tends to zero monotonically by a theorem of Sficas and Staikos [16] (see also [8]). Likewise, a theorem similar to Theorem 4.1 and 4.2 is valid for $0 < \alpha < 1$ but the results in this case are again well known.

Examples. (i) If $0 < \alpha < 1$ and $\gamma_1 > \gamma_2 > 0$ with $2\alpha + \gamma_1 < 2$ and if $k_2 t^{\gamma_2 - 3} \leqq p(t) \leqq k_1 t^{\gamma_1 - 3}$ from some $k_1, k_2 > 0$, then the conditions of Theorem 4.3 are satisfied. Since $\int_{t_0}^{\infty} (g(t))^{2\alpha} p(t)\, dt < +\infty$, there will also exist nonoscillatory solutions $y(t)$ with $\lim_{t\to\infty} y(t)/t^2 = \alpha \neq 0$ (see [9]).

(ii) If (4.6) holds and $\alpha > 1$, then any nonoscillatory solution of (4.1) is bounded and $|y(t)|$ is monotone decreasing. However, $\int_{t_0}^{\infty} t^2 p(t)\, dt = +\infty$ may fail to hold so that the results of the references will not apply and $\lim_{t\to\infty} |y(t)| \neq 0$ is possible. For example, if $g(t) = t^\delta$, $\frac{1}{2} < \delta < 1$ and $p(t) = 6t^{\alpha\delta}/t^4(1+t^\delta)^\alpha$, then $\int_{t_0}^{\infty} t^2 p(t)\, dt < +\infty$. If $\alpha \geqq 2/(2\delta - 1)$, then $\int_{t_0}^{\infty} p(t)(g(t))^{2\alpha} t^{1-\alpha}\, dt = +\infty$ and $y(t) = 1 + 1/t$ is a solution of (4.1) which does not tend to zero.

For the case $p(t) \leqq 0$, one can also obtain results similar to Theorem 3.1 guaranteeing that all nonoscillatory solutions $y(t)$ satisfy

$$\operatorname{sgn} y(t) = \operatorname{sgn} y'(t) = \operatorname{sgn} y''(t) \neq 0, \quad \text{large } t, \tag{4.14}$$

and

$$\lim_{t\to\infty} |y(t)| = \lim_{t\to\infty} |y'(t)| = +\infty. \tag{4.15}$$

THEOREM 4.4. *Let* $p(t) \leqq 0$, $\alpha > 1$ *and assume that the equation*

$$y'''(t) + \lambda p(t) \left(\frac{g(t)}{t}\right)^{\alpha} y(t) = 0 \tag{4.16}$$

has an oscillatory solution for all $\lambda > 0$. *Then any nonoscillatory solution of* (4.1) *satisfies* (4.14), (4.15).

THEOREM 4.5. *Let* $p(t)\leqq 0$, $0<\alpha<1$. *If* (a) (4.16) *is oscillatory for all* $\lambda>0$, *then any bounded solution of* (4.1) *is oscillatory. If* (b) *the equation*

$$y'''(t)+\lambda p(t)(g(t))^{\alpha}t^{-1}y(t)=0 \tag{4.17}$$

is oscillatory for all $\lambda>0$, *then any nonoscillatory solution of* (4.1) *satisfies* (4.14), (4.15).

Proof of Theorem 4.4. We argue as in Theorem 3.1 and obtain the inequality

$$u''+3uu'+u^3+k^{\alpha}\left(\frac{g(t)}{t}\right)^{\alpha}p(t)(y(t))^{\alpha-1}\geqq 0, \qquad t\geqq T_k, \tag{4.18}$$

where $u=y'/y$ and $y(t)$ satisfies $y(t)>0$, $y'(t)>0$, $y''(t)<0$ and $y'''(t)\geqq 0$ for $t\geqq T$. Hence, since $u(t)<1/(t-T)$, it follows, as in Theorem 3.1, that (4.10) is disconjugate on $[T_k,\infty)$ with $\lambda=k^{\alpha}(y(T_k))^{\alpha-1}$, and this contradiction proves the theorem.

Proof of Theorem 4.5. If $y(t)$ is a bounded nonoscillatory solution of (4.1) with $0\leqq y(t)$, $y(g(t))\leqq M$, $y'(t)>0$, $y''(t)<0$, $y'''(t)\geqq 0$ for $t\geqq T$, then we argue as in Theorem 4.4 to show that (4.16) is disconjugate with $\lambda=k^{\alpha}M^{\alpha-1}$.

If $y(t)$ is unbounded, then we use the fact that $0<y(t)\leqq ct$ for some $c>0$, and hence in (4.18) we use $(y(t))^{\alpha-1}\geqq c^{\alpha-1}t^{\alpha-1}$ to show that (4.17) with $\lambda=k^{\alpha}c^{\alpha-1}$ is disconjugate on $[T_k,\infty)$. This contradiction proves the theorem.

Example. If either

$$\lim_{t\to\infty} t^{3-\alpha}p(t)(g(t))^{\alpha}=-\infty \tag{4.19}$$

or

$$\int_{t_0}^{\infty} p(t)(g(t))^{\alpha}t^{1+\delta-\alpha}\,dt=-\infty, \quad \text{some } 0<\delta<1, \tag{4.20}$$

then (4.16) is oscillatory for all $\lambda>0$ and all $\alpha>0$. Hence, for all $\alpha>0$, any bounded solution of (4.1) is oscillatory and for $\alpha\geqq 1$ any nonoscillatory solution satisfies (4.14), (4.15). If $0<\alpha<1$ and instead of (4.20) we have $\int_{t_0}^{\infty} p(t)(g(t))^{\alpha}t^{\delta}\,dt=-\infty$, then (4.17) is oscillatory for all $\lambda>0$, and hence any nonoscillatory solution of (4.1) satisfies (4.14), (4.15).

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ORTHOGONAL POLYNOMIALS IN TWO VARIABLES. A FURTHER ANALYSIS OF THE POLYNOMIALS ORTHOGONAL OVER A REGION BOUNDED BY TWO LINES AND A PARABOLA*

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Abstract. Some new results are obtained for the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$, introduced by Koornwinder [4] which are orthogonal over a region bounded by two straight lines and a parabola. The most important results are a Rodrigues-type formula and the recurrence relations for $up_{n,k}^{\alpha,\beta,\gamma}(u, v)$ and $vp_{n,k}^{\alpha,\beta,\gamma}(u, v)$. These recurrence relations contain 5 and 9 terms, respectively. Furthermore, the quadratic norm of $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ and the value of $p_{n,k}^{\alpha,\beta,\gamma}(2, 1)$ are explicitly given.

1. Introduction. In this paper, the analysis of the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ introduced by T. H. Koornwinder [4] will be continued.

In many respects, this class of orthogonal polynomials in two variables can be compared with the important class of Jacobi polynomials. In this analysis, some properties of the Jacobi polynomials are generalized to the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$.

The polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ form an orthogonal set over a region bounded by two perpendicular straight lines, $1-u+v=0$, $1+u+v=0$ and by the parabola $u^2-4v=0$ touching these lines, with respect to the weight function $(1-u+v)^\alpha(1+u+v)^\beta(u^2-4v)^\gamma$ which is singular at the boundary of the orthogonality region. For reasons of convergence, it is required that $\alpha,\beta,\gamma>-1$ and $\alpha+\gamma+3/2$, $\beta+\gamma+3/2>0$. The main results of Koornwinder's paper are summarized in § 2.

In the subsequent sections, a further analysis is given, using as the main tools a number of partial differential operators. In [4] it is proved that the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ are eigenfunctions of a second order operator $D_1^{\alpha,\beta,\gamma}$ and a fourth order operator $D_2^{\alpha,\beta,\gamma}$, which are algebraically independent. Furthermore, two second order operators D_-^γ and $D_+^{\alpha,\beta,\gamma}$ are derived with the property that $D_-^\gamma p_{n,k}^{\alpha,\beta,\gamma}(u, v)=$ const. $p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}(u, v)$ and $D_+^{\alpha,\beta,\gamma}p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}(u, v)=$ const. $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$. Then $D_2^{\alpha,\beta,\gamma}$ is given by $D_2^{\alpha,\beta,\gamma}=D_+^{\alpha,\beta,\gamma}\circ D_-^\gamma$.

In § 4 of this paper, another pair of differential operators is derived: these operators $E_-^{\alpha,\beta}$ and $E_+^{\alpha,\beta,\gamma}$ have the property that $E_-^{\alpha,\beta}p_{n,k}^{\alpha,\beta,\gamma}(u, v)=$ const. $p_{n-1,k}^{\alpha,\beta,\gamma}(u, v)$ and $E_+^{\alpha,\beta,\gamma}p_{n-1,k}^{\alpha,\beta,\gamma+1}(u, v)=$ const. $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$. Then another fourth order operator, which has the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ as eigenfunctions, can be defined by $D_3^{\alpha,\beta,\gamma}=E_+^{\alpha,\beta,\gamma}\circ E_-^{\alpha,\beta}$. This operator is explicitly expressed as a polynomial in $D_1^{\alpha,\beta,\gamma}$ and $D_2^{\alpha,\beta,\gamma}$. The operators D_-^γ and $E_-^{\alpha,\beta}$ together play a similar role for the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ to that played by the operator d/dx for the Jacobi polynomials.

One of the first problems which arise is to find an explicit expression for $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$. We have succeeded in finding an explicit expression of the Rodrigues-type by using the second order operators D_+ and E_+. Expressing D_+ and E_+ in $(D_-)^*$ and $(E_-)^*$ respectively, we obtain a formula for $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ which is similar to the Rodrigues formula for the Jacobi polynomials, but with the two second

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order operators $(D_-)^*$ and $(E_-)^*$ instead of d/dx (§ 5). However, the expression derived by us is rather complicated, and so we have tried to find other expressions for $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$. If $\gamma=-\frac{1}{2}$ and $\gamma=+\frac{1}{2}$, the polynomials can be expressed as symmetric ([4], § 2) or antisymmetric (§ 3) products of Jacobi polynomials. In § 10, the polynomials with $\alpha,\beta=\pm\frac{1}{2}$ are expressed in terms of Jacobi polynomials. The case $\gamma=+\frac{1}{2}$ is comparable with the determinants of orthogonal polynomials treated by Karlin and McGregor [3]. The orthogonal set of 2×2 determinants of Jacobi polynomials gives $p_{n,k}^{\alpha,\beta,+1/2}(x+y, xy)$ after dividing by $(x-y)$.

In § 6, the explicit value of the quadratic norm for the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ is given. The quadratic norm is important for finding coefficients in Fourier expansions with respect to the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ and will be used for the computation of some of the coefficients in the recurrence relations (§ 9).

Without knowing an explicit expression for $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$, it is possible to find the value of $p_{n,k}^{\alpha,\beta,\gamma}(2, 1)$ by using the operators D_+ and E_+ (§ 7). The point $(u, v)=(2, 1)$ is a vertex of the orthogonality region, which probably plays a similar role for the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ to that played by the point $x=1$ for the Jacobi plynomials $P_n^{(\alpha,\beta)}(x)$. The (unproved) hypothesis is that $p_{n,k}^{\alpha,\beta,\gamma}(2, 1)$ is the absolute maximum of $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ if $\alpha\geqq\beta\geqq-\frac{1}{2}$ and $\gamma\geqq-\frac{1}{2}$. For $\gamma=-\frac{1}{2}$, this maximum property follows directly from the explicit expression of $p_{n,k}^{\alpha,\beta,-(1/2)}(u, v)$ and the maximum property of the Jacobi polynomials.

The analysis of these polynomials suggests that not all powers $\leqq(n, k)$ of u and v appear in $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$. This is proved in § 8 and it has a number of consequences. An immediate consequence is that some theorems which give alternative definitions for $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ can be derived. Another is that the number of terms in the recurrence relations is uniformly bounded, while for general polynomials in more than one variable this number depends on the degree of the polynomial. In § 9, the recurrence relations are explicitly given. For $up_{n,k}^{\alpha,\beta,\gamma}(u, v)$ and $vp_{n,k}^{\alpha,\beta,\gamma}(u, v)$, we obtain a five-term and a nine-term recurrence relation, respectively. To build up $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ using the recurrence relations we need the formula for $vp_{n,k}^{\alpha,\beta,\gamma}(u, v)$ only if $n=k$, and then six terms remain.

Finally, in § 10, two quadratic transformation formulas are given for the case $\alpha=\beta$. These formulas, together with the explicit expressions for $\gamma=+\frac{1}{2}$, $\gamma=-\frac{1}{2}$ yield explicit expressions for the cases that α and β are $+\frac{1}{2}$ or $-\frac{1}{2}$.

2. Preliminaries. In this section, the main results obtained by Koornwinder [4] are summarized.

Let $\mathcal{N}$ be the set of pairs of integers (n, k), $n\geqq k\geqq 0$, with a lexicographic ordering defined by

$$(2.1)\qquad (m, l)\leqq(n, k)\Leftrightarrow\{m<n \vee (m=n\wedge l\leqq k)\}.$$

A polynomial $q(u, v)$ is said to have *degree* $(n, k)\in\mathcal{N}$ if

$$q(u, v)=\sum_{(m,l)\leqq(n,k)} c_{m,l}u^{m-l}v^l, \quad \text{with } c_{n,k}\neq 0.$$

The region with the properties $1-u+v>0$, $1+u+v>0$ and $u^2-4v>0$, is denoted by R (cf. Fig. 1). In the region R the weight function $\mu^{\alpha,\beta,\gamma}(u, v)$ is defined by

$$\mu^{\alpha,\beta,\gamma}(u, v) = (1-u+v)^{\alpha}(1+u+v)^{\beta}(u^2-4v)^{\gamma}. \tag{2.2}$$

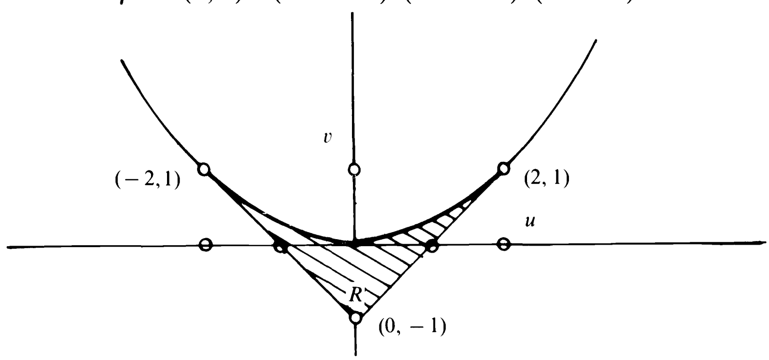

FIG. 1

DEFINITION 2.1. For $(n, k) \in \mathcal{N}$ and $\alpha,\beta,\gamma > -1$, $\alpha+\gamma+\frac{3}{2}$, $\beta+\gamma+\frac{3}{2} > 0$ the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ are given by

(i) $p_{n,k}^{\alpha,\beta,\gamma}(u, v) = u^{n-k}v^{k} +$ a polynomial of degree lower than (n, k).

(ii) $\iint_R p_{n,k}^{\alpha,\beta,\gamma}(u, v)\, u^{m-l}v^{l}\, \mu^{\alpha,\beta,\gamma}(u, v)\, du\, dv = 0$ if $(m, l) < (n, k)$.

Then $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ satisfies

$$D_1^{\alpha,\beta,\gamma} p_{n,k}^{\alpha,\beta,\gamma}(u, v) = -[n(n+\alpha+\beta+2\gamma+2) + k(k+\alpha+\beta+1)]p_{n,k}^{\alpha,\beta,\gamma}(u, v), \tag{2.3}$$

$$D_2^{\alpha,\beta,\gamma} p_{n,k}^{\alpha,\beta,\gamma}(u, v) = k(k+\alpha+\beta+1)(n+\gamma+\tfrac{1}{2})(n+\alpha+\beta+\gamma+\tfrac{3}{2})p_{n,k}^{\alpha,\beta,\gamma}(u, v), \tag{2.4}$$

$$D_-^{\gamma} p_{n,k}^{\alpha,\beta,\gamma}(u, v) = \begin{cases} k(n+\gamma+\frac{1}{2})p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}(u, v) & \text{if } k > 0, \\ 0 & \text{if } k = 0, \end{cases} \tag{2.5}$$

$$D_+^{\alpha,\beta,\gamma} p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}(u, v) = (k+\alpha+\beta+1)(n+\alpha+\beta+\gamma+\tfrac{3}{2})p_{n,k}^{\alpha,\beta,\gamma}(u, v) \quad \text{if } k > 0. \tag{2.6}$$

The operators are defined by

$$\begin{aligned} D_1^{\alpha,\beta,\gamma} = {} & (-u^2+2v+2)\frac{\partial^2}{\partial u^2} - 2u(v-1)\frac{\partial^2}{\partial u \partial v} + (u^2-2v^2-2v)\frac{\partial^2}{\partial v^2} \\ & + [-(\alpha+\beta+2\gamma+3)u + (2\beta-2\alpha)]\frac{\partial}{\partial u} \\ & + [(\beta-\alpha)u - (2\alpha+2\beta+2\gamma+5)v - (2\gamma+1)]\frac{\partial}{\partial v}, \end{aligned} \tag{2.7}$$

$$D_-^{\gamma} = \frac{\partial^2}{\partial u^2} + u\frac{\partial^2}{\partial u\, \partial v} + v\frac{\partial^2}{\partial v^2} + (\gamma+\tfrac{3}{2})\frac{\partial}{\partial v}, \tag{2.8}$$

$$\begin{aligned} D_+^{\alpha,\beta,\gamma} &= (1-u+v)^{-\alpha}(1+u+v)^{-\beta} D_-^{\gamma} \circ (1-u+v)^{\alpha+1}(1+u+v)^{\beta+1} \\ &= (1-u+v)(1+u+v)\left(\frac{\partial^2}{\partial u^2} + u\frac{\partial^2}{\partial u\, \partial v} + v\frac{\partial^2}{\partial v^2}\right) \\ &\quad + [(\alpha-\beta)(u^2-2v-2) + (\alpha+\beta+2)u(v-1)]\frac{\partial}{\partial u} \end{aligned}$$

$$(2.9)\qquad +[(\alpha+\beta+\gamma+\tfrac{7}{2})(-u^2+2v)+(\alpha-\beta)u(v-1)+(2\alpha+2\beta+\gamma+\tfrac{11}{2})v^2 +(\gamma+\tfrac{3}{2})]\frac{\partial}{\partial v}+(\alpha-\beta)(\alpha+\beta+\gamma+\tfrac{5}{2})u+(\alpha+\beta+2)(\alpha+\beta+\gamma+\tfrac{5}{2})v +(\alpha-\beta)^2+(\gamma+\tfrac{1}{2})(\alpha+\beta+2),$$

$$(2.10)\qquad D_2^{\alpha,\beta,\gamma}=D_+^{\alpha,\beta,\gamma}\circ D_-^{\gamma}.$$

Consideration of $(D_-^\gamma)^*$, the adjoint operator to D_-^γ, yields

$$(2.11)\qquad (D_-^\gamma)^*=D_-^{-\gamma}=(u^2-4v)^\gamma D_-^\gamma\circ(u^2-4v)^{-\gamma}.$$

Hence

$$(2.12)\qquad D_+^{\alpha,\beta,\gamma}=\{\mu^{\alpha,\beta,\gamma}(u,v)\}^{-1}(D_-^\gamma)^*\circ\mu^{\alpha+1,\beta+1,\gamma}(u,v).$$

The operators $D_+^{\alpha,\beta,\gamma}$ and D_-^γ are related by

$$(2.13)\qquad \iint_R (D_+^{\alpha,\beta,\gamma}p(u,v))q(u,v)\mu^{\alpha,\beta,\gamma}(u,v)\,du\,dv =\iint_R p(u,v)(D_-^\gamma q(u,v))\mu^{\alpha+1,\beta+1,\gamma}(u,v)\,du\,dv,$$

for any two polynomials $p(u, v)$ and $q(u, v)$.

Let

$$(2.14)\qquad p_n^{\alpha,\beta}(x)=\frac{2^n n!}{(n+\alpha+\beta+1)_n}P_n^{(\alpha,\beta)}(x),$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomial of order (α,β) (for Jacobi polynomials see Erdélyi [2] or Szegö [6]). Then

$$(2.15)\qquad p_{n,k}^{\alpha,\beta,-1/2}(x+y,xy)=\begin{cases} p_n^{\alpha,\beta}(x)p_k^{\alpha,\beta}(y)+p_k^{\alpha,\beta}(x)p_n^{\alpha,\beta}(y) & \text{if } n>k,\\ p_n^{\alpha,\beta}(x)p_k^{\alpha,\beta}(y) & \text{if } n=k.\end{cases}$$

3. The polynomials $p_{n,k}^{\alpha,\beta,+1/2}(u, v)$ as an antisymmetric product of Jacobi polynomials. Consider the antisymmetric product of Jacobi polynomials:

$$(3.1)\qquad f_{n,k}^{\alpha,\beta}(x,y)=p_{n+1}^{\alpha,\beta}(x)p_k^{\alpha,\beta}(y)-p_k^{\alpha,\beta}(x)p_{n+1}^{\alpha,\beta}(y),$$

where $p_n^{\alpha,\beta}(x)$ is defined by (2.14).

The polynomials $f_{n,k}^{\alpha,\beta}(x, y)$ form an orthogonal set of antisymmetric polynomials over the simplex $-1\leqq y\leqq x\leqq 1$ with respect to the weight function $((1-x)(1-y))^\alpha((1+x)(1+y))^\beta$ (cf. [3]). Then $(x-y)^{-1}f_{n,k}^{\alpha,\beta}(x, y)$ is a symmetric polynomial in x and y which can be uniquely expressed as a polynomial in $x+y=u$ and $xy=v$ (see van der Waerden [7, § 33]).

LEMMA 3.1. $(x-y)^{-1}f_{n,k}^{\alpha,\beta}(x,y)=p_{n,k}^{\alpha,\beta,+1/2}(x+y,xy)$.

Proof. Application of the definition of $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ (Definition 2.1) yields

(i) $$(x-y)^{-1}f_{n,k}^{\alpha,\beta}(x, y) = \sum_{(m,l)\leqq(n,k)} c_{m,l}\frac{x^{m+1}y^l - x^l y^{m+1}}{x-y} \quad \text{with } c_{n,k}=1$$
$$= (x+y)^{n-k}(xy)^k + \text{a polynomial in } (x+y) \text{ and } xy \text{ of degree lower than } (n, k).$$

(ii) $\{(x-y)^{-1}f_{n,k}^{\alpha,\beta}(x, y)\}$ is an orthogonal set with respect to the measure
$$(x-y)^2((1-x)(1-y))^\alpha((1+x)(1+y))^\beta\,dx\,dy =$$
$$\text{const.}\,(1-u+v)^\alpha(1+u+v)^\beta(u^2-4v)^{+1/2}\,du\,dv.$$

Hence

$$p_{n,k}^{\alpha,\beta,+1/2}(x+y, xy) = (x-y)^{-1}\{p_{n+1}^{\alpha,\beta}(x)p_k^{\alpha,\beta}(y) - p_k^{\alpha,\beta}(x)p_{n+1}^{\alpha,\beta}(y)\}. \tag{3.2}$$

4. A pair of differential operators which change *n* and *γ*. A pair of differential operators which change n and γ can be found by using (2.15) and (3.2) and the differential operators for the Jacobi polynomials. Let us define

$$D_{(x)}^{\alpha,\beta} = (1-x)^{-\alpha}(1+x)^{-\beta}\frac{\partial}{\partial x}(1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{\partial}{\partial x}; \tag{4.1}$$

then

$$D_{(x)}^{\alpha,\beta}p_n^{\alpha,\beta}(x) = c_n p_n^{\alpha,\beta}(x), \quad \text{where } c_n = -n(n+\alpha+\beta+1).$$

Hence

(4.2)
$$(x-y)^{-1}\{D_{(x)}^{\alpha,\beta} - D_{(y)}^{\alpha,\beta}\}p_{n,k}^{\alpha,\beta,-1/2}(x+y, xy) = \begin{cases}(c_n-c_k)p_{n-1,k}^{\alpha,\beta,+1/2}(x+y, xy) & \text{if } n>k,\\ 0 & \text{if } n=k,\end{cases}$$

and

(4.3)
$$\{D_{(x)}^{\alpha,\beta} - D_{(y)}^{\alpha,\beta}\}\circ(x-y)p_{n-1,k}^{\alpha,\beta,+1/2}(x+y, xy) = (c_n-c_k)p_{n,k}^{\alpha,\beta,-1/2}(x+y, xy) \quad \text{if } n>k.$$

Formulas (4.2) and (4.3) now suggest the following definition:

$$\begin{aligned}E_-^{\alpha,\beta} &= (x-y)^{-1}\{D_{(x)}^{\alpha,\beta} - D_{(y)}^{\alpha,\beta}\}\\ &= u\frac{\partial^2}{\partial u^2} + 2(v+1)\frac{\partial^2}{\partial u\,\partial v} + u\frac{\partial^2}{\partial v^2} + (\beta-\alpha)\frac{\partial}{\partial v} + (\alpha+\beta+2)\frac{\partial}{\partial u}.\end{aligned} \tag{4.4}$$

If $(E_-^{\alpha,\beta})^*$ is the adjoint operator to $E_-^{\alpha,\beta}$, then

$$(E_-^{\alpha,\beta})^* = u\frac{\partial^2}{\partial u^2} + 2(v+1)\frac{\partial^2}{\partial u\,\partial v} + u\frac{\partial^2}{\partial v^2} - (\beta-\alpha)\frac{\partial}{\partial v} - (\alpha+\beta-2)\frac{\partial}{\partial u}. \tag{4.5}$$

Note that

$$(E_-^{\alpha,\beta})^* = E_-^{-\alpha,-\beta} = (1-u+v)^\alpha(1+u+v)^\beta E_-^{\alpha,\beta}\circ(1-u+v)^{-\alpha}(1+u+v)^{-\beta}. \tag{4.6}$$

Now we can define the operator $E_+^{\alpha,\beta,\gamma}$ as

$$
\begin{aligned}
E_+^{\alpha,\beta,\gamma} &= \{\mu^{\alpha,\beta,\gamma}(u,v)\}^{-1}(E_-^{\alpha,\beta})^* \circ \mu^{\alpha,\beta,\gamma+1}(u,v) \\
&= (u^2-4v)\left(u\frac{\partial^2}{\partial u^2}+2(v+1)\frac{\partial^2}{\partial u\,\partial v}+u\frac{\partial^2}{\partial v^2}\right) \\
&\quad +[(\alpha+\beta+4\gamma+6)(u^2-4v)+8(\gamma+1)(v-1)]\frac{\partial}{\partial u} \\
&\quad +[(\beta-\alpha)(u^2-4v)+4(\gamma+1)u(v-1)]\frac{\partial}{\partial v} \\
&\quad +2(\gamma+1)(\alpha+\beta+2\gamma+3)u-4(\gamma+1)(\beta-\alpha).
\end{aligned} \tag{4.7}
$$

LEMMA 4.1. *If*

$$q_{n,k}(u,v)=u^{n-k}v^k + \textit{a polynomial of degree lower than } (n,k)$$

and

$$q_{n-1,k}(u,v)=u^{n-k-1}v^k + \textit{a polynomial of degree lower than } (n-1,k),$$

then

$$E_-^{\alpha,\beta}q_{n,k}(u,v)=\begin{cases}(n-k)(n+k+\alpha+\beta+1)u^{n-k-1}v^k + \textit{a polynomial of} \\ \textit{degree lower than } (n-1,k) \quad \textit{if } n>k, \\ \textit{a polynomial of degree equal or lower than} \\ (n-1,n-1) \quad \textit{if } n=k,\end{cases}$$

and

$$E_+^{\alpha,\beta,\gamma}q_{n-1,k}(u,v)=(n-k+2\gamma+1)(n+k+\alpha+\beta+2\gamma+2)u^{n-k}v^k + \textit{a polynomial of degree lower than } (n,k) \quad \textit{if } n>k.$$

Proof. Lemma 4.1 follows immediately from (4.4) and (4.7). □

Koornwinder proved the following.

LEMMA 4.2. *Let R be a bounded region in $\mathbb{R}^2$ such that certain polynomials $w_1(x,y), w_2(x,y), \cdots, w_k(x,y)$ are positive over R and the product $w_1 \cdot w_2, \cdots, w_k$ is zero at the boundary ∂R.*

Let $X^{\alpha_1,\cdots,\alpha_k}$ be a partial differential operator in x, and y, its coefficients being polynomials in $x, y, \alpha_1, \cdots, \alpha_k$.

Let the operator $Y^{\alpha_1,\cdots,\alpha_k}$ be defined by

$$Y^{\alpha_1,\cdots,\alpha_k}=w_1^{-\alpha_1}\cdots w_k^{-\alpha_k}(X^{\alpha_1,\cdots,\alpha_k})^* \circ w_1^{\alpha_1+i_1}\cdots w_k^{\alpha_k+i_k},$$

for certain nonnegative integers $i_1, \cdots, i_k$.

If this operator also has coefficients that are polynomials in $x, y, \alpha_1, \cdots, \alpha_k$, then

$$\iint_R p(Y^{\alpha_1,\cdots,\alpha_k}q)w_1^{\alpha_1}\cdots w_k^{\alpha_k}\,dx\,dy$$

$$=\iint_R (X^{\alpha_1,\cdots,\alpha_k}p)qw_1^{\alpha_1+i_1}\cdots w_k^{\alpha_k+i_k}\,dx\,dy,$$

for any two polynomials p and q, and for all real $\alpha_1, \cdots, \alpha_k$ *such that*

$$\iint_R w_1^{\alpha_1} \cdots w_k^{\alpha_k}\, dx\, dy < \infty.$$

Proof. For sufficiently large $\alpha_1, \alpha_2, \cdots, \alpha_k$ the equality follows from partial integration because the function $w_1^{\alpha_1} \cdots w_k^{\alpha_k}$ and its partial derivatives up to a certain order are zero at the boundary ∂R. By analytic continuation the equality follows for all $\alpha_1, \cdots, \alpha_k$ such that

$$\iint_R w_1^{\alpha_1} \cdots w_k^{\alpha_k}\, dx\, dy < \infty. \qquad \square$$

Rewriting this lemma for $E_-^{\alpha,\beta}$ and $E_+^{\alpha,\beta,\gamma}$ we obtain

$$(4.8) \qquad \iint_R p(E_+^{\alpha,\beta,\gamma} q)\mu^{\alpha,\beta,\gamma}(u, v)\, du\, dv = \iint_R (E_-^{\alpha,\beta} p) q \mu^{\alpha,\beta,\gamma+1}(u, v)\, du\, dv,$$

for any two polynomials p and q.

From Lemma 4.1, formula (4.8) and Definition 2.1 the following can be proved.

COROLLARY.

$$(4.9) \quad E_-^{\alpha,\beta} p_{n,k}^{\alpha,\beta,\gamma}(u, v) = \begin{cases} (n-k)(n+k+\alpha+\beta+1) p_{n-1,k}^{\alpha,\beta,\gamma+1}(u, v) & \text{if } n > k, \\ 0 & \text{if } n = k, \end{cases}$$

and

$$(4.10) \quad E_+^{\alpha,\beta,\gamma} p_{n-1,k}^{\alpha,\beta,\gamma+1}(u, v) = (n-k+2\gamma+1) \cdot (n+k+\alpha+\beta+2\gamma+2) p_{n,k}^{\alpha,\beta,\gamma}(u, v) \quad \text{if } n > k$$

(cf. the proof of Theorem 5.4 in [4]).

Let us define the fourth order operator

$$(4.11) \qquad D_3^{\alpha,\beta,\gamma} = E_+^{\alpha,\beta,\gamma} \circ E_-^{\alpha,\beta}.$$

The polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ are eigenfunctions of $D_3^{\alpha,\beta,\gamma}$:

$$(4.12) \quad D_3^{\alpha,\beta,\gamma} p_{n,k}^{\alpha,\beta,\gamma}(u, v) = (n-k)(n-k+2\gamma+1)(n+k+\alpha+\beta+1) \cdot (n+k+\alpha+\beta+2\gamma+2) p_{n,k}^{\alpha,\beta,\gamma}(u, v).$$

Hence $D_3^{\alpha,\beta,\gamma}$ can be uniquely expressed as a polynomial in $D_1^{\alpha,\beta,\gamma}$ and $D_2^{\alpha,\beta,\gamma}$ (cf. [4, Thm. 6.5]). By considering the eigenvalues it is clear that

$$(4.13) \qquad D_3^{\alpha,\beta,\gamma} = (D_1^{\alpha,\beta,\gamma})^2 - 4D_2^{\alpha,\beta,\gamma} - (2\gamma+1)(\alpha+\beta+1) D_1^{\alpha,\beta,\gamma}.$$

5. A Rodrigues-type formula. Using (2.6) and (4.10), $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ can be expressed in terms of polynomials of lower degree.

In (2.6) and (4.10) we write $D_+^{\alpha,\beta,\gamma}$ and $E_+^{\alpha,\beta,\gamma}$ respectively as

$$(2.11) \qquad D_+^{\alpha,\beta,\gamma} = \{\mu^{\alpha,\beta,\gamma}(u, v)\}^{-1} (D_-^{\gamma})^* \circ \mu^{\alpha+1,\beta+1,\gamma}(u, v)$$

and

$$E_{+}^{\alpha,\beta,\gamma}=\{\mu^{\alpha,\beta,\gamma}(u,v)\}^{-1}(E_{-}^{\alpha,\beta})^{*}\circ\mu^{\alpha,\beta,\gamma+1}(u,v). \tag{4.7}$$

An $(n-k)$-fold application of (4.10) and a k-fold application of (2.6) to $p_{0,0}^{\alpha+k,\beta+k,\gamma+n-k}(u,v)\equiv 1$ yields

$$\begin{aligned}(k+\alpha+\beta+1)_k(n+\alpha+\beta+\gamma+\tfrac{3}{2})_k(n-k+2\gamma+1)_{n-k}(n+k+\alpha+\beta\\ +2\gamma+2)_{n-k}p_{n,k}^{\alpha,\beta,\gamma}(u,v)\\ =(1-u+v)^{-\alpha}(1+u+v)^{-\beta}(u^2-4v)^{-\gamma}\\ \cdot\left\{\frac{\partial^2}{\partial u^2}+u\frac{\partial^2}{\partial u\,\partial v}+v\frac{\partial^2}{\partial v^2}-\left(\gamma-\frac{3}{2}\right)\frac{\partial}{\partial v}\right\}^k\\ \circ\left\{u\frac{\partial^2}{\partial u^2}+2(v+1)\frac{\partial^2}{\partial u\,\partial v}+u\frac{\partial^2}{\partial v^2}-(\beta-\alpha)\frac{\partial}{\partial v}-(\alpha+\beta+2k-2)\frac{\partial}{\partial u}\right\}^{n-k}\\ \circ(1-u+v)^{\alpha+k}(1+u+v)^{\beta+k}(u^2-4v)^{\gamma+n-k}.\end{aligned} \tag{5.1}$$

This is a Rodrigues-type formula for $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$. So far it is the only "explicit" expression for $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ in the case of general α, β, γ.

6. The quadratic norm of $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$. The quadratic norm $h_{n,k}^{\alpha,\beta,\gamma}$ of the polynomial $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ is defined by

$$h_{n,k}^{\alpha,\beta,\gamma}=\iint_R\{p_{n,k}^{\alpha,\beta,\gamma}(u,v)\}^2\mu^{\alpha,\beta,\gamma}(u,v)\,du\,dv. \tag{6.1}$$

The explicit value of $h_{n,k}^{\alpha,\beta,\gamma}$ is important for calculating the coefficients in Fourier expansions with respect to the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ (cf. § 9).

From (2.13) and (4.8) we obtain the following recurrence relations for $h_{n,k}^{\alpha,\beta,\gamma}$:

$$h_{n,k}^{\alpha,\beta,\gamma}=\frac{k(n+\gamma+\frac{1}{2})}{(k+\alpha+\beta+1)(n+\alpha+\beta+\gamma+\frac{3}{2})}h_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}, \tag{6.2}$$

and

$$h_{n,k}^{\alpha,\beta,\gamma}=\frac{(n-k)(n+k+\alpha+\beta+1)}{(n-k+2\gamma+1)(n+k+\alpha+\beta+2\gamma+2)}h_{n-1,k}^{\alpha,\beta,\gamma+1}. \tag{6.3}$$

By repeated application of (6.2) and (6.3) we find

$$h_{n,k}^{\alpha,\beta,\gamma}=\frac{k!(n-k)!(n-k+\gamma+\frac{3}{2})_k(2k+\alpha+\beta+2)_{n-k}}{(k+\alpha+\beta+1)_k(n+\alpha+\beta+\gamma+\frac{3}{2})_k(n-k+2\gamma+1)_{n-k}(n+k+\alpha+\beta+2\gamma+2)_{n-k}}\cdot h_{0,0}^{\alpha+k,\beta+k,\gamma+n-k}. \tag{6.4}$$

LEMMA 6.1.

(6.5)

$$h_{0,0}^{\alpha,\beta,\gamma}=\frac{2^{2\alpha+2\beta+4\gamma+3}}{\sqrt{\pi}}\,\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+\frac{5}{2})}\,\frac{\Gamma(\alpha+\gamma+\frac{3}{2})\Gamma(\beta+\gamma+\frac{3}{2})}{\Gamma(\alpha+\beta+2\gamma+3)}.$$

Proof. $p_{0,0}^{\alpha,\beta,\gamma}(u,v)\equiv 1$, thus

$$h_{0,0}^{\alpha,\beta,\gamma}=\iint_R (1-u+v)^{\alpha}(1+u+v)^{\beta}(u^2-4v)^{\gamma}\,du\,dv.$$

This transforms under the substitution

$$u=x+y,\quad v=xy$$

into

$$h_{0,0}^{\alpha,\beta,\gamma}=\int_{x=-1}^{1}\left\{\int_{y=-1}^{x}(1-y)^{\alpha}(1+y)^{\beta}(x-y)^{2\gamma+1}\,dy\right\}(1-x)^{\alpha}(1+x)^{\beta}\,dx.$$

By making the substitution $t=(1+x)^{-1}(1+y)$ and using

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z),$$

it follows that

$$\begin{aligned}h_{0,0}^{\alpha,\beta,\gamma}&=2^{\alpha}\frac{\Gamma(\beta+1)\Gamma(2\gamma+2)}{\Gamma(\beta+2\gamma+3)}\int_{-1}^{+1}(1-x)^{\alpha}(1+x)^{2\beta+2\gamma+2}\\&\quad\cdot{}_2F_1\left(-\alpha,\beta+1;\beta+2\gamma+3;\frac{1+x}{2}\right)dx\\&=2^{2\alpha+2\beta+2\gamma+3}\frac{\Gamma(\beta+1)\Gamma(2\gamma+2)}{\Gamma(\beta+2\gamma+3)}\int_0^1(1-s)^{\alpha}s^{2\beta+2\gamma+2}\\&\quad\cdot{}_2F_1(-\alpha,\beta+1;\beta+2\gamma+3;s)\,ds\\&=2^{2\alpha+2\beta+2\gamma+3}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\gamma+2)\Gamma(2\beta+2\gamma+3)}{\Gamma(\beta+2\gamma+3)\Gamma(\alpha+2\beta+2\gamma+4)}\\&\quad\cdot{}_3F_2(-\alpha,\beta+1,2\beta+2\gamma+3;\ \beta+2\gamma+3,\alpha+2\beta+2\gamma+4;1).\end{aligned}$$

This ${}_3F_2$ function is of type ${}_3F_2(a,b,c;1+a-b,1+a-c;1)$ with $a=2\beta+2\gamma+3$, $b=\beta+1$ and $c=-\alpha$ and so the theorem of Dixon can be applied (see Bailey [1, Chap. 3.1] or Slater [5, (2.3.3)]). This proves the lemma. □

COROLLARY. *The quadratic norm $h_{n,k}^{\alpha,\beta,\gamma}$ is equal to*

(6.6)

$$h_{n,k}^{\alpha,\beta,\gamma}=\frac{2^{4n+2\alpha+2\beta+4\gamma+3}k!(n-k)!(n-k+\gamma+\frac{3}{2})_k(2k+\alpha+\beta+2)_{n-k}}{\sqrt{\pi}(k+\alpha+\beta+1)_k(n+\alpha+\beta+\gamma+\frac{3}{2})_k(n-k+2\gamma+1)_{n-k}(n+k+\alpha+\beta+2\gamma+2)_{n-k}}$$
$$\cdot\frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)\Gamma(n-k+\gamma+1)\Gamma(n+\alpha+\gamma+\frac{3}{2})\Gamma(n+\beta+\gamma+\frac{3}{2})}{\Gamma(n+k+\alpha+\beta+\gamma+\frac{5}{2})\Gamma(2n+\alpha+\beta+2\gamma+3)}.$$

7. The value of $p_{n,k}^{\alpha,\beta,\gamma}(2,1)$. It is possible to find the value of $p_{n,k}^{\alpha,\beta,\gamma}(2,1)$ by using the operators $D_+^{\alpha,\beta,\gamma}$ and $E_+^{\alpha,\beta,\gamma}$. It is of interest to know this value because of the hypothesis that for $\alpha \geqq \beta \geqq -\frac{1}{2}$ and $\gamma \geqq -\frac{1}{2}$ the inequality

$$|p_{n,k}^{\alpha,\beta,\gamma}(u,v)| \leqq p_{n,k}^{\alpha,\beta,\gamma}(2,1)$$

is valid. This hypothesis was proved for $\gamma = -\frac{1}{2}$. If $\gamma \geqq -\frac{1}{2}$, then it is true if $\alpha = \beta = -\frac{1}{2}$. Further it holds for the polynomials $p_{n,n}^{\alpha,\beta,+1/2}(u,v)$, $p_{n,n-1}^{\alpha,\alpha,+1/2}(u,v)$ and $p_{n,0}^{+1/2,-1/2,\gamma}(u,v)$.

Considering (2.9) and (4.7) we obtain the following equalities:

$$(D_+^{\alpha,\beta,\gamma}p)(2,1) = 4(\alpha+1)(\alpha+\gamma+\tfrac{3}{2})p(2,1)$$

and

$$(E_+^{\alpha,\beta,\gamma}p)(2,1) = 8(\gamma+1)(\alpha+\gamma+\tfrac{3}{2})p(2,1),$$

for any polynomial $p(u,v)$.

Hence

$$p_{n,k}^{\alpha,\beta,\gamma}(2,1) = \frac{4(\alpha+1)(\alpha+\gamma+\frac{3}{2})}{(k+\alpha+\beta+1)(n+\alpha+\beta+\gamma+\frac{3}{2})} p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}(2,1) \tag{7.1}$$

and

$$p_{n,k}^{\alpha,\beta,\gamma}(2,1) = \frac{8(\gamma+1)(\alpha+\gamma+\frac{3}{2})}{(n-k+2\gamma+1)(n+k+\alpha+\beta+2\gamma+2)} p_{n-1,k}^{\alpha,\beta,\gamma+1}(2,1). \tag{7.2}$$

From (7.1), (7.2) and $p_{0,0}^{\alpha,\beta,\gamma}(u,v) \equiv 1$ it follows that

(7.3)

$$p_{n,k}^{\alpha,\beta,\gamma}(2,1) = \frac{2^{3n-k}(\alpha+1)_k(\gamma+1)_{n-k}(\alpha+\gamma+\frac{3}{2})_n}{(k+\alpha+\beta+1)_k(n+\alpha+\beta+\gamma+\frac{3}{2})_k(n-k+2\gamma+1)_{n-k}(n+k+\alpha+\beta+2\gamma+2)_{n-k}}.$$

Remark. The relation

$$p_{n,k}^{\alpha,\beta,\gamma}(-2,1) = (-1)^{n-k} p_{n,k}^{\beta,\alpha,\gamma}(2,1) \quad \text{(equation (10.1))}$$

immediately gives the value of $p_{n,k}^{\alpha,\beta,\gamma}(-2,1)$.

8. The coefficients in the power series of $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$. For the coefficients $a_{ij}(n,k,\alpha,\beta,\gamma)$ in the expansion

$$p_{n,k}^{\alpha,\beta,\gamma}(u,v) = \sum_{(i,j)\leqq(n,k)} a_{i,j}(n,k,\alpha,\beta,\gamma)u^{i-j}v^j \tag{8.1}$$

the following theorem holds.

THEOREM 8.1. *$a_{i,j}(n,k,\alpha,\beta,\gamma) = 0$ if $i+j > n+k$ or $i > n$.*

At this point it is useful to define the following partial ordering for $\mathcal{N} = \{(n,k) | n \geqq k \geqq 0,\ n, k \in \mathbb{N}\}$:

$$(i,j) \prec (n,k) \quad \text{iff } i \leqq n \quad \text{and} \quad i+j \leqq n+k. \tag{8.2}$$

Thus

$$(i, j) \prec (n, k) \Leftrightarrow ((i, j) \leqq (n, k) \wedge i+j \leqq n+k).$$

Theorem 8.1 is equivalent to

$$p_{n,k}^{\alpha,\beta,\gamma}(u, v) = \sum_{(i,j)\prec(n,k)} a_{i,j}(n, k, \alpha, \beta, \gamma) u^{i-j} v^j. \tag{8.3}$$

Proof of Theorem 8.1. The second statement is a consequence of the definition of $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$, because if $i > n$, then $(i, j) > (n, k)$. The first statement is trivially true for the polynomials $p_{n,n}^{\alpha,\beta,\gamma}(u, v)$, because in that case, $i+j > n+n$ implies $i > n$. It is clear from (4.7) that

$$E_+^{\alpha,\beta,\gamma} u^{m-l} v^l = \sum_{(i,j)\prec(m+1,l)} c_{i,j} u^{i-j} v^j,$$

for certain constants $c_{i,j}$. By repeated application of the operators E_+ to $p_{n,n}^{\alpha,\beta,\gamma}(u, v)$ and by using (4.10) the theorem follows. □

Corollaries of Theorem 8.1 are the next two theorems.

THEOREM 8.2. *Let*

(i) $$p(u, v) = \sum_{(m,l)\prec(n,k)} c_{m,l} u^{m-l} v^l,$$

for certain constants $c_{m,l}$, *with* $c_{n,k} = 1$, *and*

(ii) $$\iint_R p(u, v) u^{m-l} v^l \mu^{\alpha,\beta,\gamma}(u, v)\, du\, dv = 0 \quad \textit{if } (m, l) \underset{\neq}{\prec} (n, k)$$

Then

$$p(u, v) = p_{n,k}^{\alpha,\beta,\gamma}(u, v).$$

THEOREM 8.3. *Let*

(i) $$p(u, v) = \sum_{(m,l)\prec(n,k)} c_{m,l} u^{m-l} v^l,$$

for certain constants $c_{m,l}$, *with* $c_{n,k} = 1$, *and*

(ii) $$D_1^{\alpha,\beta,\gamma} p(u, v) = \lambda p(u, v) \quad \textit{for some } \lambda \in \mathbb{R}.$$

Then

$$p(u, v) = p_{n,k}^{\alpha,\beta,\gamma}(u, v)$$

and

$$\lambda = -[n(n+\alpha+\beta+2\gamma+2) + k(k+\alpha+\beta+1)].$$

Proof of Theorem 8.2. From (i) it follows that $p(u, v)$ can be uniquely expressed as

$$p(u, v) = \sum_{(m,l)\prec(n,k)} c'_{m,l} p_{m,l}^{\alpha,\beta,\gamma}(u, v) \quad \text{with } c'_{n,k} = 1.$$

Then (ii) yields

$$c'_{m,l}=(h^{\alpha,\beta,\gamma}_{m,l})^{-1}\iint_R p(u,v)p^{\alpha,\beta,\gamma}_{m,l}(u,v)\mu^{\alpha,\beta,\gamma}(u,v)\,du\,dv=0\quad \text{if } (m,l)\underset{\neq}{<}(n,k).$$

This proves the theorem. □

For the proof of Theorem 8.3 we need the following lemma.

LEMMA 8.1. *If* $(m,l)\underset{\neq}{<}(n,k)$, *then* $\lambda_{m,l}\neq\lambda_{n,k}$, *with*

$$\lambda_{m,l}=-[m(m+\alpha+\beta+2\gamma+2)+l(l+\alpha+\beta+1)].$$

Proof. The parameters α,β,γ satisfy $\alpha,\beta,\gamma>-1$, $\alpha+\gamma+\frac{3}{2}$, $\beta+\gamma+\frac{3}{2}>0$. Suppose that $(m,l)\underset{\neq}{<}(n,k)$ and $\lambda_{m,l}=\lambda_{n,k}$. Then

$$(n-m)(n+m+\alpha+\beta+2\gamma+2)=(l-k)(l+k+\alpha+\beta+1)$$

and the factors $n+m+\alpha+\beta+2\gamma+2$ and $l+k+\alpha+\beta+1$ are positive. Hence $n-m>0$ and $l-k>0$. Observe that $n+m+\alpha+\beta+2\gamma+2\geqq 2l+1+\alpha+\beta+2\gamma+2>2l+\alpha+\beta+1\geqq l+k+\alpha+\beta+1$. Thus $n-m<l-k$, contradicting the hypothesis $(m,l)<(n,k)$. □

Proof of Theorem 8.3. From (i) it follows that $p(u,v)$ can be uniquely expressed as

$$p(u,v)=\sum_{(m,l)<(n,k)} c'_{m,l}p^{\alpha,\beta,\gamma}_{m,l}(u,v)\quad\text{with } c'_{n,k}=1.$$

Then (ii) yields

$$\sum_{(m,l)<(n,k)}\lambda c'_{m,l}p^{\alpha,\beta,\gamma}_{m,l}(u,v)=\sum_{(m,l)<(n,k)}\lambda_{m,l}c'_{m,l}p^{\alpha,\beta,\gamma}_{m,l}(u,v).$$

From $c'_{n,k}=1$ it follows that

$$\lambda=\lambda_{n,k}=-[n(n+\alpha+\beta+2\gamma+2)+(k+\alpha+\beta+1)],$$

and from $\lambda_{n,k}c'_{m,l}=\lambda_{m,l}c'_{m,l}$ and Lemma 8.1 it follows that $c'_{m,l}=0$ for $(m,l)\underset{\neq}{<}(n,k)$. □

Application of $D^{\alpha,\beta,\gamma}_1$ to (8.3) and comparison of the coefficients of equal powers of u and v give the following explicit values for some of the coefficients $a_{i,j}(n,k,\alpha,\beta,\gamma)$ in (8.3), which will be used in § 9 for the computation of the coefficients in the recurrence relations:

(8.4a) $\quad a_{n,k}(n,k,\alpha,\beta,\gamma)=1,$

(8.4b) $\quad a_{n,k-1}(n,k,\alpha,\beta,\gamma)=-(\beta-\alpha)k/(2k+\alpha+\beta),$

(8.4c) $\quad a_{n,k-2}(n,k,\alpha,\beta,\gamma)=-\frac{1}{2}k(k-1)\cdot\{1-(\beta-\alpha)^2/(2k+\alpha+\beta)\}/(2k+\alpha+\beta-1),$

(8.4d) $\quad a_{n-1,k+1}(n,k,\alpha,\beta,\gamma)=-(n-k)(n-k-1)/(n-k+\gamma-\frac{1}{2}),$

(8.4e) $$a_{n-1,k}(n,k,\alpha,\beta,\gamma)=\frac{(\beta-\alpha)(n-k)}{(2n+\alpha+\beta+2\gamma+1)}\left\{\frac{2k(n-k+1)}{2k+\alpha+\beta}+\frac{(n-k-1)(k+1)}{n-k+\gamma-\frac{1}{2}}-2\right\},$$

(8.4f) $$\begin{aligned}&-2(n+k+\alpha+\beta+\gamma+\tfrac{1}{2})a_{n-1,k-1}(n,k,\alpha,\beta,\gamma)\\&=(\beta-\alpha)ka_{n-1,k}+k(k+1)a_{n-1,k+1}+2(n-k+2)\\&\quad\cdot(n-k+1)a_{n,k-2}+2(\beta-\alpha)(n-k+1)a_{n,k-1}\\&\quad+2(n-2k-\gamma+\tfrac{1}{2})k,\end{aligned}$$

$a_{n,k-3}(n,k,\alpha,\beta,\gamma)$ and $a_{n,k-4}(n,k,\alpha,\beta,\gamma)$ do not depend on n and γ.

9. The recurrence relations. For a further analysis of the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$, it is useful to have formulas for the series expansions of $up_{n,k}^{\alpha,\beta,\gamma}(u,v)$ and $vp_{n,k}^{\alpha,\beta,\gamma}(u,v)$ in terms of $p_{i,j}^{\alpha,\beta,\gamma}(u,v)$. These formulas give $p_{n+1,k}^{\alpha,\beta,\gamma}(u,v)$ and $p_{n+1,k+1}^{\alpha,\beta,\gamma}(u,v)$ as linear combinations of lower degree polynomials.

Case I. Expansion of $up_{n,k}^{\alpha,\beta,\gamma}(u,v)$. Consider the following equality:

(9.1) $$\begin{aligned}up_{n,k}^{\alpha,\beta,\gamma}(u,v)&=\sum_{(m,l)<(n,k)}a_{m,l}u^{m-l+1}v^{l}\\&=\sum_{(m,l)<(n+1,k)}a'_{m,l}\,u^{m-l}v^{l}\\&=\sum_{(m,l)<(n+1,k)}b_{m,l}(n,k,\alpha,\beta,\gamma)p_{m,l}^{\alpha,\beta,\gamma}(u,v),\end{aligned}$$

with

(9.2) $$b_{m,l}(n,k,\alpha,\beta,\gamma)=\{h_{m,l}^{\alpha,\beta,\gamma}\}^{-1}\iint_R up_{n,k}^{\alpha,\beta,\gamma}(u,v)p_{m,l}^{\alpha,\beta,\gamma}(u,v)\mu^{\alpha,\beta,\gamma}(u,v)\,du\,dv.$$

From symmetry it follows that

(9.3) $$b_{m,l}(n,k,\alpha,\beta,\gamma)=h_{n,k}^{\alpha,\beta,\gamma}\{h_{m,l}^{\alpha,\beta,\gamma}\}^{-1}b_{n,k}(m,l,\alpha,\beta,\gamma).$$

Hence $b_{m,l}(n,k,\alpha,\beta,\gamma)\neq 0$ only if $(m,l)>(n-1,k)$. And so the summation in (9.1) at most runs through $(m,l)\in\{(n+1,k),\ (n+1,k-1),\ (n+1,k-2),\ (n,k+1),(n,k),(n,k-1),(n-1,k+2),(n-1,k+1),(n-1,k)\}$. The coefficients can be computed by means of (8.4), (9.1) and (9.3). The coefficients $b_{n+1,k-1}$, $b_{n-1,k+1}$, $b_{n+1,k-2}$ and $b_{n-1,k+2}$ turn out to be zero.

For the five remaining coefficients in (9.1) we obtain

(9.4a) $b_{n+1,k}(n,k,\alpha,\beta,\gamma)=1,$

(9.4b) $$\begin{aligned}&b_{n-1,k}(n,k,\alpha,\beta,\gamma)\\&=\frac{4(n+\gamma+\frac{1}{2})(n+\alpha+\gamma+\frac{1}{2})(n+\beta+\gamma+\frac{1}{2})(n+\alpha+\beta+\gamma+\frac{1}{2})}{(2n+\alpha+\beta+2\gamma)_3(2n+\alpha+\beta+2\gamma+1)}\\&\cdot\frac{(n-k)(n-k+2\gamma)(n+k+\alpha+\beta+1)(n+k+\alpha+\beta+2\gamma+1)}{(n-k+\gamma-\frac{1}{2})(n-k+\gamma+\frac{1}{2})(n+k+\alpha+\beta+\gamma+\frac{1}{2})(n+k+\alpha+\beta+\gamma+\frac{3}{2})},\end{aligned}$$

(9.4c) $$b_{n,k+1}(n,k,\alpha,\beta,\gamma)=\frac{(n-k)(n-k+2\gamma)}{(n-k+\gamma-\frac{1}{2})(n-k+\gamma+\frac{1}{2})},$$

(9.4d) $$b_{n,k-1}(n,k,\alpha,\beta,\gamma)=\frac{4k(k+\alpha)(k+\beta)(k+\alpha+\beta)(n+k+\alpha+\beta+1)(n+k+\alpha+\beta+2\gamma+1)}{(2k+\alpha+\beta-1)_3(2k+\alpha+\beta)(n+k+\alpha+\beta+\gamma+\frac{1}{2})(n+k+\alpha+\beta+\gamma+\frac{3}{2})},$$

(9.4e) $$b_{n,k}(n,k,\alpha,\beta,\gamma)=(\beta-\alpha)(\alpha+\beta)\cdot\left\{\frac{1}{(2n+\alpha+\beta+2\gamma+1)(2n+\alpha+\beta+2\gamma+3)}+\frac{(2n+\alpha+\beta+2)(2n+\alpha+\beta+4\gamma+2)}{(2n+\alpha+\beta+2\gamma+1)(2n+\alpha+\beta+2\gamma+3)(2k+\alpha+\beta)(2k+\alpha+\beta+2)}\right\}.$$

If we define

(9.5) $$p_{n,k}^{\alpha,\beta,\gamma}(u,v)\equiv 0 \quad \text{if } n<k \quad \text{or if} \quad k<0,$$

then the following five-term formula holds for $up_{n,k}^{\alpha,\beta,\gamma}(u,v)$ for all $n\geqq k\geqq 0$:

(9.6) $$\begin{aligned} up_{n,k}^{\alpha,\beta,\gamma}(u,v)&=p_{n+1,k}^{\alpha,\beta,\gamma}(u,v)+b_{n,k+1}(n,k,\alpha,\beta,\gamma)p_{n,k+1}^{\alpha,\beta,\gamma}(u,v)\\ &\quad+b_{n,k}(n,k,\alpha,\beta,\gamma)p_{n,k}^{\alpha,\beta,\gamma}(u,v)\\ &\quad+b_{n,k-1}(n,k,\alpha,\beta,\gamma)p_{n,k-1}^{\alpha,\beta,\gamma}(u,v)\\ &\quad+b_{n-1,k}(n,k,\alpha,\beta,\gamma)p_{n-1,k}^{\alpha,\beta,\gamma}(u,v),\end{aligned}$$

with $b_{m,l}(n,k,\alpha,\beta,\gamma)$ given by (9.4).

It follows that

(9.7) $$\begin{aligned} p_{n+1,k}^{\alpha,\beta,\gamma}(u,v)&=-b_{n,k+1}(n,k,\alpha,\beta,\gamma)p_{n,k+1}^{\alpha,\beta,\gamma}(u,v)\\ &\quad+(u-b_{n,k}(n,k,\alpha,\beta,\gamma))p_{n,k}^{\alpha,\beta,\gamma}(u,v)\\ &\quad-b_{n,k-1}(n,k,\alpha,\beta,\gamma)p_{n,k-1}^{\alpha,\beta,\gamma}(u,v)\\ &\quad-b_{n-1,k}(n,k,\alpha,\beta,\gamma)p_{n-1,k}^{\alpha,\beta,\gamma}(u,v)\end{aligned}$$

if $n\geqq k\geqq 0$.

Remark. By application of the quadratic transformation formulas (10.5) and (10.6) to (9.6), repeated application of D_-^{γ} and analytic continuation, it can be proved that

(9.8)
$$p_{n,k}^{\alpha,\beta,\gamma}(u,v)=p_{n,k}^{\alpha,\beta+1,\gamma}(u,v)+Ap_{n,k-1}^{\alpha,\beta+1,\gamma}(u,v)+Bp_{n-1,k}^{\alpha,\beta+1,\gamma}(u,v)+Cp_{n-1,k-1}^{\alpha,\beta+1,\gamma}(u,v) \quad \text{if } n\geqq k\geqq 0,$$

with A, B and C being functions of n, k, α, β and γ to be determined from the coefficients of $u^{m-l}v^l$.

Case II. Expansion of $vp_{n,k}^{\alpha,\beta,\gamma}(u,v)$. Consider the equality

(9.9) $$vp_{n,k}^{\alpha,\beta,\gamma}(u,v)=\sum_{(m,l)<(n+1,k+1)} c_{m,l}(n,k,\alpha,\beta,\gamma)p_{m,l}^{\alpha,\beta,\gamma}(u,v),$$

with

$$(9.10)\quad c_{m,l}(n,k,\alpha,\beta,\gamma)=\{h_{m,l}^{\alpha,\beta,\gamma}\}^{-1}\iint_R v p_{n,k}^{\alpha,\beta,\gamma}(u,v)p_{m,l}^{\alpha,\beta,\gamma}(u,v)\mu^{\alpha,\beta,\gamma}(u,v)\,du\,dv.$$

From symmetry it follows that

$$(9.11)\qquad c_{m,l}(n,k,\alpha,\beta,\gamma)=h_{n,k}^{\alpha,\beta,\gamma}\{h_{m,l}^{\alpha,\beta,\gamma}\}^{-1}c_{n,k}(m,l,\alpha,\beta,\gamma).$$

Hence $c_{m,l}(n,k,\alpha,\beta,\gamma)\neq 0$ only if $(m,l)\succ(n-1,k-1)$. And so the summation in (9.9) at most runs through $(m,l)\in\{(n+1,k+1),\ (n+1,k),\ (n+1,k-1),\ (n+1,k-2),\ (n+1,k-3),\ (n,k+2),\ (n,k+1),\ (n,k),\ (n,k-1),\ (n,k-2),\ (n-1,k+3),\ (n-1,k+2),\ (n-1,k+1),\ (n-1,k),\ (n-1,k-1)\}$. The coefficients can be computed by means of (8.4), (9.9) and (9.10), and by comparison with the case $\gamma=-\frac{1}{2}$. The coefficients $c_{n+1,k-2}$, $c_{n-1,k+2}$, $c_{n+1,k-3}$, $c_{n-1,k+3}$, $c_{n,k+2}$ and $c_{n,k-2}$ turn out to be zero.

For the nine remaining coefficients in (9.9) we obtain

$$(9.12\text{a})\quad c_{n+1,k+1}(n,k,\alpha,\beta,\gamma)=1,$$

$$(9.12\text{b})\quad \begin{aligned} &c_{n-1,k-1}(n,k,\alpha,\beta,\gamma)\\ &\quad=\frac{2^4(n+\gamma+\frac{1}{2})(n+\alpha+\gamma+\frac{1}{2})(n+\beta+\gamma+\frac{1}{2})(n+\alpha+\beta+\gamma+\frac{1}{2})}{(2n+\alpha+\beta+2\gamma)_3(2n+\alpha+\beta+2\gamma+1)}\\ &\quad\cdot\frac{k(k+\alpha)(k+\beta)(k+\alpha+\beta)(n+k+\alpha+\beta)_2(n+k+\alpha+\beta+2\gamma)_2}{(2k+\alpha+\beta-1)_3(2k+\alpha+\beta)(n+k+\alpha+\beta+\gamma-\frac{1}{2})_3(n+k+\alpha+\beta+\gamma+\frac{1}{2})},\end{aligned}$$

$$(9.12\text{c})\quad c_{n+1,k}(n,k,\alpha,\beta,\gamma)=\frac{(\beta-\alpha)(\alpha+\beta)}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)},$$

$$(9.12\text{d})\quad \begin{aligned} &c_{n-1,k}(n,k,\alpha,\beta,\gamma)\\ &\quad=\frac{4(\beta-\alpha)(\alpha+\beta)(n+\gamma+\frac{1}{2})(n+\alpha+\gamma+\frac{1}{2})(n+\beta+\gamma+\frac{1}{2})}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)(2n+\alpha+\beta+2\gamma)_3}\\ &\quad\cdot\frac{(n+\alpha+\beta+\gamma+\frac{1}{2})(n-k)(n-k+2\gamma)(n+k+\alpha+\beta+1)(n+k+\alpha+\beta+2\gamma+1)}{(2n+\alpha+\beta+2\gamma+1)(n-k+\gamma-\frac{1}{2})(n-k+\gamma+\frac{1}{2})(n+k+\alpha+\beta+\gamma+\frac{1}{2})_2},\end{aligned}$$

$$(9.12\text{e})\quad c_{n+1,k-1}(n,k,\alpha,\beta,\gamma)=\frac{4k(k+\alpha)(k+\beta)(k+\alpha+\beta)}{(2k+\alpha+\beta-1)_3(2k+\alpha+\beta)},$$

$$(9.12\text{f})\quad \begin{aligned} &c_{n-1,k+1}(n,k,\alpha,\beta,\gamma)\\ &\quad=\frac{4(n+\gamma+\frac{1}{2})(n+\alpha+\gamma+\frac{1}{2})(n+\beta+\gamma+\frac{1}{2})(n+\alpha+\beta+\gamma+\frac{1}{2})}{(2n+\alpha+\beta+2\gamma)_3(2n+\alpha+\beta+2\gamma+1)}\\ &\quad\cdot\frac{(n-k-1)_2(n-k+2\gamma-1)_2}{(n-k+\gamma-\frac{3}{2})_3(n-k+\gamma-\frac{1}{2})},\end{aligned}$$

$$\text{(9.12g)}\quad c_{n,k+1}(n,k,\alpha,\beta,\gamma) = \frac{(\beta-\alpha)(\alpha+\beta)(n-k)(n-k+2\gamma)}{(2n+\alpha+\beta+2\gamma+1)(2n+\alpha+\beta+2\gamma+3)(n-k+\gamma-\frac{1}{2})(n-k+\gamma+\frac{1}{2})},$$

$$\text{(9.12h)}\quad \begin{aligned} c_{n,k-1}(n,k,\alpha,\beta,\gamma) &= \frac{4(\beta-\alpha)(\alpha+\beta)k(k+\alpha)(k+\beta)(k+\alpha+\beta)}{(2n+\alpha+\beta+2\gamma+1)(2n+\alpha+\beta+2\gamma+3)(2k+\alpha+\beta-1)_3(2k+\alpha+\beta)} \\ &\cdot\frac{(n+k+\alpha+\beta+1)(n+k+\alpha+\beta+2\gamma+1)}{(n+k+\alpha+\beta+\gamma+\frac{1}{2})_2}, \end{aligned}$$

$$\text{(9.12i)}\quad \begin{aligned} c_{n,k}(n,k,\alpha,\beta,\gamma) &= a_{n-1,k-1}(n,k,\alpha,\beta,\gamma) - a_{n,k}(n+1,k+1,\alpha,\beta,\gamma) \\ &\quad - c_{n+1,k}(n,k,\alpha,\beta,\gamma)a_{n,k}(n+1,k,\alpha,\beta,\gamma) - c_{n+1,k-1}(n,k,\alpha,\beta,\gamma) \\ &\quad \cdot a_{n,k}(n+1,k-1,\alpha,\beta,\gamma) - c_{n,k+1}(n,k,\alpha,\beta,\gamma)a_{n,k}(n,k+1,\alpha,\beta,\gamma). \end{aligned}$$

If $\gamma=-\frac{1}{2}$, then $c_{n,k}(n,k,\alpha,\beta,-\frac{1}{2})$ is given by

$$\text{(9.12i)}'\quad c_{n,k}(n,k,\alpha,\beta,-\tfrac{1}{2}) = \frac{(\alpha+\beta)^2(\beta-\alpha)^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)(2k+\alpha+\beta)(2k+\alpha+\beta+2)}.$$

Formula (9.9) holds, with the coefficients given by (9.12), for all $n \geqq k \geqq 0$, where the convention (9.5) is used again.

Formulas (9.6) and (9.9) together give an algorithm for calculating $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$. If $n \neq k$, then $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ can be expressed in terms of lower degree polynomials by the five-term relation (9.6). If $n=k$, then (9.9) provides a six-term relation which expresses $p_{n,n}^{\alpha,\beta,\gamma}(u,v)$ in terms of lower degree polynomials.

10. A quadratic transformation. The reflection $u \to -u$ maps the region R onto itself and transforms the weight function $\mu^{\alpha,\beta,\gamma}(u,v)$ into $\mu^{\beta,\alpha,\gamma}(u,v)$. Hence, in view of Definition 2.1, the following equality holds:

$$\text{(10.1)}\quad p_{n,k}^{\alpha,\beta,\gamma}(-u,v) = (-1)^{n-k} p_{n,k}^{\beta,\alpha,\gamma}(u,v).$$

If $\alpha=\beta$, then (10.1) becomes

$$\text{(10.2)}\quad p_{n,k}^{\alpha,\alpha,\gamma}(-u,v) = (-1)^{n-k} p_{n,k}^{\alpha,\alpha,\gamma}(u,v).$$

Formula (10.2) means that if $(n-k)$ is even, then $p_{n,k}^{\alpha,\alpha,\gamma}(u,v)$ is a polynomial in u^2 and v, and if $(n-k)$ is odd, then $u^{-1}p_{n,k}^{\alpha,\alpha,\gamma}(u,v)$ is a polynomial in u^2 and v.

Consider now the new variables

$$\text{(10.3)}\quad u'=2v,\quad v'=u^2-2v-1.$$

These variables satisfy the following properties:

(i) Each polynomial in u^2 and v is a polynomial in u' and v'.

(ii) The half region R given by $R \cap \{(u, v) | u > 0\}$ is mapped onto

$$\tilde{R} = \{(u', v') | (1+u'+v') > 0 \wedge (1-u'+v') > 0 \wedge ((u')^2 - 4v') > 0\}.$$

(iii) If $(u, v) = (2, 1)$, then $(u', v') = (2, 1)$.

(The transformation of variables $u' = -2v$ and $v' = u^2 - 2v - 1$ also satisfies (i) and (ii)).

From (10.3) we obtain

$$\begin{aligned} &u = \sqrt{1+u'+v'}, \quad v = \tfrac{1}{2}u', \\ (10.4) \qquad &(1+u+v)(1-u+v) = \tfrac{1}{4}((u')^2 - 4v'), \qquad u^2 - 4v = 1 - u' + v', \\ &du\, dv = \tfrac{1}{4}(1+u'+v')^{-1/2}\, du'\, dv'. \end{aligned}$$

If $\alpha = \beta$, the following quadratic transformation formulas hold.

THEOREM 10.1.

$$(10.5) \qquad p^{\alpha,\alpha,\gamma}_{n+k,n-k}(u, v) = 2^{-n+k} p^{\gamma,-1/2,\alpha}_{n,k}(u', v'),$$

and

$$(10.6) \qquad u^{-1} p^{\alpha,\alpha,\gamma}_{n+k+1,n-k}(u, v) = 2^{-n+k} p^{\gamma,+1/2,\alpha}_{n,k}(u', v'),$$

with u' and v' given by (10.3).

Proof.

$$p^{\alpha,\alpha,\gamma}_{n+k,n-k}(u, v) = \sum_{(i,j)<(n+k,n-k)} a_{ij} u^{i-j} v^j.$$

If $(i-j)$ is odd, then $a_{i,j} = 0$, so we can substitute $i - j = 2l$ and $i + j = 2m$. By (8.2), $(i, j) < (n+k, n-k)$ iff $(m, l) < (n, k)$. Hence

$$\begin{aligned} p^{\alpha,\alpha,\gamma}_{n+k,n-k}(u, v) &= \sum_{(m,l)<(n,k)} a'_{m,l}(u^2)^l v^{m-l} \\ &= \sum_{(m,l)<(n,k)} a''_{m,l}(u^2 - 2v - 1)^l (2v)^{m-l} \\ &= \sum_{(m,l)<(n,k)} a''_{m,l}(u')^{m-l}(v')^l, \quad \text{with } a''_{n,k} = 2^{-n+k}. \end{aligned}$$

With respect to the orthogonality the following holds:

$$\begin{aligned} 0 &= \iint_R p^{\alpha,\alpha,\gamma}_{n+k,n-k}(u, v)(2v)^{m-l}(u^2 - 2v - 1)^l \mu^{\alpha,\alpha,\gamma}(u, v)\, du\, dv \\ &= \text{const.} \iint_R p^{\alpha,\alpha,\gamma}_{n+k,n-k}(u, v)(u')^{m-1}(v')^l \mu^{\gamma,-1/2,\alpha}(u', v')\, du'\, dv', \end{aligned}$$

$$\text{if } (m, l) \underset{\neq}{<} (n, k).$$

Application of Theorem 8.2 proves (10.5). A similar proof can be given for (10.6). □

If $(u, v)=(2, 1)$, then $(u', v')=(2, 1)$; hence (10.5) and (10.6) can also be written as

$$\frac{p_{n+k,n-k}^{\alpha,\alpha,\gamma}(u, v)}{p_{n+k,n-k}^{\alpha,\alpha,\gamma}(2, 1)}=\frac{p_{n,k}^{\gamma,-1/2,\alpha}(2v, u^2-2v-1)}{p_{n,k}^{\gamma,-1/2,\alpha}(2, 1)}, \tag{10.5$'$}$$

and

$$\frac{2u^{-1}p_{n+k+1,n-k}^{\alpha,\alpha,\gamma}(u, v)}{p_{n+k+1,n-k}^{\alpha,\alpha,\gamma}(2, 1)}=\frac{p_{n,k}^{\gamma,+1/2,\alpha}(2v, u^2-2v-1)}{p_{n,k}^{\gamma,+1/2,\alpha}(2, 1)}. \tag{10.6$'$}$$

Formulas (10.5) and (10.6) in combination with (2.15) and (3.2) give an explicit expression for the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ if α and β are $+\frac{1}{2}$ or $-\frac{1}{2}$:

$$\begin{aligned} &p_{n,k}^{-1/2,-1/2,\gamma}(2xy, x^2+y^2-1)\\ &=\begin{cases} 2^{n-k}\{p_{n+k}^{\gamma,\gamma}(x)p_{n-k}^{\gamma,\gamma}(y)+p_{n-k}^{\gamma,\gamma}(x)p_{n+k}^{\gamma,\gamma}(y)\} & \text{if } k>0,\\ 2^n p_n^{\gamma,\gamma}(x)p_n^{\gamma,\gamma}(y) & \text{if } k=0,\end{cases}\end{aligned} \tag{10.7}$$

$$\begin{aligned} &p_{n,k}^{+1/2,-1/2,\gamma}(2xy, x^2+y^2-1)\\ &=2^{n-k}(x-y)^{-1}\{p_{n+k+1}^{\gamma,\gamma}(x)p_{n-k}^{\gamma,\gamma}(y)-p_{n-k}^{\gamma,\gamma}(x)p_{n+k+1}^{\gamma,\gamma}(y)\},\end{aligned} \tag{10.8}$$

$$\begin{aligned} &p_{n,k}^{-1/2,+1/2,\gamma}(2xy, x^2+y^2-1)\\ &=2^{n-k}(x+y)^{-1}\{p_{n+k+1}^{\gamma,\gamma}(x)p_{n-k}^{\gamma,\gamma}(y)+p_{n-k}^{\gamma,\gamma}(x)p_{n+k+1}^{\gamma,\gamma}(y)\},\end{aligned} \tag{10.9}$$

$$\begin{aligned} &p_{n,k}^{+1/2,+1/2,\gamma}(2xy, x^2+y^2-1)\\ &=2^{n-k}(x^2-y^2)^{-1}\{p_{n+k+2}^{\gamma,\gamma}(x)p_{n-k}^{\gamma,\gamma}(y)-p_{n-k}^{\gamma,\gamma}(x)p_{n+k+2}^{\gamma,\gamma}(y)\}.\end{aligned} \tag{10.10}$$

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GROWTH AND UNIQUENESS THEOREMS FOR AN ABSTRACT NONSTANDARD WAVE EQUATION*

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Abstract. A uniqueness theorem is proved for a solution to an abstract, nonstandard wave equation (see (1.1)). Growth estimates for a weighted norm of the solution and its kinetic energy are also given. It is then shown that solutions to the linearized equations of Cauchy elasticity, which are a special case of the abstract equation considered, are unstable for certain types of elasticities.

1. Introduction. It is our intention to study the asymptotic behavior of solutions to the equation

$$Au''(t)+B(t)u(t)=0, \qquad t\in(0,\infty), \tag{1.1}$$

for specified initial data $u(0)=u_0$, $u'(0)=v_0$. Here A and $B(t)$ are linear operators defined on a dense subdomain D, of a real Hilbert space H. The parameter t is conveniently regarded as time, and throughout the note a superscript prime denotes the (strong) derivative with respect to t. The norm and inner product on H are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$.

We assume that A is a symmetric, positive definite operator, while $B(t)$ satisfies

either

$$\langle x, B(t)y\rangle=-\langle B(t)x, y\rangle \quad \forall\, x, y\in D, \tag{1.2}$$

$$\langle x, B(t)x\rangle\leqq 0 \qquad \forall\, x\in D, \tag{1.3}$$

or

$$\langle x, B(t)x\rangle\leqq-\lambda^2\langle x, Ax\rangle \quad \forall x\in D, \tag{1.4}$$

for some real number $\lambda\neq 0$. Observe that in (1.3) and (1.4), no symmetry is imposed on B.

Since B satisfies (1.2), (1.3) or (1.4), we see that the classical wave equation does not fall into the category of equations we propose for study. Hence, we refer to the wave equation under consideration as nonstandard.

In § 2, we state and prove uniqueness and growth theorems for (1.1). In § 3, we consider a particular example of (1.1), namely, the linear dynamical theory of Cauchy elasticity. Using the results of § 2, we show that when the elasticities are governed by conditions corresponding to (1.2), (1.3) or (1.4), then the zero solution to the equations of Cauchy elasticity is unstable.

2. Growth and uniqueness theorems. Let $D'(\supseteq D)$ be a dense subdomain of H. Let $\mathscr{A}(\cdot,\cdot)$ and $\mathscr{B}_t(\cdot,\cdot)$ be the bilinear forms on D' associated with the

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operators A and $B(t)$, respectively; i.e., if $x, y \in D$, then

$$\mathscr{A}(x, y) = \langle Ax, y\rangle = \mathscr{A}(y, x),$$
$$\mathscr{B}_t(x, y) = \langle x, B(t)y\rangle.$$

$\mathscr{A}$ is a positive, symmetric bilinear form on D', and according to (1.2)–(1.4), $\mathscr{B}_t$ satisfies

either

$$\mathscr{B}_t(x, y) = -\mathscr{B}_t(y, x) \qquad \forall\, x, y \in D', \tag{2.1}$$

$$\mathscr{B}_t(x, x) \leqq 0 \qquad \forall\, x \in D', \tag{2.2}$$

or

$$\mathscr{B}_t(x, x) \leqq -\lambda^2 \mathscr{A}(x, x) \quad \forall\, x \in D'. \tag{2.3}$$

A solution u to (1.1), is a *classical* solution if $u \in C^2([0, \infty); D)$, u satisfies (1.1) identically for each t in $(0, \infty)$ and u satisfies the initial conditions $u(0) = u_0$, $u'(0) = v_0$.

u is defined to be a *weak* solution to (1:1) if $u \in C^1([0, \infty); D')$ and for each $\phi \in C^1([0, \infty); D')$ the following identity holds:

$$\mathscr{A}(\phi(t), u'(t)) = \mathscr{A}(\phi(0), v_0) + \int_0^t [\mathscr{A}(\phi', u') - \mathscr{B}_\eta(\phi, u)]\, d\eta. \tag{2.4}$$

When A is as stated in the text, $B(t)$ satisfies (1.2), (1.3) or (1.4) (or when $\mathscr{A}$ and $\mathscr{B}_t$ are as above), the initial value problem defined by (1.1) together with the Cauchy data u_0, v_0, is denoted by $\mathscr{P}$.

The first initial data we consider is the choice such that $u_0 = v_0 = 0$. The proof of the following theorem is motivated by the proof of a similar result in [8].

THEOREM 2.1. *Let u be a weak solution to* (1.1), *with $u_0 = v_0 = 0$. If either*

$\mathscr{B}_t$ satisfies (2.1) *or* (2.2) *for each t in* $(0, \infty)$ *and*

$$\liminf_{T\to\infty} \frac{\mathscr{A}(u(T), u(T))}{T^2} = 0,$$

or

$\mathscr{B}_t$ satisfies (2.3) *for each t in* $(0, \infty)$ *and*

$$\liminf_{T\to\infty} \frac{\mathscr{A}(u(T), u(T))}{e^{2\lambda T}} = 0,$$

then $u \equiv 0$ on $[0, \infty)$.

Proof. Define $F(t)$ by

$$F(t) = \mathscr{A}(u(t), u(t)), \qquad t \in [0, \infty).$$

We may take $\phi \equiv u$ in (2.4), and so obtain

$$F'(t) = F'(0) + 2\int_0^t [\mathscr{A}(u', u') - \mathscr{B}_\eta(u, u)]\, d\eta.$$

Hence

$$F''(t)=2\mathcal{A}(u'(t),\, u'(t))-2\mathcal{B}_t(u,\, u). \tag{2.5}$$

Using this relation we may obtain the identity,

$$FF''-\frac{(F')^2}{2}=2S^2-2F\mathcal{B}_t(u,\, u), \tag{2.6}$$

where S^2, which is nonnegative by the Cauchy–Schwarz inequality, is given by

$$S^2=\mathcal{A}(u,\, u)\mathcal{A}(u',\, u')-[\mathcal{A}(u,\, u')]^2. \tag{2.7}$$

Suppose $\mathcal{B}_t$ satisfies (2.1) or (2.2). Then (2.6) leads to

$$2FF''-(F')^2\geqq 0, \qquad t\in(0,\infty). \tag{2.8}$$

In particular, F is a convex function of t. Furthermore, by hypothesis, $F(0)=F'(0)=0$. Since F' is nondecreasing, either $F'\equiv 0$, from which it follows that $u\equiv 0$, or $F'(\alpha)>0$ for some $\alpha>0$. Suppose the second alternative holds. Then $F>0$ for all $t>\alpha$. We fix $\beta>\alpha$ and from (2.8) deduce that

$$(F^{1/2}(t))''\geqq 0, \qquad \beta\leqq t<\infty. \tag{2.9}$$

Consequently,

$$F(t)\geqq\frac{1}{4}\frac{[F'(\beta)]^2}{F(\beta)}(t-\beta)^2,$$

so that

$$\liminf_{t\to\infty}\frac{F(t)}{t^2}\geqq\frac{1}{4}\frac{[F'(\beta)]^2}{F(\beta)}>0, \tag{2.10}$$

which contradicts the hypotheses of the theorem. We appeal to the positive-definiteness of $\mathcal{A}$ to conclude that $u\equiv 0$ on $[0,\infty)$.

To prove the second part, suppose $\mathcal{B}_t$ satisfies (2.3). Equation (2.6) now leads to

$$FF''-\frac{(F')^2}{2}\geqq 2\lambda^2F^2, \qquad t\in(0,\infty). \tag{2.11}$$

Arguing as for (2.9) we now establish

$$(F^{1/2})''\geqq\lambda^2F^{1/2}, \qquad \beta\leqq t<\infty. \tag{2.12}$$

This inequality may now be integrated (see, e.g., [5]) to give

$$F^{1/2}(t)\geqq\frac{1}{2\lambda}\sinh\{\lambda(t-\beta)\}\{F'(\beta)/F^{1/2}(\beta)\}+\cosh\{\lambda(t-\beta)\}F^{1/2}(\beta). \tag{2.13}$$

Under the hypotheses of the theorem, $\liminf_{t\to\infty}F(t)e^{-2\lambda t}=0$ which is clearly incompatible with (2.13) unless $u\equiv 0$ on $[0,\infty)$. The theorem is thus proved.

Based upon the previous computation, we may obtain in an obvious way the following theorem.

THEOREM 2.2. *Let u be a weak solution to* (1.1).

(i) *Let* $\mathcal{B}_t$ *satisfy* (2.1) *or* (2.2) *for each* $t \in (0, \infty)$. *If* $F'(0) \geqq 0$, *then*

$$F(t) \geqq F(0)\left(1+\frac{F'(0)t}{2F(0)}\right)^2.$$

(ii) *Let* $\mathcal{B}_t$ *satisfy* (2.3) *for each* $t \in (0, \infty)$. *If* $F'(0) > -2\lambda F(0)$ *and* $u_0 \neq 0$, *then* $F(t)$ *is bounded below by an increasing exponential function of t.*

The proof of (ii) proceeds along similar lines to the proof of an analogous result of Knops and Payne in linear elasticity (see [5, p. 1240]).

Theorem 2.1 shows that any weak solution to $\mathcal{P}$ is unique provided a suitable growth condition is imposed at infinity. Theorem 2.2 shows that the zero solution to $\mathcal{P}$ is unstable, for arbitrary initial perturbations.

The bound in Theorem 2.2 is quadratic when $\mathcal{B}_t$ satisfies (2.1) or (2.2). It is possible to obtain better growth estimates, and this is done in the following theorem. However, we find it necessary to impose stronger regularity assumptions than those of Theorem 2.2, when B satisfies (1.2) or (1.3).

Motivated by physical situations, we define the kinetic energy $K(t)$ of a classical solution to (1.1), to be

$$K(t) = \tfrac{1}{2}\langle u'(t), Au'(t)\rangle. \tag{2.14}$$

THEOREM 2.3. *Let u be a classical solution to* (1.1) *with B independent of t, such that* $Bu \in C^1([0, \infty); H)$.

(i) *Suppose B satisfies* (1.2), (1.3) *or* (1.4). *If* $\langle v_0, Bu_0\rangle < 0$, *then* $K(t)$ *and* $F(t) = \langle u(t), Au(t)\rangle$ *are bounded below on* $[0, \infty)$ *by increasing polynomial functions of time, of order three and five, respectively.*

(ii) *Suppose B satisfies* (1.2). *If* $\langle v_0, Bu_0\rangle < 0$, *then* $K(t)$ *and* $F(t)$ *are bounded below on* $[0, \infty)$ *by an increasing exponential function of time.*

Proof. The proof follows from a modification of a technique due to Murray and Protter [11] (see also Murray [10]).

Introduce the function v by

$$v(t) = f^{-1}(t)u(t), \tag{2.15}$$

for a strictly positive twice continuously differentiable, real-valued function, f, to be specified. Then by direct substitution, we see that

$$A(f''v + 2f'v' + fv'') + fBv = 0. \tag{2.16}$$

We now form the inner product with $f^{-2}(f''v + fv'')$ and discard the resulting nonnegative term, to obtain

$$2\frac{f'}{f^2}\langle f''v + fv'', Av'\rangle + \left\langle \frac{f''}{f}v + v'', Bv\right\rangle \leqq 0. \tag{2.17}$$

The first choice of $f(t)$ is

$$f(t) = t + a, \tag{2.18}$$

for an arbitrary positive number a. Relation (2.17) then becomes

$$2\frac{\langle v'', Av'\rangle}{(t+a)} + \langle v'', Bv\rangle \leqq 0.$$

Noting that under the hypotheses of part (i), $\langle v', Bv'\rangle \leqq 0$, this inequality can be integrated to give

$$\frac{\langle v', Av'\rangle}{(t+a)} + \langle v', Bv\rangle \leqq \langle v'(0), Av'(0)\rangle a^{-1} + \langle v'(0), Bv(0)\rangle. \tag{2.19}$$

The first term on the left of (2.19) is now discarded together with the term $-\langle u, Bu\rangle/(t+a)^3$ which occurs when (2.19) is rewritten in terms of u. The result is

$$-\langle u', Bu\rangle \geqq H_0(t+a)^2, \tag{2.20}$$

where H_0 is given by

$$a^2H_0 = -\langle v_0, Bu_0\rangle + \left[\frac{\langle u_0, Bu_0\rangle - \langle v_0, Av_0\rangle}{a}\right] + 2\frac{\langle v_0, Au_0\rangle}{a^2} - \frac{\langle u_0, Au_0\rangle}{a^3}. \tag{2.21}$$

The term Bu is now substituted from (1.1) in (2.20), to yield

$$K'(t) \geqq H_0(t+a)^2,$$

and so

$$K(t) \geqq K(0) + H_0(t^3/3 + at^2 + a^2t). \tag{2.22}$$

If $\langle v_0, Bu_0\rangle < 0$, then we choose a so large that $H_0 > 0$ and the first part of (i) follows.

To prove the second part, we observe that from (1.1),

$$\langle u, Au''\rangle = -\langle u, Bu\rangle. \tag{2.23}$$

Hence

$$F''(t) = 4K(t) - 2\langle u, Bu\rangle. \tag{2.24}$$

Combining either (1.2), (1.3), or (1.4) with (2.22), in (2.24), we can obtain

$$F(t) \geqq F(0) + F'(0)t + 2K(0)t^2 + \frac{2}{3}H_0\left(\frac{t^5}{10} + \frac{at^4}{2} + a^2t^3\right). \tag{2.25}$$

Again, if $\langle v_0, Bu_0\rangle < 0$, we can select a such that $H_0 > 0$, and (i) follows.

To prove (ii), suppose B satisfies (1.2) and define $f(t)$ by

$$f(t) = e^{\gamma t} \tag{2.26}$$

for a positive number γ, to be specified. From (2.17), we have

$$2\gamma\langle v'' + \gamma^2 v, Av'\rangle + \langle v'', Bv\rangle \leqq 0.$$

Integrating this inequality and using the fact that B satisfies (1.2), we obtain

$$\gamma\langle u'-\gamma u, A(u'-\gamma u)\rangle+\gamma^3\langle u, Au\rangle+\langle u', Bu\rangle \leqq -e^{+2\gamma t}I_0, \tag{2.27}$$

where

$$I_0=\gamma^2F'(0)-2\gamma K(0)-2\gamma^3F(0)-\langle v_0, Bu_0\rangle. \tag{2.28}$$

We next use the inequality

$$2\gamma^2\langle u', Au\rangle \leqq 2\gamma^3\langle u, Au\rangle+\frac{\gamma}{2}\langle u', Au'\rangle$$

in (2.27) and substitute for Bu from (1.1), to find

$$\langle u', Au''\rangle \geqq e^{2\gamma t}I_0+(\gamma/2)\langle u', Au'\rangle.$$

Hence, a further integration yields

$$K(t)\geqq e^{\gamma t}K(0)+I_0\frac{e^{\gamma t}}{\gamma}(e^{\gamma t}-1). \tag{2.29}$$

If $\langle v_0, Bu_0\rangle<0$, we may choose γ such that $I_0>0$, and the first part of (ii) follows from (2.29).

A growth estimate for $F(t)$ is immediately obtained from (2.24) and (2.29). The result is

$$\begin{aligned} F(t)\geqq{}& \frac{I_0}{4\gamma^3}e^{2\gamma t}+\left(K(0)-\frac{I_0}{\gamma}\right)\gamma^{-2}\,e^{\gamma t}\\ &+\left(F'(0)-\frac{K(0)}{\gamma}+\frac{I_0}{2\gamma^2}\right)t+F(0)-\frac{K(0)}{\gamma^2}+\frac{3I_0}{4\gamma^3}. \end{aligned} \tag{2.30}$$

Again, if $\langle v_0, Bu_0\rangle<0$, we can select γ to ensure $I_0>0$ and the proof of the theorem is complete.

Theorems 2.2 and 2.3 obtain growth estimates for a measure of a solution to $\mathcal{P}$ when $u_0\neq 0$. We now consider the case $u_0=0$, $v_0\neq 0$.

THEOREM 2.4. *Let u be a weak solution to* (1.1) *and suppose $\mathcal{B}_t$ satisfies* (2.1), (2.2) *or* (2.3). *Suppose also that A is a self-adjoint linear operator bounded below by a positive multiple of the identity and $D'\subseteq D(A^{1/2})$ ($D(A^{1/2})$ = domain of $A^{1/2}$). If $u_0=0$, $v_0\neq 0$, then*

$$F(t)=\langle A^{1/2}u(t), A^{1/2}u(t)\rangle\to+\infty,$$

as $t\to\infty$.

Proof. Employing (2.4), we see that

$$F'(t)=2\int_0^t[\langle A^{1/2}u', A^{1/2}u'\rangle-\mathcal{B}_\eta(u, u)]\,d\eta. \tag{2.31}$$

Since A is self-adjoint and $A\geqq\mu I>0$ for some $\mu>0$, we may define a new norm, $\|\cdot\|_A$, on $D(A^{1/2})$ by

$$\|x\|_A=\|A^{1/2}x\| \quad \forall x\in D(A^{1/2}),$$

where $A^{1/2}$ is the square root of A. Then, since $u(t)=\int_0^t u'(\eta)\,d\eta$, (in an inner product sense),

$$\|u(t)\|_A \leqq \int_0^t \|u'(\eta)\|_A\,d\eta \leqq t^{1/2}\left(\int_0^t \|u'(\eta)\|_A^2\,d\eta\right)^{1/2}.$$

Hence

$$\int_0^t \|u'\|_A^2\,d\eta \geqq t^{-1}\|u\|_A^2, \tag{2.32}$$

and so using (2.32) together with (2.1), (2.2) or (2.3) in (2.31) we see that

$$F'(t) \geqq (2/t)F(t), \qquad t>0, \tag{2.33}$$

which leads to

$$\frac{d}{dt}\left[\frac{F}{t^2}\right] \geqq 0, \qquad t>0. \tag{2.34}$$

Integrating (2.34),

$$F(t) \geqq t^2 \lim_{\varepsilon\to 0}\,[F(\varepsilon)/\varepsilon^2].$$

Noting that $F(0)=F'(0)=0$, we use L'Hôpital's rule to deduce that

$$F(t) \geqq 2t^2K(0).$$

Since $v_0 \neq 0$, it follows that $F(t)$ is bounded below by an increasing quadratic function of time.

An interesting example of (1.1) with B satisfying (1.2), (1.3) or (1.4) occurs in the linear dynamical theory of Cauchy elasticity. In the next section we present this example explicitly.

3. Cauchy elasticity. Let Ω be a bounded domain of $\mathbb{R}^3$, with boundary $\partial\Omega$ smooth enough to admit applications of the divergence theorem. The standard mixed initial boundary value problem for a linear Cauchy elastic material is then

$$\begin{aligned} &\rho(x)\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x_j}\left(a_{ijkh}\frac{\partial u_k}{\partial x_h}\right), \quad \text{in } \Omega\times(0,T),\\ &u_i(x,0)=f_i(x),\\ &\frac{\partial u_i}{\partial t}(x,0)=g_i(x),\\ &u_i(x,t)=h_i(x,t), \qquad x\in\partial\Omega^*, \qquad t\in[0,T),\\ &n_j a_{ijkh}(x)\frac{\partial u_k}{\partial x_h}(x,t)=l_i(x,t), \qquad x\in\partial\Omega_c^*, \qquad t\in[0,T), \end{aligned} \tag{3.1}$$

where T is a positive real number, u_i are the Cartesian components of displacement, ρ is the density and a_{ijkh} are the elasticities of the material, $\partial\Omega^*$ and $\partial\Omega_c^*$ are an arbitrary subset of $\partial\Omega$ and its complement, and the standard summation convention is employed.

We are interested in the uniqueness and growth questions to (3.1), and therefore we restrict attention to zero boundary data. It is easily seen that (3.1) is a special case of (1.1), with the particular choices

$$H=[L^2(\Omega)]^3,$$

$$D=\left\{u \,|\, u\in[C^2(\Omega)\cap C^1(\bar{\Omega})]^3,\ u_i=0 \text{ on } \partial\Omega^*,\ n_j a_{ijkh}\frac{\partial u_k}{\partial x_h}=0 \text{ on } \partial\Omega_c^*\right\},$$

and

$$D'=\left\{u \,|\, u\in[C^1(\bar{\Omega})]^3,\ u_i=0 \text{ on } \partial\Omega^*,\ n_j a_{ijkh}\frac{\partial u_k}{\partial x_h}=0 \text{ on } \partial\Omega_c^*\right\}.$$

Conditions (1.2), (1.3) and (1.4) correspond to the elasticities satisfying

either

$$a_{ijkh}=-a_{khij}, \tag{3.2}$$

$$\int_\Omega a_{ijkh}\xi_{ij}\xi_{kh}\,dx \leqq 0 \quad \forall\ \xi_{ij}, \tag{3.3}$$

or

$$\int_\Omega a_{ijkh}\xi_{ij}\xi_{kh}\,dx \leqq -\mu^2\int_\Omega \rho\xi_{ij}\xi_{ij}\,dx, \tag{3.4}$$

for all ξ_{ij}, and some nonzero, real constant μ.

Theorems 2.1–2.4 now give us the following corollaries.

COROLLARY 3.1. *Let u be a weak solution to* (3.1) (*as defined by* (2.4)) *with* $f_i=g_i=h_i=l_i=0$. *If*

either

the elasticities satisfy (3.2) *and* (3.3) *and*

$$\liminf_{t\to\infty}\left[\int_\Omega \rho u_i(x,t)u_i(x,t)\,dx/t^2\right]=0, \tag{3.5}$$

or

the elasticities satisfy (3.4) *and*

$$\liminf_{t\to\infty}\left[\int_\Omega \rho u_i(x,t)u_i(x,t)\,dx/e^{2\mu t}\right]=0, \tag{3.6}$$

then $u\equiv 0$ *on* $\Omega\times[0,\infty)$.

A uniqueness theorem for a classical solution to (3.1), under (3.2), was given earlier by Murray [10]. She required the solution and certain of its derivatives to be bounded, although she only required the bound to hold on compact subsets of the real line, whereas we need a growth condition at infinity, of the form (3.5) or (3.6). Another uniqueness theorem under a similar condition to (3.3) was given by Hayes and Knops [3], the condition they required being

$$a_{ijkh}a_ia_jb_kb_h<0 \quad \forall\ a_i,\ b_i\neq 0.$$

COROLLARY 3.2. *Let u be a weak solution to* (3.1) *with* $h_i = l_i = 0$.
(i) *If either*

the elasticities satisfy (3.2) *or* (3.3) *and*

$$F'(0) = 2\int_\Omega \rho f_i g_i \, dx > 0,$$

or

the elasticities satisfy (3.2), (3.3) *or* (3.4) *with* $f_i \equiv 0$ *and* $g_i \not\equiv 0$, *then*

$$F(t) = \int_\Omega \rho u_i(x, t) u_i(x, t) \, dx$$

is bounded below by an increasing quadratic function of time.

(ii) *If the elasticities satisfy* (3.4), $f_i \not\equiv 0$ *and* $F'(0) > -2\mu F(0)$, *then* $F(t)$ *is bounded below by an increasing exponential function of time, for large enough time.*

Let u be a classical solution to (3.1) *with* $h_i = l_i = 0$. *Suppose* $(\partial^3 u_i)/(\partial t\, \partial x_j\, \partial x_k)$ *is continuous on* $\Omega \times [0, T)$ *and*

$$\langle v_0, Bu_0 \rangle = \int_\Omega a_{ijkh} g_{i,j} f_{k,h} dx < 0.$$

Then

(i) *if the elasticities satisfy* (3.3) *or* (3.4), $F(t)$ *and*

$$K(t) = \frac{1}{2}\int_\Omega \rho \frac{\partial u_i}{\partial t}(x, t) \frac{\partial u_i}{\partial t}(x, t) \, dx$$

are bounded below by increasing polynomial functions of time, of order three and five, respectively;

(ii) *if the elasticities satisfy* (3.2), *both* $F(t)$ *and* $K(t)$ *are bounded below by an increasing exponential function of time.*

4. Conclusions. The results given here may be compared with those for (1.1) when B is symmetric (see, e.g., Knops and Payne [6], Levine [9]). The case when B is symmetric corresponds to the theory of Green elasticity when it is well known that extra restrictions can be placed on the linearized strain energy in order that the material be Lyapunov stable in the measure $\int_\Omega \rho u_i u_i \, dx$ for *any* initial data. However, in the two cases we have considered, of Cauchy elasticity, i.e., when a_{ijkh} satisfy (3.2), (3.3) or (3.4), we can see no way of making the material stable for arbitrary initial data by imposing additional restrictions on the a_{ijkh}.

It is of mathematical interest to note that Corollaries 3.1 and 3.2 still hold if we replace condition (3.2) by the weaker, minor symmetry condition

$$a_{ijkh} = -a_{kjih}, \tag{4.1}$$

provided the material is homogeneous. If the material is inhomogeneous, we need, in addition to (4.1), the extra symmetries

$$\frac{\partial}{\partial x_j}(a_{ijkh}) = \frac{\partial}{\partial x_j}(a_{ihkj}). \tag{4.2}$$

These results are analogous to the uniqueness theorem of Horgan [4], in Green elasticity.

Some of the results in this note can be proved for (3.1) with the boundary conditions replaced by "ambiguous" ones.

Such problems have attracted much recent attention, e.g., Knops and Payne [7], Duvaut and Lions [1] and Fichera [2]. However, the details of this work will be given in a future article.

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ON HYPERGEOMETRIC SERIES WELL-POISED IN $SU(n)$*

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Abstract. By exploiting the known relationship between well-poised hypergeometric series and the combinatorial aspects of the representation theory of $SU(2)$, we define a generalization of the well-poised concept for multidimensional series adapted to $SU(n)$ symmetry. The analogue to Whipple's theorem is demonstrated for $SU(3)$, and the analogue to the well-poised ${}_4F_3(-1)$ theorem is given for all $SU(n)$.

1. Introduction. Hypergeometric series well-poised in $SU(2)$. Well-poised hypergeometric series, defined by the expression

$$(1.1)\qquad {}_pF_{p-1}\left(\begin{matrix} a & b_1 & b_2 & \cdots & b_{p-1} \\ & 1+a-b_1 & 1+a-b_2 & \cdots & 1+a-b_{p-1}\end{matrix}\,\middle|\, \pm 1\right),$$

have been the most extensively studied of all forms of hypergeometric series and have provided the richest store of summation theorems. The earliest example of a summation theorem for a well-poised series of the type (1.1), in fact, was found by Kummer [9] in 1836:

$$(1.2)\qquad {}_2F_1\left(\begin{matrix} a & b \\ & 1+a-b\end{matrix}\,\middle|\, -1\right) = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)} \quad \text{for } \operatorname{Re} b<1.$$

Subsequently, Dixon [6] found the sum of well-poised ${}_3F_2$ in the form

$$(1.3)\qquad \begin{aligned} &{}_3F_2\left(\begin{matrix} a & b & c \\ & 1+a-b & 1+a-c\end{matrix}\,\middle|\, 1\right) \\ &\quad = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+\frac{1}{2}a-b-c)\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)}\end{aligned}$$

for $\operatorname{Re}(a-2b-2c)>-2$, from which (1.2) may be obtained as the asymptotic limit as $c\to-\infty$. Dougall [7] then found the theorem that

$$(1.4)\qquad \begin{aligned} &{}_7F_6\left(\begin{matrix} a & 1+\frac{1}{2}a & b & c & d & e & f \\ & \frac{1}{2}a & 1+a-b & 1+a-c & 1+a-d & 1+a-e & 1+a-f\end{matrix}\,\middle|\, 1\right) \\ &= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-f)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)\Gamma(1+a-b-f)} \\ &\quad\cdot \frac{\Gamma(1+a-b-c-f)\Gamma(1+a-b-d-f)\Gamma(1+a-c-d-f)}{\Gamma(1+a-c-f)\Gamma(1+a-d-f)\Gamma(1+a-b-c-d-f)},\end{aligned}$$

provided that the series terminates and that

$$(1.5)\qquad 1+2a=b+c+d+e+f.$$

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Whipple [13], [14] was the first to introduce the term "well-poised" for series of the form (1.1) and to study them systematically. He obtained a considerable number of theorems concerning such series, among them what is now known as Whipple's theorem which relates well-poised $_7F_6$ and Saalschützian $_4F_3$, which we give in terminating form:

$$(1.6)\quad \begin{aligned} &{}_7F_6\left(\begin{matrix} f-1 & \frac{1}{2}f+\frac{1}{2} & a_1 & a_2 & d_1 & d_2 & -N \\ & \frac{1}{2}f-\frac{1}{2} & f-a_1 & f-a_2 & f-d_1 & f-d_2 & f+N \end{matrix}\middle| 1\right) \\ &= \frac{\Gamma(g+N)\Gamma(g-d_1-d_2+N)\Gamma(g-d_1)\Gamma(g-d_2)}{\Gamma(g)\Gamma(g-d_1-d_2)\Gamma(g-d_1+N)\Gamma(g-d_2+N)} \\ &\quad \cdot {}_4F_3\left(\begin{matrix} f-a_1-a_2 & d_1 & d_2 & -N \\ & f-a_1 & f-a_2 & g \end{matrix}\middle| 1\right), \end{aligned}$$

where

$$(1.7)\quad g = 1+d_1+d_2-f-N,$$

and N is a positive integer. Dougall's theorem (1.4) is merely a special case of (1.6) which is realized when the Saalschützian $_4F_3$ series on the right of (1.6) is restricted to be a Saalschützian $_3F_2$ series and hence can be summed to a monomial. Whipple also found the relation

$$(1.8)\quad \begin{aligned} &{}_4F_3\left(\begin{matrix} a & 1+\frac{1}{2}a & b & c \\ & \frac{1}{2}a & 1+a-b & 1+a-c \end{matrix}\middle| 1\right) \\ &= \frac{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(\frac{1}{2}+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(\frac{1}{2}+\frac{1}{2}a-b)\Gamma(\frac{1}{2}+\frac{1}{2}a-c)\Gamma(1+a-b-c)} \end{aligned}$$

as well as the summation theorems which can be found as the various asymptotic limits of (1.6). Thus in the limit $d_2 \to \infty$, we obtain

$$(1.9)\quad \begin{aligned} &{}_6F_5\left(\begin{matrix} f-1 & \frac{1}{2}f+\frac{1}{2} & a_1 & a_2 & d_1 & +N \\ & \frac{1}{2}f-\frac{1}{2} & f-a_1 & f-a_2 & f-d_1 & f+N \end{matrix}\middle| -1\right) \\ &= \frac{\Gamma(1-f)\Gamma(1+d_1-f-N)}{\Gamma(1-f-N)\Gamma(1+d_1-f)} {}_3F_2\left(\begin{matrix} f-a_1-a_2 & d_1 & -N \\ & f-a_1 & f-a_2 \end{matrix}\middle| 1\right). \end{aligned}$$

We then take $a_1 \to \infty$ to get

$$(1.10)\quad \begin{aligned} &{}_5F_4\left(\begin{matrix} f-1 & \frac{1}{2}f+\frac{1}{2} & d_1 & a_2 & -N \\ & \frac{1}{2}f-\frac{1}{2} & f-d_1 & f-a_2 & f+N \end{matrix}\middle| 1\right) \\ &= \frac{\Gamma(1-f)}{\Gamma(1-f-N)} \frac{\Gamma(1+d_1-f-N)}{\Gamma(1+d_1-f)} {}_2F_1\left(\begin{matrix} d_1 & -N \\ & f-a_2 \end{matrix}\middle| 1\right) \\ &= \frac{\Gamma(1-f)}{\Gamma(1-f-N)} \frac{\Gamma(1+d_1-f-N)}{\Gamma(1+d_1-f)} \frac{\Gamma(f-a_2)\Gamma(f-a_2-d_1+N)}{\Gamma(f-a_2-d_1)\Gamma(f-a_2+N)}, \end{aligned}$$

which holds also in the nonterminating case, when N is not a positive integer. Then, in the limit $d_1 \to \infty$,

$$(1.11)\qquad {}_4F_3\begin{pmatrix} f-1 & \frac{1}{2}f+\frac{1}{2} & a_2 & -N \\ & \frac{1}{2}f-\frac{1}{2} & f-a_2 & f+N \end{pmatrix}\bigg| -1\bigg) = \frac{\Gamma(f+N)\Gamma(f-a_2)}{\Gamma(f)\Gamma(f-a_2+N)}.$$

In the limit $a_2 \to \infty$ we obtain a special case of (1.3); in the further limit $-N \to \infty$ we find a special case of (1.2).

Bailey subsequently introduced a transform [1] for sums over two indices by means of which further results on well-poised series have been obtained. These results are summarized by L. J. Slater [11]. Bailey's fundamental idea was a transformation of a double sum from rectilinear summation in the two-dimensional lattice of the indices to a summation carried out first over diagonal lines in the lattice, then over an index labeling the diagonals. This idea, of course, can be generalized to the case of sums over an n-dimensional lattice, and it is hoped that such multidimensional analogues of Bailey's transform will eventually contribute to our understanding of the multidimensional well-poised series which we shall define in the present paper.

The study of well-poised hypergeometric series, then, was well established before Wigner [16] and Racah [10] initiated the combinatorial theory of the irreducible representations of the compact group $SU(2)$, which has an intimate connection with the well-poised hypergeometric series and which we shall now briefly sketch.

The group $SU(2)$ has the generators

$$(1.12)\qquad \frac{1}{2}(E_{11}-E_{22}) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and basis vectors for its irreducible representations can be given as the simultaneous eigenvectors of the operator $\frac{1}{2}(E_{11}-E_{22})$,

$$(1.13)\qquad \tfrac{1}{2}(E_{11}-E_{22})|jm\rangle = m|jm\rangle,$$

and the Casimir operator

$$(1.14)\qquad [\tfrac{1}{4}(E_{11}-E_{22})^2 + \tfrac{1}{2}E_{21}E_{12} + \tfrac{1}{2}E_{12}E_{21}]|jm\rangle = j(j+1)|jm\rangle.$$

They behave under the "raising" operator E_{12} and the "lowering" E_{21} in accordance with

$$(1.15)\qquad \begin{aligned} E_{12}|jm\rangle &= [(j+m+1)(j-m)]^{1/2}|j, m+1\rangle, \\ E_{21}|jm\rangle &= [(j-m+1)(j+m)]^{1/2}|j, m-1\rangle, \end{aligned}$$

and are normalized by

$$(1.16)\qquad \langle jm'|jm\rangle = \delta_{m'm}.$$

The invariants j are positive half-integers, and m varies in integral steps between the limits $-j \leqq m \leqq j$. The relations (1.13) and (1.15) determine matrix elements of finite transformations in the group in accordance with

$$(1.17)\quad \begin{aligned} \langle jm'|e^{(1/2)(E_{12}-E_{21})\Theta}|jm\rangle &= d^j_{m'm}(\Theta) = (-1)^{j-m'}\left[\frac{(j+m)!(j+m')!}{(j-m)!(j-m')!}\right]^{1/2} \\ &\cdot \frac{1}{(m+m')!}\,{}_2F_1\left(-j+m, -j+m'|m+m'+1|-\cot^2\frac{\Theta}{2}\right) \\ &\cdot \left(\cos\frac{\Theta}{2}\right)^{m+m'}\left(\sin\frac{\Theta}{2}\right)^{2j-m'-m}, \end{aligned}$$

tensor products of which are reduced by the Wigner coefficients [8]:

$$(1.18)\quad d^{j_1}_{m_1'm_1}(\Theta)d^{j_2}_{m_2'm_2}(\Theta) = \sum_{j_3=|j_1-j_2|}^{j_1+j_2} C^{j_1\,j_2\,j_3}_{m_1'm_2'm_3'}\, d^{j_3}_{m_3'm_3}(\Theta)\, C^{j_1\,j_2\,j_3}_{m_1m_2m_3}$$

which are given by:

$$(1.19)\quad \begin{aligned} C^{j_1\,j_2\,j_3}_{m_1m_2m_3} &= \left[\frac{(2j_3+1)(j_1+m_1)!(j_2-m_2)!(j_3+m_3)! \cdot (j_3-m_3)!(j_1+j_3-j_2)!(j_2+j_3-j_1)!}{(j_1-m_1)!(j_2+m_2)!(j_1+j_2-j_3)!(j_1+j_2+j_3+1)!}\right]^{1/2} \\ &\cdot \frac{1}{(j_3-j_1-m_2)!(j_3-j_2+m_1)!} \\ &\cdot {}_3F_2(-j_1+m_1, -j_2-m_2, -j_1-j_2+j_3|j_3-j_1-m_2+1, j_3-j_2+m_1+1|1)\,\delta_{m_3,m_1+m_2}. \end{aligned}$$

The Wigner coefficient reduces to a monomial in all degenerate cases, i.e., if any of the three numbers j_i is equal to the sum of the other two or if any of the state labels m_i is equal to $\pm j_i$. In the limit $\Theta \to 0$, (1.18) becomes the completeness relation for the Wigner coefficients:

$$(1.18a)\quad \sum_{j_3=|j_1-j_2|}^{j_1+j_2} C^{j_1\,j_2\,j_3}_{m_1'm_2'm_3}\, C^{j_1\,j_2\,j_3}_{m_1m_2m_3} = \delta_{m_1'm_2}\,\delta_{m_2'm_2}.$$

When we introduce a degeneracy $m_1' = m_1 = \pm j_1$ or $m_2' = m_2 = \pm j_2$, then (1.18a) becomes an example of the summation theorem for well-poised ${}_4F_3(-1)$. When the degeneracy is removed, (1.18a) provides us with a generalization of this theorem.

The Racah, or $(6-j)$ coefficients of $SU(2)$ are defined by the relation

$$(1.20)\quad \begin{aligned} &[(2j_{12}+1)(2j_{32}+1)]^{1/2}(-1)^{2j}\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{32} \end{Bmatrix} C^{j_{12}\,j_3\,j}_{m_{12}m_3m} \\ &= \sum_{m_1m_2} C^{j_1\,j_2\,j_{12}}_{m_1m_2m_{12}}\, C^{j_3\,j_2\,j_{32}}_{m_3m_2m_{32}}\, C^{j_{32}\,j_1\,j}_{m_{32}m_1m} \end{aligned}$$

and have been given by Racah [10] in terms of Saalschützian ${}_4F_3$ (1) series:

$$(1.21)\quad \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = \frac{\Delta(abe)\Delta(acf)\Delta(cde)\Delta(bdf)(a+b+c+d+1)!(-1)^{a+b+c+d}}{(c+d-e)!(b+d-f)!(a+b-e)!(a+c-f)!(e+f-b-c)!(e+f-a-d)!} \cdot {}_4F_3(e-c-d, f-b-d, e-a-b, f-a-c \mid e+f-b-c+1, e+f-a-d+1, -a-b-c-1 \mid 1),$$

where the invariant triangle function $\Delta(abc)$ is given by

$$(1.22)\qquad \Delta(abc) = \left[\frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!}\right]^{1/2}.$$

The symmetries of ${}_3F_2(1)$ and of Saalschützian ${}_4F_3(1)$ have been studied by Thomae [12] and Whipple [13], [14], [15] and summarized by Slater [11]. The series (1.21) becomes degenerate and reduces to a monomial whenever at least one of the triples of numbers (abe), (acf), (cde), (bdf) contains a member which is equal to the sum of the other two, viz., $a+b=e$ or $b+d=f$. The $(6-j)$ coefficients satisfy the completeness relation [10]

$$(1.23)\qquad \sum_x (2e+1)(2x+1)\begin{Bmatrix} a & b & e \\ d & c & x \end{Bmatrix}\begin{Bmatrix} a & b & e' \\ d & c & x \end{Bmatrix} = \delta_{e,e'},$$

which becomes a realization of the summation theorem for well-poised ${}_5F_4$ in the degenerate cases $e=e'=a+b$ or $e=e'=c+d$. They also satisfy the Racah sum rule [10]

$$(1.24)\qquad \sum_x (2x+1)(-1)^{x+e+e'}\begin{Bmatrix} a & b & e \\ d & c & x \end{Bmatrix}\begin{Bmatrix} a & d & e' \\ b & c & x \end{Bmatrix} = \begin{Bmatrix} a & b & e \\ c & d & e' \end{Bmatrix},$$

which illustrates the summation theorem for well-poised ${}_4F_3(-1)$ in the degenerate case $a+b=e$, $a+d=e'$, and the sum rule [2], [8]

$$(1.25)\quad \begin{Bmatrix} \alpha & \beta & \gamma \\ a & b & c \end{Bmatrix}\begin{Bmatrix} \alpha & \beta & \gamma \\ a' & b' & c' \end{Bmatrix} = \sum_x (2x+1)(-1)^{x+\alpha+\beta+\gamma+a+b+c+a'+b'+c'}\begin{Bmatrix} a & x & a' \\ b' & \gamma & b \end{Bmatrix} \cdot \begin{Bmatrix} b & x & b' \\ c' & \alpha & c \end{Bmatrix}\begin{Bmatrix} c & x & c' \\ a' & \beta & a \end{Bmatrix}.$$

When we introduce a three-fold degeneracy into one of the $(6-j)$ symbols on the left, e.g.,

$$(1.26)\qquad a'+b'=\gamma,\quad b'+c'=\alpha,\quad a'+c'=\beta,$$

while the other remains nondegenerate, we find that all three $(6-j)$ coefficients in the summand on the right become monomials. Equation (1.25) then relates a Saalschützian ${}_4F_3(1)$ series on the left to a well-poised ${}_7F_6(1)$ on the right, i.e., provides an example of Whipple's theorem.

It will be noted that all of these examples involve degenerate forms of the Wigner and Racah coefficients of $SU(2)$. The combinatorial theory of $SU(2)$, then, is itself a rich source of information on how the hypergeometric theorems can be generalized. Thus, removal of the degeneracies which we have imposed on

(1.23), (1.24) and (1.25) provide immediate generalizations of the theorems (1.6), (1.10) and (1.11).

We note that well-poised hypergeometric series appear in the above relations since they all involve summation over terms weighted by the dimension factor of the irreducible representation of $SU(2)$ whose invariant label is the index of summation; i.e., sums of the familiar form

$$\sum_J (2J+1)F(J), \tag{1.27}$$

where the summand $F(J)$ contains only factors occurring in pairs of the sort

$$\frac{(J+M)!}{(J-M)!} \quad \text{or} \quad [(A-J)!(A+J+1)!]^{\pm 1}, \tag{1.28}$$

which are invariant (except for a phase) under the symmetry

$$J \to -J-1. \tag{1.29}$$

The occurrence of the dimension factor $(2J+1)$ indicates that we are dealing only with the special cases of (1.1) in which one of the parameters b_i is equal to $1+\frac{1}{2}a$. These special cases are, in fact, those for which summation theorems are most plentiful, and it is for such special cases that we shall seek multidimensional analogues in the representation theory of $SU(n)$. We shall call such cases *special well-poised hypergeometric series in* $SU(2)$ and shall define their analogues in this paper for $SU(n)$.

From the foregoing account of the relation between well-poised hypergeometric series and the combinatorial theory of $SU(2)$, we may conjecture that the combinatorial theory of $SU(n)$ will indeed provide us both with the definition of multidimensional series analogous to the well-poised hypergeometric series of $SU(2)$ theory and with a body of theorems on such series. We shall find that such is the case; unfortunately the combinatorial theory of $SU(n)$ is not yet sufficiently advanced to provide us with analogues to all the known theorems on well-poised series. We may surmise that our generalized well-poised series will involve summation over terms weighted by the Weyl dimension factor for the irreducible representations of $SU(n)$, i.e., just as the special well-poised series in $SU(2)$ has the form (1.27), so we expect the analogous special well-poised series in $SU(n)$ to have the form

$$\sum_{\sum_{i=1}^n x_i = q} \left\{ \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (A_{ij}+x_i-x_j) \right\} F(x_k), \tag{1.30}$$

where q is some nonnegative integer and

$$A_{ij} - A_{ik} = A_{kj}, \qquad i<k<j. \tag{1.31}$$

With the condition (1.31) the factor in curly brackets in (1.30) becomes proportional to the dimension of the irreducible representation of $SU(n)$ whose Young frame has rows of length $(A_{1n}+x_1-x_n-n+1)$, $(A_{2n}+x_2-x_n-n+2), \cdots$, $(A_{n-1n}+x_{n-1}-x_n-1)$. We find by inspection of the available cases of special well-poised hypergeometric series in $SU(n)$ that the condition analogous to (1.28) is that the summand $F(x_k)$ in (1.30) contain factors in n-tuplets of the form

$$\prod_{l=1}^{n} \Gamma(a_{li}+x_l) \quad \text{or} \quad \prod_{l=1}^{n} \frac{1}{\Gamma(b_{li}+x_l)}, \tag{1.32}$$

where

$$a_{ji}-a_{ki}=A_{jk}=b_{ji}-b_{ki}. \tag{1.33}$$

We may also understand this result in the following manner. The symmetry (1.29) in the $SU(2)$ case is the interchange of the "partial hooks" $p_{12} \leftrightarrow p_{22}$, where the partial hook is defined by

$$p_{in}=m_{in}+n-i. \tag{1.34}$$

Here m_{in} denotes the length of the ith row of the Young frame in an irreducible representation of $U(n)$. In the case of $U(2)$, we have $p_{12}-p_{22}=2J+1$, and the symmetry (1.29) corresponds to the reflection which generates the Weyl group S_2. In the general $U(n)$ case, then, we expect the corresponding symmetries to be described by the transformations of the Weyl group S_n, i.e., the symmetries $p_{in} \leftrightarrow p_{jn}$, which are just the permutations of the set of n partial hooks. The factors (1.32) exhibit these symmetries when we express the partial hooks in the form

$$p_{in}=A_{jn}+x_j, \qquad 1 \leqq j \leqq n-1, \tag{1.35}$$

and define

$$p_{nn} \equiv x_n = q-\sum_{i=1}^{n-1} x_i. \tag{1.36}$$

We shall find analogues to the special well-poised ${}_4F_3(-1)$ theorem for all $SU(n)$, but of Whipple's theorem (1.6), only for $SU(3)$. We shall not treat the case of hypergeometric series well-poised in $SU(1, 1)$ and other noncompact extensions of the $SU(n)$ groups, but shall merely remark that such a program could be expected to give us theorems on nonterminating series and on hypergeometric integrals, whereas series well-poised in $SU(n)$ will realize only the cases which involve terminating series.

In § 2 we define series well-poised in $SU(3)$ and give an analogue of Whipple's theorem (1.6). In § 3 we define series well-poised in $SU(n)$ and give analogues of the ${}_4F_3(-1)$ series for all $SU(n)$.

2. Hypergeometric series well-poised in $SU(3)$. We define the series:

$$W_q^{(3)}\left(\begin{matrix} A_{12} & \\ A_{13} & A_{23} \end{matrix}\left|\begin{matrix} a_{11} \cdots a_{1k} \\ a_{21} \cdots a_{2k} \\ a_{31} \cdots a_{3k} \end{matrix}\right|\begin{matrix} b_{11} \cdots b_{1j} \\ b_{21} \cdots b_{2j} \\ b_{31} \cdots b_{3j} \end{matrix}\left|\begin{matrix} z_1 \\ z_2 \\ z_3 \end{matrix}\right.\right)$$

$$\equiv q! \sum_{y_1+y_2+y_3=q} \frac{(A_{12}+y_1-y_2)}{A_{12}} \frac{(A_{13}+y_1-y_3)}{A_{13}} \frac{(A_{23}+y_2-y_3)}{A_{23}} \cdot \left(\prod_{i=1}^{k}\prod_{l=1}^{3} \frac{\Gamma(a_{li}+y_l)}{\Gamma(a_{li})}\right)\left(\prod_{i=1}^{j}\prod_{l=1}^{3} \frac{\Gamma(b_{li})}{\Gamma(b_{li}+y_l)}\right) z_1^{y_1} z_2^{y_2} z_3^{y_3} \tag{2.1}$$

to be *well-poised in* $SU(n)$ if q is a positive integer or zero and

$$\begin{aligned}&j\geqq 3,\\ &A_{12}+A_{23}=A_{13},\\ &a_{li}-a_{ri}=A_{lr}, \qquad r>l, \quad 1\leqq i\leqq k,\\ &b_{li}-b_{ri}=A_{lr}, \qquad r>l, \quad 1\leqq i\leqq j,\\ &b_{ll}=1, \qquad 1\leqq l\leqq 3.\end{aligned}\tag{2.2}$$

Similarly, we introduce the corresponding notation for "ordinary" hypergeometric series, which we term *well-poised in* $SU(2)$:

$$\begin{aligned}&W_q^{(2)}\left(A_{12}\left|\begin{matrix}a_{11}\cdots a_{lk}\\ a_{21}\cdots a_{2k}\end{matrix}\right|\begin{matrix}b_{11}\cdots b_{1j}\\ b_{21}\cdots b_{2j}\end{matrix}\left|\begin{matrix}z_1\\ z_2\end{matrix}\right.\right)\\ &=q!\sum_{y_1+y_2=q}\frac{(A_{12}+y_1-y_2)}{A_{12}}\left(\prod_{i=1}^{k}\prod_{l=1}^{2}\frac{\Gamma(a_{li}+y_l)}{\Gamma(a_{li})}\right)\\ &\quad\cdot\left(\prod_{i=1}^{j}\prod_{l=1}^{2}\frac{\Gamma(b_{li})}{\Gamma(b_{li}+y_l)}\right)z_1^{y_1}z_2^{y_2}\\ &=q!\sum_{y_1=0}^{q}\frac{(A_{12}-q+2y_1)}{A_{12}}\left(\prod_{i=1}^{k}\frac{\Gamma(a_{1i}+y_1)\Gamma(a_{2i}+q-y_1)}{\Gamma(a_{1i})\Gamma(a_{2i})}\right)\\ &\quad\cdot\left(\prod_{i=1}^{j}\frac{\Gamma(b_{1i})\Gamma(b_{2i})}{\Gamma(b_{1i}+y_i)\Gamma(b_{2i}+q-y_1)}\right)\left(\frac{z_1}{z_2}\right)^{y_1}z_2^q,\end{aligned}\tag{2.3}$$

where q is a positive integer or zero, and

$$\begin{aligned}&a_{1i}-a_{2i}=A_{12}, \qquad 1\leqq i\leqq k,\\ &b_{1i}-b_{2i}=A_{12}, \qquad 1\leqq i\leqq j,\\ &b_{11}=b_{22}=1.\end{aligned}\tag{2.4}$$

Note that this series is related to the usual definition of well-poised series by

$$\begin{aligned}&W_q^{(2)}\left(A_{12}\left|\begin{matrix}a_{11}\cdots a_{1k}\\ a_{21}\cdots a_{2k}\end{matrix}\right|\begin{matrix}b_{11}\cdots b_{1j}\\ b_{21}\cdots b_{2j}\end{matrix}\left|\begin{matrix}z_1\\ z_2\end{matrix}\right.\right)\\ &=\left(\prod_{i=1}^{k}\frac{\Gamma(a_{2i}+q)}{\Gamma(a_{2i})}\right)\left(\prod_{i=3}^{j}\frac{\Gamma(b_{2i})}{\Gamma(b_{2i}+q)}\right)(-z_2)^q\\ &\quad\cdot\frac{\Gamma(1+A_{12}-q)}{\Gamma(A_{12}+1)}\,{}_{1+k+j}F_{k+j}\cdot\left(\begin{matrix}A_{12}-q & 1+\frac{1}{2}(A_{12}-q) & -q\\ & \frac{1}{2}(A_{12}-q) & A_{12}+1\end{matrix}\right.\\ &\qquad\begin{matrix}a_{11} & \cdots & a_{1k}\\ 1+A_{12}-q-a_{11} & \cdots & 1+A_{12}-q-a_{1k}\end{matrix}\\ &\qquad\left.\begin{matrix}1+A_{12}-q-b_{13} & \cdots & 1+A_{12}-q-b_{1j}\\ b_{13} & \cdots & b_{1j}\end{matrix}\right|(-1)^{k+j}\left(\frac{z_1}{z_2}\right)\Bigg).\end{aligned}\tag{2.5}$$

In this case we have an ambiguity in the labelling process, since we may consider a given factor

$$\frac{\Gamma(a+y_1)}{\Gamma(a)}=(-1)^{y_1}\frac{\Gamma(1-a)}{\Gamma(1-a-y_1)}=(-1)^{y_1}\frac{\Gamma(1-a)}{\Gamma(1-a-q+y_2)} \tag{2.6}$$

as contributing a numerator parameter in the index y_i,

$$a_{1i}=a,$$

or a denominator parameter in the index y_2,

$$b_{2i}=1-a-q.$$

This ambiguity is not present in the definition (2.1) of series well-poised in $SU(3)$.

We may give immediately an analogue of Whipple's theorem (1.6) for the series (2.1). Let q be a nonnegative integer or zero and Δ_1, Δ_2, Δ_3, x_1, x_2 any complex numbers. Also, let $x_3=-x_1-x_2$. Then

$$W_q^{(3)}\left(\begin{matrix} x_3+\Delta_1-\Delta_2 & \\ -x_2+\Delta_1-\Delta_3 & x_1+\Delta_2-\Delta_3 \end{matrix}\,\middle\|\,\begin{matrix} 1 & x_3+\Delta_1-\Delta_2+1 \\ -x_3-\Delta_1+\Delta_2+1 & 1 \\ x_2-\Delta_1+\Delta_3+1 & -x_1-\Delta_2+\Delta_3+1 \end{matrix}\right.$$

$$\left.\begin{matrix} -x_2+\Delta_1-\Delta_3+1 & \Delta_1-q+1 & x_3+\Delta_1-q+1 & -x_2+\Delta_1-q+1 \\ x_1+\Delta_2-\Delta_3+1 & -x_3+\Delta_2-q+1 & \Delta_2-q+1 & x_1+\Delta_2-q+1 \\ 1 & x_2+\Delta_3-q+1 & -x_1+\Delta_3-q+1 & \Delta_3-q+1 \end{matrix}\,\middle|\,\begin{matrix}1\\1\\1\end{matrix}\right)$$

$$=q!\sum_{k_1,k_2,k_3}\frac{\Gamma(\Delta_1+\Delta_2+\Delta_3-k_1-k_2-k_3+1)}{\Gamma(\Delta_1+\Delta_2+\Delta_3-q+1)(q-k_1-k_2-k_3)!}$$

$$\cdot\frac{\Gamma(\Delta_2+x_1-q+1)\Gamma(\Delta_3-x_1-q+1)\Gamma(\Delta_1-q+1)}{k_1!\Gamma(\Delta_2+x_1-k_k+1)\Gamma(\Delta_3-x_1-k_1+1)\Gamma(\Delta_1-q+k_1+1)}$$

$$\cdot\frac{\Gamma(\Delta_3+x_2-q+1)\Gamma(\Delta_1-x_2-q+1)\Gamma(\Delta_2-q+1)}{k_2!\Gamma(\Delta_3+x_2-k_2+1)\Gamma(\Delta_1-x_2-k_2+1)\Gamma(\Delta_2-q+k_2+1)} \tag{2.7}$$

$$\cdot\frac{\Gamma(\Delta_1+x_3-q+1)\Gamma(\Delta_2-x_3-q+1)\Gamma(\Delta_3-q+1)}{k_3!\Gamma(\Delta_1+x_3-k_3+1)\Gamma(\Delta_2-x_3-k_3+1)\Gamma(\Delta_3-q+k_3+1)}$$

$$\equiv(-1)^q\cdot\frac{\Gamma(\Delta_1-q+1)\Gamma(\Delta_2+x_1-q+1)\Gamma(\Delta_3-x_1-q+1)}{\Gamma(\Delta_1+1)\Gamma(\Delta_2+x_1+1)\Gamma(\Delta_3-x_1+1)}$$

$$\cdot\frac{\Gamma(\Delta_2-q+1)\Gamma(\Delta_3+x_2-q+1)\Gamma(\Delta_1-x_2-q+1)}{\Gamma(\Delta_2+1)\Gamma(\Delta_3+x_2+1)\Gamma(\Delta_1-x_2+1)}$$

$$\cdot\frac{\Gamma(\Delta_3-q+1)\Gamma(\Delta_1+x_3-q+1)\Gamma(\Delta_2-x_3-q+1)}{\Gamma(\Delta_3+1)\Gamma(\Delta_1+x_3+1)\Gamma(\Delta_2-x_3+1)}$$

$$\cdot G_q\begin{pmatrix}\Delta_1 & \Delta_3-x_1 & \Delta_2+x_1\\ \Delta_2 & \Delta_1-x_2 & \Delta_3+x_2\\ \Delta_3 & \Delta_2-x_3 & \Delta_1+x_3\end{pmatrix}.$$

We observe that (2.7) relates the series $W_q^{(3)}$ well-poised in $SU(3)$ to a polynomial expressed in terms of three overlapping series, each of which is Saalschützian ${}_3F_2(1)$, and hence summable by Saalschütz' theorem. When any one of these summations is performed, the polynomial on the right of (2.7) is then expressed in terms of two overlapping Saalschützian ${}_4F_3(1)$ series. *Hence, the theorem* (2.7) *is a multidimensional analogue of Whipple's theorem* (1.6) *in that it relates well-poised and Saalschützian forms.* It does *not* contain (1.6), however, as a degenerate case or as an asymptotic limit. If we take the lim $\Delta_1 \to \infty$, we obtain the relation (1.9).

The symmetries of the series on the left of (2.7) can be read from those of the G_q polynomial, which is invariant under all permutations of rows and columns and under transposition of rows and columns. The well-poised form of this expression can be obtained by inductive construction of the solution to the recursion relation [3]

$$\begin{aligned}(x_1+\Delta_2-\Delta_3)(x_2+\Delta_3-\Delta_1)&(x_3+\Delta_1-\Delta_2)G_q(\Delta_1,\Delta_2,\Delta_3;x_1,x_2,x_3)\\ &=\Delta_1\Delta_2(x_3+\Delta_1-\Delta_2)(\Delta_2+x_1)(\Delta_1-x_2)(\Delta_1+x_3)(\Delta_2-x_3)\\ &\quad\cdot G_{q-1}(\Delta_1-1,\Delta_2-1,\Delta_3;x_1,x_2,x_3)\\ &\quad+\Delta_2\Delta_3(x_1+\Delta_2-\Delta_3)(\Delta_3+x_2)(\Delta_2-x_3)(\Delta_2+x_1)(\Delta_3-x_1)\\ &\quad\cdot G_{q-1}(\Delta_1,\Delta_2-1,\Delta_3-1;x_1,x_2,x_3)\\ &\quad+\Delta_3\Delta_1(x_2+\Delta_3-\Delta_1)(\Delta_1+x_3)(\Delta_3-x_1)(\Delta_3+x_2)(\Delta_1-x_2)\\ &\quad\cdot G_{q-1}(\Delta_1-1,\Delta_2,\Delta_3-1;x_1,x_2,x_3),\end{aligned} \tag{2.8}$$

which defines the polynomial G_q in (2.7) along with the boundary condition $G_0=1$.

This recursion relation was established in [3] as the defining relation for the invariant normalization factor of a class of tensor operators belonging to the irreducible representation $[p, q, 0]$ of $SU(3)$, and was derived from the expression for this tensor operator in terms of a coupling of elementary tensor operators with those belonging to $[p-1, q-1, 0]$. By changing the order of the coupling we obtain the equivalent recursion relation

$$\begin{aligned}x_1x_2x_3G_q(\Delta_1,\Delta_2,\Delta_3;x_1x_2x_3) &=\Delta_1\Delta_2x_3(\Delta_3-x_1)(\Delta_3+x_2)(\Delta_1+x_3)(\Delta_2-x_3)\\ &\quad\cdot G_{q-1}(\Delta_1-1,\Delta_2-1,\Delta_3;x_1+1,x_2-1,x_3)\\ &\quad+\Delta_2\Delta_3x_1(\Delta_1-x_2)(\Delta_1+x_3)(\Delta_2+x_1)(\Delta_3-x_1)\\ &\quad\cdot G_{q-1}(\Delta_1,\Delta_2-1,\Delta_3-1;x_1,x_2+1,x_3-1)\\ &\quad+\Delta_3\Delta_1x_2(\Delta_2-x_3)(\Delta_2+x_1)(\Delta_3+x_2)(\Delta_1-x_2)\\ &\quad\cdot G_{q-1}(\Delta_1-1,\Delta_2,\Delta_3-1;x_1-1,x_2,x_3+1).\end{aligned}$$

Other forms of the recursion relation can be derived by application of the symmetries of the G_q polynomial to (2.8) and (2.9).

The "natural" solution to (2.8), i.e., the one obtained by direct inductive construction, is then proportional to the well-poised form given on the left of

(2.7). If we construct the solution for $q = 1$ and $q = 2$, however, a more symmetrical form suggests itself, the right side of (2.7). That this form actually constitutes a solution to the recursion relations was conjectured in [3] and proved in [4]. The proof was established from the symmetry properties of G_q and the location of its zeros; we have found no proof which gives the Saalchützian form of (2.7) directly from the well-poised form by known transformations of hypergeometric series.

3. Hypergeometric series well-poised in $SU(n)$. The extension of the definitions (2.1) and (2.3) to the case of series well-poised in $SU(n)$ is now straightforward. We define

$$W_q^{(n)}\left(\begin{array}{cccc|ccc|ccc|c} A_{12} & & & & a_{11} & \cdots & a_{1k} & b_{11} & \cdots & b_{1j} & z_1 \\ A_{13} & A_{23} & & & \cdot & & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot & \cdot & & \cdot & \cdot \\ A_{1n} & A_{2n} & \cdots & A_{n-1n} & a_{n1} & \cdots & a_{nk} & b_{n1} & \cdots & b_{nj} & z_n \end{array}\right)$$

(3.1)

$$\equiv q! \sum_{\sum_{i=1}^n x_i = q} \left(\prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \frac{(A_{ij} + x_i - x_j)}{A_{ij}}\right)\left(\prod_{i=1}^{k} \prod_{l=1}^{n} \frac{\Gamma(a_{li} + x_l)}{\Gamma(a_{li})}\right) \cdot \left(\prod_{i=1}^{j} \prod_{l=1}^{n} \frac{\Gamma(b_{li})}{\Gamma(b_{li} + x_l)}\right)\left(\prod_{i=1}^{n} z_i^{x_i}\right)$$

to be *well-poised in $SU(n)$* if q is a positive integer or zero and

(3.2)
$$\begin{aligned} & j \geqq n, \\ & A_{ij} - A_{ik} = A_{kj}, \quad k < j, \\ & a_{ij} - a_{kj} = A_{ik}, \quad i < k, \\ & b_{ij} - b_{kj} = A_{ik}, \quad i < k, \\ & b_{ii} = 1, \quad 1 \leqq i \leqq n. \end{aligned}$$

We have one example of a summation theorem for a general $W_q^{(n)}$ series, which is analogous to the ${}_4F_3(-1)$ theorem (1.11) in the sense that this theorem expresses the completeness property of matrix elements of totally symmetric tensor operators in $SU(n)$ on maximal initial states just as the ${}_4F_3(-1)$ theorem expresses the same property of the corresponding structure in $SU(2)$, as mentioned above in connection with (1.18a). The matrix elements of totally symmetric tensor operators in $SU(n)$ have been constructed in [5]; these matrix elements can easily be seen to reduce to monomials in the case that the initial state is maximal. We can then write down a theorem on hypergeometric series well-poised in $SU(n)$ directly from the completeness relation

$$(3.3) \quad \sum_{\sum_{i=1}^n h_i = p + \sum_{i=1}^n h'_i} \left(\left\langle \begin{matrix} h'_1 \cdots h'_n \\ h'_1 \cdots h'_{n-1} \end{matrix}; \begin{matrix} p & & 0 & & \cdots & & 0 \\ & q & & 0 & \cdots & 0 & \end{matrix} \left\| \begin{matrix} h_1 \cdots h_n \\ q_1 \cdots q_{n-1} \end{matrix} \right.\right\rangle\right)^2 = 1,$$

which, of course, may be generalized by removal of the degeneracy. The relation (3.3) leads directly to the theorem

$$
W_q^{(n)}\left(\begin{array}{llll}
z_1-z_2+1 & & & \\
z_1-z_3+2 & z_2-z_3+1 & & \\
\vdots & \vdots & \ddots & \\
z_1-z_n+n-1 & z_2-z_n+n-2 & \cdots & z_{n-1}-z_n+1
\end{array}\right|
$$

$$
(3.4)\qquad \left|\begin{array}{cccc}
z_1-\omega_1+1 & z_1-\omega_2+2 & \cdots & z_1-\omega_n+n \\
z_2-\omega_1 & z_2-\omega_2+1 & \cdots & z_2-\omega_n+n-1 \\
\vdots & \vdots & & \vdots \\
z_n-\omega_1-n+2 & z_n-\omega_2-n+3 & & z_n-\omega_n+1
\end{array}\right|
$$

$$
\left|\begin{array}{ccccc|c}
1 & z_1-z_2+2 & z_1-z_3+3 & \cdots & z_1-z_n+n & 1 \\
-z_1+z_2 & 1 & z_2-z_3+2 & & z_2-z_n+n-1 & 1 \\
-z_1+z_3-1 & -z_2+z_3 & 1 & & z_3-z_n+n-2 & 1 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
-z_1+z_n-n+2 & -z_2+z_n-n+3 & -z_3+z_n-n+4 & \cdots & 1 & 1
\end{array}\right)
$$

$$
=1, \qquad n \geqq 3.
$$

Here q is a nonnegative integer; hence the series on the left of (3.4) terminates, and the relation (3.4) can immediately be extended to hold for all complex z_i, ω_i. Note that (3.4) becomes (1.11) in the case $n=2$ if we define the right side of (3.4) in accordance with

$$
(3.5)\qquad W_q^{(2)}\left(z_1-z_2+1\left|\begin{array}{c} z_1-\omega_1+1 \\ z_2-\omega_1 \end{array}\right|\begin{array}{cc} 1 & z_1-z_2+2 \\ -z_1+z_2 & 1 \end{array}\left|\begin{array}{c} 1 \\ 1 \end{array}\right.\right)=1.
$$

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A NEW APPROACH TO THE H-EQUATION OF CHANDRASEKHAR*

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Abstract. We combine a new, constructive fixed-point technique for order-reversing maps with standard constructive techniques of real analysis to give a new, complete, elementary discussion of the number and location of solutions of Chandrasekhar's H-equation. Our methods provide different viewpoints of known results as well as new information concerning the H-functions.

1. The work of Chandrasekhar (see [3], [4]) has demonstrated the importance of nonlinear integral equations of the form

$$H(x)=1+H(x)\int_0^1 \frac{x}{x+t}\psi(t)H(t)\,dt \tag{1}$$

to the theory of radiative transfer in semi-infinite atmospheres. The known function ψ is assumed to be nonnegative, bounded, and measurable on $[0, 1]$, and a positive, continuous solution H of (1) is sought.

In Chandrasekhar's introductory papers (which are summarized in [4]) the treatment of (1) was not rigorous. The first proof of the existence of a solution of (1) was given later by M. M. Crum [5], who considered the equation in the complex plane and, employing rather involved techniques of complex analysis, derived a solution H which is analytic in the half-plane Re $z>0$ and bounded in $[0, 1]$. Crum also showed that if $\int_0^1 \psi(t)\,dt \leqq 1/2$, then (1) has at most two solutions which are bounded in $[0, 1]$, and in case $\int_0^1 \psi(t)\,dt=1/2$, there is only one such solution. Busbridge [1], recognizing the need for a less formidable treatment of (1) than that of Crum, simplified Crum's arguments slightly by considering only certain holomorphic functions ψ. Fox [6] attempted to prove the existence of a solution of (1) by appealing to a simpler equation, but it was later demonstrated (see [2]) that solutions of Fox's equation were not necessarily solutions of (1). Recently C. A. Stuart [7] applied the Leray–Schauder degree theory to give a nonconstructive existence proof for (1); Stuart did not discuss the number or location of solutions.

In this article we apply only elementary tools of real analysis to give a new proof that (1) has a positive solution whenever $\int_0^1 \psi(t)\,dt \leqq 1/2$, and we establish iterative methods of approximating the minimal solution H. Our existence proofs could be shortened somewhat by appealing to Schauder's fixed-point theorem. However, we choose to avoid powerful tools and use instead constructive arguments, which, we feel, will be illuminating to scientists working with problems in which the "H-functions" arise. For completeness, we prove a comparison theorem for (1) and establish the number of solutions of (1) for arbitrary ψ.

2. We denote by $C[0, 1]$ the Banach space (with the usual supremum norm) of continuous, real-valued functions on $[0, 1]$. By a solution of (1) we shall mean a function $H \in C[0, 1]$ satisfying (1) for each $x \in [0, 1]$. Note that if H is a solution of

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(1), then $H(0)=1$ and H can have no zeros. It follows that $H(x)\geqq 1$ for each $x\in[0,1]$, that is, any solution of (1) is positive.

We begin by listing some known integral properties which a solution of (1) must satisfy. The proofs are simple; they can be found in [3, pp. 106–107].

LEMMA 1. *If H is a solution of* (1), *then either*

$$\int_0^1 \psi(t)H(t)\,dt = 1-\left[1-2\int_0^1 \psi(t)\,dt\right]^{1/2} \tag{2}$$

or

$$\int_0^1 \psi(t)H(t)\,dt = 1+\left[1-2\int_0^1 \psi(t)\,dt\right]^{1/2}. \tag{3}$$

A necessary condition that (1) *have a solution is that*

$$\int_0^1 \psi(t)\,dt \leqq \frac{1}{2}. \tag{4}$$

A function $H\in C[0,1]$ satisfies the equation

$$[H(x)]^{-1} = \left[1-2\int_0^1 \psi(t)\,dt\right]^{1/2} + \int_0^1 \frac{t}{x+t}\psi(t)H(t)\,dt \tag{5}$$

if and only if H satisfies (1) *and* (2).

Remark. In [3], after proving that a solution H of (1) satisfies either (2) or (3), Chandrasekhar claims that, in fact, H must satisfy (2). This assertion is (at least, mathematically) incorrect. As we later show, there always exists a solution H satisfying (2), but in many cases there exists a second solution H_1 satisfying (3) and not (2).

We let $\leqq$ be the natural partial ordering on $C[0,1]$, that is, if $h_1, h_2\in C[0,1]$, then $h_1\leqq h_2$ if and only if $h_1(x)\leqq h_2(x)$ for all $x\in[0,1]$. Let A be a subset of $C[0,1]$. A map $T: A\to C[0,1]$ is said to be isotone if T preserves order (that is, $h_1\leqq h_2$ implies $Th_1\leqq Th_2$), and T is said to be antitone if T reverses order ($h_1\leqq h_2$ implies $Th_2\leqq Th_1$).

Set $\rho=[1-2\int_0^1 \psi(t)\,dt]^{1/2}$ and let $A=\{h\in C[0,1] \mid h(x)\geqq\rho,\ x\in[0,1]\}$. Note that $\rho>0$ if and only if inequality holds in (4); otherwise $\rho=0$ (assuming that (4) holds). Define $S: A\to C[0,1]$ by

$$Sh(x) = 1+h(x)\int_0^1 \frac{x}{x+t}\psi(t)h(t)\,dt, \qquad h\in A,$$

and for the case $\rho>0$, define $T: A\to C[0,1]$ by

$$Th(x)=\rho+\int_0^1 \frac{t}{x+t}\psi(t)(h(t))^{-1}\,dt, \qquad h\in A.$$

Clearly S and T are both continuous maps, S is isotone, T is antitone, and H is a solution of (1) if and only if $SH=H$. Furthermore, it follows from Lemma 1 that $Th=h$ if and only if h^{-1} satisfies (1) and (2).

In what follows we denote by 1 (respectively, ρ) the function with constant value 1 (respectively, ρ).

THEOREM 1. *Equation* (1) *has exactly one solution H satisfying* (2) *if and only if* (4) *holds. Furthermore, the increasing sequence $\{S^n(1)\}_{n=0}^{\infty}$ converges to H, and if inequality holds in* (4), *the sequence $\{T^n(\rho)\}_{n=0}^{\infty}$ converges to H^{-1}. In the latter case a bound for $H^{-1}(x)-T^n(\rho)(x)$, $n=1,2,3,\cdots$, is given by*

$$|H^{-1}(x)-T^n(\rho)(x)|\leqq|T^n(\rho)(x)-T^{n+1}(\rho)(x)|, \qquad x\in[0,1]. \tag{6}$$

Proof. Step 1. If (1) has a solution, then by Lemma 1, $\int_0^1\psi(t)\,dt\leqq 1/2$. We assume first that $\int_0^1\psi(t)\,dt<1/2$, so that ρ is positive. Clearly $\rho\leqq T\rho$ and $\rho\leqq T^2\rho$, and since T is antitone, $\rho\leqq T^2\rho\leqq T\rho$. Applying T repeatedly we find that

$$\rho\leqq T^2\rho\leqq T^4\rho\leqq T^6\rho\leqq\cdots\leqq T^7\rho\leqq T^5\rho\leqq T^3\rho\leqq T\rho.$$

It is easy to verify that the bounded set

$$\{Th|\rho\leqq h\leqq T\rho\}$$

is equicontinuous (compare the proof for the set $R(E)$ in Lemma 2). It follows that the sequences of functions $\{T^{2n}\rho\}_{n=1}^{\infty}=\{T(T^{2n-1}\rho)\}_{n=1}^{\infty}$ and $\{T^{2n+1}\rho\}_{n=0}^{\infty}=\{T(T^{2n}\rho)\}_{n=0}^{\infty}$ have convergent subsequences which converge to functions u and v, respectively. It then follows from the monotonicity of the sequences $\{T^{2n}\rho\}$ and $\{T^{2n+1}\rho\}$ and from the continuity of T that $T^{2n}\rho\to u$, $T^{2n+1}\rho\to v$, $\rho\leqq u\leqq v$, $Tu=v$, and $Tv=u$.

Now u has a minimum value greater than zero, so that there exists a largest number a, $0<a\leqq 1$, with $av\leqq u$. If $a=1$, then $v\leqq u\leqq v$, that is, $u=v$. Assume $a<1$. Define T_1 (on the domain of T) by $T_1h=Th-\rho$. Then

$$\begin{aligned}u&=\rho+T_1v\geqq\rho+T_1(a^{-1}u)=\rho+aT_1u\\&=(1-a)\rho+a(\rho+T_1u)=(1-a)\rho+av\geqq bv+av=(a+b)v\end{aligned}$$

for some constant $b>0$. But this contradicts the maximality of a. Therefore $Tu=u=v$, $H\equiv u^{-1}$ is a solution of (1) (satisfying (2)), and $\{T^n\rho\}_{n=0}^{\infty}$ converges to H^{-1}. Inequality (6) follows from the fact that $T^{2k}\rho\leqq H^{-1}\leqq T^{2k+1}\rho$ for $k=1,2,3,\cdots$.

Remark. The existence of a fixed point of T could have been proved more quickly (but with the loss of additional information) by appealing to Schauder's fixed-point theorem.

Step 2. Now assume that $\int_0^1\psi(t)\,dt=1/2$. Let $\{k_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive numbers converging to 1, and consider the functions $k_n\psi$, $n=1,2,3,\cdots$. Since $\int_0^1 k_n\psi(t)\,dt=(1/2)k_n<1/2$, it follows from Step 1 that the equation

$$H(x)=1+H(x)\int_0^1\frac{x}{x+t}k_n\psi(t)H(t)\,dt$$

has a solution H_n for $n=1,2,3,\cdots$. Then for each $x\in[0,1]$, $H_n(x)\geqq 1$ and

$$\begin{aligned}(H_n(x))^{-1}&=\left[1-2\int_0^1 k_n\psi(t)\,dt\right]^{1/2}+\int_0^1\frac{t}{x+t}k_n\psi(t)H_n(t)\,dt\\&\geqq k_n\int_0^1\frac{t}{x+t}\psi(t)\,dt\geqq k_1\int_0^1\frac{t}{x+t}\psi(t)\,dt.\end{aligned}$$

Therefore there exists $\alpha > 0$ such that $(H_n(x))^{-1} \geqq \alpha$ for each $x \in [0, 1]$ and each $n = 1, 2, 3, \cdots$. Set $B = \{h \in C[0, 1] | \alpha \leqq h(x) \leqq 1,\ x \in [0, 1]\}$. Note that $H_n^{-1} \in B$ for each n. Define $T : B \to C[0, 1]$ by

$$Th(x) = \int_0^1 \frac{t}{x+t} \psi(t)(h(t))^{-1}\, dt, \qquad h \in B.$$

It is easy to verify that the set $T(B)$ is bounded and equicontinuous. Set $h_n = H_n^{-1}$, $n = 1, 2, 3, \cdots$. Then for each n,

$$\begin{aligned} h_n(x) &= \left[1 - 2\int_0^1 k_n\psi(t)\, dt\right]^{1/2} + \int_0^1 \frac{t}{x+t} k_n\psi(t)(h_n(t))^{-1}\, dt \\ &= \left[1 - 2\int_0^1 k_n\psi(t)\, dt\right]^{1/2} + k_n(Th_n)(x). \end{aligned}$$

Since $Th_n \in T(B)$ for each n, some subsequence $\{Th_{n_j}\}_{j=1}^{\infty}$ of $\{Th_n\}_{n=1}^{\infty}$ converges in $C[0, 1]$ to some point h_0. But

$$h_{n_j} = \left[1 - 2\int_0^1 k_{n_j}\psi(t)\, dt\right]^{1/2} + k_{n_j} Th_{n_j}, \tag{7}$$

and the right side of (7) converges to h_0, so that $h_{n_j} \to h_0$. Then $\{Th_{n_j}\}_{j=1}^{\infty}$ converges to both Th_0 and h_0; that is, $Th_0 = h_0$. Then h_0^{-1} satisfies (1) and (2) (and also (3), since $\int_0^1 \psi(t)\, dt = 1/2$). Therefore there exists a positive function H satisfying (1) and (2) whenever ψ satisfies (4).

Step 3. Assume (4) holds, and suppose H satisfies (1) and (2). Since $1 \leqq H$ and $1 \leqq S(1)$, it follows from the fact that S is isotone that $1 \leqq S(1) \leqq S^2(1) \leqq S^3(1) \leqq \cdots \leqq H$. Now the sequence $\{S^n(1)\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous. (The proof of the equicontinuity is not immediate; we save it until later.) Therefore there is a convergent subsequence, say $S^{n_k}(1) \to h (\leqq H)$, and, since $\{S^n(1)\}_{n=0}^{\infty}$ is nondecreasing, the entire sequence converges to h. It follows from the continuity of S that $Sh = h$. Now h must satisfy either (2) or (3), and since $0 \leqq h \leqq H$, h must satisfy (2). Therefore, for $x \in [0, 1]$,

$$\begin{aligned} h^{-1}(x) &= \left[1 - 2\int_0^1 \psi(t)\, dt\right]^{1/2} + \int_0^1 \frac{t}{x+t} \psi(t)h(t)\, dt \\ &\leqq \left[1 - 2\int_0^1 \psi(t)\, dt\right]^{1/2} + \int_0^1 \frac{t}{x+t} \psi(t)H(t)\, dt = H^{-1}(x), \end{aligned}$$

that is, $h^{-1} \leqq H^{-1}$. Together with the inequality $h \leqq H$, this implies that $h = H$. We have proved that H is the only function satisfying both (1) and (2), and that the increasing sequence $\{S^n(1)\}_{n=0}^{\infty}$ converges to H. This completes the proof of Theorem 1, except for the verification of the following statement.

LEMMA 2. *The sequence* $\{S^n(1)\}_{n=1}^{\infty}$ *is equicontinuous.*

Proof. Set $E = \{h \in C[0, 1] | 1 \leq h \leq H\}$ (H is a solution satisfying (1) and (2)). Define $R : E \to C[0, 1]$ by

$$Rh(x) = \int_0^1 \frac{x}{x+t} \psi(t)h(t)\, dt, \qquad x \in [0, 1], \quad h \in E.$$

For $x, y \in [0, 1]$,

$$|Rh(x)-Rh(y)| = \left| \int_0^1 \left(\frac{x}{x+t} - \frac{y}{y+t} \right) \psi(t) h(t)\, dt \right|.$$

Given $\varepsilon > 0$, there exists $\gamma, 0 < \gamma < 1$, such that $\int_0^\gamma \psi(t) H(t)\, dt < \varepsilon/4$ and $\int_\gamma^1 \psi(t) H(t)\, dt > 0$. Then for $x, y \in [0, 1]$,

$$\begin{aligned} \left| \int_0^\gamma \left(\frac{x}{x+t} - \frac{y}{y+t} \right) \psi(t) h(t)\, dt \right| &\leqq \int_0^\gamma \left(\left| \frac{x}{x+t} \right| + \left| \frac{y}{y+t} \right| \right) \psi(t) H(t)\, dt \\ &\leqq 2 \int_0^\gamma \psi(t) H(t)\, dt < \varepsilon/2. \end{aligned}$$

Thus, if $|x-y| < \delta \equiv (\varepsilon/2)\gamma^2 (\int_\gamma^1 \psi(t) H(t)\, dt)^{-1}$, we find that

$$\begin{aligned} |Rh(x)-Rh(y)| &\leqq \frac{\varepsilon}{2} + \int_\gamma^1 \left| \frac{x}{x+t} - \frac{y}{y+t} \right| \psi(t) H(t)\, dt \\ &\leqq \frac{\varepsilon}{2} + \int_\gamma^1 \frac{t|x-y|}{(x+t)(y+t)} \psi(t) H(t)\, dt \\ &\leqq \frac{\varepsilon}{2} + \int_\gamma^1 \frac{|x-y|}{\gamma^2} \psi(t) H(t)\, dt \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

We have proved that the set $R(E)$ is equicontinuous.

Now if $h \in E$ and $x \in [0, 1]$,

$$\begin{aligned} Rh(x) &= \int_0^1 \frac{x}{x+t} \psi(t) h(t)\, dt \leqq \int_0^1 \frac{x}{x+t} \psi(t) H(t)\, dt \\ &< \int_0^1 \psi(t) H(t)\, dt = 1 - \left[1 - 2 \int_0^1 \psi(t)\, dt \right]^{1/2} \leqq 1. \end{aligned}$$

Therefore there exists β, $0 < \beta < 1$, such that $Rh(x) < \beta$ for all $h \in E$ and $x \in [0, 1]$.

Let $\varepsilon_0 > 0$ be fixed. Since $R(E)$ is equicontinuous, we can choose $\delta_0 > 0$ such that for every $g \in R(E)$, $|g(x) - g(y)| < \|H\|^{-1}(1-\beta)\varepsilon_0$ whenever $|x-y| < \delta_0$. Now the function $S1$ is continuous and hence uniformly continuous on $[0, 1]$, so that we can find δ_1, $0 < \delta_1 \leqq \delta_0$, such that $|x-y| < \delta_1$ implies $|S1(x) - S1(y)| < \varepsilon_0$. Suppose for all positive integers k up to and including n, $S^k 1(x) - S^k 1(y)| < \varepsilon_0$ whenever $|x-y| < \delta_1$. We shall show that the same δ_1 works for ε_0 and $S^{n+1}1$. Set $h = S^n 1$, and assume $|x-y| < \delta_1$. Then

$$\begin{aligned} |Sh(x) - Sh(y)| &= |h(x)Rh(x) - h(y)Rh(y)| \\ &\leqq |h(x)Rh(x) - h(x)Rh(y)| + |h(x)Rh(y) - h(y)Rh(y)| \\ &= h(x)|Rh(x) - Rh(y)| + Rh(y)|h(x) - h(y)| \\ &< \|H\| \|H\|^{-1}(1-\beta)\varepsilon_0 + \beta\varepsilon_0 = \varepsilon_0, \end{aligned}$$

that is, $|S^{n+1}1(x)-S^{n+1}1(y)|<\varepsilon_0$ whenever $|x-y|<\delta_1$. Therefore $\{S^n 1\}_{n=1}^{\infty}$ is equicontinuous, since the same δ_1 can be chosen for fixed ε_0 and for each n.

The following corollary is a comparison result gained from our iterative procedure.

COROLLARY 1. *Suppose that ψ_1 and ψ_2 are nonnegative, bounded, measurable functions on* $[0, 1]$ *such that $\psi_1(t) \leqq \psi_2(t)$ almost everywhere in* $[0, 1]$ *and such that* $\int_0^1 \psi_i(t)\,dt \leqq 1/2$, $i=1, 2$. *Let H_i be the unique solution of equations* (1)–(2) *corresponding to $\psi=\psi_i$, $i=1,2$. Then $H_1 \leqq H_2$.*

Proof. Define $S_i : C[0, 1] \to C[0, 1]$, $i=1, 2$, by

$$S_i h(x)=1+h(x)\int_0^1 \frac{x}{x+t}\psi_i(t)h(t)\,dt, \qquad h \in C[0, 1].$$

If h_1 and h_2 are nonnegative functions in $C[0, 1]$ with $h_1 \leqq h_2$, then $S_1 h_1 \leqq S_2 h_2$. Hence $S_1 1 \leqq S_2 1$, $S_1^2 1 \leqq S_2^2 1$, and in general, $S_1^n 1 \leqq S_2^n 1$. Since the increasing sequence $S_i^n 1$ converges to H_i, $i=1, 2$, it follows that $H_1 \leqq H_2$.

3. If $\int_0^1 \psi(t)\,dt=1/2$, it follows from our results in § 2 that the function H satisfying (1) and (2) is the unique solution of (1), since, in this case, (2) and (3) reduce to the same equation. However, if $\int_0^1 \psi(t)\,dt<1/2$, equation (1) may have two distinct solutions.

THEOREM 2. *Suppose* $\int_0^1 \psi(t)\,dt<1/2$ *and let H be the unique solution of* (1)–(2). *Then equation* (1) *has a solution H_1 satisfying* (3) *if and only if*

$$\int_0^1 \frac{\psi(t)}{1-t}H(t)\,dt>1. \tag{8}$$

(*The left side of* (8) *may be* $+\infty$.) *If* (8) *holds, then H_1 is the only solution of* (1)–(3) *and is given by*

$$H_1(x)=\frac{1+kx}{1-kx}H(x), \qquad x \in [0, 1], \tag{9}$$

where k is the unique number in $(0, 1)$ *for which*

$$\int_0^1 \frac{\psi(t)}{1-kt}H(t)\,dt=1. \tag{10}$$

Proof. Note that by the monotone convergence theorem

$$\lim_{k\to 1^-}\int_0^1 \frac{\psi(t)}{1-kt}H(t)\,dt=\int_0^1 \frac{\psi(t)}{1-t}H(t)\,dt \tag{11}$$

since $(1-kt)^{-1}$ increases monotonically with k, $0<k<1$. Assume first that (8) holds. Since

$$\int_0^1 \frac{\psi(t)}{1-0\cdot t}H(t)\,dt=1-\left[1-2\int_0^1 \psi(t)\,dt\right]^{1/2}<1,$$

and since the function $f:(0, 1)\to \mathbb{R}$ defined by

$$f(k)=\int_0^1 \frac{\psi(t)}{1-kt}H(t)\,dt$$

is strictly increasing, there exists a unique $k \in (0, 1)$ for which (10) holds. Let H_1 be defined as in (9). Applying a trick used in [5], we find that for each $x \in [0, 1]$,

$$\int_0^1 \frac{x}{x+t}\psi(t)H_1(t)\,dt = \int_0^1 \frac{x}{x+t}\psi(t)\left(\frac{1+kt}{1-kt}\right)H(t)\,dt$$

$$= \frac{1-kx}{1+kx}\int_0^1 \frac{x}{x+t}\psi(t)H(t)\,dt + \frac{2kx}{1+kx}\int_0^1 \frac{H(t)}{1-kt}\psi(t)\,dt$$

$$= \frac{1-kx}{1+kx}\int_0^1 \frac{x}{x+t}\psi(t)H(t)\,dt + \frac{2kx}{1+kx}$$

$$= \frac{1-kx}{1+kx}\left[1 - \frac{1}{H(x)}\right] + \frac{2kx}{1+kx} = 1 - \frac{1}{H_1(x)},$$

that is, H_1 satisfies (1). Since H_1 must satisfy either (2) or (3) and since $H_1(x) > H(x)$, $x \in (0, 1]$, H_1 satisfies (3).

Now suppose that $H_1 \in C[0, 1]$ satisfies (1) and (3). Since $\int_0^1 \psi(t)H_1(t)\,dt > 1$ and

$$\int_0^1 \frac{\psi(t)}{1+t}H_1(t)\,dt = 1 - \frac{1}{H_1(1)} < 1,$$

there exists a unique k, $0 < k < 1$, such that

$$\int_0^1 \frac{\psi(t)}{1+kt}H_1(t)\,dt = 1.$$

Define $H_2 \in C[0, 1]$ by

$$H_2(x) = \frac{1-kx}{1+kx}H_1(x), \qquad x \in [0, 1].$$

If $x \in [0, 1]$,

$$\int_0^1 \frac{x}{x+t}H_2(t)\psi(t)\,dt = \int_0^1 \frac{x}{x+t}\,\frac{1-kt}{1+kt}H_1(t)\psi(t)\,dt$$

$$= \frac{1+kx}{1-kx}\int_0^1 \frac{x}{x+t}H_1(t)\psi(t)\,dt - \frac{2kx}{1-kx}\int_0^1 \frac{H_1(t)}{1+kt}\psi(t)\,dt$$

$$= \frac{1+kx}{1-kx}\left(1 - \frac{1}{H_1(x)}\right) - \frac{2kx}{1-kx} = 1 - \left(\frac{1+kx}{1-kx}\right)\frac{1}{H_1(x)}$$

$$= 1 - \frac{1}{H_2(x)}.$$

Therefore H_2 satisfies (1), and since H_2 must satisfy either (2) or (3) and $H_2(x) < H_1(x)$, $x \in (0, 1]$, H_2 satisfies (2). It follows that $H_2 = H$. Furthermore,

$$\int_0^1 \frac{\psi(t)}{1-kt} H(t)\, dt = \int_0^1 \frac{\psi(t)}{1-kt} \frac{1-kt}{1+kt} H_1(t)\, dt = 1.$$

We have proved that (1) has a solution H_1 satisfying (3) if and only if (8) holds. In the process we have proved that H_1 is of the form (9). Since there is at most one $k \in (0, 1)$ for which (10) holds, H_1 is the only solution of (1) satisfying (3).

We know from Theorem 2 that if $\int_0^1 \psi(t)\, dt < 1/2$, (1) has either a unique solution or exactly two solutions. Our final result is a necessary and sufficient condition for two distinct solutions to exist.

THEOREM 3. *Suppose* $\int_0^1 \psi(t)\, dt < 1/2$. *Then* (1) *has exactly two solutions if and only if*

$$\int_0^1 \frac{\psi(t)}{1-t^2}\, dt > \frac{1}{2}. \tag{12}$$

(*The left side of* (12) *may be* $+\infty$.)

Proof. For fixed ψ satisfying $\int_0^1 \psi(t)\, dt < 1/2$, let H be the unique solution of (1)–(2). Suppose first that (1) has two solutions. Then there exists a unique $k \in (0, 1)$ for which (10) holds. Then applying a technique used in [3] and [5], we find that

$$\begin{aligned}
\int_0^1 \frac{\psi(t)}{1+kt} H(t)\, dt &= \int_0^1 \left[\int_0^1 \frac{\psi(s)}{1-ks} H(s)\, ds\right] \frac{\psi(t)}{1+kt} H(t)\, dt \\
&= \int_0^1 \int_0^1 H(t)\psi(t)H(s)\psi(s)\left[\frac{1}{s+t}\left(\frac{t}{1+kt} + \frac{s}{1-ks}\right)\right] ds\, dt \\
&= \int_0^1 \left[\frac{\psi(t)}{1+kt} H(t) \int_0^1 \frac{t}{s+t} \psi(s)H(s)\, ds\right] dt \\
&\quad + \int_0^1 \left[\frac{\psi(s)}{1-ks} H(s) \int_0^1 \frac{s}{s+t} \psi(t)H(t)\, dt\right] ds \\
&= \int_0^1 \frac{\psi(t)}{1+kt} H(t)\left(1 - \frac{1}{H(t)}\right) dt + \int_0^1 \frac{\psi(s)}{1-ks} H(s)\left(1 - \frac{1}{H(s)}\right) ds \\
&= \int_0^1 \frac{\psi(t)}{1+kt} H(t)\, dt - \int_0^1 \frac{\psi(t)}{1+kt}\, dt + \int_0^1 \frac{\psi(s)}{1-ks} H(s)\, ds \\
&\quad - \int_0^1 \frac{\psi(s)}{1-ks}\, ds \\
&= \int_0^1 \frac{\psi(t)}{1+kt} H(t)\, dt + 1 - 2\int_0^1 \frac{\psi(t)}{1-k^2t^2}\, dt.
\end{aligned}$$

Therefore,

$$\frac{1}{2} = \int_0^1 \frac{\psi(t)}{1-k^2t^2}\,dt < \int_0^1 \frac{\psi(t)}{1-t^2}\,dt.$$

Conversely, assume that (1) has only one solution. Then for fixed $k \in (0, 1)$,

$$\int_0^1 \frac{\psi(t)}{1-kt} H(t)\,dt < 1,$$

and a computation similar to the one above shows that $\int_0^1 \psi(t)/(1-k^2t^2)\,dt < 1/2$. It follows that $\int_0^1 \psi(t)/(1-t^2)\,dt \leqq 1/2$, and the theorem is proved.

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SOME NONLINEAR BOUNDARY VALUE PROBLEMS*

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Abstract. Let $\Omega \subset R^m$ be a bounded domain and **L** a second order uniformly strongly elliptic partial differential operator. Let **B** be a linear boundary operator. Suppose $f(x, u)$ and $g(x, u)$ are functions on $\Omega \times R^1$ which are nonincreasing with respect to u for sufficiently large values of $|u|$. Conditions are found under which the problem $Lu = f(x, u(x))$, $x \in \Omega$, with $Bu = g(x, u(x))$, $x \in \partial\Omega$, has a generalized solution in the Sobolev space $H^1(\Omega)$. This is followed by a brief discussion of stability of positive solutions.

1. Introduction. We shall be interested in the existence and uniqueness of solutions to nonlinear elliptic boundary value problems of the type

$$\begin{aligned} \mathbf{L}u &\equiv -D_i(a_{ij}(x)D_ju) + b_i(x)D_iu + a(x)u = f(x, u(x)) \quad \text{in } \Omega, \\ &\partial u/\partial N + \sigma(x)u = g(x, u(x)) \quad \text{on } \Delta \subset \partial\Omega, \\ &u(x) = \theta(x) \qquad \text{on } \Delta' = \partial\Omega - \Delta, \end{aligned} \tag{1}$$

where **L** is a uniformly elliptic differential operator, Δ is measurable, $N(x) = (N_1(x), N_2(x), \cdots, N_m(x))$ is the unit outer normal at $x \in \partial\Omega$ and $\partial u/\partial N$ denotes the conormal derivative

$$\partial u/\partial N = N_i(x)a_{ij}(x)D_ju.$$

In a nonlinear problem of this type one is quite naturally tempted to impose instead the less restrictive boundary condition

$$\mathbf{B}u \equiv \tau(x)\,\partial u/\partial N + \sigma(x)u(x) = g(x, u(x)), \qquad x \in \partial\Omega, \tag{2}$$

where $\sigma^2 + \tau^2 > 0$ on $\partial\Omega$. However, letting Δ be that part of $\partial\Omega$ where $\tau(x) \neq 0$ we see that the condition $\sigma(x)u(x) = g(x, u(x))$ has to be satisfied on Δ'. Therefore if we wish the problem to have a solution we must certainly require that this functional equation have a solution θ on Δ'. Thus after some minor technicalities we are led to the conclusion that, in effect, the boundary condition must indeed be of the form given in (1). Nevertheless (2) does provide a convenient notation which we shall employ with the understanding that $\tau(x) \equiv 1$ on Δ, $\tau(x) \equiv 0$ on Δ', $\sigma(x) \equiv 1$ on Δ' and $g(x, u) \equiv \theta(x)$ on Δ'.

For some results we assume that for fixed x, $f(x, u)$ and $g(x, u)$ are nonincreasing functions of u. Such a monotonicity condition was assumed by D. Cohen in connection with a radiative heat transfer problem [1]. Cohen's work deals with classical solutions; in this article we will be interested in generalized solutions. However in many cases available regularity results lead us to conclude that the generalized solutions are in fact also solutions in the classical sense. Without, at this point, detailing exactly what is meant by a solution, let us suppose that the

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linear problem

$$\text{(3)}\qquad \begin{aligned} \mathbf{L}v &= f(x, u(x)) \quad \text{in } \Omega, \\ \mathbf{B}v &= g(x, u(x)) \quad \text{on } \partial\Omega \end{aligned}$$

has a unique solution v for each u chosen from some set S ($u \in S$), and suppose that $v \in S$. We then define a map $\mathbf{\Phi} : S \to S$ by saying that $\mathbf{\Phi}(u) = v$ whenever v is a solution of (3). If $f(x, u)$ and $g(x, u)$ are nondecreasing with respect to u, then $\mathbf{\Phi}$ will be an order reversing map, that is to say that if $u_1(x) \geqq u_2(x)$ a.e. in Ω (more simply denoted $u_1 \geqq u_2$), then $\mathbf{\Phi}(u_1) \leqq \mathbf{\Phi}(u_2)$. Problem (1) now consists of finding a fixed point for $\mathbf{\Phi}$. If f_u and g_u exist and $f_u(x, u) \geqq -M$, $g_u(x, u) \geqq -M$ for some positive constant M, then we can add a term Mu on both sides of the equations (1) and obtain the problem

$$\begin{aligned} \mathbf{L}^*u &= f^*(x, u(x)) \quad \text{in } \Omega, \\ \mathbf{B}^*u &= g^*(x, u(x)) \quad \text{on } \partial\Omega, \end{aligned}$$

where $f^*(x, u)$ and $g^*(x, u)$ are nondecreasing functions with respect to u. We can, as above, define a corresponding map $\mathbf{\Phi}^*$ which now will be order preserving, or what is usually called *monotone*. The latter approach to problem (1) has been taken by several authors including H. Amann, T. Laetsch, and D. H. Sattinger [2]–[4].

In this paper we shall not try to use such monotonicity methods but instead use the order reversing property. This allows us to treat problems which cannot be transformed to a type which has an order preserving type of monotonicity.

2. Preliminaries. In this section we shall state the hypotheses, explain the notation and give the definition of a (generalized) solution.

Let Ω denote a bounded domain in R^m, $m \geqq 1$, whose boundary, $\partial\Omega$, is sufficiently well behaved. For example, it may be assumed that the boundary is of class $C^{2+\alpha}$. However, unless we are dealing with the Neumann problem, this may be weakened to requiring the following hypothesis.

$\mathbf{H}_1$: $\partial\Omega$ is of class C^1 or, more generally, $\partial\Omega$ is *piecewise smooth with nonzero interior angles.*

This last phrase means (see [7]) that $\bar{\Omega} = \bigcup_{i=1}^N \bar{\Omega}_i$, where the Ω_i are mutually disjoint open subsets of Ω and each $\bar{\Omega}_i$ can be mapped homeomorphically onto the unit ball or cube by means of a Lipschitz continuous map whose Jacobian (in the distributional sense) is bounded from below by a positive constant.

Let $x = (x_1, x_2, \cdots, x_m)$ represent a point in R^m. All functions, vector spaces, numbers, etc., shall be assumed to be real.

We also assume the following hypothesis.

$\mathbf{H}_2$: $f(x, u) : \Omega \times R^1 \to R^1$ and $g(x, u) : \Delta \times R^1 \to R^1$ satisfy the Caratheodory conditions (measurable in x for each u, continuous in u for almost all x) and are bounded on bounded sets in $\bar{\Omega} \times R^1$ and $\partial\Omega \times R^1$ respectively.

For any set X we use $(\cdot\,,\cdot)_X$ to denote the inner product on $L_2(X)$. Let $H^j(\Omega)$ denote the Sobolev space of Schwartz distributions on Ω which have derivatives of order $\leqq j$ in $H^0(\Omega) = L_2(\Omega)$. The norm on this space is $\|u\|_j = \{(D^\alpha u, D^\alpha u)_\Omega\}^{1/2}$

where we sum over the multi-index α. In other words $D^\alpha = D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_m^{\alpha_m}$ and the summation is over all α with $\alpha_1+\alpha_2+\cdots+\alpha_m \leqq j$. We assume the following.

H$_3$: $a_{ij}\in L_\infty(\Omega)$, $b_i \in L_q(\Omega)$ and $a\in L_{q/2}(\Omega)$, with $q>m$, $q\geqq 2$.

Corresponding to $\mathbf{L}$ we have a bilinear functional $\mathbf{l}_0$ on $H^1(\Omega)\times H^1(\Omega)$ defined by

$$\mathbf{l}_0(u, v) = (a_{ij}D_ju, D_iv)_\Omega + (b_iD_iu, v)_\Omega + (au, v)_\Omega.$$

The above hypotheses on the coefficients and the Sobolev embedding theorem (see, e.g., [7]) lead us to conclude that $\mathbf{l}_0$ is a continuous bilinear functional on $H^1(\Omega)\times H^1(\Omega)$.

The results which follow apply to ordinary nonlinear Sturm–Liouville equations provided we interpret the boundary conditions appropriately. It is left to the reader to make the slight modifications in case $m=1$. We assume

H$_4$: L is uniformly strongly elliptic, i.e., there is a number $\nu>0$ so that

$$a_{ij}(x)\xi_i\xi_j \geqq \nu\xi_i\xi_i$$

for all $\xi=(\xi_1, \xi_2, \cdots, \xi_m)\in R^m$ and $x\in\Omega$.

Recall that for elements in certain Sobolev spaces one can, unambiguously define their boundary values by means of the so-called trace map. For domains whose boundary is a C^∞-manifold much information on such maps can be found in the work of J. Lions and E. Magenes [8]. The C^∞ boundary requirement can often be weakened. In fact if $\partial\Omega$ satisfies $\mathbf{H}_1$ and $j\leqq m/2$, $t<2(m-1)/(m-2j)$, then there is a compact continuous linear map $\gamma: H^j(\Omega)\to L_t(\partial\Omega)$ such that $\gamma u = u|_{\partial\Omega}$ whenever $u\in H^j(\Omega)\cap C(\bar\Omega)$ ([5], [6], [7] and further references in [7]). If $j>m/2$, then it follows from the Sobolev embedding theorem ([7]) that $H^j(\Omega)\subset C(\bar\Omega)$ is a compact continuous injection and hence $\gamma: H^j(\Omega)\to C(\partial\Omega)$ is obviously defined and compact continuous. It can be shown (see [7, p. 41] for discussion and further references) that under our hypothesis $\mathbf{H}_1$ we have a Green's identity, i.e., for any $v, w\in H^1(\Omega)$ we have

$$\int_\Omega vD_kw\,dx = -\int_\Omega wD_kv\,dx + \int_{\partial\Omega}(\gamma v)(\gamma w)N_k\,dS, \qquad m\geqq 2,$$

and the analogous result for $m=1$. We shall often dispense with the symbol γ. The above formula may, for example, be more concisely written as

$$(v, D_kw)_\Omega = (-D_kv, w)_\Omega + (v, wN_k)_{\partial\Omega}.$$

Let $\Delta\subset\partial\Omega$ and

$$V_\Delta = \{\eta\in H^1(\Omega) |\ \text{supp}\ \gamma\eta\subset\Delta\}.$$

This is easily seen to be a subspace of $H^1(\Omega)$. Hypothesis $\mathbf{H}_1$ implies that $V_\phi = H_0^1(\Omega)$, the closure in $H^1(\Omega)$ of the $C^\infty(\bar\Omega)$ functions which have compact support in Ω (see [7]).

We assume either of the following sets of hypotheses.

H$_5$-I: *Dirichlet problem.* $\Delta=\varnothing$, $\sigma(x)\equiv 1$, $\theta\in H^1(\Omega)\cap L_\infty(\Omega)$.

$\mathbf{H}_5$-II: *Neumann and Robin problems.* In this case we assume that $\Delta = \partial\Omega$ and $\partial\Omega$ is of class $C^{2+\alpha}$, $\alpha > 0$. Also $\sigma \in L_p(\partial\Omega)$ with $p > m-1$, and $\sigma \geqq 0$.

$\mathbf{H}_5$-III: *Mixed problem.* We assume Δ is a nonempty measurable subset of $\partial\Omega$, $a(x) \geqq \delta > 0$ and $\sigma(x) \geqq \delta > 0$ for some positive number δ, and $\sigma \in L_p(\Delta)$ where $p > m-1$. We also assume that $\theta \in H^1(\Omega) \cap L_\infty(\Omega)$.

Corresponding to the differential operator $\mathbf{L}$ and the space V_Δ, there is bilinear functional $\mathbf{l}$ on $H^1(\Omega) \times H^1(\Omega)$ given by

$$\mathbf{l}(u, v) = \mathbf{l}_0(u, v) + (\sigma u, v)_\Delta.$$

From the Sobolev embedding result quoted above it is immediate that $(\sigma u, v)_\Delta$ is a continuous bilinear functional on $H^1(\Omega) \times H^1(\Omega)$, and hence $\mathbf{l}$ is a continuous bilinear functional on $H^1(\Omega) \times H^1(\Omega)$. From now on Δ only refers to the support of τ on $\partial\Omega$.

We shall also assume the following hypothesis.

$\mathbf{H}_6$: $\mathbf{l}$ is V_Δ-coercive, i.e., there is a positive number C so that

$$\mathbf{l}(u, u) \geqq C\|u\|_1^2 \quad \text{for all } u \in V_\Delta.$$

Consider the linear nonhomogeneous problem.

$$\begin{aligned} \mathbf{L}u &= \phi \quad \text{in } \Omega, \\ \mathbf{B}u &= \psi \quad \text{on } \Delta, \\ u &= \theta \quad \text{on } \Delta', \end{aligned} \tag{4}$$

where we assume $\phi \in L_2(\Omega)$, $\psi \in L_{2(m-1)/m}(\partial\Omega)$, $\theta \in H^1(\Omega)$. We say that u is a *solution* of (4) if $u - \theta \in V_\Delta$ and

$$\mathbf{l}(u, \eta) = (\phi, \eta)_\Omega + (\psi, \eta)_\Delta \quad \text{for all } \eta \in V_\Delta. \tag{5}$$

The reason, of course, is that any classical solution satisfies (5) and has $\gamma(u-\theta) = 0$ on Δ'. Solutions as defined above are also called *generalized solutions.*

LEMMA 1. *If $\gamma u | \Delta' \geqq 0$, then $w \equiv \max(-u, 0) \in V_\Delta$.*

Proof. We first note that if $u_n \in C^1(\bar{\Omega})$ and $u_n \to u$ in $H^1(\Omega)$, then $\max(-u_n, 0) \to \max(-u, 0)$in $H^1(\Omega)$ (see [7, p. 50]). Hence if $u_n \to u$ in $H^1(\Omega)$, then, by continuity of $\gamma : H^1(\Omega) \to L_2(\partial\Omega)$,

$$\gamma \max(-u, 0) = \gamma \lim_{n\to\infty} \max(-u_n, 0) = \lim_{n\to\infty} \max(-\gamma u_n, 0) = \max(-\gamma u, 0),$$

where the first limit is taken in $H^1(\Omega)$ and the second in $L_2(\partial\Omega)$. Since $\max(-u, 0)$ vanishes on Δ' the result follows.

3. Results. Throughout this section we assume that hypotheses $\mathbf{H}_1$–$\mathbf{H}_6$ are satisfied. The spaces $H^j(\Omega)$ are endowed with a natural partial order as follows: we say that $u \geqq v$ if $u - v \geqq 0$ a.e. If u_1 and u_2 are elements of $H^1(\Omega)$, then the function $\max(u_1, u_2)$ defined by $\max(u_1, u_2)(x) = \max(u_1(x), u_2(x))$ also is a member of $H^1(\Omega)$. This follows from a fact proved in [7] that this statement is true if $u_2(x) \equiv k$, a constant function, and the simple observation that $\max(u_1, u_2) =$

$u_1 + \max(u_2 - u_1, 0)$. It also follows from Lemma 3.2 of [7, p. 51] that $D_i \max(u, k)(x) = 0$ a.e. on the set $\{x \in \Omega | u(x) \leqq k\}$. We define

$$K^j = \{u \in H^j(\Omega) | u \geqq 0\}.$$

We shall say that u is a *V_Δ-subsolution* if $u \in H^1(\Omega)$ and $\mathbf{l}(u, v) \leqq 0$ for all $v \in V_\Delta \cap K^1$.

THEOREM (Stampacchia). *Let* $\mathbf{a}$ *be a continuous real bilinear functional on a real Hilbert space* Y *with inner product* $(\cdot, \cdot)$. *Let* U *be a closed convex subset of* Y *and suppose* $\mathbf{a}$ *is coercive on* $U - U$, *i.e., there is a positive constant* c *such that* $\mathbf{a}(y, y) \geqq c(y, y)$ *for all* $y \in Y$ *which are of the form* $y_1 - y_2$, *with* $y_1, y_2 \in U$. *Let*

$$V_y = \{z \in Y | \exists \varepsilon > 0: y + \varepsilon z \in U\}.$$

Then, for each $f \in Y$, *there exists exactly one element* $y \in U$ *such that*

$$\mathbf{a}(y, z) \geqq (f, z) \quad \textit{for all } z \in V_y.$$

The proof of this theorem is rather lengthy and can be found in [9]. Although Stampacchia assumes $\mathbf{a}$ is coercive on all of Y, a careful examination of the proof reveals we only need this property on $U - U$. In fact, the proof requires only one minor change, namely, where it is shown that the functional $\mathbf{a}(u, u) - 2(f, u)$ is uniformly bounded from below for all $u \in U$. To show this, one merely notes that, for some fixed $u_0 \in U$, $\mathbf{a}(u, u) - 2(f, u)$ can be written instead as

$$\begin{aligned} &\mathbf{a}(u - u_0, u - u_0) + \mathbf{a}(u, u_0) + \mathbf{a}(u_0, u) - \mathbf{a}(u_0, u_0) - 2(f, u) \\ &\geqq c(u - u_0, u - u_0) - c_1\|u\| - c_2 \\ &\geqq c\|u\|^2 - c_3\|u\| - c_4 \geqq -(c_4 + c_3^2/4c), \end{aligned}$$

where c_1, c_2, c_3 and c_4 are constants.

LEMMA 2. *Suppose that* u_1 *and* u_2 *are two V_Δ-subsolutions; then* $w = \max(u_1, u_2)$ *is also a V_Δ-subsolution.*

Proof. Let U be the collection of all $u \in H^1(\Omega)$ which satisfy $u \leqq w$ and $u - w \in V_\Delta$. Certainly U is closed and convex, $U - U = V_\Delta$, and for any $\eta \in U$,

$$V_\eta = \{v \in H^1(\Omega) | \exists \varepsilon > 0: \eta + \varepsilon v \in U\}.$$

Note that $V_\eta \subset V_\Delta$. Pick η to be the unique element of U for which $\mathbf{l}(\eta, z) \geqq 0$ for all z in V_η. Note that $-K^1 \cap V_\Delta \subset V_\eta$ and hence η is a V_Δ-subsolution and $\eta \leqq w$. Let $\xi = \max(u_1, \eta)$; then $\xi - \eta \in V_\eta \cap K^1$. Therefore

$$\mathbf{l}(\eta, \xi - \eta) \geqq 0$$

while, by the above remarks,

$$\mathbf{l}(\xi, \xi - \eta) = \mathbf{l}(u_1, \xi - \eta) \leqq 0$$

since u_1 is a subsolution and $\xi(x) = u_1(x)$ whenever $\xi(x) \geqq \eta(x)$. Combining these two inequalities and using the facts that $\mathbf{l}$ is V_Δ-coercive and $\xi - \eta \in V_\Delta$ we have

$$\|\xi - \eta\|_1^2 \leqq C^{-1}\mathbf{l}(\xi - \eta, \xi - \eta) \leqq 0,$$

and therefore $\xi = \eta$ and consequently $u_1 \leqq \eta$. Similarly we obtain $u_2 \leqq \eta$ and so $w = \eta$, a subsolution.

LEMMA 3. *If u is a (generalized) solution of* (4) *with $\phi \geqq 0, \psi \geqq 0, \theta \geqq 0$, then $u \geqq 0$.*

Proof. Both $-u$ and 0 are V_Δ-subsolutions and therefore so is $\max(-u, 0)$. But also $\max(-u, 0) \in K^1 \cap V_\Delta$ and therefore $\mathbf{l}(\max(-u, 0), \max(-u, 0)) \leqq 0$, which, due to the fact that $\mathbf{l}$ is V_Δ-coercive, means $\max(-u, 0) = 0$.

We shall need the fact that the solution of the linear problem is bounded whenever the nonhomogeneous terms ϕ, ψ and θ are bounded. The above lemma plays an important role in the proof of this result. First we need another definition: we say that a distribution F is a member of $W_s^{-1}(\Omega)$, $1 < s$, if $F = F_0 + D_i F_i$ where $F_i \in L_s(\Omega)$ and $F_0 \in L_t(\Omega)$ with $t = sm/(m+s)$. Using the Sobolev embedding theorem one easily verifies such distributions are continuous linear functionals on $W_{s^*}^1(\Omega)$ where $1/s + 1/s^* = 1$.

LEMMA 4. *Suppose $\theta \in H^1(\Omega) \cap L_\infty(\Omega)$, $\phi \in L_\infty(\Omega)$ and $\psi \in L_\infty(\partial\Omega)$, then every solution of* (4) *is a member of $L_\infty(\Omega)$.*

Proof. In the one-dimensional case there is nothing to prove since by definition a solution is in $H^1(\Omega)$ and by the Sobolev embedding theorem $H^1(\Omega) \subset L_\infty(\Omega)$. Let us therefore proceed to the multidimensional cases. In case I, the Dirichlet problem, it is known (e.g., [7]) that the solution is in $L_\infty(\Omega)$ if $\phi \in W_q^{-1}(\Omega)$, $q > m$, and $\theta \in L_\infty(\partial\Omega)$. Since $L_\infty(\Omega) \subset W_q^{-1}(\Omega)$ these conditions are satisfied. Next consider case II, the Neumann and Robin problems. We use a regularity result of Stampacchia ([10]) which states that the generalized solution of $\mathbf{L}U = \phi$, $\partial U/\partial N = 0$, is in $L_\infty(\Omega)$ provided $\phi \in W_q^{-1}(\Omega)$. Let ξ_R be the classical solution of $\Delta\xi = 0$ in Ω with $\partial\xi/\partial N = R$ on $\partial\Omega$ and let u_R be the generalized solution of $Lu = \phi$ in Ω with $\partial u/\partial N = R$ on $\partial\Omega$ (see [7]). Letting $U_R = u_R - \xi_R$ we obtain

$$\begin{aligned} LU_R &= \phi - L\xi_R \in W_q^{-1}(\Omega) \quad &&\text{in } \Omega, \\ \partial U_R/\partial N &= 0 &&\text{on } \partial\Omega. \end{aligned}$$

By Stampacchia's result $U_R \in L_\infty(\Omega)$ and hence $u_R \in L_\infty(\Omega)$. Next choose $R \geqq \|\psi\|_\infty$ and $M \geqq \|u_R\|_\infty + \|\theta\|_\infty$. We see that

$$L(u_R + M - u) = aM \geqq 0 \quad \text{in } \Omega,$$

$$\partial(u_R + M - u)/\partial N + \sigma(u_R + M - u) \geqq R - \psi + \sigma(u_R + M) \geqq 0 \quad \text{on } \Delta,$$

$$u_R + M - u \geqq 0 \quad \text{on } \Delta'.$$

Hence by Lemma 3, $u_R + M - u \geqq 0$ and thus $u \leqq u_R + M$. Similarly $u \geqq u_{-R} - M$ and therefore $u \in L_\infty(\Omega)$. Finally, in case III, we note that for a sufficiently large positive number M,

$$\mathbf{L}(u \pm M) \gtreqless 0, \qquad \mathbf{B}(u \pm M) \gtreqless 0,$$

so that, by Lemma 3, $|u| \leqq M$ almost everywhere.

We now turn our attention to the existence of solutions to the linear problem. In settings such as these this is often based on the Lax-Milgram theorem. However we need a more general result, namely, the above theorem of Stampacchia, or, more accurately, the following corollary.

COROLLARY 5. *Suppose* $\mathbf{a}(u, v)$ *is a continuous real bilinear functional on a real Hilbert space* Y *with inner product* $(\cdot\,,\cdot)$. *Suppose* X *is a subspace of* Y *and* $\mathbf{a}$ *is* X*-coercive. Then for each continuous real functional* $\mu \in Y^*$ *and each* $y \in Y$ *there exists exactly one element* $x \in y + X$ *such that*

$$\mathbf{a}(x, z) = \mu(z) \quad \textit{for all } z \in X.$$

Proof. Let $U = y + X$. In this case $V_u = X$ for all $u \in U$. Certainly there is an element $f_\mu \in Y$ so that $\mu(z) = (f_\mu, z)$ for all $z \in Y$. Applying the theorem we find a unique $x \in X$ for which

$$\mathbf{a}(x, z) \geqq (f_\mu, z) \quad \text{for all } z \in X.$$

However if $z \in X$, then since X is a subspace, $-z \in X$ and hence we must in fact have

$$\mathbf{a}(x, z) = (f_\mu, z) = \mu(z), \quad \text{for all } z \in X.$$

LEMMA 6. *Suppose* $\phi \in L_\infty(\Omega)$, $\psi \in L_{2(m-1)/m}(\partial\Omega)$ *and* $\theta \in H^1(\Omega)$. *Then* (4) *has a unique solution in* $H^1(\Omega)$. *There exists a number* $J > 0$, *independent of* ϕ, ψ *and* θ, *such that* $\|u\|_1 \leqq J(\|\phi\|_\infty + \|\psi\|_* + \|\theta\|_1)$, *where* $\|\cdot\|_*$ *is the* $L_{2(m-1)/m}(\partial\Omega)$ *norm.*

Proof. Let us examine the various terms in (5). On the left-hand side we have a continuous V_Δ-coercive bilinear functional. On the right-hand side we have a continuous linear functional acting on η. To see this we first note that $\eta \to (\phi, \eta)_\Omega$ is obviously continuous and linear. Applying the Sobolev embedding theorem [7] we have $\gamma\eta \in L_{2(m-1)/m-2}(\partial\Omega)$, a function space into which $H^1(\Omega)$ is continuously mapped via γ. Therefore $\eta \to (\psi, \eta)_\Delta$ is a continuous linear functional. We therefore have to solve

$$\mathbf{l}(u, \eta) = \langle d, \eta\rangle \quad \text{for all } \eta \in V_\Delta,$$

where d is a continuous real linear functional on $H^1(\Omega)$. Applying the above corollary with $X = V_\Delta$ and $y = \theta$ we are done. The last part follows from $\mathbf{H}_6$:

$$C\|u-\theta\|_1^2 \leqq \mathbf{l}(u-\theta, u-\theta) = (\phi, u-\theta)_\Omega + (\psi, u-\theta)_\Delta - \mathbf{l}(\theta, u-\theta),$$

where the right-hand side is a continuous linear functional acting on $(u-\theta) \in H^1(\Omega)$.

We denote the unique solution of (4) by $\mathbf{G}(\phi, \psi, \theta)$, or, suppressing θ, by $\mathbf{G}(\phi, \psi)$.

Finding a (generalized) solution of (1) clearly amounts to finding a fixed point:

$$u = \mathbf{G}(f(x, u), g(x, \gamma u), \theta) \equiv \mathbf{A}(u). \tag{6}$$

LEMMA 7. *Suppose* $\theta \in H^1(\Omega) \cap L_\infty(\Omega)$.

(i) $\mathbf{A}$ *maps* $H^1(\Omega) \cap L_\infty(\Omega)$ *into itself.*

(ii) *If* $f(x, u)$ *and* $g(x, u)$ *are nonincreasing with respect to* u, *then* $\mathbf{A}$ *is order reversing.*

(iii) *Let* $B \subset H^1(\Omega) \cap L_\infty(\Omega)$ *be bounded with respect to the norm* $\|\cdot\|_\infty$ *and suppose that* $\mathbf{A}(B) \subset B$. *Then* $\mathbf{A}$ *is a compact continuous map on* B *with respect to the relative topology from* $H^1(\Omega)$.

Proof. (i) follows immediately from $\mathbf{H}_2$ and Lemma 4.

(ii) Suppose that $u_1 \geqq u_2$; *then* $f(x, u_1) \leqq f(x, u_2)$ and $g(x, u_1) \leqq g(x, u_2)$. Then by the linearity of G and Lemma 3 we have

$$\mathbf{A}(u_2) - \mathbf{A}(u_1) = \mathbf{G}(f(x, u_2) - f(x, u_1), g(x, u_2) - g(x, u_1), 0) \geqq 0.$$

(iii) Let $\{u_l\}$ be a sequence in B which is bounded with respect to the norm $\|\cdot\|_1$ and let $v_l = \mathbf{A}(u_l)$. Lemma 6 implies that $\{v_l\}$ is also bounded in that norm. By Sobolev's embedding theorem the sequences $\{u_l\}$ and $\{v_l\}$ are precompact with respect to the $L_2(\Omega)$ topology, so we may assume $u_l \to u$ and $v_l \to v$ in $L_2(\Omega)$. In fact we may also assume that $\{\gamma u_l\}$ and $\{\gamma v_l\}$ converge in $L_2(\partial\Omega)$. We have

$$\mathbf{I}(v_l, \eta) = (f(x, u_l), \eta)_\Omega + (g(x, u_l), \eta)_\Delta,$$

for all η in V_Δ, in particular for $\eta = v_l - v_m$. Hence

$$\begin{aligned}\|v_l - v_m\|_1^2 &\leqq \frac{1}{C}\mathbf{I}(v_l - v_m, v_l - v_m) \\ &= \frac{1}{C}\{(f(x, u_l) - f(x, u_m), v_l - v_m)_\Omega + (g(x, u_l) - g(x, u_m), v_l - v_m)_\Delta\} \\ &\leqq k\{\|v_l - v_m\|_\Omega + \|v_l - v_m\|_\Delta\},\end{aligned}$$

where k is some constant and the norms are the $L_2(\Omega)$ and $L_2(\Delta)$ norms respectively. It follows that $\{v_l\}$ is a Cauchy sequence in $H^1(\Omega)$, which proves $\mathbf{A}$ is a compact map on B. Furthermore, since f and g satisfy the Caratheodory conditions it follows that the Nemytskii operators defined through them are continuous on B and γB respectively in their L_2 topologies [13]. As above we can obtain the following inequality:

$$\begin{aligned}\|\mathbf{A}(u) - \mathbf{A}(u^*)\|_1^2 &\leqq \frac{1}{C}\{(f(x, u) - f(x, u^*), \mathbf{A}(u) - \mathbf{A}(u^*))_\Omega \\ &\qquad + (g(x, u) - g(x, u^*), \mathbf{A}(u) - \mathbf{A}(u^*))_\Delta\} \\ &\leqq \frac{1}{C}\{\|f(x, u) - f(x, u^*)\|_\Omega \|\mathbf{A}(u) - \mathbf{A}(u^*)\|_\Omega \\ &\qquad + \|g(x, u) - g(x, u^*)\|_\Delta \|\mathbf{A}(u) - \mathbf{A}(u^*)\|_\Delta\} \\ &\leqq k(\|u - u^*\|_1)\|\mathbf{A}(u) - \mathbf{A}(u^*)\|_1\end{aligned}$$

for some continuous function k, $k(0) = 0$. This proves continuity of $\mathbf{A}$.

We now prove our first existence result for (1).

THEOREM 8. *Suppose $f(x, u)$ and $g(x, u)$ are monotone nonincreasing with respect to u. Suppose there exists a function $u_0 \in L_\infty(\Omega)$ such that either*

(i) $$\mathbf{A}(u_0) \leqq u_0 \geqq \mathbf{A}(\mathbf{A}(u_0))$$

or

(ii) $$\mathbf{A}(u_0) \geqq u_0 \leqq \mathbf{A}(\mathbf{A}(u_0));$$

then (6), *and hence* (1), *has a unique solution in* $H^1(\Omega)$.

Proof. Let us suppose (i) is the case. We have $f(x, u_0(x)) \in L_\infty(\Omega)$ and $g(x, u_0(x)) \in L_\infty(\Omega)$, so by Lemma 4, $\mathbf{A}(u_0) \in L_\infty(\Omega) \cap H^1(\Omega)$. Let us define

$$\langle v_1, v_2 \rangle = \{v \in H^1(\Omega) | v_1 \leqq v \leqq v_2\}.$$

If $\mathbf{A}(u_0) \leqq u \leqq u_0$, then $\mathbf{A}(u_0) \leqq \mathbf{A}(u) \leqq \mathbf{A}(\mathbf{A}(u_0)) \leqq u_0$; hence $\mathbf{A}$ maps the closed (in $H^1(\Omega)$) convex set $\langle \mathbf{A}(u_0), u_0 \rangle$ into itself. Therefore we can apply Schauder's theorem. To show uniqueness suppose u_1 and u_2 are both solutions. Then

$$\mathbf{L}(u_1 - u_2) = f(x, u_1) - f(x, u_2),$$

$$\mathbf{B}(u_1 - u_2) = g(x, u_1) - g(x, u_2)$$

or

$$0 \leqq \mathbf{l}(u_1 - u_2, u_1 - u_2)$$
$$= (f(x, u_1) - f(x, u_2), u_1 - u_2)_\Omega + (g(x, u_1) - g(x, u_2), u_1 - u_2)_\Delta \leqq 0.$$

Since $\mathbf{l}$ is assumed to be V_Δ-coercive $u_1 = u_2$.

The hypotheses (i) or (ii) in the above theorem may not be easy to verify. In the results below we use more practical hypotheses. The monotonicity condition can also be removed as one might expect. Schauder's principle is used to show existence of a fixed point and therefore the monotonicity is not essential; it is used only to obtain a convex set which is left invariant by $\mathbf{A}$.

COROLLARY 9. *If f and g are uniformly bounded on $\bar{\Omega} \times R^1$, then* (1) *has a solution in $L_\infty(\Omega) \cap H^1(\Omega)$.*

Proof. Suppose a number M is chosen so that $-M \leqq f(x, u), g(x, u), \theta \leqq M$ for all (x, u). Let $u_0 = \mathbf{G}(M, M, M)$. $\mathbf{A}$ takes $\langle -u_0, u_0 \rangle$ into itself, and the result follows from Schauder's principle.

THEOREM 10. *Suppose $f(x, u)$ and $g(x, u)$ are uniformly bounded from below for all (x, u). Suppose there exist arbitrarily large positive numbers N such that for all numbers u_1 and u_2 with $u_1 \leqq -N \leqq u_2$ we have $f(x, u_2) \leqq f(x, u_1)$ and $g(x, u_2) \leqq g(x, u_1)$. Then* (1) *has a solution in $L_\infty(\Omega) \cap H^1(\Omega)$.*

Before proving this we note that the hypothesis is satisfied if, for example, $f(x, u)$ and $g(x, u)$ are monotone in the region where $u \leqq -N$ for some number N, and that these functions assume values no greater than $f(x, -N)$ and $g(x, -N)$ respectively in the region $u \geqq -N$.

Proof. Let M be a number so that $-M$ is a lower bound for the values $f(x, u)$ and $g(x, u)$. Choose a number $N \geqq \max(\|\mathbf{A}(0)\|_\infty, \|\mathbf{G}(-M, -M)\|_\infty)$ which satisfies the condition stated in the hypothesis of the theorem. Then $-N \leqq \mathbf{A}(0) \leqq \mathbf{A}(-N)$. Now suppose $-N \leqq u \leqq \mathbf{A}(-N)$, then $\mathbf{A}(u) \leqq \mathbf{A}(-N)$ and also $\mathbf{A}(u) = \mathrm{G}(f(x, u), g(x, u)) \geqq \mathrm{G}(-M, -M) \geqq -N$. Therefore $\mathbf{A}$ maps $\langle -N, \mathbf{A}(-N) \rangle$ into itself and we can again apply Schauder's principle. Of course a similar proof yields the following theorem.

THEOREM 11. *Suppose $f(x, u)$ and $g(x, u)$ are uniformly bounded from above for all (x, u). Suppose there exist arbitrarily large positive numbers N such that for all numbers u_1 and u_2 with $u_1 \leqq N \leqq u_2$ we have $f(x, u_2) \leqq f(x, u_1)$ and $g(x, u_2) \leqq g(x, u_1)$. Then* (1) *has a solution in $L_\infty(\Omega) \cap H^1(\Omega)$.*

We do not need f and g to be bounded either from below or from above provided we put some restriction on their growth as $u \to \pm\infty$. More specifically we will need the following:

$$f(x,u),\, g(x,u)\begin{cases}\leqq \gamma_1+\gamma_2|u|^s & \text{if } u\leqq 0,\\ \geqq -\gamma_1+\gamma_2|u|^t & \text{if } u\geqq 0,\end{cases} \tag{7}$$

with $0<st<1$. In this case we allow the functions $f(x, u)$ and $g(x, u)$ to be nonmonotone in u in some bounded interval and require them to be nonincreasing outside of this interval.

THEOREM 12. *Suppose f and g satisfy* (7) *with $st<1$, and suppose there are numbers $u_0<v_0$ such that whenever $u\leqq u_0$ and $u\leqq u_1$ then $f(x,u)\geqq f(x,u_1)$ and $g(x,u)\geqq g(x,u_1)$, and whenever $u\geqq v_0$ and $u\geqq u_1$ then $f(x,u)\leqq f(x,u_1)$ and $g(x,u)\leqq g(x,u_1)$. Then* (1) *has a solution in $L_\infty(\Omega)\cap H^1(\Omega)$.*

Proof. Without loss of generality $\gamma_1, \gamma_2\geqq 1$. We note that if $\alpha>1$, then $\mathbf{G}(\alpha,\alpha,\theta)\leqq \mathbf{G}(\alpha,\alpha,\alpha|\theta|)\leqq \alpha k$ where $k=\|\mathbf{G}(1,1,|\theta|)\|_\infty$. Choose M so large that $-M\leqq u_0$ and $\gamma_1+\gamma_2 M^s\geqq v_0$. Also, since $st<1$ it is possible to choose M so large that $\gamma_1+\gamma_2(\gamma_1+\gamma_2M^s)^t\leqq Mk^{-1}(k^t+1)^{-1}$. We claim that $\langle -M, (\gamma_1+\gamma_2M^s)k\rangle$ is mapped into itself by $\mathbf{A}$. Let $-M\leqq u(x)\leqq(\gamma_1+\gamma_2M^s)k$; then

$$\begin{aligned}\mathbf{A}(u(x))\leqq \mathbf{A}(-M)&\leqq \mathbf{G}(\gamma_1+\gamma_2M^s,\ \gamma_1+\gamma_2M^s)\\ &\leqq(\gamma_1+\gamma_2M^s)k\end{aligned}$$

and

$$\begin{aligned}\mathbf{A}(u(x))&\geqq \mathbf{A}((\gamma_1+\gamma_2M^s)k)\\ &\geqq \mathbf{G}(-\gamma_1-\gamma_2(\gamma_1+\gamma_2M^s)^tk^t,\ -\gamma_1-\gamma_2(\gamma_1+\gamma_2M^s)^tk^t)\\ &\geqq -(1+k^t)(\gamma_1+\gamma_2(\gamma_1+\gamma_2M^s)^t)k\geqq -M.\end{aligned}$$

The result follows again by Schauders theorem.

4. Coerciveness, positivity and stability. Throughout this section we assume $\mathbf{H}_1$–$\mathbf{H}_5$. In all of the above results it is assumed that $\mathbf{l}(u, v)$ is V_Δ-coercive. Consider however the very simple heat radiation problem

$$\Delta u=0 \quad \text{in } \Omega, \tag{8}$$

$$\partial u/\partial N=a-bu^4 \quad \text{on } \partial\Omega, \tag{9}$$

where a and b are positive numbers and where Stefan's "fourth power law" is used. This problem lacks $H^1(\Omega)$-coerciveness. But if we rewrite the boundary condition as

$$\frac{\partial u}{\partial N}+u=a-bu^4+u \quad \text{on } \partial\Omega, \tag{10}$$

the associated bilinear form can be seen to be $H^1(\Omega)$-coercive. More generally, we can say that a problem which initially does not possess the coerciveness may, by adding zero order terms to either or both the equation and the boundary condition, be transformed to a problem which does have the property. In fact it is well known that if $\mathbf{L}$ is uniformly elliptic and if $\sigma\geqq 0$, then the bilinear functional $\mathbf{l}(u, v)+\lambda(u, v)_\Omega$ associated with $\mathbf{L}+\lambda$ is $H^1(\Omega)$-coercive for λ sufficiently large. We shall see how coerciveness may be obtained by adding a zero order term to $\mathbf{B}$

instead. Of course whenever we modify **L** or **B** in this manner the nonhomogeneous terms $f(x, u)$ and $g(x, u)$ are also modified and so, if we wish to apply one of the above existence theorems, we must check if the modified nonhomogeneous terms satisfy the hypotheses.

In problems (8) and (10) we have two more difficulties. First of all, the modified (as well as the original) nonhomogeneous term does not satisfy any of the hypotheses of the above existence theorems. Secondly, only positive (more accurately nonnegative) solutions are of physical significance. These difficulties are simultaneously overcome by redefining the nonhomogeneous term for negative u so that it satisfies the hypothesis in theorem and then showing that the new problem has a positive solution. The nonhomogeneous term may for example be redefined to be a for $u<0$.

For the sake of simplicity we shall from now on assume that $f(x, u)$ and $g(x, u)$ are continuous on $\bar{\Omega}\times[0, \infty)$ and that the coefficients of the first order terms of **L** are zero. To emphasize this we shall use $\mathbf{L}_1 \equiv -D_i a_{ij}(x) D_j + a(x)$.

We have the associated bilinear form

$$\mathbf{l}_1(u, v) = (a_{ij}D_j u, D_i v)_\Omega + (au, v)_\Omega + (\sigma u, v)_\Delta.$$

THEOREM 13. *Suppose that $f(x, t)\geqq 0$ and $g(x, t)\geqq 0$ whenever $t<0$. Suppose either $f(x, t)>0$ a.e. in Ω for each $t<0$ or $g(x, t)+\sigma(x)>0$ a.e. on Δ for each $t<0$. If $a\geqq 0$ and $u\in H^1$, $u-\theta\in V_\Delta$ for some $\theta\geqq 0$, and*

$$l_1(u, v) = (f(x, u), v)_\Omega + (g(x, u), v)_\Delta \quad \textit{for all } v\in V_\Delta,$$

then $u\geqq 0$.

Proof. Let $u^* = \min(0, u)$; then $u^*\in H^1(\Omega)$. By Lemma 1 and the remarks at the beginning of § 3, $u^*\in V_\Delta$ and $D_j u = D_j u^*$ a.e. on the set where $u^*(x)<0$. Since $\gamma u^* = \min(0, \gamma u)$ (see proof of Lemma 1) we have

$$(a_{ij}D_j u^*, D_i u^*)_\Omega + (au^*, u^*)_\Omega + (\sigma u^*, u^*)_\Delta = (f(x, u), u^*)_\Omega + (g(x, u), u^*)_\Delta.$$

Since $\mathbf{L}_1$ is uniformly elliptic, $a\geqq 0$, $\sigma\geqq 0$ and since on the support of u^* we have $f(x, u)\geqq 0$ and $g(x, u)\geqq 0$, we may conclude that both $(f(x, u), u^*)_\Omega$ and $(g(x, u)-\sigma u, u^*)_\Delta$ are zero. In case $f(x, t)>0$ whenever $t<0$ it follows immediately that $u^*\equiv 0$. In case $g(x, t)+\sigma>0$ whenever $t<0$ we have $g(x, u)-\sigma u^* = g(x, u^*)-\sigma u^*>0$ whenever $u^*<0$. But since $(g(x, u^*)-\sigma u^*, u^*)\geqq 0$, we see that $u^*\equiv 0$ on $\partial\Omega$ (i.e. $\gamma u^*\equiv 0$). This means [7] that $u^*\in H_0^1(\Omega)$. There is a positive number k, independent of u, such that [7]

$$\|u\|_1 \leqq k\int_\Omega (D_i u)(D_i u)\,dx \quad \text{for all } u\in H_0^1(\Omega).$$

Hence

$$\|u^*\|_1 \leqq k\int_\Omega (D_i u^*)(D_i u^*)\,dx \leqq k/\nu(a_{ij}D_j u^*, D_i u^*)_\Omega$$
$$= k/\nu(a_{ij}D_j u, D_i u^*)_\Omega \leqq 0.$$

We now state the following existence and uniqueness result which, in particular, is applicable to problem (8)–(9).

THEOREM 14. *Suppose that* $a \geqq 0$, $\theta \geqq 0$, $f(x, u)$ *and* $g(x, u)$ *are continuous on* $\bar{\Omega}\times[0, \infty)$ *with* $f(x, 0) \geqq 0$ *and* $g(x, 0) \geqq 0$, *and that there is a* $\delta > 0$ *and arbitrarily large positive numbers* N *such that either*

$$\begin{cases} f(x, u_2)+\delta u_2 \leqq f(x, u_1)+\delta u_1, \\ g(x, u_2) \qquad\quad \leqq g(x, u_1), \end{cases} \tag{11}$$

or

$$\begin{cases} f(x, u_2) \qquad\quad \leqq f(x, u_1), \\ g(x, u_2)+\delta u_2 \leqq g(x, u_1)+\delta u_1, \end{cases} \tag{12}$$

whenever $u_1 \leqq N \leqq u_2$. *Then the problem*

$$\begin{aligned} \mathbf{L}_1 u &\equiv D_i a_{ij}(x) D_j u + a(x) u = f(x, u) \quad &&\text{in } \Omega, \\ \mathbf{B}u &\equiv \partial\Omega/\partial N + \sigma(x) u \qquad\ = g(x, u) \quad &&\text{on } \Delta \subset \partial\Omega, \\ &\qquad\qquad\qquad u = \theta \geqq 0 \quad &&\text{on } \Delta' \equiv \partial\Omega - \Delta, \end{aligned} \tag{13}$$

has a positive solution. If, in addition, $f(x, u)$ *and* $g(x, u)$ *are nonincreasing with respect to* u *and either* $\mathbf{l}$ *is* V_Δ*-coercive or at least one of the function* $f(x, u)$ *or* $g(x, u)$ *is strictly decreasing with respect to* u, *then there is a unique positive solution.*

Proof. First we note the operator $\mathbf{L}_1 + \delta$ with boundary operator $\mathbf{B}_1$ leads to the corresponding bilinear form

$$(a_{ij}D_j u, D_i v)_\Omega + (au, v)_\Omega + (\sigma u, v)_\Delta + \delta(u, v)_\Omega$$

which is clearly V_Δ-coercive. Next we note that the operator $\mathbf{L}_1$ with boundary condition $\mathbf{B} + \delta$ leads to the bilinear form

$$(a_{ij}D_j u, D_i v)_\Omega + (au, v)_\Omega + (\sigma u, v)_\Delta + \delta(u, v)_\Delta.$$

We claim this form, which we shall denote by $\mathbf{l}_2$, is also V_Δ-coercive. For suppose it is not. Then there exists a sequence $\{u_k\} \subset V_\Delta$ such that $\|u_k\|_1 = 1$ for each k and $\mathbf{l}_2(u_k, u_k)$ tend to zero as $k \to \infty$. From uniform ellipticity it follows that $D_i u_k \to 0$ in $L_2(\Omega)$ as $k \to \infty$. Clearly $\gamma u_k \to 0$ in $L_2(\Delta)$. Since $\gamma u_k = 0$ on Δ' this means $\gamma u_k \to 0$ in $L_2(\partial\Omega)$. It is known (e.g., [5, p. 354]) that the norm $\|w\|_1$ is equivalent to the norm

$$\|w\| = \sum_{i=1}^{m} \|D_i w\|_\Omega + \|\gamma w\|_{\partial\Omega},$$

where the norms on the right-hand side are the $L_2(\Omega)$- and $L_2(\partial\Omega)$-norms respectively. Therefore $u_k \to 0$ in $H^1(\Omega)$, a contradiction. It follows that the hypotheses of Theorem 11 can be satisfied either by modifying $\mathbf{L}_1$ and $f(x, u)$ by the term δu or by modifying $\mathbf{B}$ and $g(x, u)$ by the term δu and redefinining these nonhomogenous terms to be $f(x, 0) + u(u-1)^{-1}$ and $g(x, 0) + u(u-1)^{-1}$ respectively for $u < 0$. We therefore know there exists a solution, which by Theorem 13 must be positive and hence also a solution of the original problem. If $f(x, u)$ and $g(x, u)$ are nonincreasing in u and if u_1 and u_2 are two positive solutions, then $w \equiv u_1 - u_2$ satisfies

$$\begin{aligned} 0 \leqq \nu(D_i w, D_i w) &\leqq (a_{ij} D_j w, D_i w)_\Omega + (aw, w)_\Omega + (\sigma w, w)_\Delta \\ &= (f(x, u_1) - f(x, u_2), w)_\Omega + (g(x, u_1) - g(x, u_2), w)_\Delta \leqq 0. \end{aligned}$$

If $\mathbf{l}_1$ is V_Δ-coercive it follows immediately that $w \equiv 0$; if not, then $D_i w \equiv 0$ for each i and hence $w = \text{constant}$. If also $f(x, u)$ or $g(x, u)$ is strictly decreasing, then it follows that this constant must be zero.

Let us very briefly discuss the stability of solutions to the associated parabolic equation

$$
\begin{aligned}
\frac{\partial U}{\partial t} + \mathbf{L}_1 U &= f(x, U) \quad && \text{in } \Omega, \\
\mathbf{B} U &= g(x, U) \quad && \text{on } \Delta, \\
U &= \theta \geqq 0 \quad && \text{on } \Delta', \\
U(x, 0) &= U_0(x).
\end{aligned} \tag{14}
$$

If $U_0(x) = u(x)$, the solution to the elliptic equation (13), then we have the solution $U(x, t) = u(x)$. If the initial data is perturbed by $\varepsilon v(x)$, one might expect a solution of the form

$$U(x, t) = u(x) + \varepsilon v(x)\, e^{-\alpha t} + O(\varepsilon^2).$$

A formal calculation as in [1], then leads to the conclusion that α must be an eigenvalue of

$$
\begin{aligned}
\mathbf{L}_1 v - f_u(x, u) v &= \alpha v \qquad \Omega, \\
\mathbf{B} v - g_u(x, u) v &= 0 \qquad \Delta, \\
v \in V_\Delta \quad (\text{i.e., } v &= 0 \quad \text{on } \Delta').
\end{aligned}
$$

Therefore, following Keller and Cohen [1], [11], we say that a solution $u(x)$ of (13) is stable if the principal (i.e., least) eigenvalue of (15) is positive.

THEOREM 15. *Suppose that $a \geqq 0$, that $f(x, u)$ and $g(x, u)$ are continuous on $\bar{\Omega} \times [0, \infty)$ and $\partial\Omega \times [0, \infty)$ respectively and that $f(x, u) \geqq 0$ and $g(x, u) \geqq 0$. Suppose that $f_u(x, u)$ and $g_u(x, u)$ exist and are nonpositive on $\bar{\Omega} \times [0, \infty)$. Suppose that $\mathbf{l}_1$ is V_Δ-coercive or that* (11) *or* (12) *are satisfied for all $u_1, u_2 \geqq 0$ and some fixed $\delta > 0$. Then the unique positive solution of* (13) *is stable.*

Proof. First note that by Theorems 11, 13 and 14 equation (13) indeed has a unique positive solution. Let v be an eigenfunction corresponding to the principal eigenvalue α_1; then

$$\mathbf{l}_1(v, v) - (f_u(x, u)v, v)_\Omega - (g_u(x, u)v, v)_\Delta = \alpha_1(v, v).$$

Clearly $\alpha_1 \geqq 0$ and $\alpha_1 > 0$ if l_1 is V_Δ-coercive. Suppose $\alpha_1 = 0$; then $l_1(v, v)$ must be zero so that $\nabla v \equiv 0$ and consequently v is a nonzero constant. In view of the hypothesis either $(f_u(x, u)v, v)_\Omega < -\delta(v, v)_\Omega$ or $(g_u(x, u)v, v)_{\partial\Omega} < -\delta(v, v)_{\partial\Omega}$ must obtain. This leads to a contradiction.

In conclusion we note that if $\partial\Omega$ is sufficiently smooth one may extend the existence results in this paper to certain oblique derivative problems. One only needs to require that the associated form is V_Δ-coercive, or that it can be modified to a V_Δ-coercive form. This can be insured by imposing bounds on the coefficients of the tangential derivatives occurring in the boundary condition. The regularity result of Stampacchia which was needed in some of the proofs was extended by R. Fiorenza [12] to the case of the oblique derivative problem, and consequently those proofs also go through without any difficulty.

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A MODEL OF SINGLE SPECIES POPULATION GROWTH*

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Abstract. The integral equation

$$(*)\qquad x(t)=\int_0^L P(L-s)h(t-L-\tau+s,\,x(t-L-\tau+s))\,ds$$

can be viewed as a model of single species population growth as well as certain other biological or economic processes. Explicit bounds are determined for solutions of (*); these bounds are expressed in terms of the relative magnitude of $h(t,x)$, $\int_0^L P(L-s)\,ds$ and the norm of the initial function of the solution. These bounds are used to determine sufficient conditions for the existence of periodic solutions of (*).

1. Introduction. In this paper we shall study a class of scalar integral equations which can be viewed as a model of single species population growth.

Let $x(t)$ denote the number of individuals in a population at time t and let $P(t)$ be the proportion of the population surviving to at least age t. Clearly we must have $P(0)=1$, $P(L)=0$, where L is the maximum life span of individuals in the population, and $P(t)$ is monotonically decreasing over the interval $[0, L]$. If the assumption is made that the number of births per unit time at time t is a function of the total population at time t, say $g(x(t))$, then the size of the population is governed by the equation

$$(1.1)\qquad x(t)=\int_0^L P(L-s)g(x(t-L+s))\,ds.$$

This is the equation derived by Cooke and Yorke [1] and [2]. It was shown there that in addition to being a model of single species population growth, this equation can be used as an infectious disease model and a model of capital growth. Their principal result was the following theorem: Assume that $g(x)$ is a continuously differentiable function and that $P(t)$ is continuously differentiable, nonincreasing and nonnegative on the interval $0\leqq t\leqq L$. Let $x(t)$ be any solution of (1.1) and let $[t_0-L,\omega]$ be its maximal interval of existence, where $t_0<\omega\leqq\infty$. Then one of the following holds: $x(t)\to\infty$, $x(t)\to$ constant or $x(t)\to-\infty$ as $t\to\omega$.

In [3] Cooke used a result of Levin and Shea to extend this result to the equation

$$x(t)=\int_0^L P(L-s)g(x(t-L+s))\,ds+f(t),$$

where $f(t)$ represents an imigration into the population.

In both of these models it is assumed that births per unit time is a function of the size of the population only, and in particular, is not dependent upon the time t. It also assumes no time lag between conception and birth.

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If we assume that there is a time lag τ between conception and birth and that the number of births also varies seasonally, then the number of births per unit time at time t is given by a function of the form $h(t-\tau, x(t-\tau))$, and the size of the population is governed by the equation

$$(*)\qquad x(t)=\int_0^L P(L-s)h(t-\tau-L+s, x(t-\tau-L+s))\,ds.$$

This is the model examined in this paper.

In § 3, explicit bounds are determined for the solutions of (*); these bounds are expressed in terms of the relative magnitude of $h(t, x)$, $\int_0^L P(s)\,ds$ and the norm of the initial function of the solution. The bounds are in general unrelated to the size of the positive numbers L and τ, although these constants determine the elapsed time needed for a solution to reach certain bounds. In § 4 the results of § 3 are applied to an asymptotic fixed-point theorem to show the existence of periodic solutions of (*). The approach used is similar to that used by numerous other authors, see, e.g., [12]–[15].

Cooke and Yorke [1], [2] also proposed a form of (1.1) with $P(t)=1$, $0\leqq t\leqq L$. This model arises under the assumption that all individuals in the population have life span L. If one again postulates a time lag τ between conception and birth this model becomes

$$(1.2)\qquad x(t)=\int_0^L g(x(t-L-\tau+s))\,ds.$$

Hale [11] obtained necessary conditions for the existence of periodic solutions of (1.2) for the class of functions $G=\{g: R^1\to[0,\infty)\,|\,g(x)=0,\ x\leqq 0 \text{ and } x\geqq 1,\ g(x)>0,\ 0<x<1\}$. The results of § 4 are also valid for (1.2).

Wangersky and Cunningham [5] discussed the special case of (*) found by setting $h(t, x)=bx$ and $p(s)=(-k/1-\exp(-kL))\exp(-k(L-s))$, and a number of authors, see, e.g., [6]–[10], have discussed various forms of the model

$$\dot{x}(t)=\left[b+\int_0^L P(L-s)g(x(t-s))\,ds\right]x(t).$$

Biologically, in both models, b is the birth rate which is assumed constant, while in the second model $g(x)$ measures how a change in the population x changes the death rate. A significant advantage of (*) over these models is the fact that the birth rate is allowed to vary seasonally as well as nonlinearly with respect to the size of the population.

2. Preliminaries and basic assumptions. It will be assumed throughout that:

H_1: L and τ are positive real numbers.

H_2: $h(t, x)$ is continuous on $[0,\infty)\times R^1$, $h(t, 0)=0$, $t\geqq 0$, and $h(t, x)>0$, $t\geqq 0$, $x>0$.

H_3: $P(0)=1$, $P(s)=\int_s^L p(u)\,du$ for $0\leqq s\leqq L$, where $p(s)$ is integrable over the interval $[0, L]$ and $0\leqq p(s)\leqq K_1$ for $0\leqq s\leqq L$. Let $P_0=\int_0^L P(L-s)\,ds$.

The following assumptions will be needed for the discussion of periodicity and oscillation of solutions of (*).

H_4: There is $\omega > 0$ such that $h(t+\omega, x) = h(t, x)$ for $t \geqq 0$ and $x \in R^1$.
H_5: $h(t, x)$ satisfies H_4 and for each $x > 0$,

$$\sup \{|h(t, x) - h(0, x)| 0 \leqq t \leqq \omega\} > 0.$$

Let $I = [-L - \tau, 0]$, $J = [0, L]$, $C = C[I, R^1]$ be the Banach space of continuous real-valued functions on the interval I and $C_+ = \{\varphi \in C | 0 < \varphi(t) t \in I\}$. For $\varphi \in C$, we define $\|\varphi\| = \sup_{t \in I} |\varphi(t)|$. Let $x \in C[[-L-\tau, A], R^1]$ for some $A > 0$; then for each $t \in [0, A]$ we define $x_t \in C$ by $x_t(s) = x(t+s)$, $s \in I$.

Since the only initial functions in C which are biologically reasonable are those which satisfy $\varphi(t) > 0$, $t \in I$, we will assume henceforth, to avoid needless repetition, that the notation $x(t; \varphi\, t_0)$ refers to a solution of (*) with $\varphi \in C_+$ and $t_0 \geqq 0$. When we refer to a solution $x(t)$ of (*), it is $x(t) = x(t; \varphi, t_0)$.

3. Bounds for solutions of (*). We shall show first that for each $\varphi \in C_+$ and $t_0 \geqq 0$, the solution $x(t) = x(t; \varphi, t_0)$ of (*) satisfying $x_{t_0} = \varphi$, exists for $t \geqq t_0 - L - \tau$. If

$$\varphi(0) = \int_0^L P(L-s) h(t_0 - \tau - L + s, \varphi(s - \tau - L))\, ds, \tag{3.1}$$

then $x(t; \varphi, t_0)$ is continuous for $t \geqq t_0 - \tau - L$, while if

$$\varphi(0) \neq \int_0^L P(L-s) h(t_0 - \tau - L + s, \varphi(s - \tau - L))\, ds,$$

then $x(t; \varphi, t_0)$ has a discontinuity at $t = t_0$ and is continuous for all other values of $t \geqq t_0 - \tau - L$. In the latter case, the solution $x(t; \varphi, t_0)$ represents a valid population only for $t > t_0 + \tau + L$.

The following result can be easily established using standard arguments which we omit.

THEOREM 1. *If* H_1–H_3 *are satisfied, then every solution of* (*) *exists and is positive for* $t \geqq t_0 - L - \tau$. *The solution* $x(t; \varphi, t_0)$ *is continuous at* $t = t_0$ *if and only if*

$$\varphi(0) = \int_0^L P(L-s) h(t_0 - \tau - L + s, \varphi(s - \tau - L))\, ds$$

and is continuous at all other $t \geqq t_0 - \tau - L$.

THEOREM 2. *Assume that* H_1–H_3 *are satisfied and that there are positive constants* b, $r > 1$ *and* $B < (bP_0)^{-1/(r-1)}$ *such that*

(i) $h(t, x) \leqq bx^r$, $x \geqq B$,

(ii) $h(t, x) \leqq bB^r$, $0 < x \leqq B$;

then every solution $x(t) = x(t; \varphi, t_0)$ *of* (*) *satisfies*

$$0 < x(t) < \max \{bP_0 B^r, (bP_0)^{-1/(r-1)} [\|\varphi\| (bP_0)^{1/(r-1)}]^{r^n}\} \tag{3.2}$$

for $t_0 + (n-1)(L+\tau) < t \leqq t_0 + n(L+\tau)$ *if* $\|\varphi\| < (bP_0)^{1/(r-1)}$ *and for* $t_0 + (n-1)\tau < t \leqq t_0 + n\tau$ *if* $\|\varphi\| \geqq (bP_0)^{-1/(r-1)}$, $n = 1, 2, \cdots$.

Note. $bP_0 B^r < B$.

Proof. It follows from Theorem 1 that $x(t)$ exists and is positive for $t \geqq t_0 - L - \tau$.

We consider first $x(t)=x(t;\varphi,t_0)$ such that $\|\varphi\|<B$. For $t_0<t\leqq t_0+\tau$ we have

$$x(t)=\int_0^L P(L-s)h(t-\tau-L+s,\varphi(t-\tau-L+s))\,ds$$
$$\leqq\int_0^L P(L-s)bB^r\,ds$$
$$=bP_0B^r,$$

and since $bP_0B^r<B$, we can proceed inductively to show that $x(t)\leqq bP_0B^r$ for $t_0<t\leqq t_0+n\tau$, $n=1,2,\cdots$, or,

$$x(t)\leqq bP_0B^r,\qquad t>t_0. \tag{3.3}$$

Next we consider $x(t)=x(t;\varphi,t_0)$ such that $B\leqq\|\varphi\|<(bP_0)^{-1/(r-1)}$. For $t_0<t\leqq t_0+\tau$ we have

$$x(t)=\int_0^L P(L-s)h(t-\tau-L+s,\varphi(t-\tau-L+s))\,ds$$
$$\leqq\int_0^L P(L-s)b\|\varphi\|^r\,ds$$
$$=bP_0\|\varphi\|^r.$$

Since $\|\varphi\|\leqq(bP_0)^{-1/(r-1)}$ and $r>1$, $bP_0\leqq\|\varphi\|^{1-r}$ and it follows that $bP_0\|\varphi\|^r\leqq\|\varphi\|$. This last inequality implies that $0<x(t)\leqq\|\varphi\|$ for $t_0-L-\tau\leqq t\leqq t_0+\tau$, and if k is a positive integer such that $(k-1)\tau<L+\tau\leqq k\tau$, k iterations of the preceeding argument yields

$$x(t)\leqq bP_0\|\varphi\|^r,\qquad t_0<t\leqq t_0+L+\tau. \tag{3.4}$$

The right-hand side of (3.2) will be established by induction on $n(L+\tau)$, so let us assume that

$$x(t)\leqq(bP_0)^{\sum_{i=0}^{n-2}r^i}\|\varphi\|^{r^{n-1}}=(bP_0)^{-1/(r-1)}[\|\varphi\|(bP_0)^{1/(r-1)}]^{r^{n-1}} \tag{3.5}$$

for $t_0+(n-2)(L+\tau)\leqq t\leqq t_0+(n-1)(L+\tau)$, and that the right-hand side of (3.4) is greater than B. It follows from (3.5) and (*) that

$$x(t)\leqq\int_0^L P(L-s)b[(bP_0)^{\sum_{i=0}^{n-2}r^i}\|\varphi\|^{r^{n-1}}]^r\,ds$$
$$=(bP_0)^{\sum_{i=0}^{n-1}r^i}\|\varphi\|^{r^n}$$
$$=(bP_0)^{-1/(r-1)}[\|\varphi\|(bP_0)^{1/(r-1)}]^{r^n}$$

for $t_0+(n-1)(L+\tau)<t\leqq t_0+(n-1)(L+\tau)+\tau$. Continuing as in the proof of (3.4), we show that

$$x(t)\leqq(bP_0)^{-1/(r-1)}[\|\varphi\|(bP_0)^{1/(r-1)}]^{r^n} \tag{3.6}$$

for $t_0+(n-1)(L+\tau)<t\leqq t_0+n(L+\tau)$, and it follows by induction that (3.6) is valid for $n=1,2,\cdots$, as long as $B<(bP_0)^{-1(r-1)}[\|\varphi\|(bP_0)^{1/(r-1)}]^{r^n}$.

Since $\|\varphi\|<(bP_0)^{-1/(r-1)}$, $\|\varphi\|(bP_0)^{1/(r-1)}<1$ which, with $r>1$, implies the existence of a positive integer N such that $(bP_0)^{-1/(r-1)}[\|\varphi\|(bP_0)^{1/(r-1)}]^{r^N}\leqq B$ and we can assume that N is the smallest such positive integer. We now have

$$x(t+t_0+N(L+\tau))=x(t, x_{t_0+N(L+\tau)}, t_0+N(L+\tau)), \qquad t\geqq t_0+N(L+\tau),$$

where $\|x_{t_0+N(L+\tau)}\|\leqq B$, and it follows from (3.3) that $x(t)<bP_0B^r$, $t>t_0+N(L+\tau)$.

Finally, if $\varphi\in C_+$, and $\|\varphi\|\geqq(bP_0)^{-1/(r-1)}$, then

$$\begin{aligned} x(t) &\leqq \int_0^L P(L-s)b\|\varphi\|^r\,ds \\ &= bP_0\|\varphi\|^r \end{aligned}$$

for $t_0<t\leqq t_0+\tau$. Since $B<(bP_0)^{-1/(r-1)}\leqq\|\varphi\|$. $B<\|\varphi\|<bP_0\|\varphi\|^r$, which implies that $x(t)\leqq bP_0\|\varphi\|^r$ for $t_0-L-\tau\leqq t\leqq t_0+\tau$. Proceeding by induction, we assume that

$$x(t)\leqq(bP_0)^{\sum_{i=0}^{n-1}r^i}\|\varphi\|^{r^{n-1}}=(bP_0)^{-1/(r-1)}[\|\varphi\|(bP_0)^{1/(r-1)}]^{r^{n-1}}$$

for $t_0-L-\tau\leqq t\leqq t_0+(n-1)\tau$, which implies that

$$x(t)\leqq(bP_0)^{-1/(r-1)}[\|\varphi\|(bP_0)^{1/(r-1)}]^{r^n} \tag{3.7}$$

for $t_0+(n-1)\tau<t\leqq t_0+n\tau$ as long as the right-hand side of (3.7) is greater than B. But $\|\varphi\|\geqq(bP_0)^{-1/(r-1)}$ implies that $\|\varphi\|(bP_0)^{1/(r-1)}\geqq 1$ and as a result $(bP_0)^{-1/(r-1)}[\|\varphi\|(bP_0)^{1/(r-1)}]^{r^n}>B$, $n=1, 2, \cdots$, which completes the proof of Theorem 2.

COROLLARY. *If* H_1–H_3 *are satisfied,* $h(t,x)\leqq bx^r$, $t\geqq 0$, $x>0$, *where* $b>0$ *and* $r>1$ *and if* $\|\varphi\|<(bP_0)^{-1/(r-1)}$, *then the solution* $x(t)=x(t;\varphi,t_0)$ *of* (*) *satisfies* $x(t)\to 0$ *as* $t\to\infty$.

THEOREM 3. *Suppose that* H_1–H_3 *are satisfied and that there are positive constants* a, $d>1$ *and* $A>(aP_0)^{-1/(d-1)}$ *such that* $h(t,x)\geqq ax^d$, $x\geqq A$. *If* $\varphi\in C_+$ *satisfies* $\varphi(t)\geqq A$, $t\in I$, *then the solution* $x(t)=x(t;\varphi,t_0)$ *of* (*) *satisfies* $x(t)\to\infty$ *as* $t\to\infty$.

Proof. If $\varphi(t)\geqq A$, then it follows from (*) that

$$\begin{aligned} x(t) &\geqq \int_0^L P(L-s)aA^d\,ds \\ &= aP_0A^d \end{aligned}$$

for $t_0<t\leqq t_0+\tau$. Since $A>(aP_0)^{-1/(d-1)}$ and $d>1$, $aP_0A^d>A$, so if k is a positive integer such that $(k-1)\tau<L+\tau\leqq k\tau$, then the preceeding argument can be repeated k times to show that $x(t)\geqq aP_0A^d$ $t_0<t\leqq L+\tau$.

Following the proof of Theorem 2, we proceed by induction to establish the inequality

$$x(t)\geqq(aP_0)^{-1/(d-1)}[A(aP_0)^{1/(d-1)}]^{d^n}$$

for $t_0+(n-1)(L+\tau)<t\leqq t_0+n(L+\tau)$, $n=1, 2, \cdots$. The theorem now follows from the assumptions $d>1$ and $A>(aP_0)^{-1/(d-1)}$.

THEOREM 4. *Assume that* H_1–H_3 *are satisfied and that there are positive constants* a, b, $A_0<A<B<B_0\leqq\infty$, $d<1$ *and* $r<1$ *such that*

(i) $A<(aP_0)^{1/(1-d)}<(bP_0)^{1/(1-r)}<B<(B_0/bP_0)^{(1/r)}$,

(ii) $h(t, x)\geqq ax^d$, $A_0<x\leqq A$, $h(t, x)\geqq aA^d$, $A\leqq x<B_0$,

(iii) $h(t, x)\leqq bx^r$, $B\leqq x<B_0$, $h(t, x)\leqq bB^r$, $A_0<x\leqq B$.

Then for every $\bar{A}$, $\bar{B}$, $A_0<\bar{A}\leqq A$, $B\leqq\bar{B}<\min\{B_0, (B_0/bP_0)^{(1/r)}\}$, *there is* $T(\bar{A}, \bar{B})$ *such that if* $\bar{A}\leqq\varphi(t)\leqq\bar{B}$, $t\in I$, *then the solution* $x(t)=x(t; \varphi t_0)$ *of* (*) *satisfies*

$$aP_0A^d\leqq x(t)\leqq bP_0B^r, \qquad t\geqq t_0+T(\bar{A}, \bar{B}). \tag{3.8}$$

Note. It follows from (i) that

$$A<aP_0A^d<(aP_0)^{1/(1-d)}<(bP_0)^{1/(1-r)}<bP_0B^r<B.$$

Proof. Consider first $\varphi\in C_+$ such that $\varphi(t)\geqq A$, $t\in I$. For $t_0<t\leqq t_0+\tau$ we have, since $x(t)$ is a solution of (*),

$$x(t)\geqq\int_0^L P(L-s)aA^d\,ds=aP_0A^d.$$

Since $A<aP_0A^d$, we see that in fact $x(t)\geqq aP_0A^d$ for $t_0<t\leqq t_0+n\tau$, $n=1, 2, \cdots$, i.e.,

$$x(t)\geqq aP_0A^d, \qquad t>t_0. \tag{3.9}$$

If $A_0<\bar{A}<A$ and if $\varphi\in C_+$ and $\varphi(t)\geqq\bar{A}$, $s\in I$, then $x(t)\geqq\int_0^L P(L-s)a\bar{A}^d\,ds=aP_0\bar{A}^d$, $t_0<t\leqq t_0+\tau$, and $\bar{A}<(aP_0)^{1/(1-d)}$ implies that $\bar{A}<aP_0\bar{A}^d$. If k is a positive integer such that $(k-1)\tau\leqq L+\tau<k\tau$, then repeating the proceeding argument k times yields

$$x(t)\geqq aP_0\bar{A}^d, \qquad t_0<t\leqq t_0+L+\tau.$$

Proceeding by induction, we find that

$$x(t)\geqq(aP_0)^{1/(1-d)}[\bar{A}(aP_0)^{-1/(1-d)}]^{d^n} \tag{3.10}$$

for $t_0+(n-1)(L+\tau)<t\leqq t_0+n(L+\tau)$, as long as the right-hand side of (3.10) is not greater than A.

Since $\bar{A}<(aP_0)^{1/(1-d)}$, $\bar{A}(aP_0)^{-1/(1-d)}<1$, which along with $d<1$, implies that

$$(aP_0)^{1/(1-d)}[\bar{A}(aP_0)^{-1/(1-d)}]^{d^n}\to(aP_0)^{1/(1-d)} \tag{3.11}$$

as $n\to\infty$.

It follows from (3.11) and $A<(aP_0)_{1/(1-d)}$ that there is a positive integer N such that

$$(aP_0)^{1/(1-d)}[\bar{A}(aP_0)^{-1/(1-d)}]^{d^n}>A, \qquad n\geqq N. \tag{3.12}$$

If we set $T(\bar{A})=(N+1)(L+\tau)$ then, combining (3.9) and (3.12), we have

(3.13) $$x(t) \geqq aP_0A^d, \qquad t \geqq t_0 + T(\bar{A}).$$

To verify the second half of the inequality in (3.8) we consider first $\varphi \in C_+$ such that $\|\varphi\| < B$. For $t_0 < t \leqq t_0+\tau$ we have, again using (*),

$$x(t) \leqq \int_0^L P(L-s)bB^r\, ds$$
$$= bP_0B^r < B,$$

or, proceeding as in the first half of the proof, $x(t) \leqq bP_0B^r$ $t_0 < t \leqq t_0 + n\tau$, $n=1, 2, \cdots$, i.e.,

(3.14) $$x(t) \leqq bP_0B^r, \qquad t > t_0.$$

If $\varphi \in C_+$, $B \leqq \|\varphi\| \leqq \bar{B} < (B_0/bP_0)^{1/r}$, then, using (*), we obtain

$$x(t) \leqq \int_0^L P(L-s)b\bar{B}^r\, ds$$
$$= bP_0\bar{B}^r \qquad t_0 < t \leqq t_0+\tau.$$

If $bP_0B^r \leqq \bar{B}$, then $x(t) \leqq \bar{B}$, $t_0 - L - \tau \leqq t \leqq t_0 + \tau$, and an argument similar to that used prior to (3.10) shows that

$$x(t) \leqq (bP_0)^{1/(1-r)}[\bar{B}(bP_0)^{-1/(1-r)}]^{r^n}$$

for $t_0+(n-1)(L+\tau) < t \leqq t_0 + n(L+\tau)$, $n = 1, 2, \cdots$, as long as $(bP_0)^{1/(1-r)}[\bar{B}(bP_0)^{-1/(1-r)}]^{r^n} \geqq B$. Since $\bar{B}(bP_0)^{-1/(1-r)} > 1$ and $r < 1$,

$$(bP_0)^{1/(1-r)}[\bar{B}(bP_0)^{-1/(1-r)}]^{r^n} \to (bP_0)^{1/(1-r)}$$

as $n \to \infty$ which implies the existence of $T_1(\bar{B})$ such that

$$x(t) \leqq bP_0B^r, \qquad t \geqq t_0 + T_1(\bar{B}).$$

If $bP_0\bar{B}^r > \bar{B}$, then $bP_0\bar{B}^r < B_0$ since $\bar{B} < (B_0/bP_0)^{(1/r)}$, and $x(t) \leqq bP_0\bar{B}^r = (bP_0)^{1/(1-r)}[\bar{B}(bP_0)^{-1/(1-r)}]^r$, $t_0 - L - \tau \leqq t \leqq t_0 + \tau$. Now for $t_0 + \tau < t \leqq t_0 + 2\tau$, using (*), we have

$$x(t) \leqq bP_0(bP_0\bar{B}^r)^r = (bP_0)^{1/(1-r)}[\bar{B}(bP_0)^{-1/(1-r)}]^{r^2}$$

and since $0 < r < 1$ and $\bar{B}(bP_0)^{-1/(1-r)} > 1$ we have

$$(bP_0)^{1/(1-r)}[\bar{B}(bP_0)^{-1/(1-r)}]^r > (bP_0)^{1/(1-r)}[\bar{B}(bP_0)^{-1/(1-r)}]^{r^2}$$

from which it follows that

$$x(t) \leqq (bP_0)^{1/(1-r)}[\bar{B}(bP_0)^{-1/(1-r)}]^r$$

for $t_0 - L - \tau \leqq t \leqq t_0 + 2\tau$. This argument can be repeated $k-1$ times to yield

$$x(t) \leqq (bP_0)^{1/(1-r)}[\bar{B}(bP_0)^{-1/(1-r)}]^r$$

for $t_0 \leqq t \leqq t_0 + L + \tau$.

We can now follow the proof for the case $bP_0\bar{B}^r \leqq \bar{B}$ to show the existence of $T_2(\bar{B})$ such that $x(t) \leqq bP_0B^r$, $t \geqq t_0 + T_2(\bar{B})$.

If we set $T(\bar{B}) = \max(T_1(\bar{B}), T_2(\bar{B}))$, then, using also (3.14), we have

$$x(t) \leqq bP_0B^r, \qquad t \geqq t_0 + T(\bar{B}). \tag{3.15}$$

Inequality (3.8) follows from (3.13) and (3.15), setting $T(\bar{A}, \bar{B}) = \max(T(\bar{A}), T(\bar{B}))$, completing the proof of Theorem 4.

Although the cases $d = 1$ and $r = 1$ have not been covered in the preceeding results, the following corollary of Theorem 4 gives information about solutions of (*) when $h(t, x)$ has linear bounds over certain intervals.

COROLLARY. *Suppose there are positive constants a_0, b_0, $A_0 < A < B < B_0$ such that*

(i) $b_0 < 1/P_0 < a_0$, $A < (a_0P_0)^2A_0 < (b_0P_0)^2B_0 < B$,

(ii) $h(t, x) \geqq a_0x$, $A_0 < x \leqq A$, $h(t, x) \geqq a_0(AA_0)^{1/2}$, $A \leqq x < B_0$,

(iii) $h(t, x) \leqq b_0x$, $B \leqq X < B_0$, $h(t, x) \leqq b_0(BB_0)^{1/2}$, $A_0 < x \leqq B$.

Then for every $\bar{A}$, $\bar{B}$, $A_0 < \bar{A} \leqq A$, $B \leqq \bar{B} < B_0$, there is $T(\bar{A}, \bar{B})$ such that if $\bar{A} \leqq \varphi(t) \leqq \bar{B}$, $t \in I$, then the solution $x(t) = x(t; \varphi, t_0)$ of (*) satisfies

$$a_0P_0(AA_0)^{1/2} \leqq x(t) \leqq b_0P_0(BB_0)^{1/2}, \qquad t \geqq t_0 + T(\bar{A}, \bar{B}).$$

Proof. Set $d = r = \frac{1}{2}$ and $a = a_0A_0^{1/2}$, $b = b_0B_0^{1/2}$ in Theorem 4.

Combining Theorems 2 and 4, we are now able to provide more general conditions than those provided in Theorem 2 which imply that the population of our model will die out, i.e., approach zero, for large t.

THEOREM 5. *Assume that* H_1–H_3 *are satisfied and that there are positive constants a, b, B, $B_0 \leqq \infty$, $d > 1$ and $r < 1$ such that*

(i) $bP_0B^r < (aP_0)^{-1/(d-1)}$,

(ii) $h(t, x) \leqq ax^d$, $0 < x \leqq bP_0B^r$,

(iii) $h(t, x) \leqq bx^r$, $B \leqq x \leqq B_0$,

(iv) $h(t, x) \leqq bB^r$, $0 < x < B$.

Then, if $\|\varphi\| < (B_0/bP_0)^{1/r}$, the solution $x(t; \varphi, t_0)$ of (*) satisfies $x(t; \varphi, t_0) \to 0$ as $t \to \infty$.

Proof. It follows from Theorem 4 that if $\|\varphi\| < B_0$, then there is $T > 0$ such that $x(t) = x(t; \varphi, t_0) < bP_0B^r$, $t \geqq t_0 + T - L - \tau$. Applying the Corollary of Theorem 2 to $x(t; x_{t_0+T}, t_0 + T)$ we have, since $\|x_{t_0+T}\| < bP_0B^r < (aP_0)^{-1/(d-1)}$,

$$x(t; x_{t_0+T}, t_0 + T) \to 0 \quad \text{as } t \to \infty,$$

and this is the same as

$$x(t; \varphi, t_0) \to 0 \quad \text{as } t \to \infty.$$

4. Periodic solutions of (*). A periodic solution in a population model represents a state of equilibrium for the system described by the model and, as such, is of considerable importance in the applications of the model. An even more favorable situation is that in which a periodic solution exists and all solutions starting near it remain near in the future in some-well-defined sense, i.e., that the periodic solution be stable. It is shown here, under rather general conditions on h, that (*) has at least one periodic solution; however, nothing as yet has been obtained about the nature of the solutions starting near a periodic solution other than the information provided in § 3.

We shall need the following result due to Browder [4].

LEMMA 1. *Let S and S_1 be open convex subsets of the Banach space X, S_0 a closed convex subset of X, $S_0 \subset S_1 \subset S$, T a compact mapping of S into X. Suppose that for a positive integer n, T^n is well-defined on S_1, $\bigcup_{0 \leqq j \leqq n} T^j(S_0) \subset S_1$, while $T^n(S_1) \subset S_0$. Then T has a fixed point in S_0.*

THEOREM 6. *If H_1–H_4 and the hypotheses of Theorem 4 are satisfied, then (*) has a periodic solution, $\psi(t)$, of period $n\omega$, where n is any positive integer satisfying $n\omega > L + \tau$ and*

$$aP_0A^d \leqq \psi(t) \leqq bP_0B^r, \qquad t \geqq -L-\tau. \tag{4.1}$$

If H_5 is also satisfied, then $\psi(t)$ is not constant.

Note. It is not asserted here that $\psi(t)$ has least period $n\omega$.

Proof. Define $T: C_+ \to C_+$ by $T\varphi = x(\varphi, 0)_{n\omega}$, where n is a positive integer such that $n\omega > L + \tau$ and $x(\varphi, 0)$ is the solution $x(t; \varphi, 0)$ of (*). We also define S_0, S_1 and S by

$$S_0 = \{\varphi \in C_+ | aP_0A^d \leqq \varphi(t) \leqq bP_0B^r, t \in I\}$$

and

$$S_1 = S = \{\varphi \in C_+ | A < \varphi(t) < B, t \in I\}.$$

It was shown in the proof of Theorem 4, see (3.9) and (3.14), that if $A \leqq \varphi(t) \leqq B$, $t \in I$, then

$$aP_0A^d \leqq x(t; \varphi, 0) \leqq bP_0B^r, \qquad t > 0. \tag{4.2}$$

So for $N = 1$ we have, since $n\omega > L + \tau$,

$$\bigcup_{0 \leqq j \leqq N} T^j(S_0) \subset S_1 \quad \text{and} \quad T^N(S_1) \subset S_0, \tag{4.3}$$

and it remains to be shown that T is a compact mapping of S into C_+.

Since $h(t, x)$ is continuous on $K^* = [-L-\tau, n\omega] \times [A, B]$, it is uniformly continuous there, and there exists a continuous real-valued function $\eta(\varepsilon)$ such that

$$|h(t, x) - h(s, y)| \leqq \eta\left(\sqrt{(t-s)^2 + (x-y)^2}\right) \quad \text{for } (t, x), (s, y) \in K^*$$

and $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. In particular, if θ, $\varphi \in S$, then $|h(t, \theta(s)) - h(t, \varphi(s))| \leqq \eta(\|\theta - \varphi\|)$ for $s, t \in I$.

If θ, $\varphi \in S$ and $x(t) = x(t; \theta, 0)$, $y(t) = x(t; \varphi, 0)$ are solutions of (*), then

$$\begin{aligned} |x(t) - y(t)| \leqq \int_0^L & P(L-s)|h(t-\tau-L+s, \theta(t-\tau-L+s)) \\ & - h(t-\tau-L+s, \varphi(t-\tau-L+s))| \, ds \leqq P_0\eta(\|\theta - \varphi\|) \end{aligned} \tag{4.4}$$

for $0 < t \leqq \tau$. We can assume that $\|\theta - \varphi\| \leqq P_0\eta(\|\theta - \varphi\|)$ and thus that

$$|h(t, x(s)) - h(t, y(s))| \leqq \eta(P_0\eta(\|\theta - \varphi\|)) \tag{4.5}$$

for $s, t \in [-L-\tau, \tau]$.

It follows from (4.5), using the same argument as used to establish (4.4), that

$$|x(t)-y(t)|\leqq P_0\eta(P_0\eta(\|\theta-\varphi\|)), \qquad \tau\leqq t\leqq 2\tau.$$

If k is a positive integer such that $n\omega\leqq k\tau$, then it is clear that

$$|x(t)-y(t)|\leqq (P_0\eta)^k(\|\theta-\varphi\|), \qquad -L-\tau\leqq t\leqq n\omega, \tag{4.6}$$

where $(P_0\eta)^k$ is the composition of the function $P_0\eta$ with itself k times. It follows from (4.6) that

$$\begin{aligned}\|T\theta-T\varphi\|&=\|(\theta,0)_{n\omega}-x(\varphi,0)_{n\omega}\|\\ &\leqq (P_0\eta)^k(\|\theta-\varphi\|).\end{aligned}$$

and, since η is continuous and $\eta(\varepsilon)\to 0$ as $\varepsilon\to 0$, $(P_0\eta)^k(\varepsilon)\to 0$ as $\varepsilon\to 0$ establishing the continuity of T.

We shall show that T is a compact mapping of S by showing that $T(S)$ is uniformly bounded and that there is $K>0$ such that $x\in T(S)$ satisfies $|x(t)-x(s)|\leqq K|t-s|$ for $s,\ t\in[n\omega-L-\tau,\ n\omega]$.

It follows from (3.14) in the proof of Theorem 4 that if $x\in T(S)$, then

$$\|x\|\leqq B. \tag{4.7}$$

If $x\in T(S)$ and $s,\ t\in[n\omega-L-\tau,\ n\omega]$, $s\leqq t$, then there is $\varphi\in S$ such that $x(t)=x(t;\varphi,0)$ and, since $n\omega-L-\tau>0$,

$$\begin{aligned}x(t)-x(s)&=\int_0^L P(L-u)h(t-\tau-L+u,\ x(t-\tau-L+u))\,du\\ &\quad-\int_0^L P(L-u)h(s-\tau-L+u,\ x(s-\tau-L+u))\,du\\ &=\int_{t-\tau-L}^{t-\tau} P(t-\tau-u)h(u,x(u))\,du-\int_{s-\tau-L}^{s-\tau} P(s-\tau-u)h(u,x(u))\,du \qquad (4.8)\\ &=\int_{s-\tau}^{t-\tau} P(t-\tau-u)h(u,x(u))\,du-\int_{s-\tau-L}^{t-\tau-L} P(s-\tau-u)h(u,x(u))\,du\\ &\quad+\int_{t-\tau-L}^{s-\tau}[P(t-\tau-u)-P(s-\tau-u)]h(u,x(u))\,du.\end{aligned}$$

Since $0<x(t)<B$, it follows from H_3 and (iv) of Theorem 4 that

$$|P(t-\tau-u)-P(s-\tau-u)|h(u,x(u))\leqq bP_0B^rK_1|t-s|$$

(4.9) and

$$P(t-\tau-u)h(u,x(u))\leqq bP_0B^r.$$

Substituting these inequalities into (4.8) we have

$$|x(t)-x(s)|\leqq(2bP_0B^r+bP_0B^rK_1)|t-s|$$

for $s,\ t\in[0,\ n\omega]$.

We can now apply Lemma 1 to assert the existence of $\psi_0 \in S_0$ such that $T\psi_0 = \psi_0$. Consider now the solution $\psi(t) = x(t; \psi_0, 0)$ of (*). Since $h(t+\omega, x) = h(t, x)$, $t > 0$, and $\psi_{n\omega} = \psi_0$, it is clear that $\psi(t)$ is a periodic solution of (*) of period $n\omega$.

It is of interest to note that since $\psi(t)$ is continuous for $t > 0$ and $\psi_{n\omega} = \psi_0$, it follows that

$$\begin{aligned}\psi_0(0) &= \int_0^L P(L-s)h(n\omega - \tau - L + s, \psi(n\omega - \tau - L + s))\,ds \\ &= \int_0^L P(L-s)h(s-\tau-L, \psi_0(s-\tau-L))\,ds,\end{aligned}$$

which implies that $\psi(t)$ is continuous at $t = 0$.

The bounds given in (4.1) for $\psi(t)$ are an immediate consequence of Theorem 4. The last statement of the theorem is obvious.

Theorem 6 can be combined with the corollary of Theorem 4 to show the existence of a periodic solution of (*) when $h(t, x)$ has linear bounds over certain intervals.

COROLLARY. *If* H_1–H_4 *and the hypotheses of the corollary of Theorem* 4 *are satisfied, then* (*) *has a periodic solution,* $\psi(t)$, *of period* $n\omega$, *where* n *is any positive integer satisfying* $n\omega > L + \tau$ *and*

$$a_0 P_0 (A A_0)^{1/2} \leqq \psi(t) \leqq b_0 P_0 (B B_0)^{1/2}, \qquad t \geqq -L-\tau.$$

THEOREM 7. *Suppose* H_1–H_4 *are satisfied and that there are positive constants* a, b, $d < 1$, $r > 1$ *and* $A_0 < A < (aP_0)^{1/(1-d)} < B < \bar{B} \leqq (bP_0)^{-1/(r-1)}$ *such that*

$$\begin{array}{lll}\text{(i)} & h(t,x) \geqq ax^d & A_0 < x \leqq A, \\ \text{(ii)} & h(t,x) \leqq bB^r & 0 < x \leqq B, \\ \text{(iii)} & h(t,x) \leqq bx^r & B \leqq x \leqq \bar{B}.\end{array}$$

Then (*) *has a periodic solution* $\psi(t)$ *of period* $n\omega$ *where* n *is any positive integer such that* $n\omega > L + \tau$, *and furthermore,*

$$aP_0A^d \leqq \psi(t) \leqq bP_0B^r, \qquad t \geqq -L-\tau. \tag{4.10}$$

Proof. Define $T: C_+ \to C_+$ by $T\varphi = x(\varphi, 0)_{n\omega}$ where n is a positive integer such that $n\omega > L + \tau$, and S_0, S_1 and S by

$$S_0 = \{\varphi \in C_+ | aP_0A^d \leqq \varphi(t) \leqq bP_0B^r\},$$

$$S = S_1 = \{\varphi \in C_+ | A < \varphi(t) < B\}.$$

Since $B < (bP_0)^{-1/(r-1)}$ and $r > 1$, $bP_0B^r < B$ and thus $S_0 \subset S_1 \subset S$.

If $\varphi \in S$, then it follows from Theorem 2 that

$$x(t; \varphi, 0) \leqq \max\{bP_0B^r, (BP_0)^{-1/(r-1)}[\|\varphi\|(bP_0)^{1/(r-1)}]^{r^n}\}$$

over the intervals stated there. Since $\|\varphi\| < B < (bP_0)^{-1/(r-1)}$ and $r > 1$, we have

$$\|\varphi\|(bP_0)^{1/(r-1)} < 1 \quad \text{and} \quad bP_0B^r < B,$$

and it follows that there is a positive integer N such that

$$x(t; \varphi, 0) \leqq bP_0B^r, \qquad t \geqq N\tau.$$

We note also that if $\|\varphi\| \leqq bP_0B^r$, then

$$x(t; \varphi, 0) \leqq bP_0B^r, \qquad t \geqq 0.$$

We note from the proof of Theorem 4 that the lower bound determined in Theorem 4 for solutions of (*) was independent of the upper bound of $h(t, x)$ and thus

$$x(t; \varphi, 0) \geqq aP_0A^d, \quad t > 0 \quad \text{if} \quad \varphi \in S.$$

The preceeding argument can be summarized as

$$\cup_{0 \leqq j \leqq N} T^j(S_0) \subset S_1 \quad \text{and} \quad T^N(S_1) \subset S_0.$$

The remainder of the proof is the same as the proof of Theorem 6, noting only that the upper bounds in (4.7), (4.9) and (4.10) can be deduced from Theorem 2 rather than Theorem 4.

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ISOLATED SINGULARITIES IN STEADY STATE FLUID FLOW*

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Abstract. New results concerning isolated singularities for classical and distribution solutions of the nonlinear stationary Navier-Stokes equations are established. Also new results are established for the equations of Stokes and Oseen. Some counterexamples are given. In particular, it is shown that the nonlinear result for distribution solutions in dimension 2 is, in a certain sense, a best possible result.

1. Introduction. Letting $\mathbf{v} = (v_1, \ldots, v_N)$, $N \geqq 2$, and letting p be a scalar function, we shall study the following three systems of equations where c is a constant $\neq -1$, $\mathbf{b}$ is a constant vector, and ν is a nonzero constant:

$$\begin{aligned} &\Delta \mathbf{v} = \nabla p, \\ &\nabla \cdot \mathbf{v} = -cp, \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\Delta \mathbf{v} - (\mathbf{b} \cdot \nabla)\mathbf{v} = \nabla p, \\ &\nabla \cdot \mathbf{v} = 0, \end{aligned} \tag{1.2}$$

$$\begin{aligned} &\nu \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla p, \\ &\nabla \cdot \mathbf{v} = 0. \end{aligned} \tag{1.3}$$

If $c = 0$, equations (1.1) are called the Stokes equations and represent the slow steady flow of a viscous incompressible fluid with $\mathbf{v}$ the velocity vector and p the pressure divided by the viscosity.

Equations (1.3) are called the stationary Navier-Stokes equations and represent in general the steady flow of a viscous incompressible fluid where ν is the coefficient of viscosity. Equations (1.2) are called the Oseen equations of hydrodynamics and represent a particular linearized version of equations (1.3).

With $k = 1, 2$ or 3, we shall say the pair $(\mathbf{v}, p)$ is a classical solution of $(1.k)$ in the open set Ω if v_j and p are in $C^\infty(\Omega)$, $j = 1, \cdots, N$, and if for each x in Ω, equations $(1.k)$ are satisfied.

We shall designate the open N-ball with center x and radius r by $B(x, r)$ and shall prove the following theorems.

THEOREM 1. *Let* $(\mathbf{v}, p)$ *be a classical solution of* (1.3) *in* $B(0, r_0) - \{0\}$. *Suppose that*

(i) *there is a* $\beta > N$ *such that* $\mathbf{v}$ *is in* $L^\beta[B(0, r_0)]$;

(ii) *for* $N = 2$, $\lim_{r \to 0} |r^2 \log r|^{-1} \int_{B(0,r)} |\mathbf{v}|\, dx = 0$.

Then $(\mathbf{v}, p)$ *can be defined at* 0 *so that* $(\mathbf{v}.p)$ *is a classical solution of* (1.3) *in* $B(0, r_0)$.

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THEOREM 2. *Let* $(\mathbf{v}, p)$ *be a classical solution of* (1.2) *in* $B(0, r_0) - \{0\}$. *Suppose that*

(i) $\mathbf{v}$ *is in* $L^1[B(0, r_0)]$;

(ii) $\lim_{r\to 0} r^{-2} \int_{B(0,r)} |\mathbf{v}|\, dx = 0$ *for* $N \geqq 3$,
$\lim_{r\to 0} |r^2 \log r|^{-1} \int_{B(0,r)} |\mathbf{v}|\, dx = 0$ *for* $N = 2$.

Then $(\mathbf{v}, p)$ *can be defined at* 0 *so that* $(\mathbf{v}, p)$ *is a classical solution of* (1.2) *in* $B(0, r_0)$.

THEOREM 3. *Let* $(\mathbf{v}, p)$ *be a classical solution of* (1.1) *in* $B(0, r_0) - \{0\}$, *where* $c \neq -1$ *for* $N \geqq 2$ *and* $c \neq -1/2$ *for* $N = 2$. *Suppose that*

(i) $\mathbf{v}$ *is in* $L^1[B(0, r_0)]$;

(ii) $\lim_{r\to 0} r^{-2} \int_{B(0,r)} |\mathbf{v}|\, dx = 0$ *for* $N \geqq 3$;
$\lim_{r\to 0} |r^2 \log r|^{-1} \int_{B(0,r)} |\mathbf{v}|\, dx = 0$ *for* $N = 2$.

Then $(\mathbf{v}, p)$ *can be defined at* 0 *so that* $(\mathbf{v}, p)$ *is a classical solution of* (1.1) *in* $B(0, r_0)$.

Theorem 3 actually does not hold for the case $N = 2$, $c = -1/2$ as can be seen by the following counterexample: $v_1 = x_1^2|x|^{-2}$, $v_2 = x_1x_2|x|^{-2}$, $p = 2x_1|x|^{-2}$. The correct theorem in this case is the following.

THEOREM 4. *With* $N = 2$ *and* $c = -1/2$, *let* $(\mathbf{v}, p)$ *be a classical solution of* (1.1) *in* $B(0, r_0) - \{0\}$. *Suppose that*

(i) $\mathbf{v}$ *is in* $L^\infty[B(0, r_0)]$;

(ii) *for either* $j = 1$ *or* $j = 2$, *there is a constant called* $v_j(0)$ *such that*

$$\lim_{r\to 0} r^{-2} \int_{B(0,r)} |v_j(x) - v_j(0)|\, dx = 0.$$

Then $(\mathbf{v}, p)$ *can be defined at* 0 *so that* $(\mathbf{v}, p)$ *is a classical solution of* (1.1) *in* $B(0, r_0)$.

In the concluding section of this paper, namely § 7, we state and prove Theorems 5 and 6 concerning isolated singularities for distribution solutions of the nonlinear stationary Navier-Stokes equations with an external force. Theorem 1 is an immediate consequence of these two theorems and [3, Thm. 3].

We note that Theorem 1 for $N \geqq 3$ gives an improvement of [3, Thm. 1]. Next, we note that Theorem 3 is in a certain sense a best possible result. In particular, the conclusion of Theorem 3 does not hold for $N = 3$ if we replace (ii) by (ii′) where

$$\text{(ii}'\text{)} \quad \int_{B(0,r)} |\mathbf{v}|\, dx = O(r^2).$$

To see this, we set for $|x| \neq 0$ and $c \neq -1$,

$$v_1 = -(1 + c)|x|^{-1} + 2^{-1}\partial^2|x|/\partial x_1^2,$$

$$v_j = 2^{-1}\partial^2|x|/\partial x_1\, \partial x_j, \qquad j = 2, 3,$$

$$p = \partial|x|^{-1}/\partial x_1.$$

A similar situation prevails for $N \geqq 4$ and $N = 2$.

In particular for $N = 2$, we observe that (ii) cannot be replaced by (ii′) where

$$\text{(ii}'\text{)} \quad \int_{B(0,r)} |v^2|\, dx = o(r^2 \log r),$$

$$\int_{B(0,r)} |v_1|\, dx = O(r^2 \log r).$$

In this case, we set for $|x| \neq 0$ and $c \neq -1/2$ or -1,

$$v_1 = -(1 + c)\log|x| + 4^{-1}\, \partial^2 |x|^2 \log|x| / \partial x_1^2,$$

$$v_2 = 4^{-1}\, \partial^2 |x|^2 \log|x| / \partial x_1\, \partial x_2,$$

$$p = \partial \log|x| / \partial x_1.$$

In a similar vein, it can also be shown that Theorem 2 is in a certain sense best possible (see [2, pp. 391–392]). It is also possible to show that Theorem 1 for $N = 2$ is in a certain sense best possible; we shall deal with this and related matters in a future paper.

We shall prove Theorem 3 first, then Theorem 4, then in § 5 we shall discuss the case $c = -1$ for equations (1.1). In § 6, we prove Theorem 2 and in the concluding section of this paper, namely § 7, we prove Theorems 5 and 6 and show that Theorem 6 is in a certain sense a best possible result. Theorem 1 is an immediate corollary to Theorems 5 and 6.

In this paper, in contrast to our results in [3], we put no conditions on the pressure in the neighborhood of the singular point. As a consequence, in order to achieve removability of the singularity in the nonlinear case, we have to make stronger assumptions here (which nevertheless turn out to be best possible in dimension 2) on the velocity vector.

Also in this paper, in contrast to [3], we proceed in the nonlinear situation by first establishing new removable singularity results for the linearized Navier-Stokes equations, i.e., equations (1.1) with $c = 0$. Then using the fundamental solutions u_j^k and q_j introduced in § 2, we relate the linearized results to the nonlinear situation. The nonlinear theory is dealt with in § 7 of this paper.

2. Multiple trigonometric series and fluid flow. We shall use the theory of trigonometric series somewhat as in [7] to establish the results in this paper. In particular, we shall need a number of specific functions on the N-torus, $T_N = \{x\colon -\pi \leqq x_j < \pi, j = 1, \cdots, N\}$. To deal with this, we shall use the following notation: m will designate an integral lattice point; $(x, y) = x_1 y_1 + \cdots + x_N y_N$; $|x|^2 = (x, x)$; for a function g in the $L^1(T_N)$, we shall set

$$(2.1) \qquad g\hat{}(m) = (2\pi)^{-N} \int_{T_N} f(x)\, e^{-i(m,x)}\, dx.$$

For g in $L^1(T_N)$ and $t > 0$ we shall set

$$(2.2) \qquad g(x, t) = \sum_m g\hat{}(m)\, e^{i(m,x)}\, e^{-|m|t}.$$

The following remark is well-known (see for example [5, Lem. 2]).

Remark 1. If g is in class C^1 in a neighborhood of the point x^0, then $\lim_{t\to 0} \partial g(x^0, t)/\partial x_j = \partial g(x^0)/\partial x_j$, $j = 1, \cdots, N$. Likewise if g is in C^2 in a neighborhood of the point x^0, then $\lim_{t\to 0} \partial^2 g(x^0, t)\, \partial x_j\, \partial x_k = \partial^2 g(x^0)/\partial x_j\, \partial x_k$ for $j, k = 1, \cdots, N$.

Next, with E_N designating Euclidean N-space, and periodic meaning of period 2π in each variable, we introduce the functions

$$G, H, u_j^k \quad \text{and} \quad q_j, \qquad j, k = 1, \cdots, N, \tag{2.3}$$

which are periodic in $E_N - \bigcup_m \{2\pi m\}$, in $C^\infty[E_N - \bigcup_m \{2\pi m\}]$, and in $L^1(T_N)$.

In particular,

$$G\hat{\ }(0) = H\hat{\ }(0) = q_j\hat{\ }(0) = u_j^k\hat{\ }(0) = 0, \tag{2.4}$$

and for $m \neq 0$,

$$\begin{aligned} &G\hat{\ }(m) = -|m|^{-4}, \qquad H\hat{\ }(m) = |m|^{-2}, \\ &q_j\hat{\ }(m) = im_j|m|^{-2}, \\ &u_j^k\hat{\ }(m) = [-(1 + c)\, \delta_j^k + m_j m_k |m|^{-2}]\, |m|^{-2}, \end{aligned} \tag{2.5}$$

where δ_j^k is the Kronecker δ and c is the constant in (1.1).

To be specific, we define $H(x, t)$, $q_j(x, t)$ and $u_j^k(x, t)$ for $t > 0$ in a manner analogous to (2.2) using (2.4) and (2.5). Then in [6, p. 72], it is shown that the limits of $H(x, t)$ and $q_j(x, t)$ exist and are finite as $t \to 0$ for x in $E_N - \bigcup_m \{2\pi m\}$. Defining $H(x)$ and $q_j(x)$ respectively as these limits, it is furthermore shown in [6, p. 72] that these functions have the properties enumerated in (2.3).

Also the following two facts are established in [6, p. 72]:

$$H(x) - |x|^2/2N \quad \text{is harmonic in} \quad E_N - \bigcup_m \{2\pi m\}. \tag{2.6}$$

There is a function $H^*(x)$ in $C^\infty[B(0, 2)]$ and a positive constant α_N such that for x in $B(0, 2) - \{0\}$,

$$\begin{aligned} &H(x) - \alpha_N |x|^{2-N} = H^*(x) \qquad \text{for} \quad N \geqq 3, \\ &H(x) - \alpha_N \log |x|^{-1} = H^*(x) \quad \text{for} \quad N = 2. \end{aligned} \tag{2.7}$$

Next, we define

$$G(x) = -(2\pi)^{-N} \int_{T_N} H(x - y) H(y)\, dy \quad \text{for } x \quad \text{in } E_N - \bigcup_m \{2\pi m\}, \tag{2.8}$$

and observe from (2.6), (2.7), and the standard arguments of potential theory that G has the properties enumerated in (2.3), (2.4) and (2.5).

We furthermore have from Remark 1 that

$$\Delta G(x) = H(x) \quad \text{for } x \quad \text{in } E_n - \bigcup_m \{2\pi m\}. \tag{2.9}$$

Next, we observe from (2.8) that in dimension $N = 2$ and $N = 3$, G can be defined at 0 so that it is continuous in $B(0, 1)$. Also it follows from (2.6), (2.7) and

(2.8) that, in particular, $G(x) = O(|x|^{3-N})$ as $|x| \to 0$ for $N \geqq 4$. Consequently, we obtain from (2.7), (2.9) and well-known facts concerning removable singularities in potential theory that

there is a function $G^*(x)$ in $C^\infty[B(0, 2)]$
such that for x in $B(0, 2) - \{0\}$,

$$(2.10)\qquad \begin{aligned} &G(x) - \frac{\alpha_N |x|^{4-N}}{2(4-N)} = G^*(x) \quad \text{for} \quad N \geqq 3, \quad N \neq 4,\\ &G(x) - 2^{-1}\alpha_4 \log|x| = G^*(x) \quad \text{for} \quad N = 4,\\ &G(x) - 4^{-1}\alpha_2 |x|^2 \log|x|^{-1} = G^*(x) \quad \text{for} \quad N = 2, \end{aligned}$$

where α_N is the constant in (2.7).

Observing that

$$(2.11)\qquad u_j^k(x, t) = -\delta_j^k(1 + c)H(x, t) + \partial^2 G(x, t)/\partial x_j\, \partial x_k$$

and defining

$$(2.12)\qquad u_j^k(x) = \lim_{t \to 0} u_j^k(x, t) \quad \text{for } x \quad \text{in } E_N - \bigcup_m \{2\pi m\},$$

we see from Remark 1 and the properties previously enumerated for G and H that u_j^k does indeed satisfy the conditions stated in (2.3).

Next, we establish the following lemma.

LEMMA 1. *Let g be a periodic function in $C^\infty[E_N - \bigcup_m \{2\pi m\}]$ and in $L^1(T_N)$. Suppose furthermore that*

(i) *g is harmonic in $B(0, r_0) - \{0\}$ where $0 < r_0 < 1$;*

(ii) $\lim_{|x| \to 0} |x|^N g(x) = 0$.

Then there are constants K_j, $j = 0, \cdots, N$, and a periodic function A in $C^\infty(E_N)$ such that

$$(2.13)\qquad g(x) = K_0 H(x) + \sum_{j=1}^{N} K_j q_j(x) + A(x) \quad \textit{for } x \quad \textit{in } E_N - \bigcup_m \{2\pi m\}.$$

From the definition of $q_j(x)$ given below (2.5) and from (2.5), (2.7), and Remark 1, we have that for x in $B(0, 2) - \{0\}$ and $j = 1, \cdots, N$,

$$(2.14)\qquad \begin{aligned} &q_j(x) + (N - 2)\alpha_N x_j |x|^{-N} = \partial H^*(x)/\partial x_j \quad \text{for} \quad N \geqq 3,\\ &q_j(x) + \alpha_N x_j |x|^{-2} = \partial H^*(x)/\partial x_j \quad \text{for} \quad N = 2. \end{aligned}$$

Consequently, it follows from (i) and (ii), the hypothesis of the lemma, from well-known facts concerning removable singularities in potential theory, and from (2.7) and (2.14) that there is a function $A(x)$ in $C^\infty[B(0, r_0/2)]$ and constants $K_0, K_1, \cdots, K_N$ such that

$$(2.15)\qquad A(x) = g(x) - K_0 H(x) - \sum_{j=1}^{N} K_j q_j(x) \quad \text{for } x \quad \text{in } B(0, r_0/2) - \{0\}.$$

We use the right-hand side of (2.15) to define $A(x)$ in $E_N - \bigcup_m \{2\pi m\}$ and we define $A(2\pi m) = A(0)$. The conclusion to the lemma then follows immediately from periodicity, (2.3), and the conditions asserted for g in the first sentence of the lemma.

Next, we state the following remark.

Remark 2. Suppose that g satisfies the conditions in the hypothesis of Lemma 1. Suppose furthermore $\lim_{|x|\to 0} |x|^{N-1}g(x) = 0$. Then (2.13) holds with $K_1 = K_2 = \cdots = K_N = 0$.

Remark 2 follows immediately from Lemma 1 and the details of the proof can be left to the reader.

3. Proof of Theorem 3. From the first equation in (1.1), we have that

$$\Delta v_j = \partial p/\partial x_j \quad \text{for } x \text{ in} \quad B(0, r_0) - \{0\} \tag{3.1}$$

for $j = 1, \cdots, N$. Combining this fact with the second equation in (1.1) we obtain $-c\,\Delta p = \Delta p$. Since $c \neq -1$, we conclude that

$$\begin{aligned} &p \text{ is harmonic and } v_j \text{ is biharmonic} \\ &\text{in } B(0, r_0) - \{0\}, \quad j = 1, \cdots, N. \end{aligned} \tag{3.2}$$

But then from the well-known mean value theorem for biharmonic functions, we have the existence of constants C_1 and C_2 such that

$$\begin{aligned} |x|^2\,\Delta v_j(x) = C_1|x|^{-N} \int_{B(x,|x|/2)} v_j(y)\,dy + C_2 v_j(x) \\ \text{for} \quad 0 < |x| < r_0/2 \quad \text{and} \quad j = 1, \cdots, N. \end{aligned} \tag{3.3}$$

Now $B(x, |x|/2) \subset B(0, 3|x|/2)$. We consequently conclude from (ii) in the hypothesis of Theorem 3 and from (3.1) and (3.3) that

$$\begin{aligned} &\lim_{|x|\to 0} |x|^{N-2}[|x|^2\,\partial p(x)/\partial x_j - C_2 v_j(x)] = 0 \quad \text{for} \quad N \geqq 3 \\ &\lim_{|x|\to 0} |\log|x||^{-1}[|x|^2\,\partial p(x)/\partial x_j - C_2 v_j(x)] = 0 \quad \text{for} \quad N = 2 \end{aligned} \tag{3.4}$$

and $j = 1, \cdots, N$.

From (3.4) in conjunction with (ii) of Theorem 3, we next obtain that as $r \to 0$,

$$\int_{B(0,r)} |x|^2|\partial p(x)/\partial x_j|\,dx = \begin{cases} o(r^2) & \text{for} \quad N \geqq 3, \\ o(r^2 \log 1/r) & \text{for} \quad N = 2, \end{cases} \tag{3.5}$$

for $j = 1, \cdots, N$.

Letting $N = 2$ and fixing j, we have from (3.2) that $\partial p/\partial x_j$ is harmonic in $B(0, r_0) - \{0\}$. Consequently, there are constants $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and a function $\psi(x)$ harmonic in $B(0, r_0)$ such that

$$\partial p(x)/\partial x_j = \psi(x) + a_0 \log|x|^{-1} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^{-n} \tag{3.6}$$

for x in $B(0, r_0) - \{0\}$ where $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ and where the series in (3.6) converges uniformly with respect to θ for $0 < r_1 \leqq r \leqq r_2 < r_0$.

From (3.6), we obtain for $0 < \varepsilon < r < r_0$ that modulo a good term

$$(3.6') \qquad \int_{B(0,r)-B(0,\varepsilon)} |x|^2 \, \partial p(x)/\partial x_j \cos n\theta \, dx = a_n \pi \int_\varepsilon^r \rho^{3-n} \, d\rho.$$

We consequently conclude from (3.5) that $a_n = 0$ for $n \geqq 3$. Likewise $b_n = 0$ for $n \geqq 3$. But this implies

$$(3.7) \qquad |\nabla p| = O(|x|^{-2}) \quad \text{as} \quad |x| \to 0 \quad \text{for} \quad N = 2.$$

Using the theory of spherical harmonics for $N \geqq 3$ and proceeding in a similar manner (see [4, p. 94]), we obtain a slightly better result, namely

$$(3.8) \qquad |\nabla p| = O(|x|^{1-N}) \quad \text{as} \quad |x| \to 0 \quad \text{for} \quad N \geqq 3.$$

From (3.7) and (3.8), we obtain

$$(3.9) \qquad p = \begin{cases} 0(|x|^{-1}) & \text{as} \quad |x| \to 0 \quad \text{for} \quad N = 2, \\ 0(|x|^{2-N}) & \text{as} \quad |x| \to 0 \quad \text{for} \quad N \geqq 3. \end{cases}$$

Next, we choose r_1 and r_2 so that

$$(3.10) \qquad 0 < r_1 < r_2 < \min(r_0, 1)$$

and select a function λ such that

$$(3.11) \qquad \begin{aligned} &\lambda \quad \text{is} \quad C_0^\infty[B(0, r_2)] \quad \text{and} \\ &\lambda = 1 \quad \text{in} \quad B(0, r_1). \end{aligned}$$

We then define for $j = 1, \cdots, N$,

$$(3.12) \qquad \begin{aligned} v_j' &= \begin{cases} \lambda v_j & \text{in} \quad B(0, r_2) - \{0\}, \\ 0 & \text{in} \quad T_N - B(0, r_2), \end{cases} \\ p' &= \begin{cases} \lambda p & \text{in} \quad B(0, r_2) - \{0\}, \\ 0 & \text{in} \quad T_N - B(0, r_2). \end{cases} \end{aligned}$$

We then extend v_j' and p' by periodicity to $E_N - \bigcup_m \{2\pi m\}$ and observe in particular from (3.2), (3.9) and (3.12) that the conditions in the hypothesis of Lemma 1 are met for p'. We consequently conclude from Lemma 1 and Remark 2 that there are constants K_0, K_1 and K_2 and a periodic function A in $C^\infty(E_N)$ such that

$$(3.13) \qquad p' = \begin{cases} K_0 H + \sum_{j=1}^{2} K_j q_j + A & \text{for} \quad N = 2, \\ K_0 H + A & \text{for} \quad N \geqq 3 \end{cases} \qquad \text{for } x \quad \text{in } E_N - \bigcup_m \{2\pi m\}.$$

Next, we define

$$(3.14) \qquad P_j(x) = (2\pi)^{-N} \int_{T_N} p'(x - y) q_j(y) \, dy \quad \text{for } x \quad \text{in } E_N - \bigcup_m \{2\pi m\}.$$

Since

$$p' \text{ and } v_j' \text{ are in } C^\infty[E_N - \bigcup_m \{2\pi m\}] \quad \text{for} \quad j = 1, \cdots, N, \tag{3.15}$$

it is easy to infer from (2.3) and (2.14) that for $j = 1, \cdots, N$,

$$P_j \text{ is in } C^\infty[E_n - \bigcup_m \{2\pi m\}]. \tag{3.16}$$

Next, we define

$$w_j(x) = v_j'(x) + P_j(x) \quad \text{for } x \quad \text{in } E_N - \bigcup_m \{2\pi m\} \quad \text{and} \quad j = 1, \cdots, N. \tag{3.17}$$

From (3.15), (3.16) and (3.17), we have that for $j = 1, \cdots, N$,

$$w_j \text{ is in } C^\infty[E_N - \bigcup_m \{2\pi m\}]. \tag{3.18}$$

From (3.1), (3.11), (3.12) and Remark 1, we have

$$\lim_{t \to 0} \sum_m [|m|^2 \hat{v_j'}(m) + i m_j \hat{p'}(m)] \exp[i(m, x) - |m|t] = 0 \tag{3.19}$$

$$\text{for } x \text{ in } B(0, r_1) - \{0\}.$$

Consequently, we infer from (2.2), (3.14), (3.17) and (3.19) that

$$\lim_{t \to 0} \Delta w_j(x, t) = 0 \quad \text{for } x \quad \text{in } B(0, r_1) - \{0\}. \tag{3.20}$$

But then we obtain from Remark 1, (3.15), (3.16) and (3.17) that

$$w_j \text{ is harmonic in } B(0, r_1) - \{0\} \quad \text{for} \quad j = 1, \cdots, N. \tag{3.21}$$

From (3.14) and the properties associated with p' and q_j, we see that for x in $B(0, r_1) - \{0\}$,

$$|P_j(x)| \leqq (2\pi)^{-N} \int_{B(0,1)} |p'(x - y)| |q_j(y)| \, dy + O(1). \tag{3.22}$$

Writing $B(0, 1) = B(0, |x|/2) \cup [B(0, 2|x|) - B(0, |x|/2)] \cup [B(0, 1) - B(0, 2|x|)]$ for $|x|$ small, it is an easy matter to infer from (2.14), (3.9), (3.22) that as $|x| \to 0$,

$$P_j(x) = \begin{cases} O(\log |x|^{-1}) & \text{for} \quad N = 2 \text{ and } 3, \\ O(|x|^{3-N}) & \text{for} \quad N \geqq 4. \end{cases} \tag{3.23}$$

Next, we split the rest of the proof up into two cases, namely $N \geqq 3$ and $N = 2$.

First let us assume that $N \geqq 3$. We then infer from condition (ii) in the hypothesis of Theorem 3, from (3.17), and from (3.23) that

$$\int_{B(0,r)} |w_j(x)| \, dx = o(r^2) \quad \text{as} \quad r \to 0. \tag{3.24}$$

Using the theory of spherical harmonics exactly as in [4, p. 94] in conjunction with (3.21) and (3.24) enables us to conclude that for $j = 1, \cdots, N$,

$$w_j \text{ can be defined at 0 so that it is harmonic in } B(0, r_1). \tag{3.25}$$

From (3.17), we see that for $m \neq 0$,

$$w_j\hat{}(m) = v_j'\hat{}(m) + im_j p'\hat{}(m)|m|^{-2}. \tag{3.26}$$

From the second equation in (1.1) and from Remark 1, we also have that

$$\lim_{t \to 0} \sum_{j=1}^{N} \frac{\partial v_j'(x,t)}{\partial x_j} = -cp'(x) \quad \text{for } x \quad \text{in } B(0,r_1) - \{0\}. \tag{3.27}$$

We consequently conclude from (3.25), (3.26), (3.27) and Remark 1 that

$$\sum_{j=1}^{N} \frac{\partial w_j(x)}{\partial x_j} = -cp'(x) - p'(x) + p'\hat{}(0) \quad \text{for } x \quad \text{in } B(0,r_1) - \{0\}. \tag{3.28}$$

We therefore infer from (3.25) and (3.28) that

(3.29) p' can be defined at 0 so that it is harmonic in $B(0, r_1)$.

We next use (3.14) to define P_j at 0 and infer from (3.29),

(3.30) P_j is in $C^\infty[B(0, r_1)]$.

But then from (3.17), (3.21) and (3.30) we have that

(3.31) v_j' can be defined at 0 so that it is in $C^\infty[B(0, r_1)]$.

But from (3.11) and (3.12), we have that $v_j = v_j'$ and $p = p'$ in $B(0, r_1) - \{0\}$. This fact in conjunction with (3.29) and (3.31) concludes the proof of Theorem 3 for the case $N \geqq 3$.

We now assume that $N = 2$ and $c \neq -1/2$ and $c \neq -1$. We then infer from condition (ii) in the hypothesis of Theorem 3, from (3.17), and from (3.23) that

$$(r^2 \log r^{-1})^{-1} \int_{B(0,r)} |w_j(x)|\, dx = O(1) \quad \text{as} \quad r \to 0. \tag{3.32}$$

From (3.21), we see that w_j can be expressed in a manner similar to the right-hand side of (3.6). We conclude consequently from (3.32) that there is a function $A'(x)$ in $C^\infty[B(0, r_1)]$ and a constant a' such that $w_j(x) = A'(x) + a' \log|x|^{-1}$ in $B(0, r_1) - \{0\}$. But then it follows from Remark 2 that there is a periodic function A_j in $C^\infty(E_N)$ and constant a_j such that

$$w_j = a_j H + A_j \quad \text{for } x \quad \text{in } E_N - \bigcup_m \{2\pi m\}. \tag{3.33}$$

Next we note from (2.3), (2.4) and (2.5) that in $E_N - \bigcup_m (\{2\pi m\}$,

$$\begin{aligned} -\partial G(x)/\partial x_j &= (2\pi)^{-N} \int_{T_N} H(x-y) q_j(y)\, dy, \\ -\partial^2 G(x)/\partial x_j\, \partial x_k &= (2\pi)^{-N} \int_{T_N} q_k(x-y) q_j(y)\, dy. \end{aligned} \tag{3.34}$$

We consequently infer from (3.13), (3.14), (3.17), (3.33) and (3.34) that

$$v_j' = a_j H + K_0 \frac{\partial G(x)}{\partial x_j} + \sum_{k=1}^{2} K_k \frac{\partial^2 G(x)}{\partial x_j\, \partial x_k} + A_j'' \quad \text{in} \quad E_2 - \bigcup_m \{2\pi m\}, \tag{3.35}$$

where A_j'' is a periodic function in $C^\infty(E_2)$.

From (2.10), we obtain that for $j = 1, 2$

$$\partial^2 G(x)/\partial x_j^2 - \alpha_2 2^{-1} \log |x|^{-1} = O(1) \quad \text{as} \quad |x| \to 0, \tag{3.36}$$

and

$$\begin{aligned} \partial G(x)/\partial x_j &= O(1) \quad \text{as} \quad |x| \to 0, \\ \partial^2 G(x)/\partial x_1\, \partial x_2 &= O(1) \quad \text{as} \quad |x| \to 0. \end{aligned} \tag{3.37}$$

We consequently conclude from (2.7), (3.35), (3.36), (3.37) and (ii) of Theorem 3,

$$a_j = -K_j/2 \quad \text{for } j = 1, 2. \tag{3.38}$$

Using the fact that $\Delta G = H$ and that $q_j = \partial H/\partial x_j$ except at the integral lattice points in the plane, we conclude from (3.13), (3.35) and (3.38) after a short computation that in $E_2 - \bigcup_m \{2\pi m\}$,

$$\frac{\partial v_1'}{\partial x_1} + \frac{\partial v_2'}{\partial x_2} + cp' = \left(c + \frac{1}{2}\right) \sum_{j=1}^{2} K_j \frac{\partial H}{\partial x_j} + (c + 1)K_0 H + A'', \tag{3.39}$$

where A'' is a periodic function in $C^\infty(E_2)$.

From the second equation in (1.1), and from (3.11) and (3.12), we obtain from (3.39) that

$$\left(c + \frac{1}{2}\right) \sum_{j=1}^{2} K_j \frac{\partial H}{\partial x_j} = -(c + 1)K_0 H - A'' \tag{3.40}$$

$$\text{for } x \text{ in } B(0, r_1) - \{0\}.$$

Since $q_j = \partial H(\partial x_j)$ in $B(0, r_1) - \{0\}$, we obtain from (2.7), (2.14) and (3.40) that for x_1 positive and small

$$-\left(c + \frac{1}{2}\right) K_1 \alpha_2 x_1 x_1^{-2} = -(c + 1)K_0 \alpha_2 \log x_1^{-2} + O(1). \tag{3.41}$$

Since $c \neq -1/2$ and since $\alpha_2 \neq 0$, it follows from (3.41) first that $K_1 = 0$ and then since $c \neq -1$ that $K_0 = 0$. In a similar manner, it follows from (3.40) that $K_2 = 0$. We consequently conclude from (3.13), (3.35) and (3.38) that

$$\begin{aligned} v_j' &= A_j'', & j = 1, 2, \\ p' &= A \quad \text{in} \quad E_2 - \bigcup_m \{2\pi m\}. \end{aligned} \tag{3.42}$$

Since A_1'', A_2'' and A are periodic functions in $C^\infty(E_2)$, we see from (3.42) that v_1', v_2' and p' can be defined at 0 so that they are in $C^\infty[B(0, r_1)]$. But from (3.11) and (3.12), we see that in $B(0, r_1)$ they are respectively equal to v_1, v_2 and p. The proof of Theorem 3 is therefore complete.

4. Proof of Theorem 4. We shall suppose that condition (ii) in the hypothesis of Theorem 4 holds for v_1, i.e.,

$$\lim_{r \to 0} r^{-2} \int_{B(0,r)} |v_1(x) - v_1(0)|\, dx = 0. \tag{4.1}$$

A similar proof will prevail in case (4.1) holds for v_2.

Since obviously v_1 and v_2 satisfy condition (ii) in Theorem 3 and since for $N = 2$ in the proof of Theorem 3 we only started using the fact that $c \neq -1/2$ after (3.41), we see that (3.1) through (3.9) hold for $\mathbf{v}$ and p. In particular, we have the following:

$$
\text{(4.2)} \quad
\begin{aligned}
&\text{(i)} \quad p \text{ is harmonic and } v_1 \text{ and } v_2 \text{ are biharmonic in } B(0, r_0) - \{0\}. \\
&\text{(ii)} \quad \Delta v_j = \partial p/\partial x_j \quad \text{in} \quad B(0, r_0) - \{0\}, \qquad j = 1, 2. \\
&\text{(iii)} \quad |\nabla p| = O(|x|^{-2}) \quad \text{as} \quad |x| \to 0. \\
&\text{(iv)} \quad p = O(|x|)^{-1}) \quad \text{as} \cdot |x| \to 0.
\end{aligned}
$$

observe that if g is biharmonic in a neighborhood of the origin

$$g(0) = (\pi r^2)^{-1} \int_{B(0,r)} g(x)\, dx - r^2 8^{-1} \Delta g(0)$$

for r small. Consequently, we infer from (4.2 (i)) that for $0 < |x| < r_0/2$,

$$
\text{(4.3)} \quad
\begin{aligned}
&(|x|/2)^2 8^{-1} \Delta v_1(x) + [v_1(x) - v_1(0)] \\
&\qquad = (\pi |x|^2 2^{-2})^{-1} \int_{B(x,|x|/2)} [v_1(y) - v_1(0)]\, dy.
\end{aligned}
$$

Observing that $B(x, |x|/2) \subset B(0, 3|x|/2)$, we conclude from (4.1) and (4.3) that

$$\text{(4.4)} \qquad |x|^2 \Delta v_1(x) + 32[v_1(x) - v_1(0)] = o(1) \quad \text{as} \quad |x| \to 0.$$

But then we obtain from (4.1), (4.2 (ii)) and (4.4) that

$$\text{(4.5)} \qquad \lim_{r \to 0} r^{-2} \int_{B(0,r)} |x|^2 |\partial p(x)/\partial x_1|\, dx = 0.$$

Next, we obtain from (4.2 (i)) and (iv) and well-known facts in potential theory that

$$\text{(4.6)} \quad p(x) = \xi(x) + a_0 \log|x| + a_1 x_1 r^{-2} + b_1 x_2 r^{-2} \quad \text{for } x \quad \text{in } B(0, r_0) - \{0\},$$

where ξ is harmonic in $B(0, r_0)$.

But then

$$\text{(4.7)} \quad \frac{\partial p(x)}{\partial x_1} = \frac{\partial \xi(x)}{\partial x_1} + a_0 x_1 r^{-2} + a_1[r^2 - 2x_1^2] r^{-4} - 2b_1 x_1 x_2 r^{-4} \qquad \text{for} \quad x \quad \text{in } B(0, r_0) - \{0\}.$$

From (4.7), we obtain that for $0 < r < r_0$,

$$\text{(4.8)} \qquad 8(\pi r^2)^{-1} \int_{B(0,r)} x_1 x_2 \left[\frac{\partial p(x)}{\partial x_1} - \frac{\partial \xi}{\partial x_1} \right] dx = -2b_1.$$

Since ξ is harmonic in $B(0, r_0)$, we obtain from (4.5) and (4.8) that

$$\text{(4.9)} \qquad b_1 = 0.$$

Next, we observe once again from (4.7) that

$$(\pi r^2)^{-1} \int_{B(0,r)} x_1^2 \left[\frac{\partial p(x)}{\partial x_1} - \frac{\partial \xi(x)}{\partial x_1} \right] dx = a_1(-4^{-1})$$

and once again we use (4.5) and (4.8) to obtain that

$$a_1 = 0. \tag{4.10}$$

We next choose r_1 and r_2 as in (3.10), λ as in (3.11) and define v_j' and p' as in (3.12). Likewise, we define P_j and w_j' as in (3.14) and (3.17) respectively and observe that (3.18) and (3.21) hold. From (4.6), (4.9), (4.10) and Remark 2, however. we now have that

$$p' = K_0 H + A \quad \text{in} \quad E_2 - \bigcup_m \{2\pi m\}, \tag{4.11}$$

where A is a periodic function in $C^\infty(E_2)$.

From (4.11) and (3.14) we obtain that for $m \neq 0$, $\hat{P_j}(m) = [K_0|m|^{-2} + \hat{A}(m)]im_j|m|^{-2}$. Consequently $\sum_m |\hat{P_j}(m)| < \infty$, and we conclude that

(4.12) P_j can be defined at 0 so that it is continuous at $B(0, r_1)$.

But then from condition (i) in the hypothesis of Theorem 4 and from (4.12), (3.17) and (3.21), we conclude that

(4.13) w_j can be defined at 0 so that it is harmonic $B(0, r_1)$.

But from (3.17) and the second equation in (1.1), we have that

$$\sum_{j=1}^{2} \frac{\partial w_j(x)}{\partial x_j} = \frac{1}{2} p(x) - p(x) + \hat{p'}(0) \quad \text{for } x \quad \text{in } B(0, r_1) - \{0\}. \tag{4.14}$$

We conclude from (4.13) and (4.14) that

(4.15) p can be defined at 0 so that it is in $C^\infty[B(0, r_1)]$.

But then P_j is in $C^\infty[B(0, r_1)]$ and we obtain from this fact in conjunction with (4.13) and (3.17) that v_1 and v_2 can be defined at 0 so that they are in $C^\infty[B(0, r_1)]$. This, along with (4.15), concludes the proof of Theorem 4.

5. The case c = −1. In this section we discuss the case $c = -1$ in (1.1), i.e., the system

$$\begin{aligned} \Delta \mathbf{v} &= \nabla p, \\ \nabla \cdot \mathbf{v} &= p. \end{aligned} \tag{5.1}$$

In particular, we shall show that an isolated singularity at 0 is not removable for classical solutions of the system (5.1) even if $\mathbf{v}$ is in $C^{1+\mathrm{Lip}}[B(0, r_0)]$ for $r_0 > 0$. To see this set

$$v_j(x) = x_j|x|, \qquad j = 1, \cdots, N,$$

and

$$p(x) = (N + 1)|x|.$$

Then an easy computation shows that $(\mathbf{v}, p)$ is a classical solution of (5.1) in

$B(0, r_0) - \{0\}$ for any $r_0 > 0$. Also it is clear that v_{jx_k} exists at 0 (and is zero) and that v_{jx_k} is Lipschitz continuous in $B(0, r_0)$ for $j, k = 1, \cdots, N$. On the other hand we see that $v_1(x)$ is not in $C^2[B(0, r_0)]$, i.e.,

$$v_{1x_1x_1}(x_1, 0, \cdots, 0) = \begin{cases} 2, & x_1 > 0, \\ -2, & x_1 < 0, \end{cases}$$

and our assertion concerning the system (5.1) is established.

6. Proof of Theorem 2. The proof of Theorem 2 at first has points in common with that given for Theorem 3. In particular, we choose r_1 and r_2 as in (3.10), λ as in (3.11), and define v'_j and p' as in (3.12). We then extend v'_j and p' by periodicity to $E_n - \bigcup_m \{2\pi m\}$ and observe in particular that

(6.1) $$v'_j \text{ and } p' \text{ are in } C^\infty[E_N - \bigcup_m \{2\pi m\}].$$

Next, we define

(6.2) $$V_j(x) = \sum_{k=1}^N (2\pi)^{-N} b_k \int_{T_N} v'_j(x - y) q_k(y)\, dy \qquad \text{for } x \quad \text{in } E_n - \bigcup_m \{2\pi m\},$$

and observe that

(6.3) $$V_j \text{ is in } C^\infty[E_N - \bigcup_m \{2\pi m\}].$$

Also we observe that

(6.4) $$\hat{V_j}(m) \begin{cases} i\hat{v'_j}(m) \sum_{k=1}^N b_k m_k |m|^{-2} & \text{for} \quad m \neq 0, \\ 0 & \text{for} \quad m = 0. \end{cases}$$

Next, we set

(6.5) $$W_j = v'_j + V_j \quad \text{in} \quad E_N - \bigcup_m \{2\pi m\}$$

and observe from (6.1), (6.3) and (6.4) that

$$\Delta W_j = \Delta v'_j - \sum_{k=1}^N b_k \frac{\partial v'_j}{\partial x_k} \quad \text{in} \quad E_N - \bigcup_m \{2\pi m\}.$$

From the first equation in (1.2) and this last fact, we conclude that

(6.6) $$\Delta W_j = \partial p / \partial x_j \quad \text{in} \quad B(0, r_1) - \{0\}.$$

But

(6.7) $$p \text{ is harmonic in } B(0, r_1) - \{0\}.$$

We therefore obtain from (6.6) that

(6.8) $$W_j \text{ is biharmonic in } B(0, r_1) - \{0\}.$$

Consequently, there are constants C_1 and C_2 such that the analogue of (3.3) holds for $0 < |x| < r_1/2$ with W_j replacing v_j. But then from (ii) in Theorem 2 and from (6.6), we conclude that the analogue of (3.4) holds with W_j replacing v_j. We use (ii) in Theorem 2 once again and obtain that (3.5) holds. Proceeding exactly as before using (3.6), (3.6′), (3.7) and (3.8), we obtain (3.9) which we record here as

$$(6.9) \qquad p = \begin{cases} O(|x|^{-1}) & \text{as } |x| \to 0 \quad \text{for} \quad N = 2, \\ O(|x|^{2-N}) & \text{as } |x| \to 0 \quad \text{for} \quad N \geqq 3. \end{cases}$$

From (6.1) and (6.9), we have that p' is in $L^1(T_N)$. We therefore obtain from (6.7), (6.9), Lemma 1 and Remark 2 that there are constants K_0, K_1 and K_2 and a periodic function A in $C^\infty(E_N)$ such that

$$(6.10) \qquad p' = \begin{cases} K_0 H + \sum_{j=1}^{2} K_j q_j + A & \text{for} \quad N = 2, \\ K_0 H + A & \text{for} \quad N \geqq 3. \end{cases}$$

Next, we define P_j as in (3.14) and set

$$(6.11) \qquad w_j = W_j + P_j.$$

From (3.14) and (6.1), we have that

$$(6.12) \qquad P_j \text{ is in } C^\infty[E_N - \bigcup_m \{2\pi m\}]$$

and furthermore that

$$(6.13) \qquad \hat{P_j}(m) = \begin{cases} i m_j \hat{p'}(m) |m|^{-2} & \text{for} \quad m \neq 0, \\ 0 & \text{for} \quad m = 0. \end{cases}$$

We conclude from (6.6), (6.11), (6.12) and (6.13) that

$$(6.14) \qquad w_j \text{ is harmonic in } B(0, r_1) - \{0\}.$$

It follows from (2.3), (2.4), (2.5), (6.10), and (6.13) that

$$(6.15) \qquad P_j = \begin{cases} \dfrac{-K_0 \partial G}{\partial x_j} - \sum_{k=1}^{2} K_k \dfrac{\partial^2 G}{\partial x_j \partial x_k} + A_j & \text{for} \quad N = 2, \\ -K_0 \, \partial G / \partial x_j + A_j & \text{for} \quad N \geqq 3 \end{cases} \qquad \text{in } E_N - \bigcup_m \{2\pi m\}$$

where A_j is a periodic function in $C^\infty(E_N)$.

Now, in particular, it follows from (6.4), condition (ii) in Theorem 2, and [4, Lem. 4] that $\mathbf{V} = (V_1, \cdots, V_N)$ meets condition (ii) in Theorem 2. Consequently, it follows from (6.5) that $\mathbf{W} = (W_1, \cdots, W_N)$ meets condition (ii) in Theorem 2.

But then it follows from (6.11) and (6.15) that as $r \to 0$,

$$(6.16)\qquad \begin{aligned} r^{-2}\int_{B(0,r)} |w_j|\,dx &= o(1) \qquad \text{for} \quad N \geqq 3,\\ |r^2 \log r|^{-1}\int_{B(0,r)} |w_j|\,dx &= O(1) \quad \text{for} \quad N = 2. \end{aligned}$$

We also note from (6.1), (6.3), (6.5), (6.11) and (6.12) that

$$(6.17)\qquad w_j \text{ is in } C^\infty[E_N - \bigcup_m \{2\pi m\}].$$

From (6.14), (6.16) and an analogue of (3.6) and (3.6′), we obtain that for $N = 2$, $w_j(x) = O(\log |x|^{-1})$ as $|x| \to 0$. This fact in conjunction with (6.14), (6.17) and Remark 2 enables us to conclude that there is a constant K_0' and a periodic function A' in $C^\infty(E_2)$ such that

$$(6.18)\qquad w_j = K_0'H + A' \quad \text{in} \quad E_2 - \bigcup_m \{2\pi m\} \quad \text{for} \quad N = 2.$$

For $N \geqq 3$, a better situation prevails. From (6.14), (6.16) and an argument involving spherical harmonics as in [4, p. 94] we can conclude that w_j can be defined at 0 so that w_j is harmonic in $B(0, r_1)$. We consequently conclude from (6.17) that

(6.19) for $N \geqq 3$, w_j can be defined in $\bigcup_m \{2\pi m\}$ so that it is a periodic function in $C^\infty(E_N)$.

Next, we note from (6.5) and (6.11) that

$$(6.20)\qquad v_j' = w_j - P_j - V_j \quad \text{in} \quad E_N - \bigcup_m \{2\pi m\}.$$

Now, it follows from (6.2) and (2.14) that

$$(6.21)\qquad V_j \text{ is in } L^\gamma(T_N) \quad \text{for} \quad 1 < \gamma < N/(N-1).$$

From (6.15), we have that

$$(6.22)\qquad \begin{aligned} &P_j \text{ is in } L^\gamma(T_2) \quad \text{for} \quad 1 < \gamma < \infty \quad \text{and} \quad N = 2,\\ &P_j \text{ is in } L^\gamma(T_N) \quad \text{for} \quad 1 < \gamma < N/(N-3) \quad \text{and} \quad N \geqq 3, \end{aligned}$$

We consequently conclude from (6.18), (6.19), (6.20), (6.21) and (6.22) that

$$(6.23)\qquad v_j' \text{ is in } L^\gamma(T_N) \quad \text{for} \quad 1 < \gamma < N/(N-1).$$

From (6.2), (6.3), (6.23) and [3, Lem. 2] we next obtain that

$$(6.24)\qquad V_j \text{ is in } L^\gamma(T_N) \quad \text{for} \quad 1 < \gamma < N/(N-2) \quad \text{and} \quad N \geqq 2.$$

But then we obtain from (6.18), (6.20), (6.22) and (6.24) that

$$(6.25)\qquad v_j' \text{ is in } L^\gamma(T_2) \quad \text{for} \quad 1 < \gamma < \infty \quad \text{and} \quad N = 2.$$

In a similar manner, we obtain from (6.24) that

$$(6.26)\qquad v'_j \text{ is in } L^\gamma(T_N) \quad \text{for} \quad 1 < \gamma < N/(N-2) \quad \text{for} \quad N \geqq 3.$$

But then using (6.22), (6.3) and [3, Lem. 2] once again but this time in conjunction with (6.26), we obtain that

(6.27) $\quad V_j$ is in $L^\gamma(T_N)$ for $1 < \gamma < N/(N-3)$ for $N \geqq 3$.

We infer once again from (6.19), (6.20), (6.22), and (6.27) that

(6.28) $\quad v'_j$ is in $L^\gamma(T_N)$ for $1 < \gamma < N/(N-3)$ for $N \geqq 3$.

Next, it follows from [1, p. 261] (as in [3, pp. 341–347]) and from (6.25) and (6.28) that

(6.29)
$$\sum_{m \neq 0} \hat{v'_j}(m) m_R m_k |m|^{-2} e^{i(m,x)}$$
is the Fourier series of a function in $L^\gamma(T_N)$, where $1 < \gamma < \infty$ for $N = 2$ and $1 < \gamma < N/(N-3)$ for $N \geqq 3$ and $j, k, R = 1, \cdots, N$.

From (6.3), (6.4), Remark 1 and [6, Thm. 2], we obtain from (6.29) that

(6.30) $\quad \partial V_j/\partial x_R$ is in $L^\gamma(T_N)$, where $1 < \gamma < \infty$ for $N = 2$ and $1 < \gamma < N/(N-3)$ for $N \geqq 3$, and where $j, R = 1, \cdots, N$.

Next, we observe from the second equation in (1.2) that

(6.31)
$$\sum_{j=1}^{N} \frac{\partial v'_j}{\partial x_j} = 0 \quad \text{in} \quad B(0, r_1) - \{0\}.$$

Also we observe from (6.13) that

(6.32)
$$\sum_{j=1}^{N} \frac{\partial P_j}{\partial x_j} = -[p' - \hat{p'}(0)] \quad \text{in} \quad E_N - \bigcup_m \{2\pi m\}.$$

From (6.10), (6.19), (6.20), (6.30), (6.31), (6.32), we obtain that

(6.33)
p' is in $L^\gamma(T_N)$,
where $1 < \gamma < 2$ for $N = 2$,
and $1 < \gamma < N/(N-3)$ for $N \geqq 3$.

We now subdivide the proof of the theorem into two cases, namely Case I where $N \geqq 3$ and Case II where $N = 2$.

For Case I, we use (6.33) in conjunction with (6.10) and obtain that $K_0 H$ must be $L^\gamma(T_N)$ for $1 < \gamma < N/(N-3)$. From (2.7), we see that this is impossible unless $K_0 = 0$. We consequently obtain from (6.10) that

(6.34) $\quad p'$ can be defined in $\bigcup_m \{2\pi m\}$ so that it is a periodic function in $C^\infty(E_N)$.

Next, we observe from (6.13) and [3, Lem. 1] that a similar situation to (6.34) holds for P_j. We record this as

(6.35) $\quad P_j$ can be defined in $\bigcup_m \{2\pi \mathrm{m}\}$ so that it is a periodic function in $C^\infty(E_N)$.

Next we observe from (6.2) and [3, Lem. 2] that

(6.36) if S is a positive integer and $S < N$ and if v'_j is in $L^\gamma(T_N)$ for $1 < \gamma < N/(N - S)$, then V_j is in $L^\alpha(T_N)$ for $1 < \alpha < N/(N - S - 1)$.

We consequently conclude from (6.19), (6.20), (6.35), (6.28) and (6.36) that

$$V_j \text{ is in } L^\gamma(T_N) \quad \text{for} \quad 1 < \gamma < \infty. \tag{6.37}$$

From (6.37) and (6.20), it follows that

$$v'_j \text{ is in } L^\gamma(T_N) \quad \text{for} \quad 1 < \gamma < \infty. \tag{6.38}$$

We use a similar bootstrap argument in conjunction with (6.38) and [3, Lemmas 1 and 3] to obtain finally that

(6.39) v'_j can be defined in $\bigcup_m \{2\pi m\}$ so that it is a periodic function in $C^\infty(E_N)$.

The conclusion in the statement of Theorem 2 follows immediately from (6.34) and (6.39), and the proof of Theorem 2 for $N \geqq 3$ is complete.

We now consider Case II where $N = 2$. From an argument similar to that used to obtain (6.29), we observe from (6.33) that $p'^{\wedge}(m)m_j m_k|m|^{-2}$ is the Fourier coefficient of a function in $L^\gamma(T_2)$ where $1 < \gamma < 2$. We consequently conclude from (6.12) and (6.13) that

$$\partial P_j/\partial x_R \text{ is in } L^\gamma(T_\gamma) \quad \text{for} \quad 1 < \gamma < 2 \quad \text{and} \quad j, R = 1, 2. \tag{6.40}$$

It then follows from (6.18), (6.20), (6.30) and (6.40) that

$$\partial v'_j/\partial x_R \text{ is in } L^\gamma(T_2) \quad \text{for} \quad 1 < \gamma < 2 \quad \text{and} \quad j, R = 1, 2. \tag{6.41}$$

Next, we set

$$f_j = \sum_{k=1}^{2} b_k \frac{\partial v'_j}{\partial x_k} \quad \text{in} \quad E_2 - \bigcup_m \{2\pi m\}, \tag{6.42}$$

and observe from (6.41) that

$$f_j \text{ is in } L^\gamma(T_2) \quad \text{for} \quad 1 < \gamma < 2. \tag{6.43}$$

Next, with $c = 0$ in (2.5), $v = 1$, and f_j defined by (6.42), we define $U_j(x) = \sum_{k=1}^{2} (2\pi)^{-2} \int_{T_2} u_j^k(x - y) f(y)\, dy$ and $Q(x) = \sum_{k=1}^{2} (2\pi)^{-2} \int_{T_2} q_k(x - y) f_k(y)\, dy$ for x in $E_2 - \bigcup_m \{2\pi m\}$. Then it follows from (6.1) and (6.42) that

$$U_j \text{ and } Q \text{ are in } C^\infty[E_2 - \bigcup_m \{2\pi m\}]. \tag{6.44}$$

From (6.42), we have that $f_j^{\wedge}(0) = 0$. We consequently observe that for all m

$$\begin{aligned} -|m|^2 U_j^{\wedge}(m) - i m_j Q^{\wedge}(m) &= f_i^{\wedge}(m), \\ \sum_{k=1}^{2} i m_k U_k^{\wedge}(m) &= 0, \end{aligned} \tag{6.44'}$$

and conclude that

$$\text{(6.45)}\qquad \begin{aligned} &\Delta U_j - \partial Q/\partial x_j = f_j, \\ &\sum_{k=1}^{2} \partial U_k/\partial x_k = 0 \quad \text{in} \quad E_2 - \bigcup_m \{2\pi m\}. \end{aligned}$$

From (6.42) and (6.43), we next observe that

$$\text{(6.46)}\qquad \hat{f_j}(m) = \hat{v_j'}(m) \sum_{k=1}^{2} i m_k b_k.$$

It follows therefore from (2.5) (with $c = 0$), from (6.44′), and from (6.46) that for $m \neq 0$,

$$\text{(6.47)}\qquad \hat{U_j}(m) = \sum_{R=1}^{2} b_R i m_R |m|^{-2} \sum_{k=1}^{2} \hat{v_k'}(m)[m_j m_k |m|^{-2} - \delta_j^k].$$

But then it follows from (6.25), (6.29), [3, Lem. 3] and (6.34) that

(6.48) U_j can be defined in $\bigcup_m \{2\pi m\}$ so that it is a periodic function in $C^\alpha(E_2)$ for $0 < \alpha < 1$.

From (1.2), (6.42) and (6.45), we have that

$$\text{(6.49)}\qquad \begin{aligned} &\Delta(v_j' - U_j') - \partial(p' - Q')/\partial x_j = 0, \\ &\sum_{k=1}^{2} \partial(v_k' - U_k)/\partial x_k = 0 \quad \text{in} \quad B(0, r_1) - \{0\}. \end{aligned}$$

It follows from condition (ii) in Theorem 2, from (6.49), (6.48) and from Theorem 3 that

(6.50) there are periodic functions ψ_1, ψ_2 and ξ in $C^\infty(E_2)$ such that in $E_2 - \bigcup_m \{2\pi m\}$,

$$v_j' = \psi_j + U_j, \qquad j = 1, 2,$$

$$p' = \xi + Q.$$

But then it follows from (6.48) and (6.50) that

(6.51) v_j' can be defined $\bigcup_m \{2\pi m\}$ so that it is a periodic function in $C^\alpha(E_2)$ for $0 < \alpha < 1$.

As a consequence, it follows from (6.51) that the Fourier series in (6.29) is the Fourier series of a periodic function in $C^\alpha(E_2)$ for $0 < \alpha < 1$. But then it follows from (6.47) and (6.51) that U_j is in $C^{1+\alpha}(E_2)$. As a consequence, it follows from (6.50) that α can be replaced by $(1 + \alpha)$ in (6.51). Iterating this argument, we conclude that

(6.52) v_j' can be defined in $\bigcup_m \{2\pi m\}$ so that it is a periodic function in $C^\infty(E_2)$.

From (6.42), we see that the analogue of (6.52) holds for f_j. Since $Q(x) = \sum_{k=1}^{2} (2\pi)^{-2} \int_{T_2} q_k(x - y) f_k(y)\, dy$ in $E_2 - U_m\{2\pi m\}$ with f_j given by (6.42), it follows from [3, Lem. 1] that the analogue of (6.52) holds also for Q. But then it follows from (6.50) that a similar situation prevails for p'. We record this as

(6.53) p' can be defined in $\cup_m \{2\pi m\}$ so that it is a periodic function in $C^\infty(E_2)$.

The conclusion to Theorem 2 for $N = 2$ follows immediately from (6.52) and (6.53), and the proof of Theorem 2 is complete.

7. Isolated singularities for distribution solutions. In this section, we deal with the system of equations

$$\begin{aligned} \nu\Delta \mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla p &= -\mathbf{f}, \\ \nabla\cdot\mathbf{v} &= 0. \end{aligned} \tag{7.1}$$

Let Ω be an open set in E_N and let $\mathbf{f}$ be locally in $L^1(\Omega)$. We shall say (following for the most part the notation in [3, p. 336]) that $(\mathbf{v}, p)$ is a distribution solution of (7.1) in Ω if $\mathbf{v}$ is locally in $L^2(\Omega)$, p is locally in $L^1(\Omega)$, and the following holds:

$$\begin{aligned} &\int_\Omega \left[\nu v_j \Delta\phi + \sum_{k=1}^N v_j v_k \frac{\partial\phi}{\partial x_k} + \frac{p\partial\phi}{\partial x_j}\right] dx = -\int_\Omega \phi f_j\, dx, \qquad j = 1, \cdots, N, \\ &\sum_{k=1}^N \int_\Omega v_k \frac{\partial \phi}{\partial x_k}\, dx = 0 \qquad\qquad \text{for } \phi \text{ in } C_0^\infty(\Omega). \end{aligned} \tag{7.2}$$

In this section, we intend to establish the following two theorems.

THEOREM 5. *Let f be locally in $L^1[B(0, r_0)]$ and suppose that*

(i) *there is a $\beta > 2$ such that $\mathbf{v}$ is locally in $L^\beta[B(0, r_0)]$;*
(ii) *p is locally in $L^1[B(0, r_0) - \{0\}]$;*
(iii) *$(\mathbf{v}, p)$ is a distribution solution of (7.1) in $B(0, r_0) - \{0\}$;*
(iv) *$\{r^{-N}\int_{B(0,r)} |\mathbf{v}|^\beta\, dx\}^{1/\beta} = o(r^{-(N-1)/2})$ as $r \to 0$;*
(v) *for $N = 2$, $\int_{B(0,r)} |\mathbf{v}|\, dx = o(r^2|\log r|)$ as $r \to 0$.*

Then p is locally in $L^1[B(0, r_0)]$, and $(\mathbf{v}, p)$ is a distribution solution of (7.1) in $B(0, r_0)$.

THEOREM 6. *For $N = 2$, Theorem 5 is also true if condition (iv) is deleted.*

To establish Theorem 5, we choose r_1 and r_2 as in (3.10) and λ as in (3.11). Next, we assign p a value at 0 (say 1) and then define for $j = 1, \cdots, N$,

$$\begin{aligned} &v'_j = \lambda v_j, \qquad p' = \lambda p, \qquad f''_j = \lambda f_j \quad \text{in} \quad B(0, r_2), \\ &v'_j = p' = f''_j = 0 \quad \text{in} \quad T_N - B(0, r_2). \end{aligned} \tag{7.3}$$

Next, we find a function λ_j which is in $C_0^\infty[B(0, 2) - \bar{B}(0, 1)]$ and is such that $\int_{T_N} [f''_j - \lambda_j]\, dx = 0$. We then define

$$f'_j = f''_j - \lambda_j \quad \text{in} \quad T_N \tag{7.4}$$

and extend v'_j, p' and f'_j by periodicity to all of E_N. We note in particular that

$$f'_j = f_j \quad \text{in} \quad B(0, r_1) \tag{7.5}$$

and that

$$f_j'\hat{}(0) = 0, \qquad j = 1, \cdots, N. \tag{7.6}$$

Next, we define for $j, k = 1, \cdots, N$,

$$v_{jk} = v_j' v_k' \tag{7.7}$$

and observe that

$$v_{jk} \text{ is in } L^{\beta/2}(T_N). \tag{7.8}$$

Also, we observe that as $r \to 0$,

$$\left\{ r^{-N} \int_{B(0,r)} |v_{jk}|^{\beta/2} \, dx \right\}^{2/\beta} = o(r^{-(N-1)}). \tag{7.9}$$

Next, using (2.5) with $c = 0$, we define U_j and Q to be periodic functions in E_N which are also in $L^1(T_N)$ and which have the following Fourier coefficients:

$$\nu U_j\hat{}(m) = \sum_{k=1}^{N} u_j^k\hat{}(m) \left[\sum_{R=1}^{N} i m_R v_{kR}\hat{}(m) - f_k'\hat{}(m) \right], \tag{7.10}$$

$$Q\hat{}(m) = \sum_{k=1}^{N} q_k\hat{}(m) \left[\sum_{R=1}^{N} i m_R v_{kR}\hat{}(m) - f_k'\hat{}(m) \right]. \tag{7.11}$$

It is not difficult to see from (2.5), (6.29), (7.8) and [3, Lem. 2] that there exists a γ such that $\gamma > 1$ and such that both U_j and Q are in $L^\gamma(T_N)$. (γ of course depends on N and β.)

Next we observe from (7.9) and the fact that $\beta > 2$ that

$$r^{-N} \int_{B(0,r)} |v_{jk}| \, dx = o(r^{-(N-1)}).$$

Now from this observation, in conjunction with (2.5), (7.9), [3, Lemmas 12 and 13] and [4, Lem. 4], we conclude that

$$r^{-N} \int_{B(0,r)} |U_j| \, dx = \begin{cases} o(r^{-(N-2)}) & \text{for} \quad N \geqq 3, \\ o(|\log r) & \text{for} \quad N = 2 \end{cases} \tag{7.12}$$

and

$$r^{-N} \int_{B(0,r)} |Q| \, dx = o(r^{-(N-1)}) \quad \text{for} \quad N \geqq 2. \tag{7.13}$$

Next, we set for $t > 0$,

$$\begin{aligned} U_j(x, t) &= \sum_m U_j\hat{}(m) \exp\,[i(m, x) - |m|t], \\ Q(x, t) &= \sum_m Q\hat{}(m) \exp\,[i(m, x) - |m|t]. \end{aligned} \tag{7.14}$$

We define $v_{kR}(x, t)$ and $f_k'(x, t)$ in a similar manner using $v_{kR}\hat{}(m)$ and $f_k'\hat{}(m)$. Observing that $-|m|^2 \nu U_j\hat{}(m) - \mathrm{i}m_j Q\hat{}(m) \doteq \sum_{R=1}^{N} \mathrm{i}m_R v_{jR}\hat{}(m) - f_j'\hat{}(m)$ and that

$\sum_{R=1}^{N} im_k U_k(m) = 0$, we obtain from (7.14) that for $t > 0$ and x in E_N,

$$\nu\Delta U_j(x,t) - \frac{\partial Q(x,t)}{\partial x_j} = \sum_{R=1}^{N} \frac{\partial v_{jR}(x,t)}{\partial x_R} - f_j'(x,t) \tag{7.15}$$

and

$$\sum_{k=1}^{N} \frac{\partial U_k(x,t)}{\partial x_k} = 0. \tag{7.16}$$

From (7.15), we observe that for $t > 0$,

$$\begin{aligned}\int_{T_N} \left[\nu U_j(x,t)\,\Delta A(x) + Q(x,t)\frac{\partial A(x)}{\partial x_j} \right] \\ = -\int_{T_N} \left[\sum_{R=1}^{N} v_{jR}(x,t)\frac{\partial A(x)}{\partial x_R} + f_j'(x,t)A(x) \right] dx\end{aligned} \tag{7.17}$$

for every periodic function A in $C^\infty(E_N)$.

As is well-known, $U_j(x,t)$, $Q(x,t)$, $v_{jR}(x,t)$ and $f_j'(x,t)$ tend respectively in the L^1-norm over T_N to $U_j(x)$, $Q(x)$, $v_{jR}(x)$ and $f_j'(x)$. We consequently conclude from (7.17) that

$$\begin{aligned}\int_{T_N} \left[\nu U_j(x)\,\Delta A(x) + Q(x)\frac{\partial A(x)}{\partial x_j} \right] dx \\ = -\int_{T_N} \left[\sum_{k=1}^{N} v_{jk}(x)\frac{\partial A(x)}{\partial x_k} + f_j'(x)A(x) \right] dx.\end{aligned} \tag{7.18}$$

Next, we observe that for every ϕ in $C_0^\infty[B(0,r_1) - \{0\}]$, there is a periodic function A in $C^\infty[E_N]$ such that $A = \phi$ in $B(0,r_1) - \{0\}$. We consequently conclude from (7.3), (7.5), (7.7) and (7.18) that

$$\int_{B(0,r_1)} \left[\nu U_j\,\Delta\phi + \frac{Q\partial\phi}{\partial x_k} \right] dx = -\int_{B(0,r_1)} \left[\sum_{k=1}^{N} v_j v_k \frac{\partial\phi}{\partial x_k} + f_j\phi \right] dx \tag{7.19}$$

for ϕ in $C_0^\infty[B(0,r_1) - \{0\}]$.

Proceeding in a similar manner, we obtain from (7.16) that

$$\int_{B(0,r_1)} \left[\sum_{k=1}^{N} U_k\,\partial\phi/\partial x_k \right] dx = 0 \quad \text{for } \phi \text{ in } C_0^\infty[B(0,r_1) - \{0\}]. \tag{7.20}$$

Setting $\Omega = B(0,r_1) - \{0\}$ in (7.2), we see from (iii) in Theorem 5, (7.19) and (7.20) that for $j = 1, \cdots, N$,

$$\int_{B(0,r_1)} \nu(v_j - U_j)\,\Delta\phi + (p - Q)\,\partial\phi/\partial x_j = 0 \tag{7.21a}$$

and

$$\int_{B(0,r_1)} \left[\sum_{k=1}^{N} (v_k - U_k)\,\partial\phi/\partial x_k \right] = 0 \tag{7.21b}$$

for ϕ in $C_0^\infty(B(0,r_1) - \{0\}]$.

It is an easy matter using the method of mollifiers to conclude from (7.21) there is a pair $(\mathbf{W}, P)$ such that the following three facts hold:

(7.22) $$W_j \text{ and } P \text{ are in } C^\infty[B(0,r_1) - \{0\}] \quad \text{for} \quad j = 1, \cdots, N;$$

(7.23) $$\begin{aligned} &\nu \Delta W_j(x) - \partial P(x)/\partial x_j = 0, \\ &\sum_{k=1}^{N} \partial W_k(x)/\partial x_k = 0 \qquad \text{for } x \text{ in } B(0,r_1) - \{0\}; \end{aligned}$$

(7.24) $$\begin{aligned} v_j - U_j &= W_j, \qquad & j = 1, \cdots, N, \\ p - Q &= P, \qquad & \text{almost everywhere in } B(0,r_1). \end{aligned}$$

Next, from (iv) in Theorem 5, we observe that

(7.25) $$\lim_{r \to 0} r^{-2} \int_{B(0,r)} |\mathbf{v}|\, dx = 0 \quad \text{for} \quad N \geqq 3.$$

We consequently obtain from (7.12), (7.24), (7.25), (v) in Theorem 5 that

(7.26) $$r^{-N} \int_{B(0,r)} |W_j|\, dx = \begin{cases} o(r^{-(N-2)}) & \text{for} \quad N \geqq 3, \\ o(|\log r|) & \text{for} \quad N = 2. \end{cases}$$

But then it follows from (7.22), (7.23), (7.26) and Theorem 3 that

(7.27) W_j and P can be defined at 0 so that W_j and P are in $C^\infty[B(0,r_1)]$ for $j = 1, \cdots, N$.

Now Q is in $L^1[B(0,r_1)]$. So we conclude from (7.24) and (7.27) that p is in $L^1[B(0,r_1/2)]$. But this fact in conjunction with (ii) in Theorem 5 tells us that

(7.28) $$p \text{ is in } L^1[B(0,r_1)].$$

Also from (7.13), (7.24) and (7.27), we have that as $r \to 0$,

(7.29) $$r^{-N} \int_{B(0,r)} |p|\, dx = o(r^{-(N-1)}) \quad \text{for} \quad N \geqq 2.$$

From (7.28) and (7.29), we see that the conditions concerning p in the hypothesis of [3, Thm. 2] are met in $B(0,r_1)$. This, plus (i), (iii), and (iv) of Theorem 5 in conjunction with [3, Thm. 2] tells us that

(7.30) $(\mathbf{v}, p)$ is a distribution solution of (7.1) in $B(0,r_1)$.

Next, we observe that for every ϕ in $C_0^\infty[B(0,r_0)]$, there are functions ψ and η such that ψ is in $C_0^\infty[B(0,r_1)]$ and η is in $C_0^\infty[B(0,r_0) - \{0\}]$ and such that $\phi = \psi + \eta$. This fact in conjunction with (iii) of Theorem 5 and (7.30) tells us that $(\mathbf{v}, p)$ is a distribution solution of (7.1) in $B(0,r_0)$. Also (7.28) in conjunction with (ii) of Theorem 5 tells us that p is locally in $L^1[B(0,r_0)]$ and the proof of Theorem 5 is complete.

Before proving Theorem 6, we state the following lemma.

LEMMA 2. *Let* $(\mathbf{v}, p)$ *be a classical solution of* (1.1) *in* $B(0, r_0) - \{0\}$ *where* $N = 2$ *and* $c \neq -1$. *Suppose that*

(i) $\mathbf{v}$ *is in* $L^1[B(0, r_0)]$;

(ii) $\lim_{r \to 0} r^{-1} \int_{B(0,r)} |\mathbf{v}|\, dx = 0$.

Then there are K_0, K_1, K_2, a_1, a_2 *and functions* A, A_1 *and* A_2 *in* $C^\infty[B(0, r_0)]$ *such that*

$$v_j(x) = a_j \log |x|^{-1} + K_0 \frac{\partial |x|^2 \log |x|^{-1}}{\partial x_j} + \sum_{k=1}^{2} K_k \frac{\partial^2 |x|^2 \log |x|^{-1}}{\partial x_j\, \partial x_k} + A_j(x),$$

$$4^{-1} p(x) = K_0 \log |x|^{-1} - \sum_{k=1}^{2} K_k x_k |x|^{-2} + A(x)$$

for x *in* $B(0, r_0) - \{0\}$ *and* $j = 1, 2$.

To prove Lemma 2, we proceed precisely as in the proof of Theorem 3 and observe that the conclusion of Lemma 2 is given exactly by (3.13) and (3.35). We leave the filling in of the details to the reader and consider the proof of Lemma 2 complete.

To establish Theorem 6, we shall show that for $N = 2$, conditions (i), (ii), (iii) and (v) of Theorem 5 imply that condition (iv) holds every γ such that $2 < \gamma \leqq 3$.

To do this we proceed exactly as in the proof of Theorem 5 and observe that everything is valid from (7.1) through (7.24) except (7.9), (7.12) and (7.13). However, it is easy to see from (7.8) and [3, Lem. 13] that instead of (7.12), we have

$$r^{-2} \int_{B(0,r)} |U_j|\, dx = o(r^{-1}) \quad \text{as} \quad r \to 0. \tag{7.31}$$

But then it follows from condition (v) in Theorem 5, (7.24) and (7.31) that

$$r^{-2} \int_{B(0,r)} |W_j|\, dx = o(r^{-1}) \quad \text{as} \quad r \to 0. \tag{7.32}$$

We next set $\mathbf{W} = (W_1, W_2)$ and observe from (7.22) and (7.23) that with $c = 0$, $(\mathbf{W}, P)$ is a classical solution of (1.1) in $B(0, r_1) - \{0\}$. From (7.24) and (7.31), we also see that $\mathbf{W}$ meets conditions (i) and (ii) in Lemma 2. We have consequently from Lemma 2 that for $j = 1, 2$,

$$W_j \text{ is } L^\gamma[B(0, r_1/2)] \quad \text{for} \quad 1 < \gamma < \infty. \tag{7.33}$$

Next, we observe that $\int_{T_2} f'_k(x - y) H(y)\, dy$ is in $L^\gamma(T_N)$ for $1 < \gamma < \infty$, and we conclude from (7.8), (7.10) and [3, Lem. 2] that if $2 < \beta < 4$,

$$U_j \text{ is in } L^{2\beta/(4-\beta)}[B(0, r)] \quad \text{for} \quad 0 < r < r_1/2. \tag{7.34}$$

But then from (7.24), (7.33) and (7.34) we conclude that

$$v_j \text{ is } L^{2\beta/(4-\beta)}[B(0, r)] \quad \text{for} \quad 0 < r < r_1/2. \tag{7.35}$$

We then use [3, (3.36), (3.37) and (3.38)] and conclude with no difficulty from a bootstrap argument that

$$v_j \text{ is } L^6[B(0,r)] \quad \text{for} \quad 0 < r < r_1/2. \tag{7.36}$$

From (7.36), we see that for $0 < r < r_1/2$,

$$r^{-2}\int_{B(0,r)} |\mathbf{v}|^3\,dx \leqq \left\{\int_{B(0,r)} |\mathbf{v}|^6\,dx\right\}^{1/2} \pi^{1/2} r^{-1}.$$

Therefore $\{r^{-2}\int_{B(0,r)}|\mathbf{v}|^3\}^{1/3} = o(r^{-1/3})$. However this fact implies in particular that condition (iv) of Theorem 5 holds for every γ such that $2 < \gamma \leqq 3$, and the proof of Theorem 6 is complete.

In conclusion, we shall show that Theorem 6 (and therefore Theorem 5 in dimension 2) is in a certain sense best possible. In particular, we shall show that condition (v) cannot be replaced by

$$\text{(v}') \qquad \begin{aligned} &r^{-2}\int_{B(0,r)} |v_1|\,dx = O(\log r), \\ &r^{-2}\int_{B(0,r)} |v_2|\,dx = o(\log r) \quad \text{as} \quad r \to 0. \end{aligned}$$

To see this, we set for $|x| \neq 0$,

$$\begin{aligned} v_1(x) &= (2\nu)^{-1}[x_1^2|x|^{-2} + 2^{-1} - \log|x|], \\ v_2(x) &= (2\nu)^{-1}x_1x_2|x|^{-2}, \\ p(x) &= x_1|x|^{-2}, \\ f_1(x) &= v_1(x)\,\partial v_1(x)/\partial x_1 + v_2(x)\,\partial v_1(x)/\partial x_2, \\ f_2(x) &= v_1(x)\,\partial v_2(x)/\partial x_1 + v_2(x)\,\partial v_2(x)/\partial x_2. \end{aligned} \tag{7.37}$$

Clearly, v_1 and v_2 meet condition (v'). Also we observe that $\mathbf{f}$ and p are in $L^1[B(0,1)]$, $\mathbf{v}$ is in $L^6[B(0,1)]$, and that (7.1) is satisfied in the classical sense in $B(0,1) - \{0\}$. Consequently (7.2) holds with $\Omega = B(0,1) - \{0\}$.

Therefore all the conditions in the hypothesis of Theorem 6 hold provided (v) is replaced by (v'). If the conclusion of Theorem 6 held also, we would have that (7.2) holds with $\Omega = B(0,1)$. It is not difficult to see from (7.37) that this implies in particular that

$$\int_{B(0,1)} \left[\nu v_1\,\Delta\phi + p\frac{\partial\phi}{\partial x_1}\right] dx = 0 \quad \text{for } \phi \quad \text{in } C_0^\infty[B(0,1)]. \tag{7.38}$$

Observing from (7.37) that $\nu v_1 = -\log|x| + 4^{-1}\,\partial^2|x|^2\log|x|/\partial x_1^2$, we obtain after an easy computation that the integral on the left in (7.38) is given by

$$\int_{B(0,1)} \log|x|^{-1}\,\Delta\phi(x)\,dx \quad \text{for } \phi \quad \text{in } C_0^\infty[B(0,1)]. \tag{7.39}$$

From another easy computation, we obtain furthermore that the integral in (7.39) is equal to $-2\pi\phi(0)$. Consequently we see from (7.38) and (7.39) that if the conclusion to Theorem 6 held for the functions defined by (7.37), we would have that

$\phi(0) = 0$ for every ϕ in $C_0^\infty[B(0, 1)]$. This however is a manifest contradiction, and our assertion is established.

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SOLUTION OF INTERFACE PROBLEMS BY HOMOGENIZATION. I*

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Abstract. The problem of an elliptic differential equation with nonsmooth coefficients is studied. It is assumed that the coefficients are double periodic with the period H and the right-hand side is smooth. The behavior of the solution for $H \to 0$ is investigated and theorems about limiting solution are proved. The study is related to the homogenization ideas used in an intuitive way in the theory of nuclear reactors and theory of composite materials.

1. Introduction.

1.1. Motivation. In various fields there are problems which lead to a special kind of partial differential equation. We show its simplest form. The problem deals with the boundary value problem on Ω for the self-adjoint elliptic differential equation

$$-\sum_{i=1}^{2} \frac{\partial}{\partial x_i} a^H(x_1, x_2) \frac{\partial u^H}{\partial x_i} + C^H(x_1, x_2) u^H = f, \tag{1.1.1}$$

where the functions $a^H(x_1, x_2)$ and $C^H(x_1, x_2)$ are piecewise constant (or piecewise smooth) and periodic (or "nearly" periodic) with period H which is small compared to the inverse of the first and second derivatives of f, and the diameter of the domain Ω.

Let us mention some areas where we can find this kind of problem.

(a) Problems related to the study of composite materials. See, e.g., [1], [13], [17], [18], [19], [31].

(b) In reactor computations we encounter it when subassemblies create cells of the reactor. (See e.g. [11, p. 255]). We refer here to [6], [29], [30], [33], [38].

(c) In transformer computations this problem arises when we study the electrical field of conductors and insulations in oil. In chemistry this problem is important in connection with the polymer studies. See, e.g., [39].

(d) We find it in studies related to the flow of electrical current in the brain, the diffusion of metabolics in the tissues, etc.

1.2. Numerical aspects. The problem described by (1.1.1) or some similar equation is the problem of interfaces where the solution has many singularities. Singularities of the solution create very severe computational difficulties. For dealing with singularities caused by interfaces or unsmooth boundaries see e.g. [2], [3], [4], [12], [20], [37] and many others. If the number of singularities is large, then it is impossible to consider them individually. So the question is how to solve such problems. The practical approach here is the *homogenization*, which, e.g., plays an essential role in reactor computations. See, e.g., [14], [30], [38].

1.3. The theoretical aspect. Obviously one natural question is how the solution behaves when $H \to 0$. If $a^H \to a_0$ in the space L_2 strongly, then the solution converges to the solution of the differential equation with coefficients a_0.

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A quite different situation occurs when $a^H \rightharpoonup a_0$—only weakly. Let us mention here some studies related to this or analogous problems in [10] or [35], [36].

Another question arises in connection with the numerical solution, namely, whether it is possible to use ideas of a small parameter (H). These problems are in general unsolved. Some particular results are contained in this paper. For a survey see [7], [40].

The situation is quite different when ordinary differential equations are investigated instead of partial DE's. In the case of ODE's the problem is then much simpler and has been studied in different aspects, e.g., see [26], [32].

1.4. Outline of the results. This paper is the first of a series. It deals with the elliptic equation

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{i,j}^H \frac{\partial u^H}{\partial x_j} = f \tag{1.4.1}$$

with

$$a_{i,j}^H(x_1, x_2) = a_{i,j}^1\left(\frac{x_1}{H}, \frac{x_2}{H}\right) \tag{1.4.2}$$

and $a_{i,j}^1(x_i, x_2)$ is a doubly periodic function of period one. We are interested here in the solution on the entire plane R_2 only.

1. Provided that f is smooth (and has properties which lead to the finite energy of the solution) then $u^H \to U$ in L_2 on every bounded domain with the rate H (when u^H and U were properly normalized) (Theorems 4.3.2, 5.3.2). Here U is the solution of the limiting differential equation with constant coefficients

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} b_{i,j} \frac{\partial U}{\partial x_j} = f. \tag{1.4.3}$$

The coefficients $b_{i,j}$ are computable by solving *a periodic problem* related to the differential operator in (1.4.1). Hence it is possible to determine these coefficients $b_{i,j}$ by the numerical study of a single unit cell only.

Further (Theorems 4.3.4, 5.3.4)

$$\int_{R_2} \left[\sum_{i,j=1}^{2} a_{i,j}^H \frac{\partial u^H}{\partial x_i} \frac{\partial u^H}{\partial x_j}\right] dx \to \int_{R_2} \left[\sum_{i,j=1}^{2} b_{i,j} \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j}\right] dx. \tag{1.4.4}$$

If $a_{1,1} = a_{2,2}$ $_{1,1} = a_{2,2}$ and $a_{1,2} = a_{2,1} = 0$, then, in general, $b_{1,1} \neq b_{2,2}$ and $b_{1,2} \neq 0$.

2. With U obtained from (1.4.3) and V being the solution of the equation

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} b_{i,j} \frac{\partial V}{\partial x_j} = \Lambda U, \tag{1.4.5}$$

where Λ is an operator of degree 3 with constant coefficients (which are obtained as a result of the analysis of the single cell only), we see that an approximate solution can be constructed which converges in energy with the rate H^2 (see Theorem 4.3.3). It is important that the functions U and V are smooth and, e.g., the finite element method may be very well used to find them (see, e.g., [5] and others). The single (unit) cell problem with all the interfaces has to be solved by special techniques of the finite element method. See, e.g., [8].

3. Section 5 introduces illustrative examples and summarizes the approach.

2. Basic notations and auxiliary results.

2.1. Domains. Let R_2 denote the two-dimensional Euclidian space $[x \equiv (x_1, x_2) \in R_2]$ with the norm $\|\cdot\|$.

Further let Ω be a given domain. We shall assume that Ω is bounded and its boundary $\partial\Omega$ is composed by a finite number of arcs, which have all continuous derivatives. Some special domains will be of importance later.

Let now $H > 0$ be given. Then we define for any integral $k \equiv (k_1, k_2)$,

$$S_k^H = S_{(k_1,k_2)}^H = \{x \mid |x_1 - Hk_1| < \tfrac{1}{2}H, |x_2 - Hk_2| < \tfrac{1}{2}H\}.$$

The domain S_k^H will be called a cell. If $H = 1$, then we will write S_k instead of S_k^1. Obviously $\bigcup_k S_k^H = R_2$.

Denote further $\Gamma^H = \bigcup_k \partial S_k^H = \bigcup_k [\{x \mid x_1 = k_1 H,\} \cup \{x \mid x_2 = k_2 H\}]$ again omitting H when $H = 1$.

Let

$$\partial S_k^H = \bigcup_{i=1}^{4} I_k^{H,i},$$

where $I_k^{H,i}$ are the sides of S_k^H shown in Fig. 2.1.1. The index H will be omitted if $H = 1$. We shall also use the notation $I^H = \{t \mid |t| < \tfrac{1}{2}H\}$.

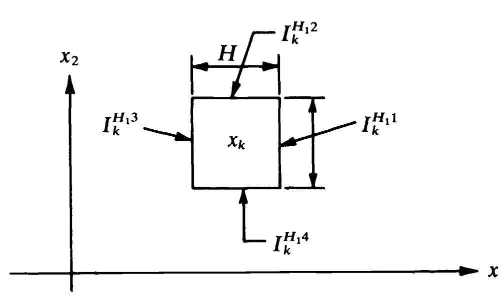

FIG. 2.1.1

For $p > 0$ we shall use the notation $Q_{p;k}^H = S_k^{pH}$ and Q_p will be used instead of $Q_{p;0}^1$.

2.2. Sobolev spaces. Consider a domain Ω (resp. R_2) and let $\mathscr{E}(\bar{\Omega})$ (resp. $\mathscr{E}(R_2)$) be the space of all (real) infinitely differentiable functions on Ω (resp. R_2) such that all the derivatives have continuous extension to $\partial\Omega$. Furthermore denote by $\mathscr{D}(\Omega) \subset \mathscr{E}(\bar{\Omega})$ (resp. $\mathscr{D}(R_2)$) the subspace consisting of all functions with compact support in Ω (resp. R_2); supp f denotes the support of f.

Denote by $\mathscr{E}_H(R_2) \subset \mathscr{E}(R_2)$ the set of all H-periodic functions, i.e., $u \in \mathscr{E}_H(R_2)$ if and only if

(2.2.1) $$u(x + kH) \equiv u(x_1 + k_1 H, x_2 + k_2 H) = u(x_1, x_2)$$

for any integer and $\mathscr{E}_H(S_k^H)$ is the restriction of $\mathscr{E}_H(R_2)$ on S_k^H.[1] Obviously $\mathscr{D}(S_k^H) \subset \mathscr{E}_H(S_k^H)$.

Let $\mathscr{E}(\Gamma^H)$ (resp. $\mathscr{D}(\Gamma^H)$) be the restrictions of $\mathscr{E}(R_2)$ (resp. $\mathscr{D}(R_2)$) on Γ^H and $\mathscr{E}(\partial\Omega)$ be the restriction of $\mathscr{E}(\bar{\Omega})$ on $\partial\Omega$.

As usual $L_2(\Omega)$ will be the space of square integrable functions u on Ω with the norm

$$\|u\|^2_{L_2(\Omega)} = \int_\Omega u^2\, dx \tag{2.2.2}$$

and $dx = dx_1\, dx_2$. The scalar product will be denoted as $(u, v)_{L_2(\Omega)}$. Sometimes we shall use the notation $L_2(\Omega) = H^0(\Omega)$. Further let for $\alpha > 0$,

$$\|u\|^2_{L_{2,\alpha}(R_2)} = \int_{R_2} u^2(1+\|x\|^2)^{-\alpha}\, dx \tag{2.2.3}$$

with obvious definition for $(u, v)_{L_{2,\alpha}(R_2)}$.

In an obvious sense we shall use the notation $\|u\|_{L_2(\Gamma^H)}$, $\|u\|_{L_2(\partial S_k^H)}$, etc., e.g.,

$$\|u\|_{L_2(\partial S_0^H)} = \oint_{\partial S_0^H} u^2\, ds. \tag{2.2.4}$$

Later we shall use also the norm $(1 < \alpha < \infty)$, $\|u\|_{L_\alpha(\partial S_k^H)}$, $\|u\|_{l_\alpha(\Gamma^H)}$ etc., where

$$\|u\|^\alpha_{L_\alpha(\partial S_K^H)} = \oint_{\partial S_0^k} |u|^\alpha\, ds. \tag{2.2.5}$$

Suppose now that $l \geqq 1$ is an integer. The Sobolev space H^l, (l integral) will be defined as the closure of $\mathscr{E}(\bar{\Omega})$ in the norm $\|\cdot\|_{H^l(\Omega)}$, where

$$\|u\|^2_{H^l(\Omega)} = \sum_{0 \leqq |\alpha| \leqq l} \|D^\alpha u\|^2_{L_2(\Omega)}, \tag{2.2.6}$$

$$D^\alpha = \frac{\partial^{\alpha_1+\alpha_2}}{\partial x_1^{\alpha_2}\, \partial x_2^{\alpha_2}}, \quad \alpha = (\alpha_1, \alpha_2), \quad |\alpha| = \sum_{i=1}^2 \alpha_i, \tag{2.2.7}$$

α_i are nonnegative integers. In a similar way the space $H^l(R_2)$ may be introduced, where the closure is taken of the subset of $\mathscr{E}(R_2)$ of functions where (2.2.6) is finite. We denote $H_0^l(\Omega) \subset H^l(\Omega)$ (resp. $H_H^l(S_0^H) \subset H^l(S_0^H)$), the closure of $\mathscr{D}(\Omega)$ (resp. $\mathscr{E}_H(S_0^H)$).

We shall introduce also the spaces $L^1(\Omega)$, $L_H^1(S_0^H)$, etc. These spaces are the closure of $\mathscr{E}(\bar{\Omega})$, etc. in the norm $\|\cdot\|_{L^1(\Omega)}$ where

$$\|u\|^2_{L^1(\Omega)} = \sum_{i=1}^2 \left\|\frac{\partial u}{\partial x_i}\right\|^2_{L_2(\Omega)}. \tag{2.2.8}$$

The expression (2.2.8) creates a norm *of classes* if functions modulo constant functions are considered. In a completely analogous way as closure of the proper subspaces we introduce the spaces $L^1(R_2)$, $L_0^1(\Omega)$, $L_H^1(S_k^H)$, etc. Further let $\mathscr{E}_p(R_2)$

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[1] More precisely, $\mathscr{E}_H(S_k^H)$ is the space of all $u \in \mathscr{E}_H(R_2)$ being restricted to S_k^H.

be the subspace of functions such that

(2.2.9) $$\int_{Q_p} u\,dx = 0$$

and $L^{1;p}(R_2)$ be the closure of $\mathscr{E}_p(R_2)$ in the norm $\|\cdot\|_{L^1(R_2)}$. (Note that obviously we deal here with norms and not seminorms as we did in $L^1(R_2)$.)

Further we shall denote $L^0_{\mathrm{Loc}}(R_2)[L^1_{\mathrm{Loc}}(R_2),\ H^1_{\mathrm{Loc}}(R_2)]$ the space of all functions u whose restriction to Ω belongs to $L_2(\Omega)$, $[L^2(\Omega), H^1(R_2)]$ for any bounded Ω. We remark that if $u\in L^1_{\mathrm{Loc}}(R_2)$, then $u\in L^0_{\mathrm{Loc}}(R_2)$ also.

Let us make some comments now.

1. Sobolev (see [34]) has shown that the spaces $L^1(R_2)$ and $L^1_0(R_2)$ are identical, i.e., that $\mathscr{D}(R_2)$ is dense in $L^1(R_2)$.

2. In general if $u\in L^1(R_2)$, then u need not be integrable. Nevertheless if $u\in L^1(R_2)$, then $u\in L_{2,\alpha}(R_2)$ with $\alpha>1$. For more see [24].

3. A similar situation occurs in the case of the space $L^{1;p}(R_2)$, i.e., if $u\in L^{1;p}(R_2)$, then

(2.2.10) $$\|u\|_{L_{2,\alpha}(R_2)}\leqq C(\alpha)\|u\|_{L^1(R_2)}.$$

For more see [24], [25].

Next let $u^H_k\equiv u^H_{(k_1,k_2)}\in L^1(S^H_k)$, resp. $[u^H_k\in H^1(S^H_k)]$. Then we shall denote by $\{u^H_k\}$ the sequence of functions u^H_k, and write $\{u^H_k\}\in {}^H\hat{L}^1_{\mathrm{Loc}}(R_2)$ [resp. ${}^H\hat{H}^1_{\mathrm{Loc}}(R_2)$]). Further ${}^H\hat{L}^1(R_2)\subset {}^H\hat{L}^1_{\mathrm{Loc}}(R_2)$ so that

(2.2.11) $$\|\{u^H_k\}\|^2_{L^1(R_2)}=\sum_k \|u^H_k\|^2_{L^1(S^H_k)}<\infty$$

and analogously for ${}^H\hat{H}^1(R_2)$. $\{u^H_k\}$ could be understood as a function defined on R_2 in an obvious way. We shall say that the sequence $\{u_k\}$ *coincides* if there exists $u^H\in L^1_{\mathrm{Loc}}(R_2)$ so that $u^H=u^H_k$ for $x\in S^H_k$. If $\{u^H_k\}\in {}^H\hat{L}^1(R_2)$ and $\{u^H_k\}$ coincides, then

$$\|\{u^H_k\}\|_{{}^HL^1(R_2)}=\|u^H\|_{L^1(R_2)}\ .$$

We shall often write $u^H=\{u^H_k\}\in L^1_{\mathrm{Loc}}(R_2)$, etc.

We introduce $\mathscr{E}(\partial\Omega)$ as the restriction of $\mathscr{E}(\bar{\Omega})$ on $\partial\Omega$ and define $H^{\frac12}(\partial\Omega)$ as a closure of $\mathscr{E}(\partial\Omega)$ in the norm $\|\cdot\|_{H^{\frac12}(\partial\Omega)}$ where for $u\in\mathscr{E}(\partial\Omega)$

(2.2.12) $$\|u\|_{H^{\frac12}(\partial\Omega)}=\inf_{\substack{v\in\mathscr{E}(\Omega)\\ u=v \text{ on } \partial\Omega}}\|v\|_{H^1(\Omega)}.$$

Similarly we define $L^{1/2}(\partial\Omega)$ and $L^{1/2}(\Gamma^H)$, $H^{1/2}(\Gamma^H)$, etc. The structure of the space $H^{1/2}(\partial\Omega)$ is well known. For $\partial\Omega$ smooth see, e.g., [27] and for $\partial\Omega$ piecewise smooth see, e.g., [9] and [15]. We shall speak about traces of $u\in L^1(\Omega)$ (resp. $H^1(\Omega)$) on $\partial\Omega$ in the usual sense. Then we have

(2.2.13) $$\|u\|_{L^{1/2}(\partial\Omega)}\leqq\|u\|_{L^1(\Omega)}$$

and analogously for $H^{1/2}(\partial\Omega)$.

Now we shall mention some well-known facts about the introduced spaces.

THEOREM 2.2.1. *Let* $u \in L^{1/2}(\partial\Omega)$. *Then* $u \in L_q(\partial\Omega)$ *for any* $1 \leqq q < \infty$. *If* $\oint_{\partial\Omega} u\, ds = 0$, *then*

$$\|u\|_{L_q(\partial\Omega)} \leqq C\|u\|_{L^{1/2}(\partial\Omega)}.^2 \tag{2.2.14}$$

THEOREM 2.2.2. *Let* $\{u_k^H\} \in {}^H\hat{L}^1_{\text{Loc}}(R_2)$ *be given. For any pair* k^1, k^2 *such that* $I^H = \partial S^H_{k^1} \cap \partial S^H_{k^2} \neq \phi$ *let* $u^H_{k^1} = u^H_{k^2}$ *on* I *(in the sense of traces). Then* $\{u_k^H\}$ *coincides, i.e.,* $\{u_K^H\} \in L^1_{\text{Loc}}(R_2)$.

2.3. Bilinear form. We introduce a theorem which is a generalization of the well-known Lax–Milgram theorem.

THEOREM 2.3.1. *Suppose*

(I) H_1 *and* H_2 *are two real Hilbert spaces with the scalar product* $(\cdot\,,\cdot)_{H_1}$ *and* $(\cdot\,,\cdot)_{H_2}$ *respectively.*

(II) $B(u, v)$ *is a bilinear form on* $H_1 \times H_2$, $(u \in H_1, v \in H_2)$ *such that*

$$|B(u, v)| \leqq C_1\|u\|_{H_1}\|v\|_{H_2}, \tag{2.3.1}$$

$$\inf_{\substack{u\in H_1\\ \|u\|_{H_1}=1}} \sup_{\substack{v\in H_2\\ \|v\|_{H_2}\leqq 1}} |B(u, v)| \geqq C_2 > 0, \tag{2.3.2}$$

$$\sup_{u\in H_1} |B(u, v)| > 0, \qquad v \neq 0, \tag{2.3.3}$$

where $C_1 < \infty$.

(III) $f \in H_2^1$, *i.e.,* f *is a linear functional on* H_2.

Then

(IV) *there exists a unique element* $u_0 \in H_1$ *such that*

$$B(u_0, v) = f(v) \tag{2.3.4}$$

for any $v \in H_2$.

$$\|u_0\|_{H_1} \leqq \frac{\|f\|_{H_2}}{C_2}. \tag{2.3.5}$$

For proof see, e.g., [5].

Let us show some examples which will be useful later. Let $H_1 = L^1(R_2) = H_2$,

$$B(u, v) = \int_{R_1} \left[\sum_{i=1}^{2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\right] dx. \tag{2.3.6}$$

Obviously this bilinear form satisfies (2.3.1), (2.3.2) and (2.3.3). Let now $f \in L^0(R_2)$, $\operatorname{supp} f \subset Q_q$, $q > 0$ and

$$\int_{r_2} f\, dx = 0. \tag{2.3.7}$$

Then

$$F(v) = \int_{R_2} fv\, dx \tag{2.3.8}$$

is a continuous functional on H_2 and so there is an unique $u_0 \in L^1(R_2)$ (in $L^1(R_2)$,

[2] C is a generic constant with different values on different places.

i.e., up to a constant) such that

(2.3.9) $$B(u_0, v) = F(v) \quad \forall v \in L^1(R_2),^3$$

and therefore,

(2.3.10) $$-\Delta u_0 = f$$

(in the usual generalized sense). Condition (2.3.7) is a necessary condition for the continuity of F.

Take now $H_1 = H_2 = L^{1;p}(R_2)$. Then the F of (2.3.8) will be continuous for any f satisfying $\int_{R_2} f^2(1+\|x\|)^\alpha\, dx < \infty$ with $\alpha > 1$. By Theorem 2.3.1 there is a unique $u_0^p \in L^{1;p}(R_2)$ such that

(2.3.11) $$B(u_0^p, v) = F(v) \quad \forall v \in L^{1;p}(R_2).$$

Let us assume now that supp $f \in Q_p$ and f satisfies (2.3.7). Then as we said we may find u_0 (resp. u_0^p) such that (2.3.9) (resp. (2.3.11)) holds. Because $u_0 \in L^1(R_2)$ and $u_0 \in L_{2,\alpha}$ we may choose a representative $\bar{u}_0$ for u_0 so that $\bar{u}_0 \in L^{1;p}(R_2)$. We shall often identify $\bar{u}_0$ with u_0. Since (2.3.9) holds for all $v \in L^1(R_2)$ we see that $\bar{u}_0 = u_0^p$ in this case. This simple observation will be useful later, when $\bar{u}_0 \in L^1(R_2)$, $f \in L_2(R_2)$, supp $f \in Q_q$ with $\int_{Q_q} f\, dx = 0$ will be given such that

$$B(\bar{u}_0, v) = \int_{R_2} vf\, dx$$

for any $v \in L^{1;p}(R_2)$. Then we shall identify $\bar{u}_0$ and u_0 where u_0 is the solution of $-\Delta u_0 = f$.

2.4. The regularity problem. Let $a_{i,j} = a_{j,i}$, $i, j = 1, 2$, be measurable functions on R_2 which are H-periodic with $H = 1$ and satisfy

(2.4.1) $$\sum_{i,j=1}^{2} a_{i,j}(x)\xi_i\xi_j \geqq \alpha(\xi_1^2 + \xi_2^2)$$

with $\alpha > 0$ and

(2.4.2) $$|a_{i,j}(x)| < \beta < \infty.$$

On $L^1(R_2) \times L^1(R_2)$ define the bilinear form

(2.4.3) $$B(u, v) = \int_{R_2} \left[\sum_{i,j=1}^{2} a_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \right] dx.$$

Assume that $u \in L^1_{\text{Loc}}(R_2)$ and $f \in L_2(Q_q)$, $q \geqq 3$, and u is such that

(2.4.4) $$B(u, v) = \int_{R_2} fv\, dx$$

for any $v \in \mathcal{D}(R_2)$ with supp $v \in Q_q$. If in addition functions a_{ij} were smooth, then $u \in H^2(Q_{q_1})$, $q_1 < q$ and so $\partial u/\partial x_i \in H^1(Q_{q_1})$ and therefore the traces of $\partial u/\partial x_i$ on ∂Q_{q_1} would exist and u would be continuous on Q_{q_1}. (See, e.g., [27].)

[3] $\forall v$ means "for every v."

In general, conditions (2.4.1) and (2.4.2) are not sufficient for the above conclusions. Therefore we shall restrict ourselves to the case when u is continuous on Q_{q_1}, the traces of $\partial u/\partial x_i$ exist on ∂Q_{q_1} and for some $p>1$ $(q_1<q)$,

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{L_p(\partial Q_{q_1})} \leqq C[\|u\|_{L^1(Q_q)}+\|f\|_{L_2(Q_q)}] \tag{2.4.5}$$

with C depending on α, β, q, q_1, p. The coefficients which guarantee (2.4.5) for u satisfying (2.4.4) and the continuity of u will be called regular.

With this regularity assumption we may write for any $v\in L^1(Q_{q_1})$,

$$\int_{Q_{q_1}} \left[\sum_{i,j=1}^{2} a_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\right]dx - \int_{Q_{q_1}} fv\,dx = \oint_{\partial Q_{q_1}} \xi v\,ds \tag{2.4.6}$$

with $\xi\in L_p(\partial Q_{q_1})$ and

$$\|\xi\|_{L_p(\partial Q_{q_1})} \leqq C[\|u\|_{L^1(Q_q)}+\|f\|_{L_2(Q_q)}], \tag{2.4.7}$$

and so ξ may be defined on every side of the Q_{q_1} separately.

The question is now—What conditions of $a_{i,j}$ will guarantee the regularity assumption? This question is likely not solved. Nevertheless, the coefficients which occur in applications satisfy mostly the regularity condition, e.g., the case when the coefficients $a_{i,j}$ are smooth (as we already mentioned). Another case is when the $a_{i,j}$ are piecewise smooth. Here it is sufficient to assume that in S_0 there are a finite number of domains with piecewise smooth boundaries without turning points and the coefficients $a_{i,j}$ are smooth on these domains. This conclusion follows from [5], [16], [21], [22] and [23]. In the application the most important case is when the $a_{i,j}$ are smooth or piecewise constant.

3. Systems of particular solutions. Through this entire section we shall assume $H=1$. Further $a_{i,j}$ will be H-periodic, measurable functions defined on R_2, satisfying (2.4.1) and (2.4.2) and will be regular (see § 2.4). Further $B(u, v)$ will be the bilinear form (2.4.3) defined on $L^1_{\text{Loc}}(R_2)\times\mathcal{D}(R_2)$ and

$$B_k(u, v) = \int_{S_k} \left[\sum_{i,j=1}^{2} a_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\right]dx.$$

Obviously,

$$B(u, v) = \sum_k B_k(u, v),$$

and because of the periodicity of $a_{i,j}$,

$$B(u, v) = \sum_k B_0(u(x-k), v(x-k)).$$

Because we assume $v\in\mathcal{D}(R_2)$, the sum is finite.

3.1. Particular solution for linear functions. It is obvious that if $u=C$, then $B(u, v)=0$ for any $v\in\mathcal{D}(R_2)$.

Let us define now $u_k^{[1;i]}\in L^1(S_k)$, $i=1, 2$, $k\equiv(k_1, k_2)$, as follows. For $x\equiv(x_1, x_2)\in S_k$ let

$$u_k^{[1;i]}(x) = k_i + (x_i - k_i) - \chi^{[1;i]}(x-k), \tag{3.1.1}$$

where $\chi^{[1;i]} \in L^1_H(S_0)$ (i.e., $\chi^{[1;i]}$ is an H-periodic ($H=1$) function such that

$$B_0(\chi^{[1;i]}, v) = B_0(\chi_i, v) \tag{3.1.2}$$

for any $v \in L^1_H(S_0)$.[4] To determine the function uniquely we shall assume that

$$\int_{S_0} u^{[1;i]}\, dx = \int_{S_0} x_i\, dx = 0. \tag{3.1.3}$$

Bilinear form B_0 defined on $L^1_H(S_0) \times L^1_H(S_0)$ satisfies all assumptions of Theorem 2.3.1. Further $B_0(x_i, v)$ is a continuous linear functional on $L^1_H(S_0)$. Therefore by using Theorem 2.3.1, the function $\chi^{[1;i]}$ exists and is uniquely determined (in $L^1_H(S_0)$, i.e., up to a constant function).

LEMMA 3.1.1. *The system of functions defined by* (3.1.1) *coincides and so*

$$u^{[1,i]} = \{u_k^{[1,i]}\} \in L^1_{\text{Loc}}(R_2).$$

Proof. We shall use Theorem 2.2.2. The trace of $u_k^{[1,i]}$ exists on ∂S_k. Therefore we may write for $|x_2| < \frac{1}{2}$,

$$\begin{aligned} u_k^{[1,i]}(k_1+\tfrac{1}{2}, k_2+x_2) = (k_1+\tfrac{1}{2})\delta(1, i), \\ +(k_2+x_2)\delta(2, i) - \chi^{[1;i]}(\tfrac{1}{2}, \chi_2), \end{aligned} \tag{3.1.4}$$

where $\delta(i, j)$ is the Kronecker symbol (i.e., $\delta(i, j) = 1$ for $i = j$ and $\delta(i, j) = 0$ for $i \neq j$). Similarly

$$\begin{aligned} u_{(k_1+1,k_2)}^{[1;i]}(k_1-\tfrac{1}{2}, k_2+\chi_2) = (k_1+\tfrac{1}{2})\delta(1, i) \\ +(k_2+x_2)\delta(2, i) - \chi^{[1,i]}(-\tfrac{1}{2}, x_2). \end{aligned} \tag{3.1.5}$$

Because $\chi^{[1;i]} \in L^1_H(S_0)$ we have

$$u_{(k_1+1,k_2)}^{[1;i]}(k_1-\tfrac{1}{2}, k_2+x_2) = u_{(k_1,k_2)}^{[1;i]}(k_1+\tfrac{1}{2}, k_2+x_2). \tag{3.1.6}$$

In the same manner one could show that for $|x_1| < \frac{1}{2}$,

$$u_{(k_1,k_2+1)}^{[1;i]}(k_1+x_1, k_2-\tfrac{1}{2}) = u_{k_1,k_2}^{[1;i]}(k_1+x_1, k_2+\tfrac{1}{2})$$

etc. so that all assumptions of Theorem 2.2.2 are satisfied. Therefore the lemma is proved.

THEOREM 3.1.1. *For* $\{u_k^{[1,i]}\}$ *defined in* (3.1.1) *we have*

$$u^{[1;i]} = \{u_k^{[1,i]}\} \in L^1_{\text{Loc}}(R_2), \tag{3.1.7}$$

$$B(u^{[1;i]}, v) = 0 \tag{3.1.8}$$

for any $v \in \mathscr{D}(R_2)$.

Proof. Let us show that

$$B(u^{[1,i]}, v) = 0 \tag{3.1.9}$$

for any $v \in \mathscr{D}(R_2)$. In fact $\partial u^{[1,i]}/\partial x_j$, $j = 1, 2$, is a periodic function and therefore for $x \in S_0$,

$$B_k(u_k^{[1;i]}, v) = B_0(u_0^{[1;i]}, v(x+k)). \tag{3.1.10}$$

[4] We shall not explicitly distinguish between $\chi \in L^1_H(R_2)$ and $\chi \in L^1_H(S_0)$.

Hence

$$(3.1.11)\qquad \begin{aligned} B(u^{[1,i]}, v) &= \sum_k B_k(u_k^{[1,i]}, v) \\ &= B_0\Big(u_0^{[1,i]}, \sum_k v(x+k)\Big). \end{aligned}$$

Denote

$$(3.1.12)\qquad \psi(x) = \sum_k v(x+k).$$

Because $v \in \mathscr{D}(R_2)$ the sum in (3.1.12) is finite and obviously $\psi(x) \in L_H^1(S_0)$. So we have

$$B(u^{[1;i]}, \varphi) = B_0(u_0^{[1;i]}, \psi) = B_0(x_i - \chi^{[1;i]}, \psi) = 0$$

because of (3.1.2).

Let us study the function $u^{[1;i]}$ further. We have

$$(3.1.13)\qquad \|u^{[1;i]}\|_{L^1(Q_3)} \leqq 3\|u^{[1;i]}\|_{L^1(S_0)}.$$

Therefore using (2.4.6) and (3.1.8) and the regularity assumption we have

$$(3.1.14)\qquad \begin{aligned} B_0(u^{[1;i]}, v) &= \oint_{\partial S_0} \xi^{[1;i]} v\, ds \\ &= \sum_{j=1}^{4} \int_{I_0^\gamma} \xi_j^{[1;i]} v_j\, ds \end{aligned}$$

with $\xi_j^{[1;i]}$ (resp. v_j) being the restrictions on I_0^j and

$$(3.1.15)\qquad \|\xi_j^{[1;i]}\|_{L_\alpha(I_0^j)} \leqq C\|u^{[1;i]}\|_{L^1(S_0)}{}^5$$

for some $\alpha > 1$.

We saw in the proof of Theorem 3.1.1 that $B_0(u^{[1;i]}, v) = 0$ for any $v \in L_H^1(S_0)$. Therefore defining $z_i(x_1, x_2) = \rho(x_i)$ with $\rho(t) \in \mathscr{D}(I)^6$,

$$(3.1.16)\qquad B_0(u^{[1;i]}, z_j) = 0.$$

Therefore

$$(3.1.17)\qquad \xi_1^{[1;i]} = -\xi_3^{[1;i]}$$

and similarly

$$(3.1.18)\qquad \xi_2^{[1;i]} = -\xi_4^{[1;i]}.$$

Let us define now

$$(3.1.19)\qquad q_j^{[1;i]} = \int_{I_0^j} \xi_j^{[1;i]}\, ds, \qquad i, j = 1, 2.$$

These coefficients will play a major role later. Let $\omega_i(x_1, x_2) = x_i + \frac{1}{2}$. Then $\omega_i \in L^1(S_0)$ and using (3.1.16) and (3.1.7) we easily see that

$$(3.1.20)\qquad q_j^{[1;i]} = B_0(u^{[1;i]}, \omega_j) = B_o(u^{[1;i]}, x_j) = B_0(u^{[1;i]}, u^{[1;j]}).$$

[5] $I = \{t \mid -\frac{1}{2} < t < \frac{1}{2}\}$.

[6] See Fig. 2.1.1.

Remarks. The regularity assumption was not necessary here for defining the coefficients $q_j^{[1;i]}$. They could be defined directly by (3.1.20), but the regularity assumption will be used later.

3.2. Particular solution for quadratic functions. In § 3.1 we introduced the function $u^{[1;i]}$ (with the leading term x_i) which satisfies Theorem 3.1.1. In this section functions $u^{[2;i,j]}$, $i \geqq 0$, $j \geqq 0$ integers $i+j=2$, with the leading term $x_2^i x_2^j$ will be introduced. Similarly as in § 3.1, the function $u^{[2;i,j]}$ will be defined on S_k. So for $x \in S_k$ let

$$(3.2.1)\quad u^{[2;i,j]}(x) = x_1^i x_2^j - [(ik_1^{\|[i-1]\|} k_2^{\|[j]\|} + i\omega_{m(1i;i,j)}(x-k)\chi^{[1;i]}(x-k)]$$
$$- [jk_1^{\|[i]\|} k_2^{\|[j-1]\|} + j\omega_{m(2;i,j)}(x-k)\chi^{[1;2]}(x-k)] - \chi^{[2;i,j]},$$

where

$$(3.2.2)\quad \begin{aligned} &\|[i]\| = i \quad \text{for } i \geqq 0, \qquad \|[i]\| = 0 \quad \text{for } i < 0, \\ &m(1;\, i, j) = \min(2, j+1), \\ &m(2;\, i, j) = \max(1, j) \end{aligned}$$

and $\chi^{[2;i,j]} \in L_1^H(R_2)$, i.e., $\chi^{[2;i,j]}$ is an H-periodic ($H=1$) function which will be determined later. See (3.2.24). We shall assume that the function $\chi^{[2;i,j]}$ is so determined that

$$(3.2.3)\quad \int_{S_0} u^{[2;i,j]}\, dx = \int_{S_0} x_1^i\, x_2^j\, dx.$$

Functions $\chi^{[1;i]}$ and ω_i have been introduced in § 3.1. It is easy to verify in an analogous way as in § 3.1 that $\{u_k^{[2;i,j]}\}$ coincides. Therefore we may define $u^{[2;i,j]} = \{u_k^{[2;i,j]}\} \in L_{\text{Loc}}^1(R_2)$.

Let us show now that the function $\chi^{[2;i,j]}$ may be chosen so that for any $v \in \mathcal{D}(R_2)$,

$$(3.2.4)\quad B[u^{[2;i,j]}, v] = \nu_{i,j} \int v\, dx$$

holds, with $\nu_{i,j}$ constants which will be determined later. Let us first prove two lemmas.

LEMMA 3.2.1. *Let $\varphi \in \mathcal{D}(R_2)$, $x \in S_0$ and*

$$(3.2.5)\quad \sum_k \varphi(x+k) = 0.$$

Then

$$(3.2.6)\quad \sum k_i \varphi(x+k) \in L_H^1(R_2).$$

Proof. Denoting

$$(3.2.7)\quad \psi_i(x) = \sum_k k_i \varphi(x+k)$$

and letting $m \equiv (m_1, m_2)$, m_i be integers we get

$$\psi_i(x+m) = \sum_k k_i\varphi(x+k+m)$$

$$= \sum_k (k_i + m_i)\varphi(x+k+m) - \sum_k m_i\varphi(x+k+m)$$

$$= \psi_i(x) - m_i \sum_k \varphi(x+k) = \psi_i(x).$$

Lemma 3.2.1 may be stated obviously in a slightly different way. Given $\varphi_j \in \mathcal{D}(R_2)$, $j = 1, 2$, and

$$\sum_k \varphi_1(x+k) = \sum_k \varphi_2(x+k), \tag{3.2.8}$$

then

$$\sum_k k_i\varphi_1(x+k) = \sum k_i\,\varphi_2(x+k) + w(x), \tag{3.2.9}$$

where $w \in L^1_H(R_2)$.

Let us prove now the following lemma.

LEMMA 3.2.2. *Let* $\psi \in \mathscr{E}_H(S_0)$. *Then there is* $\varphi \in \mathcal{D}(R_2)$ *with* supp $\varphi \in Q_3$ *such that for* $x \in S_0$,

$$\sum_k \varphi(x+k) = \psi(x) \tag{3.2.10}$$

and

$$\|\varphi\|_{H^1(R_2)} \leqq C\|\psi\|_{H^1(S_0)}. \tag{3.2.11}$$

Proof. For $0 \leqq t \leqq 1$ let $\rho(t)$ be a C^∞ function:

$$\begin{aligned} &\rho(t) = 0 \quad \text{for } 0 \leqq t \leqq \tfrac{1}{4},\\ &\rho(t) = 1 \quad \text{for } \tfrac{3}{4} \leqq t \leqq 1,\\ &\rho(1-t) = 1 - \rho(t),\\ &0 \leqq \rho(t) \leqq 1. \end{aligned} \tag{3.2.12}$$

Further for $-\infty < t < \infty$ let

$$\begin{aligned} &\varkappa(t) = 1 && \text{for } -\tfrac{3}{4} \leqq t \leqq \tfrac{3}{4},\\ &\varkappa(t) = \rho(t+\tfrac{3}{2}) && \text{for } -\tfrac{3}{2} \leqq t \leqq -\tfrac{1}{2},\\ &\varkappa(t) = \rho(\tfrac{3}{2}-t) && \text{for } \tfrac{1}{2} \leqq t \leqq \tfrac{3}{2},\\ &\varkappa(t) = 0 && \text{for } t \leqq -\tfrac{3}{2},\\ &\varkappa(t) = 0 && \text{for } t \leqq \tfrac{3}{2}. \end{aligned} \tag{3.2.13}$$

Let us extend the function ψ periodically and denote it with the same symbol. Take

$$\varphi(x)=\tfrac{1}{4}\psi(x)\varkappa(x_1)\varkappa(x_2). \tag{3.2.14}$$

It is easy to check that (3.2.14) satisfies (3.2.10) and (3.2.11).

We return now to (3.2.4). On S_k we have

$$\begin{aligned} u^{[2;i,j]}(x) = {} & k_1^i k_2^j + ik_1^{|[i-1]|}k_2^{|[j]|}(x_1-k_1) \\ & + jk_1^{|[i]|}k_2^{|[j-1]|}(x_2-k_2)+\tfrac{1}{2}|[i-1]|ik_1^{|[i-2]|}k_2^{|[j]|}(x_1-k_1)^2 \\ & + ijk_1^{|[i-1]|}k_2^{|[j-1]|}(x_1-k_1)(x_2-k_2) \\ & + \tfrac{1}{2}|[j-1]|jk_1^{|[i]|}k_2^{|[j-2]|}(x_2-k_2)^2 \\ & - [(ik_1^{|[i-1]|}k_2^{|[j]|}+i\omega_{m(1;i,j)}(x-k))\chi^{[1;1]}(x-k)] \\ & - [(jk_1^{|[i]|}k_2^{|[j-1]|}+j\omega_{m(2;i,j)}(x-k))^{[1;2]}(x-k)] \\ & - \chi^{[2;i,j]}(x-k). \end{aligned} \tag{3.2.15}$$

Therefore

$$\begin{aligned} B(u^{[2;i,j]}, v) &= \sum_k B_k(u^{[2;i,j]}, v_k) \\ &= B_0\Big(x_1-\chi^{[1;1]}, i\sum_k k_1^{|[i-1]|}k_2^{|[j]|}v(x+k)\Big) \\ &\quad + B_0\Big(x_2-\chi^{[1;2]}, j\sum_k k_1^{|[i]|}k_2^{|[j-1]|}v(x+k)\Big) \\ &\quad + B_0\Big(\tfrac{1}{2}|[i-1]|ix_1^2+ijx_1x_2+\tfrac{1}{2}|[i]|\,|[j-1]|x_2^2 \\ &\qquad - (i\omega_{m(1;i,j)}(x)\chi^{[1;1]}+j\omega_{m(2;i,j)}(x)\chi^{[1;2]} \\ &\qquad - \chi^{[2;i,j]}, \sum_k v(x+k))\Big). \end{aligned} \tag{3.2.16}$$

In (3.2.16) we used the fact that $i+j=2$. This assumption also gives

$$ik_1^{|[i-1]|}k_2^{|[j]|} = \begin{cases} 2k_1 & \text{for } i=2, \\ k_2 & \text{for } i=1, \\ 0 & \text{for } i=0. \end{cases} \tag{3.2.17}$$

Other analogous terms have a similar form.

Using (3.1.2), (3.2.17) and Lemma 3.2.1 we have

$$B_0\Big(x_1-\chi^{[1;1]}, i\sum_k k_1^{|[i-1]|}k_2^{|[j]|}v(x+k)\Big)=F_1^{(i,j)}\Big(\sum_k v(x+k)\Big), \tag{3.2.18}$$

i.e., $F_1^{(i,j)}$ depends only on $\sum_k v(x+k)$.

Analyze now the functional $F_1^{(i,j)}$ defined by (3.2.18) more. Let $\xi(x)=1$, $x \in R_2$, and $\bar{v}(x)=\frac{1}{4}\varkappa(x_1)\varkappa(x_2)$ be defined as in (3.2.13). We have

$$F_1^{(2,0)}(\xi) = B_0\Big(x_1-\chi^{[1;1]}, 2\sum_k k_1\bar{v}(x+k)\Big)$$
$$(3.2.19) \qquad = B_0[x_1-\chi^{[1;1]}, 1-2\rho(x_1+\tfrac{1}{2})]$$
$$= -B_0[x_1-\chi^{[1;1]}, 2\rho(x_1-\tfrac{1}{2})] = -2q_1^{[1;1]}.$$

In general for $i+j=2$, we have

$$F_1^{(i,j)} = B_0\Big(x_1-\chi^{[1;1]}, i\sum_k k_1^{\|[i-1]\|}k_2^{\|[j]\|}\Big)$$
$$(3.2.20) \qquad = -iq^{[1;1]}_{m(1;i,j)}),$$
$$F_2^{(i,j)} = B_0\Big(x_2-\chi^{[1;2]}, j\sum_k k_1^{\|[i]\|}k_2^{\|[j-1]\|}\bar{v}(x+k)\Big)$$
$$(3.2.21) \qquad = -jq^{[1;2]}_{m(2;i,j)}.$$

Therefore we see that

$$(3.2.22) \qquad K(v) = B_0\Big(x_1-\chi^{[1;1]}, i\sum_k k_1^{\|[i-1]\|}k_2^{\|[j]\|}v(x+k) + iq^{[1;1]}_{m(1;i,j)}\int_{S_0}\psi\,dx\Big)$$

with

$$(3.2.23) \qquad \psi = \sum_k v(x+k)$$

depends on ψ only (see (3.2.18)) and for $\psi=1$, we have $K(\psi)=0$. Therefore we may assume that $\int_{S_0}\psi\,dx=0$. Under this assumption $\|\psi\|_{H^1_H(S_0)} \leqq C\|\psi\|_{L^1(S_0)}$. Now using Lemma 3.2.2, for any given $\psi \in \mathscr{E}_H(S_0)$ we may construct v so that (3.2.23) holds and $\|v\|_{L^1(R_2)} \leqq C\|\psi\|_{L^1_H(S_0)}$. It follows immediately that K is a continuous linear functional on $L^1_H(S_0)$.

The same situation occurs with expressions for $F_2^{(i,j)}$.

Going back to (3.2.16) we see that

$$B(u^{[2;i,j]}, v) + [iq^{[1;1]}_{m(1;i,j)} + jq^{[1;2]}_{m(2;i,j)}]\int_{S_o}\psi\,dx$$

is a linear functional on $L^1_H(S_0)$ when (3.2.23) is used. Hence using Theorem 2.3.1 we may construct $\chi^{[2;i,j]} \in L^1_H(S_0)$ so that

$$(3.2.24) \qquad B(u^{[2;i,j]}, v) = -[iq^{[1;1]}_{m(1;i,j)} + jq^{[1;2]}_{m(2;i,j)}]\int_{R_2} v\,dx$$

for any $v \in \mathscr{D}(R_2)$. This condition serves as definition of $\chi^{[2;i,j]} \in L^1_H(S_0)$.

So we have proved the following theorem.

THEOREM 3.2.1. *For* $\{u_k^{[2;i,j]}\}$ *defined in* (3.2.1) *with* $\chi^{[2;i,j]}$ *determined by* (3.2.24) *and* (3.2.3) *we have*

$$\{u_k^{[2;i,j]}\} = u^{[2;i,j]} \in L^1_{\mathrm{Loc}}(R_2) \tag{3.2.25}$$

and

$$B(u^{[2;i,j]}, v) = -[iq^{[1;1]}_{m(1;i,j)} + jq_{m(2;i,j)}] \int_{R_2} v\,dx \tag{3.2.26}$$

for any $v \in \mathcal{D}(R_2)$.

Similarly as in § 3.1 we may write

$$\begin{aligned} B_0(u^{[2;i,j]}, v) + [iq^{[1;1]}_{m(1;i,j)} + jq^{[1;2]}_{m(2;i,j)}] \int_{S_0} v\,dx &= \oint_{\partial S_0} \xi^{[2;i,j]} v\,ds \\ &= \sum_{l=1}^{4} \int_{I_0^l} \xi_l^{[2;i,j]} v_l\,ds \end{aligned} \tag{3.2.27a}$$

with $\xi_l^{[2;i,j]}$ resp. v_l being functions on I_0^l and

$$\|\xi_l^{[2;i,j]}\|_{L_\alpha(I_0^l)} \leqq C \|u^{[2;i,j]}\|_{L^1(S_0)}. \tag{3.2.27b}$$

Let us introduce

$$r_l^{[2;i,j]} = \int_{I_0^l} \xi_l^{[2;i,j]}\,ds; \tag{3.2.28}$$

then it is easy to see that

$$\begin{aligned} r_1^{[2;i,j]} &= -r_3^{[2;i,j]}, \\ r_2^{[2;i,j]} &= -r_4^{[2;i,j]}. \end{aligned} \tag{3.2.29}$$

3.3. The structure of the particular solutions. In §§ 3.1 and 3.2 particular solutions with leading linear and quadratic terms have been constructed. It is also possible to create, in an analogous, but more laborious way, the particular solution with leading terms of higher order. It is important, especially from a computational point of view, that these solutions may be constructed by finding periodic solutions based on the knowledge of the solution of lower order terms.

We derived the particular solution for $H = 1$ only. It is obvious that knowing the particular solution for $H = 1$, the general case $H \neq 1$ is obtainable by a simple transformation. E.g. denoting $u_k^{[1;i],H}$ the particular solution for a given H, then we have

$$u_k^{[1;i],H}(x) - x_i - H\chi^{[1;i]}\left(\frac{x}{H}\right), \tag{3.3.1}$$

and analogously,

$$\begin{aligned} u_k^{[2;i,j],H}(x) = x_1^i x_2^j &- H^2\left[ik_1^{|[i-1]|}k_2^{|[j]|} + i\omega_{m(1;i,j)}\left(\frac{x}{H} - k\right)\chi^{[1;1]}\left(\frac{x}{H}\right)\right] \\ &- H^2\left[jk_1^{|[i]|}k_2^{|[j-1]|} + j\omega_{m(2;i,j)}\left(\frac{x}{H} - k\right)\chi^{[1;2]}\left(\frac{x}{H}\right)\right] \\ &- H^2\chi^{[2;i,j]}\left(\frac{x}{H}\right) \end{aligned} \tag{3.3.2}$$

for $x \in S_k^H$.

4. Homogenization problem.

4.1. Introduction. Let $a_{i,j}$, $i, j=1,2$, be H-periodic ($H=1$), measurable functions on R_2 satisfying (2.4.1) and (2.4.2) and the regularity conditions (see § 2.4). Further let $a_{i,j}^H(x)=a_{i,j}(x/H)$. Obviously functions $a_{i,j}^H$ satisfy (2.4.1) and (2.4.2), too, with the constants α and β independent of H. We may define on $L^1(R_2)\times L^1(R_2)$ the bilinear form

$$B^H(u, v)=\int_{R_2}\left(\sum_{i,j} a_{i,j}^H\frac{\partial u}{\partial x_i}\frac{\partial r}{\partial x_j}\right)dx. \tag{4.1.1}$$

This form satisfies conditions (2.3.1) and (2.3.2) with C_1 and C_2 independent of H.

Given $f\in\mathcal{D}(R_2)$ with compact support and $\int_{R_2}f\,dx=0$, there exists a unique $u^H\in L^{1,p}(R_2)$ so that

$$B^H(u^H, v)=\int_{R_2} fv\,dx \tag{4.1.2}$$

for any $v\in L^1(R_2)$ and, in addition,

$$\|u^H\|_{L^1(R_2)}\leqq C, \tag{4.1.3}$$

where C depends on f but is independent of H. The solution u^H obviously depends on H. The main topic in this section is to study how the solution behaves for $H\to 0$. Let us prove a lemma now which will be useful later.

LEMMA 4.1.1. *Let f be the same function as above. We define f^H so that $f^H(x)=f(kH)$ for $x\in S_k^H$. Further let $\tilde{u}^H\in L^{1,p}(R_2)$ be such that*

$$B^H(\tilde{u}^H, v)=\int_{R_2} f^Hv\,dx \quad \forall v\in L^{1,p}(R_2). \tag{4.1.4}$$

Then

$$\|\tilde{u}^H-u^H\|_{L^1(R_2)}\leqq CH^2, \tag{4.1.5}$$

where C depends on f but not on H.

Proof. Denote $Q=\operatorname{supp}(f-f^H)$. By the assumption Q is a bounded set.

Let us show that

$$\left|\int_{R_2}(f^H-f)v\,dx\right|=\left|\int_Q(f^H-f)v\,dx\right|\leqq CH^2\|v\|_{L^1(R_2)}. \tag{4.1.6}$$

Obviously,

$$\int_{R_2}(f^H-f)v\,dx=\sum_{k,S_k^H\cap Q\neq\varnothing}\int_{S_k^H}(f^H-f)v\,dx \tag{4.1.7}$$

and

$$|f^H-f|\leqq CH \tag{4.1.8}$$

and

$$\left|\int_{S_k^H}(f^H-f)\,dx\right|\leqq CH^H. \tag{4.1.9}$$

On S_k^H we may write

(4.1.10) $$v = \bar{v}_k^H + d_k^H,$$

where

(4.1.11) $$d_k^H = H^{-2} \int_{S_k^H} v\, dx$$

and

(4.1.12) $$\int_{S_k^H} \bar{v}_k^H\, dx = 0.$$

Therefore

(4.1.13) $$\|\bar{v}_k^H\|_{L^1(S_k^H)} = \|v\|_{L^1(S_k^H)}$$

and

(4.1.14) $$|d_k^H| \leqq H^{-1} \|v\|_{L_2(S_k^H)}$$

and

(4.1.15) $$\|\bar{v}_k^H\|_{L_2(S_k^H)} \leqq CH \|\bar{v}^H\|_{L^1(S_k^H)}.$$

Therefore we have

(4.1.16) $$\begin{aligned}\left|\int_{S_k^H} (f^H - f) v\, dx\right| &\leqq |d_k^H| \left|\int_{S_k^H} (f^H - f)\, dx\right| + \left|\int_{S_k^H} (f^H - f) \bar{v}_k^H\, dx\right| \\ &\leqq CH^3 [\|v\|_{L_2(S_k^H)} + \|v\|_{L^1(S_k^H)}].\end{aligned}$$

Because (4.1.7) holds we have by Schwarz's inequality and (2.2.10),

(4.1.17) $$\left|\int_{R_2} (f - f^H) v\, dx\right| \leqq CH^2 \|v\|_{L^1(R_2)}$$

for any $v \in L^{1,p}(R_2)$. Therefore using Theorem 2.3.1 we get (4.1.5).

Let us remark that (4.1.17) holds also when $|D^\alpha f| < C/(1 + \|x\|^\beta)$, $\beta > 1$, $|\alpha| = 2$ and $|D^\alpha f| \leqq C$, $\alpha = |1|$.

4.2. Analysis of the limiting equation. It will be seen later that the operator

(4.2.1) $$Au = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} q_j^{[1;i]} \frac{\partial u}{\partial x_j}$$

and the bilinear form

(4.2.2) $$\hat{B}(u, v) = \int_{R_2} \left(\sum_{i,j=1}^{2} q_i^{[1;j]} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \right) dx$$

will play an important role. Let us prove the following theorem.

THEOREM 4.2.1. *There exists $\alpha_0 > 0$ so that*

(4.2.3) $$\sum_{i,j=1}^{2} q_i^{[1;j]} \xi_i \xi_j \geqq \alpha_0 (\xi_1^2 + \xi_2^2)$$

for any pair (ξ_1, ξ_2) and $q_i^{[1;j]}$ defined by (3.1.19).

Proof. Using (3.1.20), we see that

$$\sum_{i,j=1}^{2} q_i^{[1;j]}\xi_i\xi_j = B_0\left(\sum_{i=1}^{2} \xi_i u^{[1;i]}, \sum_{i=1}^{2} \xi_i u^{[1;i]}\right). \tag{4.2.4}$$

Because of (2.4.1) and (4.2.4),

$$\sum q_i^{[1;j]}\xi\xi_2 \geqq \alpha \left\| \sum_{i=1}^{2} \xi_i u^{[1;i]} \right\|^2_{L^1(S_0)}, \tag{4.2.5}$$

so (4.2.3) follows if we are able to show that the $u^{[1;i]}$ are linearly independent functions. But this is obvious because x_i are linearly independent functions and $x_i \notin L^1_H(S_0)$ and $\chi^{[1;i]} \in L^1_H(S_0)$. Theorem 4.2.1 is completely proved.

The bilinear form (4.2.2) satisfies all assumptions of the Theorem 2.3.1 with $H_1 = H_2 = L^1(R_2)$ or $H_1 = H_2 = L^{1,p}(R_2)$. Consequently the following theorem holds.

THEOREM 4.2.2. *Let $f \in H^k(R_2)$, $k \geqq 0$, $\operatorname{supp} f \in Q_S$ and $\int_{R_2} f\,dx = 0$. Then there exists exactly one $u \in L^1(R_2)$ such that*

$$\hat{B}(u, v) = \int_{R_2} fv\,dx \tag{4.2.6}$$

for any $v \in L^1(R_2)$. Furthermore u has the following additional properties:

1. $u \in H^l_{\mathrm{Loc}}(R_2)$ *for* $l = 0, 1, \cdots, k+2$;
2. *on $R_2 - Q_{2S}$ has all continuous derivatives and*

$$|D^\alpha u(x)| \leqq K \frac{1}{1+\|x\|^{|\alpha|+1}}, \qquad |\alpha| \geqq 1, \tag{4.2.7}$$

where K depends on f but not on x. If $f \in \mathcal{D}(R_2)$, then (4.2.7) *holds for all $x \in R_2$.*

The second part of the theorem follows from the well-known theorems about the solution of the problem $-\Delta u = f$ on R_2, because a simple transformation of the coordinates transforms the operator A given by (4.2.1) into a Laplace operator.

4.3. The homogenization problem. Let $f \in \mathcal{D}(R_2)$ and $\int_{R_2} f\,dx = 0$. Denote by $U \in L^{1,p}(R_1)$ the function introduced in § 4.2, i.e., let

$$\hat{B}(U, v) = \int_{R_2} fv\,dx \quad \forall v \in L^1(R_2). \tag{4.3.1}$$

Because of Theorem 4.2.2, U has derivatives of all orders and

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} q_j^{[1;i]} \frac{\partial U}{\partial x_j} = f. \tag{4.3.2}$$

Next we construct ${}^1V^H = \{{}^1V^H_k\} \in {}^H\hat{L}^1(R_2)$ so that for $x \in S^H_k$,

$$\begin{aligned} {}^1V^H_k(x) &= \sum_{l=0}^{2} \frac{1}{l!} \sum_{i+j=l}^{i} \frac{\partial^l U}{\partial x_1^i\,\partial x_2^j}(kH)(x_1 - k_1H)^i(x_2 - k_2H)^j \\ &= \sum_{0 \leqq i+j \leqq 2} b^H_{i,j}(k)(x_1 - Hk_1)^i(x_2 - Hk_2)^j \end{aligned} \tag{4.3.3}$$

with

$$b^H_{i,j}(k)=\binom{i+j}{i}\frac{i}{(i+j)!}\frac{\partial^{i+j}U}{\partial x_1^i\,\partial x_2^j}(kH). \tag{4.3.4}$$

We see immediately the following lemma.

LEMMA 4.3.1.

$$\|U-{}^1V^H\|_{L_2(R_2)}\leqq CH^3, \tag{4.3.5}$$

$$\|U-{}^1V^H\|_{^H\hat{\mathcal{L}}^1(R_2)}\leqq CH^2. \tag{4.3.6}$$

Remark. A sufficient condition for this lemma is that all third derivatives $D^\alpha U$, $|\alpha|=3$ are majorized by the function $C(1+\|x\|)^{-\beta}$ with $\beta>1$.

We set $W^H=\{W^H_k\}$ and ${}^1W^H=\{{}^1W^H_k\}$ so that

$$\begin{aligned} W^H_k(x)=b^H_0(k)+\sum_{i=1}^{2} b^H_0(k)u^{[1;i],H}(x-kH)\\ +\sum_{i,j=1}^{2} b^H_{i,j}(k)u^{[2;i,j],H}(x-kH), \end{aligned} \tag{4.3.7}$$

$${}^1W^H_k(x)=b^H_0(k)+\sum_{i=1}^{2} b^H_i(k)u^{[1;i],H}(x-kH). \tag{4.3.8}$$

In (4.3.7) and (4.3.8) the notation

$$b^H_0=b^H_{0,0},\quad b^H_1=b^H_{1,0},\quad b^H_2=b^H_{0,1} \tag{4.3.9}$$

has been used with $b^H_{i,j}$ defined by (4.3.4).

Let us state some simple lemmas.

LEMMA 4.3.2.

$$\|W^H-{}^1W^H\|_{^H\hat{\mathcal{L}}^1(R_2)}\leqq CH, \tag{4.3.10}$$

$$\|W^H-{}^1W^H\|_{L_2(R_2)}\leqq CH^2, \tag{4.3.11}$$

$$\|{}^1V^H-W^H\|_{L_2(R_2)}\leqq CH. \tag{4.3.12}$$

C depends on f but is independent of H. The lemma is easy to check using (3.1.3) and (3.2.3).

Remark. It is sufficient for Lemma 4.3.2 that $|D^\alpha U|$, $|\alpha|=1,2,3$, is majorized by the function $C(1+\|x\|)^{-\beta}$, $\beta>1$.

In general $\|{}^1V^H-W^H\|_{^H\hat{\mathcal{L}}^1(R_2)}\not\to 0$ for $H\to 0$ and $\|W^H\|_{^H\hat{\mathcal{L}}^1(R_2)}<C$ independently of H.

In general the system $\{W^H_k\}$ does not coincide. Nevertheless it is easy to check that there exists a case when the system coincides.

LEMMA 4.3.3. *Let*

$$\Phi(x)=\sum_{0\leqq i+j\leqq 2} p_{i,j}x_1^i x_2^j, \tag{4.3.13}$$

$p_{i,j}$ *constants, and let* b^H_0, b^H_i, $b^H_{i,j}$ *in* (4.3.7) *be given by* (4.3.4) *and* (4.3.9) *using* Φ

instead of U. Then $\{W_k^H\}$ *coincides and*

$$B^H(W^H, v) = \left[-\sum_{i+j=2} p_{i,j}[iq^{[1;1]}_{m(1;i,j)} + jq^{[1;2]}_{m(2;i,j)}] \right] \int_{R_2} v\,dx \quad \forall v \in \mathscr{D}(R_2). \tag{4.3.14}$$

Lemma 4.3.3 follows immediately from Theorems 3.1.1 and 3.2.1.

Define the auxiliary functions

$$\beta_k^H(x) = \varkappa^{[2H/9]}([(x_1 - (k_1 + \tfrac{1}{2})H)^2 + (x_2 - (k_1 + \tfrac{1}{2})H)^2]^{1/2}), \tag{4.3.15}$$

$$\gamma_k^H(x) = \varkappa^{[2H/9]}([(x_1 - (k_1 + \tfrac{1}{2})H)^2 + (x_2 - k_2 H)^2]^{1/2}), \tag{4.3.16}$$

$$\delta_k^H(x) = \varkappa^{[2H/9]}([(x_1 - k_1 H)^2 + (x_2 - (k_2 + \tfrac{1}{2})H)^2]^{1/2}), \tag{4.3.17}$$

where

$$\varkappa^{[\alpha]}(t) = \varkappa(t/\alpha) \tag{4.3.18}$$

with $\varkappa(t)$ defined by (3.2.13). It is easy to check that

$$\sum_k (\beta_k^H(x) + \gamma_k^H(x) + \delta_k^H(x)) = 1 \tag{4.3.19}$$

on Γ^H.

Denote now $[r \equiv (x_1, x_2)]$,

$$\Phi_r^H(x) = \sum_{l=0}^{2} \frac{1}{l!} \sum_{i+j=l} \frac{\partial^l U}{\partial x_1^i\, \partial x_2} (rH)(x_1 - r_1 H)^i (x_2 - r_2 H)^j \tag{4.3.20}$$

with r_1, r_2 integral. Further let

$${}^2V^{H;r} = \{{}^2V_k^{H;r}\}, \tag{4.3.21}$$

where ${}^2V_k^{H;r}$ is defined by (4.3.7) using (4.3.9) and (4.3.4) after replacing U in (4.3.4) by Φ_r^H.

In S_k^H we construct the function ${}^3V_k^H$ so that

$$\begin{aligned} {}^3V_k^H = &\sum_{i,j=0}^{1} \beta^H_{(k_1-i,k_2-j)}[W_k^H - {}^2V^{H;(k_1-i,k_2-j)}] \\ &+ \sum_{i=0}^{1} \gamma^H_{(k_1-i,k_2)}[W_k^H - {}^2V_k^{H;(k_1-i,k_2)}] \\ &+ \sum_{j=0}^{1} \delta^H_{(k_1,k_2-j)}[W_k^H - {}^2V_k^{H;(k_1,k_2-j)}]. \end{aligned} \tag{4.3.22}$$

The system $\{{}^2V_k^{H;r}\}$ coincides as shown in Lemma 4.3.3. Therefore it is easy to check that $\{W_k^H - {}^3V_k^H\}$ coincides.

Let us estimate $\|{}^3V^H\|_{{}^H\mathcal{L}^1(R_2)}$. It is easy to see that

$$\|W_k^H - {}^2V^{H;(k_1-i,k_2-j)}\|_{{}^H\mathcal{L}^1(S_k^H)} \leqq C(k)H^3 \tag{4.3.23}$$

and

$$\|W_k^H - {}^2V^{H;(k_1-i,k_2-j)}\|_{L_2(S_k^H)} \leqq C(k)H^4, \tag{4.3.24}$$

where

(4.3.25)
$$0 < C(k) \leqq C \max_{x \in Q^H_{3;k}, |\alpha|=3} (|D^\alpha U|).$$

Similar estimates hold for $W^H_k - {}^2V^{H;(k_1-i,k_2)}$ and $W^H_k - {}^2V^{H;(k_1,k_2-j)}$.
We have

(4.3.26)
$$|\beta^H_k| \leqq 1,$$

(4.3.27)
$$\left|\frac{\partial \beta^H_k}{\partial x_i}\right| \leqq CH^{-1},$$

and similar bounds hold for γ^H_k and δ^H_k. Inequalities (4.3.23), (4.3.24), (4.3.26), (4.3.27) and analogous ones for the other terms in (4.3.22) give

(4.3.28)
$$\|{}^3V^H_k\|_{L^1(S^H_k)} \leqq C(k)H^3.$$

Using (4.2.7) and Schwarz's inequality we obtain

(4.3.29)
$$\|{}^3V^H_k\|_{{}^H\hat{L}^1(R_2)} \leqq CH^2,$$

where C depends on f but is independent of H.

We have now proved the following lemma.

LEMMA 4.3.4. *Let $W^H = \{W^H_k\}$ be defined by* (4.3.7). *There exists the function ${}^3V^H \in {}^H\hat{L}^1(R_2)$ so that*

(4.3.30)
$$\|{}^3V^H\|_{{}^H\hat{L}^1(R_2)} \leqq CH^2$$

and $W^H - {}^3V^H \in L^1(R_2)$, i.e., the difference coincides.

Remark. It is possible to weaken the assumption on U so that

$$|D^\alpha U| \leqq C(1+\|x\|)^{-\beta}, \quad \beta > 1, \quad |\alpha| = 1, 2, 3.$$

Lemma 4.3.4 and Theorem 2.3.1 then yield the following lemma.

LEMMA 4.3.5. *Let W^H be defined by* (4.3.7). *Then there exists ${}^4V^H = \{{}^4V^H_k\}$ so that*

(4.3.31)
$${}^5V^H = \{{}^5V^H_k\} = \{W^H_k - {}^4V^H_k\} \in L^1(R_2)$$

and

(4.3.32)
$$B^H({}^5V^H, v) = \sum_k B^H_k(W^H_k, v)$$

for any

$$v \in L^{1,p}(R_2)$$

and

(4.3.33)
$$\|{}^4V^H\|_{{}^H\hat{L}^1(R_2)} \leqq CH^2$$

with C independent of H.

Let $v \in \mathscr{D}(R_2)$ be given. Then using the regularity condition we obtain

$$\sum_k B_k^H(W_k^H, v) = \sum_k \oint_{\partial S_k^H} v\xi_k^H \, ds - \sum_{i+j=2} \sum_k p_{i,j}(k)[iq^{[1;1]}_{m(1;i,j)} + jq^{[1;2]}_{m(2;i,j)}] \int_{S_k^H} v \, dx, \tag{4.3.34}$$

where for $i+j=2$,

$$p_{i,j}(k) = \tfrac{1}{2}\binom{2}{i} \frac{\partial^2 U}{\partial x_1^i \, \partial x_2^j}(kH). \tag{4.3.35}$$

We have

$$\sum_k \oint_{\partial S_k^H} v\xi_k^H \, ds = \sum_k \int_{I_k^{H;1}} [\xi^{H;1}_{(k_1,k_2)} - \xi^{H;3}_{(k_1+1,k_2)}] \, ds + \sum_k \int_{I_k^{H;2}} [\xi^{H;2}_{(k_1,k_2)} - \xi^{H;4}_{(k_1,k_2+1)}] \, ds. \tag{4.3.36}$$

Applying Lemma 4.3.3 we obtain

$$\begin{aligned} \xi^{H;1}_{(k_1,k_2)} - \xi^{H;3}_{(k_1+1,k_2)} &= \Big[\frac{\partial U}{\partial x_1}(kH) - \Big(\frac{\partial U}{\partial x_1}((k_1+1)H_1 k_2 H) \\ &\quad - H\frac{\partial^2 U}{\partial x_1^2}((k_1+1)H;\, k_2H)\Big)\Big]\xi_1^{[1;1];H} + \Big[\frac{\partial U}{\partial x_2}(kH) \\ &\quad - \Big(\frac{\partial U}{\partial x_2}((k_1+1)H,\, k_2H) - H\frac{\partial^2 U}{\partial x_1 \, \partial x_2}((k_1+1)H,\, k_2H)\Big)\Big]\xi_1^{[1;2];H} \\ &\quad + \tfrac{1}{2}\sum_{i+j=2}\binom{2}{i}\Big[\frac{\partial^2 U}{\partial x_1^i \, \partial x_2^j}(kH) - \frac{\partial^2 U}{\partial x_1^i \, \partial x_2^j}((k_1+1)H,\, k_2H\Big]\xi^{[2;i,j];H}, \\ &= \Big[\frac{H^2}{2}\frac{\partial^3 U}{\partial x_1^3}((k_1+\tfrac{1}{2})H,\, k_2H) + H^3 R_1^H((k_1+\tfrac{1}{2})H,\, k_2H)\Big]\xi_1^{[1;1];H} \\ &\quad + \Big[\frac{H^2}{2}\frac{\partial^3 U}{\partial x_1^2 \, \partial x_2}((k_1+\tfrac{1}{2})H,\, k_2H) \\ &\qquad + H^3 R_2^H((k_1+\tfrac{1}{2})H, k_2H)\Big]\xi^{[1;2];H} \\ &\quad - \tfrac{1}{2}\sum_{i+j=2}\Big[H\binom{2}{i}\frac{\partial^3 U}{\partial x_1^{i+1} \, \partial x_2^j}((k_1+\tfrac{1}{2})H,\, k_2H) \\ &\qquad + H^2 R^H_{i,j}((k_1+\tfrac{1}{2})H, k_2H)\Big]\xi^{[2;i,j];H} \end{aligned} \tag{4.3.37}$$

where

$$|R_i^H((k_1+\tfrac{1}{2})H,\, k_2H)| \leqq C \max_{x \in Q^H_{3,k}, |\alpha|=3,4} (|D^\alpha U|), \tag{4.3.38}$$

and the same bound holds for $R^H_{i,j}$.

An analogous expression holds for the term $\xi^{H,2}_{k_1,k_2}-\xi^{H,4}_{k_1,k_2+1}$.
Denote

$$K_1^H(k)=\int_{I_k^{H;1}}[\xi^{H,1}_{(k_1,k_2)}-\xi^{H,3}_{k_1+1,k_2}]\,ds, \tag{4.3.39}$$

$$K_2^H(k)=\int_{I_k^{H;2}}[\xi^{H,1}_{(k_1,k_2)}-\xi^{H,3}_{k_1,k_2+1}]\,ds. \tag{4.3.40}$$

Similar to the above, we obtain (as in (4.3.37))

$$\begin{aligned}K_1^H(k)=\tfrac{1}{2}H^3\Big[&\frac{\partial^3 U}{\partial x_1^3}((k_1+\tfrac12)H,k_2H)q_1^{[1;1]}\\&+\frac{\partial^3 U}{\partial x_1^2\,\partial x_2}((k_1+\tfrac12)H,k_2H)q_1^{[1;2]}\\&-\sum_{i+j=2}\binom{2}{i}\frac{\partial^3 U}{\partial x_1^{i+1},\partial x_2^{j}}((k_1+\tfrac12)H,k_2H)r_1^{[2;i,j]}\Big]\\&+H^4\bar{R}_1^H((k_1+\tfrac12)H,k_2H),\end{aligned} \tag{4.3.41}$$

and R_1^H has the same bound as in (4.3.38).
Analogously,

$$\begin{aligned}K_2^H(k)=\tfrac{1}{2}H^3\Big[&\frac{\partial^3 U}{\partial x_1\,\partial x_2^2}(k_1H,(k_2+\tfrac12)H)q_2^{[1;1]}\\&+\frac{\partial^3 U}{\partial x_2^3}(k_1H,(k_2+\tfrac12)H)q_2^{[1;2]}\\&-\sum_{i+j=2}\binom{2}{i}\frac{\partial^3 U}{\partial x_1^{i}\,\partial x_2^{j+1}}(k_1H,(k_1+(\tfrac12))H)r_2^{[2;i,j]}\Big]\\&+H^4\bar{R}_2^H(k_1H,(k_2+\tfrac12)H).\end{aligned} \tag{4.3.42}$$

Now we may construct ${}^6V^H=\{{}^6V_k^H\}$ so that

$$\begin{aligned}B_k^H({}^6V_k^H,v)+[K_1^H(k_1,k_2)+K_1^H(k_1-1,k_2)&\\+K_2^H(k_1,k_2)+K_2^H(k_1,k_2-1)]\int_{S_k^H}v\,dx&\\=\int_{I_k^{H;1}}[\xi^{H;1}_{(k_1,k_2)}-\xi^{H;3}_{(k_1+1,k_2)}]v\,ds&\\+\int_{I_k^{H;3}}[\xi^{H;1}_{(k_1-1,k_2)}-\xi^{H;3}_{k_1,k_2}]v\,ds&\\+\int_{I_k^{H;2}}[\xi^{H;2}_{(k_1,k_2)}-\xi^{H;4}_{(k_1,k_1+1)}]v\,ds&\\+\int_{I_k^{H;4}}[\xi^{H;2}_{(k_1,k_2-1)}-\xi^{H;4}_{k_1,k_2}]v\,ds&\end{aligned} \tag{4.3.43}$$

for any $v \in L^1(S_k^H)$. Using (4.3.37) and an expression for $\xi_k^{H;l}$ based on (3.1.1) and (3.2.1) we obtain

$$\|{}^6V_k^H\|_{L^1(S_k^H)} \leqq C(k)H^3 \tag{4.3.44}$$

with

$$|C(k)| \leqq C \max_{x \in Q_3^H;k, |\alpha|=3} (|D^\alpha U|), \tag{4.3.45}$$

and hence,

$$\|{}^6V^H\|_{{}^HL^1(R_3)} \leqq CH^2. \tag{4.3.46}$$

Coming back to (4.3.34), using (4.3.35) and (4.3.2) we have for any $v \in \mathscr{D}(R_2)$,

$$\begin{aligned}\sum_k B_k^H(W_k^H, v) = &\int_{R_2} fv\,dx + \sum_k B^H_{(2k_1,2k_2+1)}({}^6V^H_{2k_1,2k_2+1}, v)\\ &+ H\sum_k \Big[\frac{\partial^3 U}{\partial x_1^3}(2k_1H, (2k_2+1)H)(q_1^{[1;1]} - r_1^{[2;2,0]})\\ &+ \frac{\partial^3 U}{\partial x_1^2\,\partial x_2}(2k_1H, (2k_2+1)H)(q_1^{[1;2]} - 2r_1^{[2;1,1]} - r_2^{[2;2,0]})\\ &+ \frac{\partial^3 U}{\partial x_1\,\partial x_2^2}(2k_1H, (2k_2+1)H)(q_2^{[1;1]} - r_1^{[2;0,2]} - 2r_2^{[2;1,1]})\\ &+ \frac{\partial^3 U}{\partial x_2^3}(2k_1H, (2k_2+1)H)(q_2^{[1;2]} - r_2^{[2;0,2]})\Big] \int_{S^H_{(2k_1,2k_2+1)}} v\,dx.\end{aligned} \tag{4.3.47}$$

Let us denote

$$\begin{aligned}2\Lambda = &(q_1^{[1;1]} - r_1^{[2;2,0]})\frac{\partial^3}{\partial x_1^3}\\ &+ (q_1^{[1;2]} - 2r_1^{[2;1,1]} - r_2^{[2;2,0]})\frac{\partial^2}{\partial x_1^2\,\partial x_2}\\ &+ (q_2^{[1;1]} - 2r_2^{[2;1,1]} - r_1^{[2;0,2]})\frac{\partial^3}{\partial x_1\,\partial x_2^2}\\ &+ (q_2^{[1;2]} - r_2^{[2;0,2]})\frac{\partial^2}{\partial x_2^3}.\end{aligned} \tag{4.3.48}$$

Therefore it is easy to see that we may construct ${}^7V^H$ so that

$$\begin{aligned}\sum_k B_k^H(W_k^H, v) = &\int_{R_2} f^H v\,dx\\ &+ H\int_{R_2} (\Lambda U)v\,dx + \sum_k B^H_{(2k_1,2k_2+1)}({}^6V^H_{2k_1,2k_2+1}, v)\\ &+ \sum_k B_k^H({}^7V_k^H, v)\end{aligned} \tag{4.3.49}$$

with

$$\|{}^7V_k^H\|_{H\hat{L}^1(R_2)} \leqq CH^2.$$

Using Lemma 4.1.1 and Lemma 4.3.5 we get for $v \in L^{1,p}(R_2)$,

$$\sum_k B_k^H(W^H - u^H, v) = H\int_{R_2} \Lambda Uv\, dx + F(v), \tag{4.3.50}$$

where $|F(v)| \leqq CH^2\|v\|_{L^1(R_2)}$ and the operator Λ is given by (4.3.48).

Applying Theorem 3.2.1 and Lemma 4.3.2 and taking into consideration Lemma 4.3.5 and Theorem 4.2.2 we get the following theorem.

THEOREM 4.3.1. *Let* $U \in L^{1;p}(R_2)$ *satisfy* (4.3.1) (*resp.* (4.3.2)) *and* u^H *satisfy* (4.1.2). *Further let* ${}^1W^H$ *be given by* (4.3.8). *Then*

$$\|u^H - {}^1W^H\|_{H\hat{L}^1(R_2)} \leqq CH. \tag{4.3.51}$$

Using Lemma 4.3.1 and (4.3.12) we get the following theorem.

THEOREM 4.3.2. *Let* u^H *and* U *be as in Theorem* 4.3.1. *Then*

$$\|U - u^H\|_{L_{2,\alpha}(R_2)} \leqq CH,^7 \qquad \alpha > 1. \tag{4.3.52}$$

The proof of Theorems 4.3.1 and 4.3.2 goes as follows. Using Lemma 4.3.5 and (4.3.50), we construct $z^H = \{z_k^H\}$ so that $W^H - z^H \in L^1(R_2)$ and

$$\|z^H\|_{H\hat{L}^1(R_2)} \leqq CH \tag{4.3.53}$$

and

$$B^H(u^H - (W^H - z^H), v) = 0 \quad \forall v \in L^{1,p}(R_2). \tag{4.3.54}$$

Further we have

$$B^H(u^H, v) = \int_{R_2} fv\, dx. \tag{4.3.55}$$

So that, based on the analysis in § 2.3, we may claim that $u^H = W^H - z^H$ (in $L^1(R_2)$ or after proper normalization in $L^{1,p}(R_2)$).

Let us further remark that in (4.3.50), which is the basis of Theorem 4.3.1, we assumed that f has compact support. Essential only was that $U \in L^1(R_2)$, $|D^\alpha U| \leqq K(1+\|x\|)^{-\beta}$ for $\beta > 1$ and $|\alpha| = 2, 3$ and $|D^\alpha f| \leqq K(1+\|x\|)^{-\beta}$ for $\beta > 1$ with $|\alpha| = 2$.

Let us define $\tilde{f} = \Lambda U$. Then

$$|D^\alpha \tilde{f}| \leqq C(1+\|x\|)^{-|\alpha|-4}, \qquad |\alpha| \geqq 0. \tag{4.3.56}$$

Further, for $i \geqq j$, $i + j = 3$ we shall define

$$\int_{R_2} \frac{\partial^3 U}{\partial x_1^i\, \partial x_2^j} v\, dx = -\int_{R_2} \frac{\partial^2 U}{\partial x_1^{i-1}\, \partial x_2} \frac{\partial v}{\partial x_1} dx, \tag{4.3.57}$$

and for $j \geqq i$,

$$\int_{R_2} \frac{\partial^3 U}{\partial x_1^i\, \partial x_2^j} v\, dx = -\int_{R_2} \frac{\partial^2 U}{\partial x_1^i\, \partial x_2^{j-1}} \frac{\partial v}{\partial x_2} dx. \tag{4.3.58}$$

[7] For $L_{2,\alpha}(R_2)$ see (2.2.5).

Because of Theorem 4.2.2 all second derivatives are square integrable. Let us take $\tilde{U} \in L^1(R_2)$ so that

$$\hat{B}(\tilde{U}, v) = \int_{R_2} \Lambda U v \, dx, \tag{4.3.59}$$

and replace the terms with third derivatives by terms with second derivatives of U using (4.3.57) and (4.3.58).

Obviously the right-hand side of (4.3.59) is a continuous functional on $L^1(R_2)$ and Theorem 2.3.1 assures the existence and uniqueness of $\tilde{U}$.

Because the operator A in (4.2.1) has constant coefficients, the function $\tilde{U}$ has second and third derivatives which are majorized by the function $C(1+\|x\|)^{-4}$.

Let us denote by ${}^1\tilde{W}^H$ the function which is defined by (4.3.8) using $\tilde{U}$ instead of U. Also, $\tilde{u}^H$ is the function which satisfies (4.3.2) with $\tilde{f}$ instead of f (use (4.3.57) and (4.3.58)). Then Theorem 4.3.1 holds, i.e.,

$$\|{}^1\tilde{u}^H - {}^1\tilde{V}^H\|_{{}^H\hat{L}^1(R_2)} \leqq CH, \tag{4.3.60}$$

and we get using (4.3.50) also,

$$\sum_k B_k^H(W^H - H^H\tilde{W}^1 - u^H, v) = \bar{F}(v) \tag{4.3.61}$$

with

$$|\bar{F}(v)| \leqq CH^2\|v\|_{L^1(R_2)} \quad \forall v \in L^{1,p}(R_2). \tag{4.3.62}$$

Therefore

$$\|u^H - W^H + H^1\tilde{W}^H\|_{{}^H\hat{L}^1} \leqq CH^2. \tag{4.3.63}$$

We have proved the following theorem.

THEOREM 4.3.3. *Let U and u^H be functions as in Theorem* 4.3.1. *Further let $\tilde{U} \in L^1(R_2)$ be such that*

$$\hat{B}(\tilde{U}, v) = \int_{R_2} \Lambda U v \, dx \quad \forall v \in L^1(R_2) \tag{4.3.64}$$

and let W^H be the function defined by (4.3.7), *and ${}^1\tilde{W}^H$ the function defined by* (4.3.8) *using $\tilde{U}$ instead of U. Then*

$$\|u^H - (W^H - H^1\tilde{W}^H)\|_{{}^H\hat{L}^1(R_2)} \leqq CH^2, \tag{4.3.65}$$

$$\|u^H - (W^H - H^1\tilde{W}^H)\|_{L_{2,\alpha}(R_2)} \leqq CH^2, \qquad \alpha > 1. \tag{4.3.66}$$

Let us make some additional comments about this theorem.

1. We analyze the one single cell first.
2. We solve the differential equation of the homogenized problem (4.3.2) essentially twice (for H^2 convergence) and we use this solution for the construction of function W^H.
3. In the case when the coefficients $a_{i,j}$ are constant, then the homogenized problem (4.3.2) is identical with the original and $\Lambda = 0$.
4. The function $H^1\tilde{W}^H$ is of magnitude H. Therefore homogenization without corrections (operator Λ) leads to the accuracy of order H only, and the error in the norm $\|\cdot\|_{L_{2,\alpha}(R_2)}$ is not better than in the norm $\|\cdot\|_{{}^H\hat{L}^1(R_2)}$.

Let us analyze now the problem of the accumulated energy. We are interested in

$$\mathcal{T}(f) = B^H(u^H, u^H). \tag{4.3.67}$$

Going back to (4.2.4) we have

$$B_0\left(\sum_{i=1}^{2} \xi_i u^{[1;i]}, \sum_{i=1}^{2} \xi_i u^{[1;i]}\right) = \sum_{i,j=1}^{2} q_i^{[1;j]} \xi_i \xi_j, \tag{4.3.68}$$

and therefore,

$$\begin{aligned} B_k({}^1W_k^H, {}^1W_k^H) &= \int_{S_k^H} \sum_{i,j=1}^{2} q_i^{[1,j]} \frac{\partial U}{\partial x_i}(kH) \frac{\partial U}{\partial x_j}(kH)\, dx \\ &= \int_{S_k^H} \sum_{i,j=1}^{2} q_i^{[1,j]} \frac{\partial U}{\partial x_i}(x) \frac{\partial U}{\partial x_j}(x)\, dx + H^2 \int_{S_k} R(x)\, dx. \end{aligned} \tag{4.3.69}$$

Using Theorem 4.2.2 we obtain

$$|R(x)| \leqq C(1+\|x\|)^{-2}. \tag{4.3.70}$$

Hence we get

$$\left|\sum_k B_k^H({}^1W_k^H, {}^1W_k^H) - \hat{B}(U, U)\right| \leqq CH^2. \tag{4.3.71}$$

On the other hand, using Theorem 4.3.1 we obtain

$$\begin{aligned} B^H(u^H, u^H) &= \sum_k B_k^H(u^H, u^H) \\ &= \sum_k B_k^H({}^1W_k^H, {}^1W_k^H) + R \end{aligned} \tag{4.3.72}$$

with $|R| \leqq CH$.

Hence we have the following theorem.

THEOREM 4.3.4. *Let* $f \in \mathcal{D}(\Omega)$ *and* $\int f\, dx = 0$. *Further let* $u^H \in L^1(R_2)$ *satisfy* (4.1.2) *and* $U \in L^1(R_2)$ *satisfy* (4.3.1). *Then*

$$B^H(u^H, u^H) = \hat{B}(U, U) + R(H), \tag{4.3.73}$$

where

$$|R(H)| \leqq CH. \tag{4.3.74}$$

5. Examples.

5.1. General algorithms. We now describe the general approach based on the homogenization principle. Let us be interested in the solution of the problem

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{j,i}\left(\frac{x}{H}\right) \frac{\partial u^H}{\partial x_j} = f \tag{5.1.1}$$

when $a_{i,j}(x)$ are bounded double periodic functions with period 1 and

$$\sum a_{i,j}(x)\xi_i\xi_j \geqq \gamma(\xi_1^2+\xi_2^2) \tag{5.1.2}$$

with $\gamma > 0$ (and satisfying the regularity condition (see § 2.4)). In addition, assume that f has compact support and

$$\int_{R_2} f\,dx = 0. \tag{5.1.3}$$

Then there exists exactly one weak solution $u^H \in L^1(R_2)$ with

$$\int_{Q_1} u^H\,dx = 0. \tag{5.1.4}$$

This solution is the only $u^H \in L^1(R_2)$ such that

$$B(u^H, v) = \int_{R_2} \left(\sum_{i,j=1}^{2} a_{i,j}\left(\frac{x}{H}\right)\frac{\partial u^H}{\partial x_i}\frac{\partial v}{\partial x_j}\right) dx = \int fv\,dx \tag{5.1.5}$$

for any $v \in L^1(R_2)$.

Now the homogenization approach is the following.

1. Find periodic functions (with period 1) $\chi^{[1;i]} \in L^1(S_0)$, $l = 1, 2$,

$$\int_{S_0} \chi^{[1;i]}\,dx = 0 \tag{5.1.6}$$

such that with

$$B_0(u, v) = \int_{S_0} \left[\sum_{i,j=1}^{2} a_{i,j}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\right] dx, \tag{5.1.7}$$

$$B_0(\chi^{[1;i]}, v) = B_0(x_i, v) \tag{5.1.8}$$

for any $v \in L_H^1(S_0)$ (see (3.1.2)). Function $\chi^{[1;i]}$ exists and is uniquely determined by (5.1.8) which is the weak form of the equation

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{j,i}(x)\frac{\partial \chi^{[1;k]}}{\partial x_j} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} a_{i,k}. \tag{5.1.9}$$

2. Find the constants $q_j^{[1;i]}$, $i, j = 1, 2$, so that (see (3.1.20))

$$q_j^{[1;i]} = B_0(x_i - \chi^{[1;i]}, x_j). \tag{5.1.10}$$

Because of (5.1.8) we have also

$$q_j^{[1;i]} = B_0(u^{[1;i]}, u^{[1;j]}), \tag{5.1.11}$$

where

$$u^{[1;i]} = x_i - \chi^{[1;i]}. \tag{5.1.12}$$

3. Find $U \in L^1(R_2)$ so that

$$\int_{Q_1} U\,dx = 0 \tag{5.1.13}$$

and

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} q_j^{[1;i]} \frac{\partial U}{\partial x_j} = f. \tag{5.1.14}$$

Denoting (see (4.2.2))

$$\hat{B}(u, v) = \int_{R_2} \left(\sum_{i,j=1}^{2} q_i^{[1;j]} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \right) dx, \tag{5.1.15}$$

we see that U is determined also by the condition

$$\hat{B}(U, v) = \int_{R_2} fv\,dx \tag{5.1.16}$$

and (5.1.13). Function U exists and is uniquely determined.

Now Theorem 4.3.2 claims that

$$\|U - u^H\|_{L_{2,\alpha}(R_2)} \leqq CH \tag{5.1.17}$$

for any $\alpha > 1$ (for $L_{2,\alpha}(R_2)$ see (5.2.5)) (i.e., U approximates u^H in $L_2(\Omega)$ for any bounded Ω). Function U does not approximate u^H in $L^1(R_2)$. But defining

$$V^H(x) = U(x) - H \sum_{i=1}^{2} \frac{\partial U}{\partial x_i}(x) \chi^{[1;i]}(x/H), \tag{5.1.18}$$

we get

$$\|u^H - V^H\|_{L^1(R_2)} \leqq CH. \tag{5.1.19}$$

Claim (5.1.19) follows from Theorem 4.3.1 because it is easy to see that

$$\|V^H - {}^1W^H\|_{{}^H\hat{L}^1(R_2)} \leqq CH.$$

Function V^H approximates u^H and its derivatives in contrast to function U which approximates u^H but not its derivatives.

5.2. A concrete example. Let us show a concrete example which may be easily solved. Let $a_{1,1}(x) = a_{2,2}(x) = a(x)$ and $a_{1,2} = a_{2,1} = 0$ with

$$a(x) = p_1 > 0 \quad \text{for } |x_1| < \tfrac{1}{4},$$
$$a(x) = p_2 > 0 \quad \text{for } \tfrac{1}{4} \leqq |x_1| < \tfrac{1}{2}.$$

In this very special case functions $\chi^{[1;i]}$ may be easily found. We have $\chi^{[1;2]} = 0$. In fact, it follows immediately from the observation that

$$\int_{S_0} a(x_1) \frac{\partial v}{\partial x_2}\,dx = 0 \tag{5.2.1}$$

for any $v \in L_H^1(S_0)$.

Let us show now that

$$\chi^{[1;1]} = \begin{cases} \dfrac{p_1-p_2}{p_1+p_2} x_1 & \text{for } |x_1| < \frac{1}{4}, \\[2mm] \dfrac{p_1-p_2}{p_1+p_2} (x_1 - \frac{1}{2}) & \text{for } \frac{1}{4} < x_1 < \frac{1}{2}, \\[2mm] \dfrac{p_1-p_2}{p_1+p_2} (x_1 + \frac{1}{2}) & \text{for } -\frac{1}{2} < x_1 < -\frac{1}{4}. \end{cases} \tag{5.2.2}$$

By simple computation we get

$$\begin{aligned} &\int_{S_0} a\left[\frac{\partial \chi^{[1;1]}}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial \chi^{[1;1]}}{\partial x_2}\frac{\partial v}{\partial x_2}\right] dx \\ &\quad = (p_1 - p_2)\int_{-\frac{1}{2}}^{+\frac{1}{2}} [v(\tfrac{1}{4}, x_2) - v(-\tfrac{1}{4}, x_2)]\, dx_2. \end{aligned} \tag{5.2.3}$$

It is also easy to check that

$$\int_{S_0} a\frac{\partial v}{\partial x_1}\, dx = (p_1 - p_2)\int_{-\frac{1}{2}}^{+\frac{1}{2}} (v(\tfrac{1}{4}, x_1) - v(-\tfrac{1}{4}, x_2))\, dx_2. \tag{5.2.4}$$

Because it is obvious that

$$\int_{S_0} \chi^{[1;1]}(x)\, dx = 0, \tag{5.2.5}$$

equalities (5.2.3), (5.2.4), (5.2.5) are proving together that $\chi^{[1;1]}$ is given by expression (5.2.2).

It is also easy to compute $q_j^{[1;i]}$. We get $q_1^{[1;2]} = q_2^{[1;1]} = 0$ and

$$qq_1^{[1;1]} = \frac{2p_1p_2}{p_1+p_2} = \left[\left(\frac{1}{p_1} + \frac{1}{p_2}\right)\frac{1}{2}\right],$$

$$q_2^{[1;2]} = \frac{p_1+p_2}{2}.$$

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SOLUTION OF INTERFACE PROBLEMS BY HOMOGENIZATION. II*

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Abstract. The paper studies the effect of the boundary on the homogenization problem analyzed in [1].

1. Introduction. This paper is the second in the series of papers dealing with homogenization problems. The first part [1] studied the case when the domain was the entire plane, i.e., the case when no boundary is present. This paper analyzes the simplest case of a domain with boundary, namely, a half-plane and shows new features created by the presence of boundary.

All notations will be the same as in [1], which will be referred to as I. The references to the equations and sections in I will be made by prefix I, e.g., (I.3.2.1) means equation (3.2.1) in I. On the other hand, (3.2.1) means equation (3.2.1) of this paper.

2. Basic notations and auxiliary results.

2.1. Domains and spaces. As in I, R_2 denotes two-dimensional Euclidian space $[x \equiv (x_1, x_2) \in R_2]$. Further let for $t \geqq 0$, $R^+(t) = \{x \in R_2 | x_2 > t\}$ and $R^+ \equiv R^+(0)$. The notations S_k^H, Γ^H, etc., have the same meaning as in I.

For $k \equiv (k_1, k_2)$, k_i integral and $H > 0$ let

$$R_{k_1}^{+H}(t) = \{x \in R^+(t) \,|\, |x_1 - Hk_1| \leqq \tfrac{1}{2}H\}$$

with $R_{k_1}^{+H} \equiv R_{k_1}^{+H}(0)$.

In I we introduced the spaces $\mathscr{E}(\Omega)$, $\mathscr{E}(R_2)$, $H^l(R_2)$, etc. In the same vein we shall use notation $\mathscr{E}(R^+)$, $\mathscr{D}(R^+)$, $H^l(R^+)$, $H_0^l(R^+)$, etc. In I the set of H-periodic functions has been denoted by $\mathscr{E}_H(R_2)$, $L_H^1(S_k^H)$, etc. The same notation will be used here. In addition, $\mathscr{E}_H^+(R^+) \subset \mathscr{E}(R^+)$ will be the set of all H-periodic functions in x_1, i.e., $u \in \mathscr{E}_H^+(R^+)$ iff

$$u(x_1 + k_1 H, x_2) = u(x_1, x_2)$$

for k_1 any integer. The restriction of $\mathscr{E}_H^+(R^+)$ on $R_{k_1}^{+H}$ will be denoted by $\mathscr{E}_H(R_{k_1}^H)$. All other notations have analogous self-explanatory meanings.

In I we introduced the space $\mathscr{E}_p(R_2) \subset \mathscr{E}(R_2)$ as a subspace of functions satisfying equations (I.2.2.9):

$$\text{(I.2.2.9)} \qquad \int_{Q_p} u \, dx = 0.$$

The space $\mathscr{E}_p(R^+) \subset \mathscr{E}(R^+)$ is the space of functions such that instead of (I.2.2.9), equation (2.1.1) is satisfied:

$$\text{(2.1.1)} \qquad \int_{Q_p \cap R^+} u \, dx = 0.$$

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We introduced the space $L^{1,p}(R_2)$ as a completion of $\mathscr{E}_p(R_2)$ and introduced the inequality

$$\|u\|_{L_{2,\alpha}(R_2)} \leqq C(\alpha)\|u\|_{L^1(R_2)} \quad \text{(I.2.2.10)}$$

[1]

which holds for any $u \in L^{1,p}(R_2)$. This inequality holds also for the space $L_0^1(R^+)$; i.e., we have

$$\|u\|_{L_{2,\alpha}(R^+)} \leqq C(\alpha)\|u\|_{L_0^1(R^+)} \quad (2.1.2)$$

for any $u \in L_0^1(R^+)$.

2.2. The bilinear form. In § I.2.3 an application of Theorem I.2.3.1 has been studied. Let us study now another example of a bilinear form satisfying the hypothesis of Theorem I.2.3.1. Let $a_{i,j}^H$, $i, j = 1, 2$, be measurable H-periodic in x_1 functions defined on R_0^{+H} ([i.e., $R_{k_1}^{+H}$ for $k_1 = 0$). Assume that

$$\sum_{i,j=1}^{2} a_{ij}^H(x)\xi_i\xi_j \geqq \alpha(\xi_1^2 + \xi_2^2) \quad (2.2.1)$$

and

$$|a_{i,j}^H(x)| \leqq \beta \quad (2.2.2)$$

with $\alpha > 0$ and $\beta < \infty$ independent of H. Further let us define the bilinear form

$$B^H(u, v) = \int_{R_0^{+H}} \left[\sum_{i,j=1}^{2} a_{i,j}^H \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right] dx. \quad (2.2.3)$$

Because of (2.2.1) and (2.2.2), the bilinear form is defined on $L_H^1(R_0^{+H}) \times L_H^1(R_0^{+H})$ and satisfies all assumptions of Theorem I.2.3.1. Bilinear form (2.2.3) satisfies the assumptions of Theorem I.2.3.1 also when some other spaces are used. Let the space $L_H^{1,\gamma}(R_0^+)$ be defined in the following way. Let

$$\|u\|_\gamma^2 = \int_{R_0^{+H}} e^{2\gamma x_2} \left[\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial x_2} \right)^2 \right] dx. \quad (2.2.4)$$

and denote by $L_H^{1,\gamma}(R_0^{+H})$ the completion in (2.2.4) of the set $M \subset \mathscr{E}_H(R_0^{+H})$ of functions with (2.2.4) finite. Now the following theorem holds.

THEOREM 2.2.1. *For* $|\gamma| \leqq \alpha\pi/(2\sqrt{2}\beta H)$ *and* $H_1 \times H_2 = L_H^{1,\gamma}(R_0^{+H}) \times L_H^{1,-\gamma}(R_0^{+H})$, *bilinear form* (2.2.3) *satisfies all assumptions of Theorem* I.2.3.1 *with* C_1 *and* C_2 *independent of* H.

Proof. 1. The norm $\|\cdot\|_\gamma$ may be expressed also in a different way. Because $u \in L_H^{1,\gamma}(R_0^{+H})$ is H-periodic in variable x_1 we may write

$$\begin{aligned} u(x) = &\sum_{k=0}^{\infty} \psi_k(x_2) \cos \frac{2\pi}{H} kx_1 \\ &+ \sum_{k=1}^{\infty} \varphi_k(x_2) \sin \frac{2\pi}{H} kx_1 \end{aligned} \quad (2.2.5)$$

[1] As in I, C is a generic constant with different values on different places.

and

$$\|u\|_\gamma^2=\frac{H}{2}\left[\sum_{k=1}^{\infty}\int_0^{\infty} e^{2\gamma x_2}\left[\psi_k'^2(x_2)+\left(\frac{2\pi}{H}k\right)^2\psi_k^2(x_2)\right]dx\right.$$
$$\left.+\sum_{k=1}^{\infty}\int_0^{\infty} e^{2\gamma x_2}\left[\varphi'^2(x_2)+\left(\frac{2\pi}{H}k\right)^2\varphi^2(x_2)\right]dx\right] \tag{2.2.6}$$
$$+H\int_0^{\infty} e^{2\gamma x_2}\psi_0'^2(x_2)\,dx$$

2. Let $u\in L_H^{1,\gamma}(R_0^{+H})$, $v\in L_H^{1,-\gamma}(R_0^{+H})$. Then using the Schwarz inequality we get

$$|B^H(u,v)|=\left|\int_{R^{+H}}\left[\sum_{i,j=1}^{2} a_{i,j}^H e^{\gamma x_2}\frac{\partial u}{\partial x_i}e^{-\gamma x_2}\frac{\partial u}{\partial x_j}\right]\right|dx \tag{2.2.7}$$
$$\leqq C\|u\|_\gamma\|v\|_{-\gamma}.$$

So the first condition of Theorem I.2.3.1 is satisfied.

3. Let $u\in L_H^{1,\gamma}(R_0^{+H})$ be given. Using (2.2.5), we see that the function u is characterized by the functions φ_k and ψ_k and the norm of u is given by (2.2.6). Let us construct $v\in L_H^{1,-\gamma}(R_0^{+H})$ in the dependence on H in the following way. Function v will be given by $\bar\varphi_k$ and $\bar\psi_k$ so that

$$\bar\psi_k=e^{2\gamma x_2}\psi_k(x_2),\qquad k=1,2,\cdots,$$
$$\bar\varphi_k=e^{2\gamma x_2}\varphi_k(x_2),\qquad k=1,2,\cdots, \tag{2.2.8}$$
$$\bar\psi_0=\int_0^{x_2} e^{2\gamma t}\psi_0'(t)\,dt.$$

Equations (2.2.8) yield

$$v=e^{2\gamma x_2}(u-\psi_0)+\bar\psi_0, \tag{2.2.9}$$

and therefore,

$$\frac{\partial v}{\partial x_1}=e^{2\gamma x_2}\frac{\partial u}{\partial x_1},$$
$$\frac{\partial v}{\partial x_2}=e^{2\gamma x_2}\frac{\partial u}{\partial x_2}+2\gamma e^{2\gamma x_2}(u-\psi_0). \tag{2.2.10}$$

So we get

$$|B^H(u,v)|=\int_{R_0^{+H}}\left[\sum_{i,j=1}^{2} a_{i,j}\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\right]e^{2\gamma x_2}\,dx$$
$$+\int_{R_0^{+H}}\left[\sum_{i=1}^{2} a_{i,2}\frac{\partial u}{\partial x_i}2\gamma e^{2\gamma x_2}(u-\psi_0)\right]dx \tag{2.2.11}$$
$$\geqq\alpha\|u\|_\gamma^2-2\gamma\beta\|u\|_\gamma\left[\int_{R_0^{+H}}(u-\psi_0)^2 e^{2\gamma x_2}\,dx\right]^{1/2}.$$

But

$$\int_{R_0^{+H}} (u-\psi_0)^2 e^{2\gamma x_2}\,dx = \frac{H}{2}\sum_{k=1}^{\infty} e^{2\gamma x_2}(\psi_k^2+\varphi_k^2)\,dx \tag{2.2.12}$$
$$\leqq \left(\frac{H}{2\pi}\right)^2 \|u\|_\gamma^2.$$

Therefore,

$$|B^H(u,v)| \geqq \left[\alpha - 2\gamma\beta\sqrt{2}\frac{H}{2\pi}\right]\|u\|_\gamma^2.$$

On the other hand, for $|\gamma| \leqq C/H$,

$$\|v\|_{-\gamma} \leqq C\|u\|_\gamma \tag{2.2.13}$$

with C independent of H and u. So for any $u \in L_H^{1,\gamma}(R_0^{+H})$ we have constructed a $v \in L_H^{1,-\gamma}(R_0^{+H})$ so that (2.2.13) holds and

$$|B^H(u,v)| \geqq \frac{\alpha}{2}\|u\|_\gamma^2. \tag{2.2.14}$$

So we have

$$|B^H(u,v)| \geqq \frac{\alpha}{2}\|u\|_\gamma \|v\|_{-\gamma}\frac{\|u\|_\gamma}{\|v\|_{-\gamma}} \tag{2.2.15}$$
$$\geqq C\|u\|_\gamma\|v\|_{-\gamma}$$

with C independent of H and u. Inequality (2.2.15) gives (I.2.3.2). Changing the role of u and v, inequality (2.3.15) yields (I.2.3.3). The theorem is proved.

Let us remark that Theorem 2.2.1 holds when the spaces $L_H^{1,\gamma}(R_0^{+H})$ will be replaced by $L_{H,0}^{1,\gamma}(R_0^{+H}) \subset L_H^{1,\gamma}(R_0^{+H})$ which is the subspace of functions whose traces are zero when $x_2 = 0$.

The importance of Theorem 2.2.1 is shown in the following observation. Let $u_0 \in L_{H,0}^{1}(R_0^{+H})$ be such that

$$B^H(u_0, v) = \int_{R_0^{+H}} fv\,dx$$

for every $v \in L_{H,0}^1(R_0^{+H})$ and $\|f\|_{L_2(R_0^{+H})} \leqq C$ with $f(x) = 0$ for $x_2 > kH$. (k, C is independent of H.) Then $\partial u_0/\partial x_1$, $\partial u_0/\partial x_2$ have boundary layer character because by Theorem 2.2.1, $\|u_0\|_\gamma \leqq C$ with $\gamma = C/H$.

3. Homogenization problem for R^+.

3.1. Introduction. In § 4 of I the homogenization problem has been discussed when $\Omega = R_2$, i.e., when Ω has no boundary. This section will be devoted to the case when Ω is a half-plane, i.e., when the boundary occurs in the simplest way. Here, the boundary may be located differently with respect to the position of the cells.

We may restrict ourselves to the case $\Omega = R^+$; i.e., ∂R^+ will intersect the cells $S_{(k_1,0)}^H$ in the middle. Because many steps of the analysis will be the same as in §I.4, we shall be briefer and we shall concentrate on these parts which are essential to the study of R^+.

Let us formulate our problem. As in § I.4.1, let

$$B^H(u, v) = \int_{R^+} \left(\sum_{i,j=1}^{2} a_{i,j}^H \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \right) dx \tag{3.1.1}$$

and

$$\hat{B}(u, v) = \int_{R^+} \left(\sum_{i,j=1}^{2} q_i^{[1;j]} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \right) dx. \tag{3.1.2}$$

We integrate here over R^+ instead of R_2 as was done in § I.4. Instead of the spaces $L^1(R_2)$ and $L^{1,p}(R_2)$ we shall deal here with $L_0^1(R^+)$. Let $f \in \mathscr{E}(R^+)$ and $f(x) = 0$ for $|x| > p$. In contrast to the case studied in § I.4, we do not assume that $\int f\, dx = 0$. Similar to Theorem I.4.2.2, there exists a unique R^+, $U \in L_0^1(R^+)$, such that

$$\hat{B}(U, v) = \int_{R^+} fv\, dx \tag{3.1.3}$$

for any $v \in L_0^1(R^+)$, and U has all its derivatives extending to the boundary and (I.4.2.7) holds here too. This follows from the well-known theory of elliptic partial differential equations. Further, we may extent the function U into R_2 preserving the existence of all its derivatives and (I.4.2.7). Similarly there exists (and it is uniquely determined) a $u^H \in L_0^1(R^+)$ such that

$$B^H(u^H, v) = \int_{R^+} fv\, dx. \tag{3.1.4}$$

3.2. Particular solutions. Through this entire section we shall assume $H = 1$. Let

$$B_{k_1}(u, v) = \int_{R_k^{+1}} \left[\sum_{i,j=1}^{2} a_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right] dx. \tag{3.2.1}$$

Obviously

$$B(u, v) = \sum_{k_1} B_{k_1}(u, v), \tag{3.2.2}$$

and because of the assumed periodicity of $a_{i,j}$ we have

$$B(u, v) = \sum_{k_1} B_0(u(x_1 - k_1, x_2), v(x_1 - k_1, x_2)). \tag{3.2.3}$$

We have introduced the space $L^1(R^+)$ resp. $L_0^1(R^+)$, $L^1(R_k^{+H})$, etc. Further we introduced the spaces $L_H^1(R_k^{+H})$, etc., as spaces of periodic functions. In § 2.2 the spaces $L_H^{1,\gamma}(R_k^{+H})$ and $L_{H,0}^{1,\gamma}(R_k^{+H})$ have been defined. We shall also introduce the space $L^{1,\gamma}(R^+)$ resp. $L_0^{1,\gamma}(R^+)$ in the obvious manner, namely,

$$\|u\|_{L^{1,\gamma}(R^+)}^2 = \int_{R^+} e^{2\gamma x_2} \left[\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial x_2} \right)^2 \right] dx.$$

In § I.3.1 the function $u_k^{[1;i]}$ and in § I.3.2 the function $u_k^{[2;i,j]}$ have been introduced. Let us introduce here functions which will play an important role. Let us first prove the following lemma.

LEMMA 3.2.1. *There exist the function* $\mu_1^{[1;i]} \in L_H^1(R^+)$ *and numbers* $\lambda^{[1,i]}$, $i=1, 2$ (*resp.* $\mu^{[2;i,j]} \in L_H^1(R^+)$ *and* $\lambda^{[2;i,j]}$, $i, j = 1, 2$), *such that*

$$\mu^{[1;i]}(x_1, 0) = \chi^{[1;i]}(x_1, 0) + \lambda^{[1;i]} \tag{3.2.4}$$
$$(\textit{resp.} \quad \mu^{[2;i,j]}(x_1, 0) = \chi^{[2;i,j]}(x_1, 0) + \lambda^{[2;i,j]}),$$

$$\|\mu^{[1;i]}\|_{L^{1,\gamma}(R_0^+)} \leqq C\|\chi^{[1;i]}\|_{L^1(S_0)} \tag{3.2.5}$$
$$(\textit{resp.} \quad \|\mu^{[2;i,j]}\|_{L^{1,\gamma}(R_0^+)} \leqq C\|\chi^{[2;i,j]}\|_{L^1(S_0)})$$

(*with* $\chi^{[1;i]}$ *resp.* $\chi^{[2;i,j]}$ *defined in* I.3.1, *resp.* I.3.2, and γ *given in Theorem* 2.2.1). *In addition*

$$|\lambda^{[1;i]}| \leqq C\|\chi^{[1;i]}\|_{L^1(S_0)} \tag{3.2.6}$$
$$(\textit{resp.} \quad |\lambda^{[2;i,j]}| \leqq C\|\chi^{[2;i,j]}\|_{L^1(S_0)})$$

and

$$\int_{-1/2}^{+1/2} (\mu^{[1;i]}(x_1, x_2))^2 \, dx_1 \leqq C e^{-2\gamma x_2} \|\chi^{[1;i]}\|^2_{L^1(S_0)} \tag{3.2.7}$$
$$(\textit{resp.} \quad \int_{-1/2}^{+1/2} (\mu^{2;i,j]}(x_1, x_2))^2 \, dx_1 \leqq C e^{-2\gamma x_2} \|\chi^{[2;i,j]}\|^2_{L^1(S_0)}),$$

$$B(\mu^{[1;i]}, v) = 0 \tag{3.2.8}$$
$$(\textit{resp.} \quad B(\mu^{[2;i,j]}, v) = 0)$$

for any $v \in \mathcal{D}(R^+)$.

Proof. Let $\xi(x) = 1 - \rho(4x_2)$, where $\rho(t)$, $0 \leqq t \leqq 1$ is defined by (I.3.2.16) and $\rho(t) = 1$ for $t > 1$. Obviously $\xi(x_1, 0) = 1$ and $\xi(x_1, x_2) = 0$ for $x_2 \geqq \frac{3}{16}$.

It is easy to see that

$$\|\xi(x)\chi^{[1;i]}\|_{L^1(R_0^{+H})} \leqq C\|\chi^{[1;i]}\|_{L^1(S_0)} \tag{3.2.9}$$

because of the normalization given in (I.3.1.3). The same estimate holds for $\xi(x)\chi^{[2;i,j]}$. The estimate (3.2.9) also yields

$$\|\xi(x)\chi^{[1;i]}\|_{L^{1,\gamma}(R_0^{+H})} \leqslant C\|\chi^{[1;i]}\|_{L^1(S_0)} \tag{3.2.10}$$

and analogous estimates for $\xi(x)\chi^{[2;i,j]}$ hold. Using Theorems I.2.3.1 and 2.2.1, we may find $w^{[1;i]} \in L_{H,0}^{1,\gamma}(R_0^{+H})$ so that

$$B(w^{[1;i]}, v) = B(\xi\chi^{[1;i]}, v) \tag{3.2.11}$$

for any

$$v \in L_{H,0}^{1,-\gamma}(R_0^{+H})$$

and

$$\|w^{[1;i]}\|_{L_{H,0}^{1,\gamma}(R_0^{+H})} \leqslant C\|\chi^{[1;i]}\|_{L^1(S_0)}. \tag{3.2.12}$$

Using (I.2.2.6) and (3.2.12), we may find $\lambda^{[1;i]}$ satisfying (3.2.6) so that

$$\int_{-1/2}^{1/2} (w^{[1;i]}(x_1, x_2) - \lambda^{[1;i]})^2 \, dx \leqq C e^{-2\gamma x_2} \|\chi^{[1;i]}\|^2_{L^1(S_0)}. \tag{3.2.13}$$

Now we may define

$$\mu^{[1;i]} = \xi\chi^{[1;i]} - w^{[1;i]} + \lambda^{[1;i]},$$

and $\mu^{[1;i]}$ satisfies all properties of the Lemma 3.2.1. Analogously we get the function $\mu^{[2;i,j]}$. Let us prove now the following lemma.

LEMMA 3.2.2. *There exists a function $v \in L^{1,\gamma/2}_{H,0}(R^+)$ (γ as in Lemma 3.2.1) such that*

$$\varphi = \{\varphi_k\} = \{[k + w_1(x_1 - k)]\mu^{[1;2]} + \nu\} \in L^1_{\text{Loc}}(R^+), \tag{3.2.14}$$

$$\|\nu\|_{L^{1,\gamma/2}(R_0^+)} \leqq C\|\chi^{[1;2]}\|_{L^1(S_0)}, \tag{3.2.15}$$

$$\nu(x_1, 0) = \sigma, \tag{3.2.16}$$

$$|\sigma| \leqq C\|\chi^{[1;2]}\|_{L^1(S_0)}, \tag{3.2.17}$$

$$\int_{-1/2}^{1/2} (\nu(x_1, x_2))^2 \, dx_1 \leqq C e^{-2\gamma x_2} \|\chi^{[1;2]}\|^2_{L^1(S_0)} \tag{3.2.18}$$

and

$$B(\varphi, v) = 0 \tag{3.2.19}$$

for any $v \in \mathcal{D}(R^+)$.

Proof. The proof is similar to the one in § I.3.2 and in the proof of the Lemma 3.2.1. We have

$$\begin{aligned} B(\varphi, v) &= \sum_k B_k(\varphi_k, v_k) \\ &= B_0(\mu^{[1;2]}, \sum kv(x_1 + k, x_2)) + B_0(w_1\mu^{[1;2]}, \sum_k v(x_1 + k, x_2)) \\ &\quad + B_0(\nu, \sum_k v(x_1 + k, x_2)). \end{aligned} \tag{3.2.20}$$

We have (just as in § I.3.2),

$$B_0\Big(\mu^{[1;2]}, \sum_k kv(x_1 + k, x_2)\Big) = F^{(2)}\Big(\sum_k v(x_1 + k, x_2)\Big). \tag{3.2.21}$$

Let us study now this functional. First, it is easy to check that if $\psi \in L^{1,-\gamma/2}_{H,0}(R^+)$, then

$$v = \psi\tfrac{1}{2}\varkappa(x_1) \in L_0^{1,-\gamma/2}(R^+)$$

and

$$\|v\|_{L^{1,-\gamma/2}(R^+)} \leqq C\|\psi\|_{L^{1,-\gamma/2}(R^+)}, \tag{3.2.22}$$

and as is Lemma I.3.2.2 we get

$$
\begin{aligned}
|F^{(2)}(\psi)| &\leqq |B_0(\mu^{[1;2]}, \sum_k kv(x_1+k, x_2))| \\
&\leqq C\|\mu^{[1;2]}\|_{L^{1,\gamma/2}(R_0^+)}\|\psi\|_{L^{1,-\gamma/2}(R_0^+)}.
\end{aligned} \tag{3.2.23}
$$

We have also

$$\|\omega_1\mu^{[1;2]}\|_{L^{1,\gamma/2}(R_0^+)} \leqq C\|\chi^{[1;2]}\|_{L^1(S_0)},$$

and hence,

$$B(\omega_1\mu^{[1;2]}, \psi) \leqq C\|\chi^{[1;2]}\|_{L^1(S_0)}\|\psi\|_{L_0^{1,-\gamma/2}(R_0^+)}. \tag{3.2.24}$$

Therefore there exists $w \in L_{H,0}^{1,\gamma/2}(R_0^+)$,

$$
\begin{aligned}
&\|w\|_{L^{1,\gamma/2}(R_0^+)} \leqq C\|\chi^{[1;2]}\|_{L^1(S_0)}, \\
&\bar{\varphi} = \{(k + w_1(x_1 - k))\mu^{[1;2]} + w\} \in L^1_{\mathrm{Loc}}(R^+)
\end{aligned} \tag{3.2.25}
$$

and $B(\bar{\varphi}, v) = 0$ for any $v \in \mathscr{D}(R^+)$. As in the proof of Lemma 3.2.1, we find σ so that $w + \sigma = \nu$ with ν and σ satisfying all properties of the lemma.

3.3. Homogeneous problem. Just as in § I.4 we may construct the function W^H and W using (I.4.3.7) and (I.4.3.8) with U defined by (3.1.3).

All steps we did in the § I.4 are valid here, too, when space $L^{1,p}(R_2)$ is replaced by $L_0^1(R^+)$. Unfortunately the step in connection with the function ${}^3V^H$ will not lead to a function of $L_0^1(R^+)$. To correct this deficiency we have to subtract function ${}^8V^H \in L^1(R^+)$ such that

$${}^8V^H(x_1, 0) = (W^H - {}^3V^H)(x_1, 0) \tag{3.3.1}$$

and

$$B({}^8V^H, v) = 0 \tag{3.3.2}$$

for any $v \in \mathscr{D}(R^+)$.

Let us study the structure of ${}^8V^H$. For this purpose let ${}^9V^{H,r} = \{{}^9V_k^{H,r}\}$ be defined so that

$$
\begin{aligned}
{}^9V_k^{H,r}(x) &= \Big(\frac{\partial U}{\partial x_2}(rH, 0) + \frac{\partial^2 U}{\partial x_1\,\partial x_2}(rH, 0)(k-r)H)(-H\mu^{[1;2]}(x/H))\Big) \\
&\quad + \frac{\partial^2 U}{\partial x_1\,\partial x_2}(rH, 0)H^2(-\tfrac{1}{2}\mu^{[1;1]}(x/H) \\
&\quad - \omega_1(x_1/H - k)\mu^{[1;2]}(x/H) - \mu^{[2;1,1]}(x/H)) \\
&\quad + \frac{1}{2}\frac{\partial^2 U}{\partial x_2^2}(rH, 0)H^2(-\mu^{[1;2]}(x/H) - \mu^{[2;0,2]}(x/H)) \\
&= -H\frac{\partial U}{\partial x_2}(rH, 0)\mu^{[1;2]}(x/H) - H^2\frac{\partial^2 U}{\partial x_1\,\partial x_2}(rH, 0) \\
&\quad \cdot [((k-r) + w_1(x_1/H - k))\mu^{[1;2]}(x/H) + \mu^{[2;1,1]}(x/H)]
\end{aligned} \tag{3.3.3}
$$

$$-\frac{1}{2}\frac{\partial^2 U}{\partial x_2^2}(rH,0)H^2[\mu^{[1;2]}(x/H)+\mu^{[2;0,2]}(x/H)].$$

We see that ${}^9V^{H,r}\in L^1_{\mathrm{Loc}}(R^+)$. Denote ${}^{10}V^H=\{{}^9V_k^{H,k}\}$. Construct ${}^{11}V^H=\{{}^{11}V_k^H\}$ analogously as the function ${}^3V^H$; i.e., let

$$(3.3.4)\qquad {}^{11}V^H \mp \sum_{i=0}^{1}\gamma_{k-i,0}^H[{}^9V_k^{H,k}-{}^9V_k^{H,k-i}].$$

Then we get by analogous reasoning, using Lemmas 3.2.1 and 3.2.2, that for $\gamma\leqq\alpha\pi/(2\sqrt{2}\beta)$,

$$(3.3.5)\qquad \|{}^{11}V_k^H\|_{{}^H\hat{L}^{1,\gamma/4H}(R^+)}\leqq CH^2$$

and

$${}^{10}V^H-{}^{11}V^H\in L^{1,\gamma/4H}(R^+).$$

Therefore there exists ${}^{12}V^H\in{}^H\hat{L}^1(R^+)$ such that

$$(3.3.6)\qquad \|{}^{12}V^H\|_{{}^H\hat{L}^1(R^+)}\leqq CH^2,$$

$$(3.3.7)\qquad {}^{13}V^H={}^{10}V^H-{}^{12}V^H\in L^1(R^+),$$

$$(3.3.8)\qquad B^H({}^{13}V^H,v)=\sum_k B_k^H({}^9V_k^{H,k},v)$$

for any $v\in L_0^1(R^+)$ and

$$(3.3.9)\qquad {}^{13}V^H(x_1,0)=({}^{10}V^H-{}^{11}V^H)(x_1,0).$$

Let us find now ${}^{14}V^H\in L_0^1(R^+)$ such that

$$(3.3.10)\qquad B^H({}^{13}V^H-{}^{14}V^H,v)=0$$

for any $v\in L_0^1(R^+)$. Let $v\in L_0^1(R^+)$ such that $v(x)=0$ for $x_1=(\frac{1}{2}+k)H$, k arbitrary integer. Because of the construction of ${}^{10}V^H$, we see that

$$(3.3.11)\qquad \sum_k B_k^H({}^9V_k^{H,k},v)=0.$$

Let $v\in L_0^1(R^+)$. Denote

$$(3.3.12)\qquad w_k^H(x)=v(x)\varkappa((x_1/H-(k-\tfrac{1}{2}))2)$$

with $\varkappa$ defined in (I.3.2.13). Then

$$(3.3.13)\qquad \begin{aligned}\sum_k B_k^H({}^9V_k^{H,k},v)&=\sum_k(B_k^H({}^9V_k^{H,k},w_k^H)+B_{k-1}^H({}^9V_{k-1}^{H,k-1},w_B^H))\\&=\sum_k B^H({}^9V_k^{H,k}-{}^9V_k^{H,k-1},w_k^H).\end{aligned}$$

We get also

$$(3.3.14)\qquad \begin{aligned}&|B_k^H({}^9V_k^{H,k}-{}^9V_k^{H,k-1},w_k^H)|\\&\qquad\leqq C(k)\|{}^9V_k^{H,k}-{}^9V_k^{H,k-1}\|_{L^{1,\gamma/4H}(R_k^{+H})}\|v\|_{L^1(R_k^{+H})}.\end{aligned}$$

This yields

$$\left|\sum_k B_k^H({}^9V_k^{H,k}, v)\right| \leqq CH^2\|v\|_{L^1(R^+)} \tag{3.3.15}$$

and using Theorem I.2.3.1, we see that there exists ${}^{14}V^H \in L_0^1(R^+)$ with

$$\|{}^{14}V^H\|_{L^1(R^+)} \leqq CH^2 \tag{3.3.16}$$

and satisfying (3.3.10).

So there exists ${}^{15}V^H \in {}^H\hat{L}^1(R^+)$,

$$\|{}^{15}V^H\|_{{}^H\hat{L}^1(R^+)} \leqq CH^2 \tag{3.3.17}$$

so that

$$\begin{gathered} {}^{10}V^H - {}^{15}V^H \in L^1(R^+), \\ B^H({}^{10}V^H - {}^{15}V^H, v) = 0 \end{gathered} \tag{3.3.18}$$

for any $v \in L_0^1(R^+)$ and

$$({}^{10}V^H - {}^{15}V^H)(x_1, 0) = ({}^{10}V^H - {}^{11}V^H)(x_1, 0). \tag{3.3.19}$$

Going back to the definition of ${}^{10}V^H$ and ${}^{11}V^H$, we shall get by simple computation

$$\begin{aligned} ({}^{10}V^H - {}^{11}V^H)(x_1, 0) &= (W^H - {}^3V^H)(x_1, 0) \\ &= H\frac{\partial U}{\partial x_2}(x_1, 0)\lambda^{[1;2]} + H^2 g(x_1), \end{aligned} \tag{3.3.20}$$

where function g depends on U and H with

$$\int_{-\infty}^{+\infty} (g^2 + g'^2)\, dx_1 < C \tag{3.3.21}$$

and C independent of H. Equations (3.3.18), (3.3.19) and (3.3.20) describe the structure of the function ${}^8V^H$ introduced at the beginning of this section. This analysis of ${}^8V^H$ leads to the following theorems.

THEOREM 3.3.1. *Let U satisfy* (3.1.3) *with $f \in \mathscr{E}(R^+) \cap \mathscr{D}(R_2)$ and u^H satisfy* (3.1.4). *Further let ${}^1W^H$ be given by* (I.4.3.8). *Then*

$$\|u^H - {}^1W^H\|_{{}^H\hat{L}^1(R^+)} \leqq CH^{1/2}, \tag{3.3.22}$$

$$\|u^H - {}^1W^H\|_{{}^H\hat{L}^1(R^+(z))} \leqq CH + CH^{1/2}\, e^{-\beta z/H}, \tag{3.3.23}$$

where $\beta > 0$ is independent of H (and $R^+(z)$ is introduced in § 2.1).

THEOREM 3.3.2. *Let u^H and U be as in the previous theorem. Then*

$$\|U - u^H\|_{L_{2,\alpha}(R^+)} \leqq CH, \qquad \alpha < 1. \tag{3.3.24}$$

THEOREM 3.3.3. *Let u^H and U be the same as in Theorem* 3.3.1. *Further let $\tilde{U} \in L^1(R^+)$ be such that*

$$\hat{B}(\tilde{U}, v) = \int_{R^+} \Lambda U\, dx \tag{3.3.25}$$

for any $v \in L_0^1(R^+)$ *and*

$$\tilde{U}(x_1, 0) = \lambda^{[1,2]} \frac{\partial U}{\partial x_2}(x_1, 0). \tag{3.3.26}$$

Let W^H *be given by* (I.4.3.7) *and* ${}^1\tilde{W}^H$, *the function defined by* (I.4.3.8) *using* $\tilde{U}$ instead of U. Then

$$\|u^H - (W^H - H^1 \tilde{W}^H)\|_{{}^H\hat{L}^1(R^+(z))} \leqq CH^2 + CH^{1/2} e^{-\beta z/H}. \tag{3.3.27}$$

THEOREM 3.3.4. *Let the assumptions of Theorem* 3.3.1 *hold. Then*

$$B^H(u^H, u^H) = \hat{B}(U, U) + R(H), \tag{3.3.28}$$

with

$$R(H) \leqq CH.$$

3.4. The general problem. The existence of the boundary influenced the analysis of the problem. We have been able to proceed because of our assumption about the boundary, namely, we only investigated the case of a half-plane. The question is whether the theorems hold for a general domain also, provided that the solution U is smooth. The problem remains open. Of course some weaker results are valid in the case of the general domain. So, e.g., it is possible to prove that (3.3.22) holds in general. The validity of (3.3.23) is not clear. Using the maximum principle which is valid for the equation, estimate (3.3.24) holds for $\alpha = 0$ if the domain is bounded. In some of the next papers we shall study these problems more thoroughly.

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AN INVERSE EIGENVALUE PROBLEM OF ORDER FOUR*

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Abstract. In this paper coefficients $A(s) \in c^{\infty}[0, 1]$, $B(s) \in c^{\infty}[0, 1]$ are constructed so that given positive numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ are the first n eigenvalues and given positive numbers $\rho_1, \cdots, \rho_n$ are the first n normalization constants for the first n eigenfunctions for the fourth order self-adjoint eigenvalue problem $y^{(4)} + (Ay^{(1)})^{(1)} + By - \lambda y = 0$, $y(0) = y^{(1)}(0) = y(1) = y^{(1)}(1) = 0$. The solution is determined from the spectral function for the eigenvalue problem.

1. Introduction. In the usual fourth order, self-adjoint eigenvalue problem, Coddington and Levinson [4], real coefficients $A(s) \in C^{(1)}[0, 1]$, $B(s) \in C[0, 1]$, and self-adjoint boundary conditions $\sum_{j=1}^{4} [M_{ij}y^{(j-1)}(0) + N_{ij}y^{(j-1)}(1)] = 0$, $i = 1, 2, 3, 4$, are given with M_{ij}, N_{ij} being real constants, $i, j = 1, 2, 3, 4$; then eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots$ are sought so that the problem

$$y^{(4)}(s) + (A(s)y^{(1)}(s))^{(1)} + B(s)y(s) - \lambda y(s) = 0,$$

$$\sum_{j=1}^{4} [M_{ij}y^{(j-1)}(0) + N_{ij}y^{(j-1)}(1)] = 0, \qquad i = 1, 2, 3, 4,$$

has a nontrivial solution when $\lambda = \lambda_i$, $i = 1, 2, 3, \cdots$. The inverse eigenvalue problem to be considered here is that of assuming that a set of positive real numbers is given and then seeking to determine real coefficients $A(s)$ and $B(s)$ in a fourth order, self-adjoint, linear equation, and real constants M_{ij}, N_{ij}, $i, j = 1, 2, 3, 4$, so that the given set of real numbers is the set of eigenvalues for a fourth order, self-adjoint eigenvalue problem.

Interest in this fourth order, inverse eigenvalue problem is fairly recent. However, inverse second order problems have been considered by a number of authors. Extensive work has been done by Borg [3], Marcenko [12], Krein [6], [7], Levinson [9], and Gel'fand and Levitan [5]. Roughly speaking, in each of their papers an infinite set of real numbers is given and then a function $q(s)$ and corresponding boundary conditions are sought so that the infinite set of real numbers is the set of eigenvalues for the eigenvalue problem

$$y^{(2)} + (\lambda - q(s))y = 0,$$

$$y(0) + hy^{(1)}(0) = 0,$$

$$y(1) + Hy^{(1)}(1) = 0,$$

where h and H are determined from the given set of real numbers. It was noted by Borg [3], that knowledge of all of the eigenvalues for one eigenvalue problem was not enough to determine $q(s)$ uniquely; and that both for uniqueness and for existence proofs an additional condition had to be assumed. Accordingly, in [3],

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[12], [6], [7], [9] knowledge of two alternating sequences of eigenvalues is assumed, while Gel'fand and Levitan [5] assume knowledge of a sequence of eigenvalues, and the corresponding sequence of normalization constants, i.e., the spectral function. (The normalization constants are squares of the L^2 norms of eigenfunctions corresponding to the sequence of eigenvalues.) Furthermore, Levitan [10], [11] has shown that given the two sequences of eigenvalues as in [3], [12], [6], [7], [9], the normalization constants associated with either sequence, i.e., the spectral function, may be constructed. Hence if either two alternating sequences of eigenvalues are known or if the spectral function is known, the approach given by I. M. Gel'fand and B. M. Levitan may be used to find the unknown coefficient, $q(s)$ and boundary conditions in the second order inverse problem.

Some work has been done for the fourth order problem by Barcelon [1], [2]. Following the approach of M. Krein, V. Barcelon shows that uniqueness and construction of coefficients can be obtained for a fourth order problem from the knowledge of three, distinct, interlacing sequences of eigenvalues. Other work has been done by McKenna [13] in which the author parameterizes the coefficients in a fourth order self-adjoint equation and attempts to vary the parameters to force the first four eigenvalues for an associated eigenvalue problem to be chosen real numbers. The results here are largely negative; i.e., the parameters cannot be chosen so as to make the given four real numbers the first four eigenvalues for the problem.

The present paper attempts to generalize the work of Gel'fand and Levitan [5] to the fourth order case. That is, roughly speaking, it will be assumed that eigenvalues and normalization constants are known, and then the corresponding eigenvalue problem (i.e., coefficients of a fourth order, self-adjoint differential equation and self-adjoint boundary conditions) will be sought. In order to make the analysis more elementary and to exhibit basic ideas more clearly, it will be assumed that only the *first n* eigenvalues and *first n* normalization constants are known. The assumption that only the "first part" of the spectral function is known means that solutions (i.e., coefficients of the fourth order equation and self-adjoint boundary conditions) are not unique. It does allow us, however, to choose the remaining eigenvalues and normalization constants, that is, the remainder of the spectral function to vary the resultant boundary conditions; also, it enables us to represent the coefficients in the fourth order differential equation as finite sums of other functions which, in turn, are solutions of a finite set of nonhomogeneous linear equations.

The particular problem to be solved is the following. We assume we are given $2n$ positive real numbers $\lambda_1<\lambda_2<\lambda_3<\cdots<\lambda_n, \rho_1, \rho_2, \rho, \cdots, \rho_n$. We then seek to find a set of coefficients $A(s)\in C^{(1)}[0, 1]$ and $B(s)\in C[0, 1]$ and functions y_{λ_i}, $i=1, 2, \cdots, n$, such that $\lambda_1<\lambda_2<\cdots<\lambda_n$ are the first n eigenvalues, and y_{λ_i}, $i=1, \cdots, n$, are the corresponding eigenfunctions, for the eigenvalue problem

$$\begin{gathered} y^{(4)}+(Ay^{(1)})^{(1)}+By-\lambda y=0, \\ y(0)=y^{(1)}(0)=y(1)=y^{(1)}(1)=0. \end{gathered} \tag{1}$$

Further, it is required that $\rho_i=\int_0^1[y_{\lambda_i}(s)]^2\,ds$. The boundary conditions $y(0)=y'(0)=y(1)=y'(1)=0$ are chosen for ease of solution. It is possible that for the given finite set of eigenvalues and normalization constants, more general boundary conditions may be obtained; this is discussed in Remark 2 of § 3 and in § 4.

The analysis of this paper is divided into three sections. In § 2 the set of eigenvalues and normalization constants is completed (judiciously) and a set of functions $\{y_{\lambda_i}\}_{i=1}^{\infty}$ is generated so that each y_{λ_i}, $i=1,2,\cdots$, has the form

$$y_{\lambda_i}(s)=Z_{\lambda_i}(s)+\int_0^s K(s,t)Z_{\lambda_i}(t)\,dt,$$

with each $Z_{\lambda_i}(s)$, $i=1,2,\cdots$, being a given, known function. The function $K(s,t)$, $0\leqq t\leqq s\leqq 1$, is determined from the assumption that $\{y_{\lambda_i}\}_{i=1}^{\infty}$ is a complete orthogonal set on $0\leqq s\leqq 1$ with $\rho_i=\int_0^1[y_{\lambda_i}(s)]^2\,ds$. Under this assumption it is shown that $K(s,t)$ is a solution of an integral equation of the form

$$f(s,t)+\int_0^s K(s,z)f(t,z)\,dz+K(s,t)=0,$$

where $f(s,t)$ is represented as a *finite* sum of known functions. It is also shown that $y_{\lambda_i}(0)=y_{\lambda_i}^{(1)}(0)=0$ and $y_{\lambda_i}(1)=y_{\lambda_i}^{(1)}(1)=0$.

In § 3, the coefficients $A(s)$ and $B(s)$ are defined, and it is shown that each y_{λ_i} satisfies

$$y_{\lambda_i}^{(4)}+(Ay_{\lambda_i}^{(1)})^{(1)}+By_{\lambda_i}-\lambda_i y_{\lambda_i}=0,$$

$i=1,2,\cdots$. Further, it is noted that since the set $\{y_{\lambda_i}\}_{i=1}^{\infty}$ is complete and orthogonal and each element of the set satisfies the same set of self-adjoint boundary conditions then the given $\lambda_1<\lambda_2<\cdots<\lambda_n$ set of positive real numbers are eigenvalues (an in particular the *first n* eigenvalues) for the problem

$$y^{(4)}+(Ay^{(1)})^{(1)}+By-\lambda y=0,$$
$$y(0)=y^{(1)}(0)=y(1)=y^{(1)}(1)=0.$$

Section 4 is devoted to a discussion of the existence of self-adjoint boundary conditions satisfied by a set of functions $\{y_{\lambda_i}\}_{i=1}^{\infty}$ which is orthogonal and such that for fixed $A(s)$ and $B(s)$ each y_{λ_i} satisfies an equation of the form

$$y_{\lambda_i}^{(4)}+(Ay_{\lambda_i}^{(1)})^{(1)}+By_{\lambda_i}-\lambda_i y_{\lambda_i}=0.$$

This discussion is relevant when more general boundary conditions are desired.

2. Construction of a complete, orthogonal set of functions. In this section a complete, orthogonal sequence, $\{y_{\lambda_i}\}_{i=1}^{\infty}$, of functions will be generated; each of the functions y_{λ_i}, $i=1,2,\cdots$, will also be shown to satisfy the boundary conditions $y_{\lambda_i}(0)=y_{\lambda_i}^{(1)}(0)=y_{\lambda_i}(1)=y_{\lambda_i}^{(1)}(1)=0$. These functions will later be shown to be the entire set of eigenfunctions for the eigenvalue problem (1).

To this end, the following associated eigenvalue problem will be considered. Let $\lambda_1^*, \lambda_2^*, \cdots$ be the eigenvalues for the eigenvalue problem

$$Z^{(4)} - \lambda Z = 0,$$

(2)
$$Z(0) = Z^{(1)}(0) = Z(1) = Z^{(1)}(1) = 0.$$

Let

$$Z_\lambda(s) = \frac{[\sin(\lambda^{1/4}s) - \sinh(\lambda^{1/4}s)]}{2\lambda^{3/4}} + \alpha \frac{[\cosh \lambda^{1/4}s - \cos \lambda^{1/4}s]}{2\lambda^{3/4}},$$

where

$$\alpha = \frac{\sinh \lambda^{1/4} - \sin \lambda^{1/4}}{\cosh \lambda^{1/4} - \cos \lambda^{1/4}}.$$

Then for each λ, $Z_\lambda(s)$ satisfies the fourth order differential equation $Z_\lambda^{(4)} - \lambda Z_\lambda = 0$, and the initial values, $Z_\lambda(0) = Z_\lambda^{(1)}(0) = 0$, $Z_\lambda^{(2)}(0) = \alpha/\lambda^{1/4}$ and $Z_\lambda^{(3)}(0) = -1$. Furthermore, $Z_\lambda(1) = 0$. Also, $Z_{\lambda_i^*}$ is the associated eigenfunction for each eigenvalue λ_i^* of problem (2). Furthermore, let ρ_i^* be the normalization constant for $Z_{\lambda_i^*}$, that is,

$$\rho_i^* = \int_0^1 [Z_{\lambda_i^*}]^2 \, ds.$$

It should be noted, at this point, that in order to determine the coefficients $A(s)$ and $B(s)$ in (1), more than the given first n eigenvalues $\lambda_1, \cdots, \lambda_n$ and corresponding normalization constants $\rho_1, \cdots, \rho_n$ will be needed. In fact, for the formulation being presented here, the entire set of eigenvalues and associated normalization constants is needed. These are chosen judiciously to ease the solution of the problem and insure that the y_{λ_i} satisfy the chosen boundary conditions. Consequently, the remaining eigenvalues and normalization constants are defined as follows. Let the remaining eigenvalues λ_i, $i = n+1, n+2, \cdots$, be defined by $\lambda_i = \lambda_i^*$, $i = n+1, n+2, \cdots$; and similarly, let the remaining normalization constants ρ_i, $i = n+1, \cdots$, be defined by $\rho_i = \rho_i^*$, $i = n+1, n+2, \cdots$. Since the known eigenvalues $\lambda_1, \cdots, \lambda_n$ are required to be the first n eigenvalues, the above choice for the remaining eigenvalues implicitly assumes that $\lambda_n < \lambda_{n+1}^*$. This assumption is without loss of generality since an additional, finite number, of eigenvalues, say $\lambda_{n+1} < \cdots < \lambda_{n+m}$ with $\lambda_n < \lambda_{n+1}$, may be added to the list of known eigenvalues until $\lambda_{n+m} < \lambda_{n+m+1}^*$.

The set of functions $\{y_{\lambda_i}\}_{i=1}^\infty$ is now defined in terms of the functions $\{Z_{\lambda_i}\}_{i=1}^\infty$ and an unknown function $K(s, t)$ as

(3)
$$y_{\lambda_i}(s) = Z_{\lambda_i}(s) + \int_0^s K(s, t) Z_{\lambda_i}(t) \, dt.$$

The formulation defines $y_{\lambda_i}(s)$ as equal to $Z_{\lambda_i}(s)$ plus the "error" term, $\int_0^s K(s, t) Z_{\lambda_i}(t)\, dt$. The function $K(s, t)$ is the same for each $i = 1, 2, \cdots$ and will be completely determined by requiring that the sequence of functions $\{y_{\lambda_i}\}_{i=1}^\infty$ is a complete, orthogonal set. In addition, under the assumption that $K(s, t)$ is sufficiently differentiable, this formulation for $y_{\lambda_i}(s)$ automatically gives the left-hand boundary conditions of (1), namely that $y_{\lambda_i}(0)=y_{\lambda_i}^{(1)}(0)=0$.

Again, the function $K(s,t)$ is determined from the assumption that $\{y_{\lambda_i}(s)\}_{i=1}^{\infty}$ is a complete, orthogonal set on $0 \leqq s \leqq 1$. In fact, the following theorem is true.

THEOREM 1. *Let $Z_{\lambda_i}(s)$, $Z_{\lambda_i^*}(s)$, and $y_{\lambda_i}(s)$, $0 \leqq s \leqq 1$, $i=1,2,\cdots$, be defined as above. Suppose, in addition, that $K(s,t)$ is continuous in $0 \leqq t \leqq s \leqq 1$. Then $\{y_{\lambda_i}(s)\}_{i=1}^{\infty}$ is a complete, orthogonal set on $0 \leqq s \leqq 1$, with normalization constants $\{\rho_i\}_{i=1}^{\infty}$, $\rho_i=\rho_i^*$, $i=n+1,n+2,\cdots$, if and only if $K(s,t)$ satisfies the integral equation*

$$f(s,t)+\int_0^s f(p,t)K(s,p)\,dp+K(s,t)=0, \tag{4}$$

where

$$f(s,t)=\sum_{i=1}^{n}\left[\frac{Z_{\lambda_i}(s)Z_{\lambda_i}(t)}{\rho_i}-\frac{Z_{\lambda_i^*}(s)Z_{\lambda_i^*}(t)}{\rho_i^*}\right]. \tag{5}$$

Remark 1. The function $f(s,t)$ is represented as a finite sum precisely because of the judicious choice of the remaining eigenvalues $\lambda_n, \lambda_{n+1},\cdots$ and remaining normalization constants $\rho_n, \rho_{n+1},\cdots$. This representation of $f(s,t)$ makes the solution of the integral equation easier. One could, however, choose the remaining eigenvalues and normalization constants so that asymptotically they approach the sequences $\{\lambda_i^*\}_{i=1}^{\infty}$ and $\{\rho_i^*\}_{i=1}^{\infty}$, respectively. In general, this would result in expressing $f(s,t)$ as an infinite sum, thus making the solution of the integral equation more difficult.

Remark 2. The proof of Theorem 1 uses the following two well-known concepts: (i) the sequence $\{y_{\lambda_i}\}_{i=1}^{\infty}$, with $\rho_i=\int_0^1[y_{\lambda_i}(s)]^2\,ds$, $i=1,2,\cdots$, satisfies Parseval's equality on $0 \leqq s \leqq 1$ if and only if for every two functions $f(s)$, $g(s)\in L^2(0,1)$,

$$\sum_{i=1}^{\infty}\frac{[\int_0^1 f(s)y_{\lambda_i}(s)\,ds][\int_0^1 g(s)y_{\lambda_i}(s)\,ds]}{\rho_i}=\int_0^1 f(s)g(s)\,ds;$$

(ii) the sequence $\{y_{\lambda_i}\}_{i=1}^{\infty}$, where $\int_0^1[y_{\lambda_i}(s)]^2\,ds=\rho_i$, $i=1,2,\cdots$, is a complete, orthogonal sequence on $0 \leqq s \leqq 1$ if and only if $\{y_{\lambda_i}\}_{i=1}^{\infty}$ satisfies Parseval's equality.

Proof of Theorem 1. Assume first that $K(s,t)$ is a solution of the integral equation (4). Recall that, by hypothesis, $K(s,t)$ is continuous on $0 \leqq t \leqq s \leqq 1$. Then let h, g be arbitrary functions in $L^2[0,1]$. It will be shown that Parseval's equality holds (hence, $\{y_{\lambda_i}\}_{i=1}^{\infty}$ is a complete orthogonal sequence), i.e.,

$$\sum_{i=1}^{\infty}\frac{[\int_0^1 h(p)y_{\lambda_i}(p)\,dp][\int_0^1 g(q)y_{\lambda_i}(q)\,dq]}{\rho_i}=\int_0^1 h(p)g(p)\,dp.$$

More specifically, since it is known that $\{Z_{\lambda_i^*}\}_{i=1}^{\infty}$ is a complete, orthogonal sequence, we shall show that

$$I=\sum_{i=1}^{\infty}\frac{[\int_0^1 h(p)y_{\lambda_i}(p)\,dp][\int_0^1 g(q)y_{\lambda_i}(q)\,dq]}{\rho_i} - \sum_{i=1}^{\infty}\frac{[\int_0^1 h(p)Z_{\lambda_i^*}(p)\,dp][\int_0^1 g(q)Z_{\lambda_i^*}(q)\,dq]}{\rho_i^*}=0.$$

When this is shown then we will have proved that if $K(s, t)$ is a solution of (4), then $\{y_{\lambda_i}\}_{i=1}^{\infty}$ is a complete orthogonal sequence with the given normalization constants.

In order to show that $I=0$, we first recall that $Z_{\lambda_i}=Z_{\lambda_i^*}$, and $\rho_i=\rho_i^*$, for $i=n+1, n+2, \cdots$. We then observe that

$$\int_0^1 h(p)y_{\lambda_i}(p)\,dp=\int_0^1 h(p)Z_{\lambda_i}(p)\,dp+\int_0^1 Z_{\lambda_i}(t)\left[\int_t^1 h(p)K(p, t)\,dp\right]dt,$$

and that a similar expression holds when h is replaced by g. Then, after a suitable rearrangement of terms, and use of the fact that $\{Z_{\lambda_i^*}\}_{i=1}^{\infty}$ is a complete, orthogonal set, it can be shown, after a lengthy but elementary calculation, that

$$\begin{aligned}I=&\int_0^1\int_0^p\Big\{[h(p)g(q)+g(p)h(q)]\\&\cdot\left[f(p, q)+\int_0^p f(q, t)K(p, t)\,dt+K(p, q)\right]\Big\}\,dq\,dp\\&+\int_0^1\int_0^q\Big\{[h(p)g(q)+g(p)h(q)]\\&\cdot\int_0^p K(p, t)\,dt\left[f(q, t)+\int_0^q f(t, s)K(q, s)\,ds+K(q, t)\right]\Big\}\,dp\,dq.\end{aligned}$$

Since $K(s, t)$ satisfies the integral equation (4), we then have $I=0$.

It remains to show that in order for $\{y_{\lambda_i}\}_{i=1}^{\infty}$ to be a complete, orthogonal sequence, with normalization constants $\{\rho_i\}_{i=1}^{\infty}$, it is *necessary* that $K(s, t)$ satisfy the integral equation (4). Assume then that $\{y_{\lambda_i}\}_{i=1}^{\infty}$ is a complete, orthogonal sequence, and again that h and g are arbitrary functions in $L^2[0, 1]$. This implies that $I=0$ and, hence, if we let

$$J(p, q)=f(p, q)+\int_0^p f(q, t)K(p, t)\,dt+K(p, q),$$

we have

$$0=I=\int_0^1 dp\int_0^p dq\left[J(p, q)+\int_0^q J(p, t)K(q, t)\,dt\right][h(p)g(q)+h(q)g(p)].$$

Since h and g are arbitrary functions in $L^2[0, 1]$, it is then true that

$$J(p, q)+\int_0^q J(p, t)K(q, t)\,dt=0, \qquad 0\leqq q\leqq p\leqq 1.$$

For fixed but arbitrary p, $0 \leqq p \leqq 1$, this is a homogeneous Volterra integral equation with kernel $K(q, t)$. The only solution of this equation is the zero solution, Yosida [14]. Hence, since p was arbitrary, $J(p, q) = 0$ for $0 \leqq q \leqq p \leqq 1$, i.e., $K(p, q)$ must satisfy

$$f(p, q) + \int_0^p f(q, t)K(p, t)\, dt + K(p, q) = 0.$$

We have shown that under the assumptions that y_{λ_i} has the form given in (3) and that $\{y_{\lambda_i}\}_{i=1}^{\infty}$ is orthogonal and complete, with normalization constants $\{\rho_i\}_{i=1}^{\infty}$, we must have that $K(s, t)$ satisfies the integral equation (4). It must now be shown that (4) actually has a solution. We show first that (4) has at most one solution. Then, the proof of existence of a solution is constructive; that is, it will be shown that $K(s, t)$ can be obtained by solving an appropriate set of linear equations. We have the following theorems.

THEOREM 2. *Let $f(s, t)$ be defined as in* (5) *and suppose that $K(s, t)$ is continuous in t, $0 \leqq t \leqq s \leqq 1$, for each fixed s, $0 \leqq s \leqq 1$. Then, there exists at most one solution of the integral equation*

$$f(s, t) + \int_0^s f(p, t)K(s, p)\, dp + K(s, t) = 0. \tag{4}$$

Proof. For fixed s, the integral equation (4) is a nonhomogeneous Fredholm equation in $K(s, t)$ with kernel $f(p, t)$. In order to show that (4) has at most one solution, one need only show that for each s the homogeneous equation

$$\int_0^s h(p)f(p, t)\, dp + h(t) = 0, \qquad 0 \leqq t \leqq s,$$

has only the zero solution (i.e., $h \equiv 0$). Therefore, fix s, assume that $h(t)$ is continuous on $0 \leqq t \leqq s$ and that

$$\int_0^s h(p)f(p, t)\, dp + h(t) = 0.$$

Multiply by $h(t)$ and integrate from 0 to s to obtain

$$\int_0^s [h(t)]^2\, dt + \int_0^s \int_0^s h(t)f(p, t)h(p)\, dp\, dt = 0.$$

Since $\{Z_{\lambda_i^*}\}_{i=1}^{\infty}$ is a complete, orthogonal set, we have from Parseval's equality,

$$\int_0^s [h(t)]^2\, dt = \sum_{i=1}^{\infty} \frac{[\int_0^s h(t)Z_{\lambda_i^*}(t)\, dt]^2}{\rho_i^*}.$$

If we substitute this last equation into the above integral equation and recall that $Z_{\lambda_i} = Z_{\lambda_i^*}$ and $\rho_i = \rho_i^*$ for $i = n+1, n+2, \cdots$, we obtain the result that

$$\sum_{i=1}^{\infty} \frac{[\int_0^s h(t)Z_{\lambda_i}(t)\, dt]^2}{\rho_i} = 0$$

or that $\int_0^s h(t)Z_{\lambda_i}(t)\, dt = 0$, $i = 1, 2, \cdots$. It can now be shown that this implies that $h(t) \equiv 0$, $0 \leqq t \leqq s$, if we can show that $\{Z_{\lambda_i}(t)\}_{i=1}^{\infty}$ is a complete set on $0 \leqq t \leqq s$. In fact it will be shown that $\{Z_{\lambda_i}\}_{i=1}^{\infty}$ is a complete set on $0 \leqq t \leqq 1$, since then $\{Z_{\lambda_i}\}_{i=1}^{\infty}$ is a complete set on $[0, s]$ for every s, $0 \leqq s \leqq 1$.

In order to show that $\{Z_{\lambda_i}\}_{i=1}^{\infty}$ is a complete set on $[0, 1]$, it will be shown that if $g(t) \in L^2(0, 1)$ and if $\int_0^1 g(t)Z_{\lambda_i}(t)\,dt = 0$, $i = 1, 2, \cdots$, then $\int_0^1 g(t)Z_{\lambda_i^*}(t) = 0$ for $i = 1, 2, \cdots$, and hence, $g(t) = 0$, a.e., in $[0, 1]$. To do this, we observe that we already know that if $\int_0^1 g(t)Z_{\lambda_i}(t)\,dt = 0$, $i = 1, 2, \cdots$, then $\int_0^1 g(t)Z_{\lambda_i^*}(t)\,dt = 0$, $i = n+1, n+2, \cdots$. Consider, then, the case where $i = 1, \cdots, n$. Again using Parseval's equality, we have that

$$0 = \int_0^1 g(t)Z_{\lambda_i}(t)\,dt = \sum_{j=1}^{\infty} \frac{[\int_0^1 g(t)Z_{\lambda_j^*}(t)\,dt][\int_0^1 Z_{\lambda_i}(t)Z_{\lambda_j^*}(t)\,dt]}{\rho_j^*}$$
$$= \sum_{j=1}^{n} \frac{[\int_0^1 g(t)Z_{\lambda_j^*}(t)\,dt][\int_0^1 Z_{\lambda_i}(t)Z_{\lambda_j^*}(t)\,dt]}{\rho_j^*}, \qquad i = 1, 2, \cdots, n.$$

If $\lambda_i = \lambda_j^*$ for some $i, j = 1, \cdots, n$, then $\int_0^1 g(t)Z_{\lambda_j^*}(t)\,dt = 0$ and we can delete λ_i from consideration and λ_j^* from the sum on the right-hand side above. Therefore, we assume without loss that $\lambda_i \neq \lambda_j^*$ for any $i, j = 1, \cdots, n$. The above set of finite sums then gives n homogeneous linear equations in the n unknowns $\int_0^1 g(t)Z_{\lambda_j^*}(t)\,dt$. The components, $(1/\rho_j^*)\int_0^1 Z_{\lambda_i}Z_{\lambda_j^*}\,dt$, $i, j = 1, \cdots, n$, of the coefficient matrix may be simplified by making use of the boundary conditions satisfied by Z_{λ_i}, $Z_{\lambda_j^*}$, $i, j = 1, \cdots, n$, so that

$$\frac{1}{\rho_j^*}\int_0^1 Z_{\lambda_i}Z_{\lambda_j^*}\,dt = \frac{Z_{\lambda_i}^{(1)}(1)Z_{\lambda_j^*}^{(2)}(1)}{(\lambda_i - \lambda_j^*)(\rho_j^*)}.$$

For each $i = 1, \cdots, n$, $Z_{\lambda_i}^{(1)}(1) \neq 0$ since λ_i is not an eigenvalue for (2); and for each $j = 1, \cdots, n$, $Z_{\lambda_j^*}^{(2)}(1) \neq 0$ (see [8, pp. 327–328]). The coefficient matrix thus has a determinant equal to $[\prod_{i=1}^n (1/\rho_i^*)Z_{\lambda_i}^{(1)}(1)Z_{\lambda_i^*}^{(2)}(1)] \det(1/(\lambda_i - \lambda_j^*))$. The determinant can be shown to be nonzero by induction; hence, the n linear homogeneous equations have only the zero solution, i.e., $\int_0^1 g(t)Z_{\lambda_i^*}(t)\,dt = 0$, $i = 1, \cdots, n$. This completes the proof of Theorem 2.

Now that it is known that the integral equation (4) has at most one solution, one can proceed by *any* means whatsoever to find some solution of (4). Once a solution is obtained, it will be known that that is *the* solution. Accordingly, we have the following theorem.

THEOREM 3. *Assume that $K(s, t)$ is continuous in t, $0 \leqq t \leqq s$, for each s, $0 \leqq s \leqq 1$. Then the solution of* (4) *has the form*

$$K(s, t) = \sum_{i=1}^{n} F_i(s)Z_{\lambda_i}(t) - G_i(s)Z_{\lambda_i^*}(t), \tag{6}$$

where $F_i(s)$, $G_i(s)$, $i = 1, \cdots, n$, are solutions of the nonhomogeneous set of $2n$ linear equations

$$0 = Z_{\lambda_i}(s) + \rho_i F_i(s) + \sum_{j=1}^{n} [F_j(s)(Z_{\lambda_j}, Z_{\lambda_i})(s) - G_j(s)(Z_{\lambda_j^*}, Z_{\lambda_i})(s)], \qquad i = 1, \cdots, n,$$

$$0 = Z_{\lambda_i^*}(s) + \rho_i^* G_i(s) + \sum_{j=1}^{n} [F_j(s)(Z_{\lambda_j}, Z_{\lambda_i^*})(s) - G_j(s)(Z_{\lambda_j^*}, Z_{\lambda_i^*})(s)], \qquad i = 1, \cdots, n, \tag{7}$$

where $(Z_{\lambda_j}, Z_{\lambda_i})(s) = \int_0^s Z_{\lambda_j}(t)Z_{\lambda_i}(t)\,dt$, etc.

Proof of Theorem 3. The proof consists merely of assuming the form (6) for $K(s, t)$, substituting this form into the integral equation (4), and then setting the coefficients of $Z_{\lambda_i}(t)$ and $Z_{\lambda_i^*}(t)$, $i = 1, \cdots, n$, equal to zero. This results in the set of $2n$ linear, nonhomogeneous equations (7) in $F_i(s)$, $G_i(s)$, $i = 1, \cdots, n$. This set of equations in $F_i(s)$, $G_i(s)$, $i = 1, \cdots, n$, has a unique nontrivial solution if the coefficient matrix is nonsingular for all s, $0 \leqq s \leqq 1$.

The proof that the coefficient matrix is nonsingular is by contradiction. Assume that the coefficient matrix, $\Gamma(s)$, is singular for some s, say s_0, $0 \leqq s_0 \leqq 1$. Then there exists a nonzero vector $C = (C_1, \cdots, C_n, C_{n+1}, \cdots, C_{2n})$ such that $\Gamma(s_0)C^T = 0$, or more specifically, so that

$$C_i\rho_i + \sum_{j=1}^{n} [C_j(Z_{\lambda_j}, Z_{\lambda_i})(s_0) - C_{j+n}(Z_{\lambda_j^*}, Z_{\lambda_i})(s_0)] = 0, \qquad i = 1, \cdots, n,$$

and

$$C_{i+n}\rho_i^* + \sum_{j=1}^{n} [C_j(Z_{\lambda_j}, Z_{\lambda_i^*})(s_0) - C_{j+n}(Z_{\lambda_j^*}, Z_{\lambda_i^*})(s_0)] = 0, \qquad i = 1, \cdots, n.$$

Taking the scalar product of C and $\Gamma(s_0)C^T$ we have that $C\Gamma(s_0)C^T = 0$ or

$$\sum_{j=1}^{n} \left[\sum_{i=1}^{n} C_j[\rho_i\,\delta_{ij} + (Z_{\lambda_j}, Z_{\lambda_i})(s_0)]C_i + \sum_{i=1}^{n} C_{n+j}(\rho_i^*\,\delta_{ij} - (Z_{\lambda_j^*}, Z_{\lambda_i^*})(s_0))C_{n+i} \right] = 0,$$

where δ_{ij} is the Kronecker delta. However, the $n \times n$ matrix with components $\rho_i\,\delta_{ij} + (Z_{\lambda_j}, Z_{\lambda_i})(s_0)$, $i, j = 1, \cdots, n$, is positive definite for all s_0, $0 \leqq s_0 \leqq 1$, and the $n \times n$ matrix with components $\rho_i^*\,\delta_{ij} - (Z_{\lambda_j^*}, Z_{\lambda_i^*})(s_0)$, $i, j = 1, \cdots, n$, is positive definite for all s_0, $0 \leqq s_0 < 1$, and has all components zero when $s_0 = 1$. Hence $\Gamma(s)$ can be singular only when $s_0 = 1$ and in addition one must have $C_i = 0$, $i = 1, \cdots, n$. But if $s_0 = 1$ and $C_i = 0$, $i = 1, \cdots, n$, the components C_i, $i = n+1, \cdots, 2n$, must satisfy the homogeneous, linear set of n equations

$$\sum_{j=1}^{n} C_{j+n}(Z_{\lambda_j^*}, Z_{\lambda_i})(1) = 0.$$

The coefficient matrix for this set of equations was noted to be nonsingular in the proof of Theorem 2. Hence, $C_{j+n} = 0$, $j = 1, \cdots, n$. This contradiction proves the result.

Remark 1. If the form (6) for $K(s, t)$ is substituted into the form (3) for $y_{\lambda_i}(s)$, one obtains the expression

$$y_{\lambda_i} = Z_{\lambda_i}(s) + \sum_{j=1}^{n} [F_j(s)(Z_{\lambda_j}, Z_{\lambda_i})(s) - G_j(s)(Z_{\lambda_j^*}, Z_{\lambda_i})(s)], \qquad i = 1, \cdots, n.$$

The first set of n equations in (7) then yields that $y_{\lambda_i} = -\rho_i F_i(s)$, $i = 1, \cdots, n$. Similarly, if $y_{\lambda_i^*}$ is defined as

$$y_{\lambda_i^*} = Z_{\lambda_i^*} + \int_0^s K(s, t)Z_{\lambda_i^*}(t)\,dt, \qquad i = 1, \cdots, n,$$

then the second set of n linear equations in (7) yields that

$$y_{\lambda_i^*} = -\rho_i^* G_i(s), \qquad i = 1, \cdots, n.$$

Remark 2. Since the coefficient matrix in (7) is nonsingular for all s, $0 \leqq s \leqq 1$, and since $Z_{\lambda_i}(1) = Z_{\lambda_i^*}(1) = Z_{\lambda_i^*}^{(1)}(1) = 0$, $Z_{\lambda_i}(0) = Z_{\lambda_i^*}^{(1)}(0) = Z_{\lambda_i^*}(0) = Z_{\lambda_i^*}^{(1)}(0) = 0$, $i = 1, \cdots, n$, it can be shown that $G_i(0) = G_i^{(1)}(0) = G_i(1) = 0$, $i = 1, \cdots, n$, and $F_i(0) = F_i^{(1)}(0) = F_i(1) = F_i^{(1)}(1) = 0$, $i = 1, \cdots, n$. Hence y_{λ_i}, $i = 1, \cdots, n$, all satisfy the same four boundary conditions $y_{\lambda_i}(0) = y_{\lambda_i}^{(1)}(0) = y_{\lambda_i}(1) = y_{\lambda_i}^{(1)}(1) = 0$. Furthermore, since for $i = n+1, n+2, \cdots$,

$$y_{\lambda_i}(s) = Z_{\lambda_i^*}(s) + \sum_{j=1}^{n} F_j(s)(Z_{\lambda_j}, Z_{\lambda_i^*})(s) - G_j(s)(Z_{\lambda_j^*} Z_{\lambda_i^*})(s),$$

we have, in addition, that for $i = n+1, n+2, \cdots$,

$$y_{\lambda_i}(0) = y_{\lambda_i}^{(1)}(0) = y_{\lambda_i}(1) = y_{\lambda_i}^{(1)}(1) = 0.$$

Remark 3. The form for $K(s, t)$ given by (6) and the fact that $F_i(s)$ and $G_i(s)$, $i = 1, \cdots, n$ are solutions of equations (7), yield that $K(s, t)$ is infinitely continuously differentiable in both variables s and t, $0 \leqq t \leqq s \leqq 1$.

Remark 4. The form (6) for $K(s, t)$ and the fact that

$$Z_{\lambda_i}(0) = Z_{\lambda_i}^{(1)}(0) = Z_{\lambda_i^*}(0) = Z_{\lambda_i^*}^{(1)}(0) = 0,$$

yields that $K(s, 0) = 0$ and $(\partial/\partial t)K(s, t)|_{t=0} = 0$ for $0 \leqq s \leqq 1$. This will be used in § 3.

Before proceeding to § 3, where it will be shown that each y_{λ_i}, $i = 1, 2, \cdots$, satisfies an equation of the form $y_{\lambda_i}^{(4)} + (A y_{\lambda_i}^{(1)})^{(1)} + B y_{\lambda_i} - \lambda_i y_{\lambda_i} = 0$, it would be of interest to examine the role of the set $\{Z_{\lambda_i}^*\}_{i=1}^{\infty}$ and the functions Z_{λ_i}, $i = 1, \cdots, n$. We note first that the proof of Theorem 1 requires only that the set $\{Z_{\lambda_i^*}\}_{i=1}^{\infty}$ be a complete, orthogonal set. The presented proofs of Theorems 2 and 3 do make use of the fact that $Z_\lambda(s)$ satisfies $Z^{(4)} - \lambda Z = 0$, and that $Z_\lambda(0) = Z_\lambda^{(1)}(0) = Z_\lambda(1) = 0$ while $Z_{\lambda_i^*}^{(1)}(1) = 0$, $i = 1, 2, \ldots, n$. All that is needed, however, is that $\{Z_{\lambda_i^*}\}_{i=1}^{\infty}$ is a complete orthogonal set and that there is no nontrivial linear combination of the set $\{Z_{\lambda_i}\}_{i=1}^{n}$ which is either identically zero or is orthogonal to every member of the set $\{Z_{\lambda_i^*}\}_{i=1}^{n}$. The conclusions (made in the remarks) that $y_{\lambda_i}(0) = y_{\lambda_i}^{(1)}(0) = y_{\lambda_i}(1) = y_{\lambda_i}^{(1)}(1) = 0$, $i = 1, \cdots, n$, that $F_i(0) = F_i^{(1)}(0) = F_i(1) = F_i^{(1)}(1) = G_i(0) = G_i^{(1)}(0) = G_i(1) = 0$, $i = 1, \cdots, n$, and similarly that $K(s, 0) = 0$ and $K_t(s, t)|_{t=0} = 0$ do depend on the boundary conditions satisfied by the Z_λ's, but not on the fact that each Z_λ satisfies the corresponding differential equation $Z_\lambda^{(4)} - \lambda Z = 0$.

3. Determining the differential equation. In this section we seek to determine $A(s)$ and $B(s)$ so that each y_{λ_i} satisfies a corresponding fourth order differential equation $y_{\lambda_i}^{(4)} + (A y_{\lambda_i}^{(1)})^{(1)} + B y_{\lambda_i} - \lambda_i y_{\lambda_i} = 0$. It will be seen that $A(s)$ and $B(s)$ are infinitely differentiable on $0 \leqq s \leqq 1$. Then, since the set $\{y_{\lambda_i}\}_{i=1}^{\infty}$ is a complete set of functions in $L^2(0, 1)$ and each y_{λ_i}, $i = 1, 2, \cdots$, satisfies the same boundary conditions, then $\lambda_1, \lambda_2, \cdots$ is the complete set of eigenvalues (with corresponding eigenfunctions $y_{\lambda_1}, y_{\lambda_2}, \cdots$, and corresponding normalization constants $\rho_1, \rho_2, \cdots$) for the eigenvalue problem (1). Hence, $\lambda_1, \cdots, \lambda_n$ is the set of *first n* eigenvalues for the eigenvalue problem (2).

In order to determine $A(s)$ and $B(s)$, one first substitutes the form (3) for y_{λ_i} into the expression $y_{\lambda_i}^{(4)}+(Ay_{\lambda_i}^{(1)})^{(1)}+By_{\lambda_i}-\lambda_i y_{\lambda_i}$. Rewriting $\lambda_i y_{\lambda_i}$ as $\lambda_i y_{\lambda_i}= Z_{\lambda_i}^{(4)}(s)+\int_0^s K(s,t)Z_{\lambda_i}^{(4)}(t)\,dt$ and integrating by parts, one can obtain for the entire expression,

$$
\begin{aligned}
&y_{\lambda_i}^{(4)}(s)+(A(s)y_{\lambda_i}^{(1)}(s))^{(1)}+B(s)y_{\lambda_i}(s)-\lambda_i y_{\lambda_i}(s)\\
&\quad=\int_0^s \{K_{ssss}(s,t)-K_{tttt}(s,t)+(A(s)K_s(s,t))_s+B(s)K(s,t)\}Z_{\lambda_i}(t)\,dt\\
&\qquad+Z_{\lambda_i}(s)\left[2\frac{d^3}{ds^3}(K(s,s))+2(K_{tt}-K_{ss})_t|_{t=s}+B(s)+A(s)K_s|_{t=s}\right.\\
&\qquad\qquad\left.+\frac{d}{ds}\{AK(s,s)+2(K_{ss}-K_{tt})|_{t=s}\}\right]\\
&\qquad+Z_{\lambda_i}^{(1)}(s)\left[2(K_{ss}-K_{tt})|_{t=s}+AK(s,s)+\frac{d}{ds}\left\{A+4\frac{d}{ds}K(s,s)\right\}\right]\\
&\qquad+Z_{\lambda_i}^{(2)}(s)\left[A(s)+4\frac{d}{ds}K(s,s)\right]+[-K(s,t)Z_{\lambda_i}^{(3)}(t)+K_t(s,t)Z_{\lambda_i}^{(2)}(t)\\
&\qquad\qquad-K_{tt}(s,t)Z_{\lambda_i}^{(1)}(t)+K_{ttt}(s,t)Z_{\lambda_i}(t)]|_{t=0}.
\end{aligned}\tag{8}
$$

(The same equation may be obtained for $y_{\lambda_i^*}$ by replacing λ_i by λ_i^* on both sides of the above equation.) The contribution at $t=0$ (last bracket term in (8)) from integration by parts, is zero since $K(s,0)=K_t(s,0)=0$ (by Remark 4 following Theorem 3) and since $Z_{\lambda_i}(0)=Z_{\lambda_i}^{(1)}(0)=0$.

Having obtained the expression (8), we can state the following theorem.

THEOREM 4. *Let $K(s,t)$ be the solution of* (4) *given by* (6) *and* (7). *Let $y_{\lambda_i}(s)$ have the form* (3), $i=1,2,\cdots$. *Let*

$$
\begin{aligned}
A(s)&=-4\frac{d}{ds}K(s,s),\\
B(s)&=-AK_s|_{s=t}+2(K_{ss}-K_{tt})_t|_{t=s}-2\frac{d^3}{ds^3}K(s,s).
\end{aligned}\tag{9}
$$

Then, y_{λ_i} satisfies the fourth order differential equation $y_{\lambda_i}^{(4)}+(A(s)y_{\lambda_i}^{(1)}(s)^{(1)})+B(s)y_{\lambda_i}(s)-\lambda_i y_{\lambda_i}(s)=0$, $i=1,2,\cdots$.

Proof of Theorem 4. The proof consists first in showing that the above given choice for $A(s)$ and $B(s)$ insures that each bracketed term, $[\cdot]$, in (8) is equal to zero. The choice for $A(s)$ and $B(s)$ and the fact that $Z_{\lambda_i}(0)=Z_{\lambda_i}^{(1)}(0)=K(s,0)=K_t(s,t)|_{t=0}$, yields that we need only show, given $A(s)$ and $B(s)$ as in (9), that

$$2(K_{ss}-K_{tt})(s,t)|_{t=s}+AK(s,s)=0.$$

This may be shown to be true more easily by establishing the following notation. Let $\mathcal{F}$, Z, $\mathcal{Z}$ be the $1\times 2n$ row vectors

$$\mathcal{F}=(F_1,\cdots,F_n,G_1,\cdots,G_n),$$

$$Z=(Z_{\lambda_i},\cdots,Z_{\lambda_n},Z_{\lambda_1^*},\cdots,Z_{\lambda_n^*}),$$

$$\mathcal{Z}=(Z_{\lambda_i},\cdots,Z_{\lambda_n},-Z_{\lambda_1^*},\cdots,-Z_{\lambda_n^*}).$$

Let Γ be the $2n\times 2n$ matrix $\Gamma=R+A$ where R and A are the matrices

$$R=\begin{bmatrix}\rho_1 & & & & 0\\ & \ddots & \rho_n & & \\ & & & \rho_1^* & \\ 0 & & & \ddots & \rho_n^*\end{bmatrix},\qquad A=\begin{bmatrix}A_{11} & A_{12}\\ -A_{12} & -A_{22}\end{bmatrix}$$

and A_{ij}, $i,j=1,2$, are $n\times n$ matrices and the i,jth components of A_{11},A_{12},A_{22} are $(Z_{\lambda_i},Z_{\frac{1}{2}}l_j)(s)$, $-(Z_{\lambda_i},Z_{\lambda_j^*})(s)$, $(Z_{\lambda_i^*},Z_{\lambda_j^*})(s)$, respectively. Let $\tilde{\Gamma}$, $\tilde{R}$ be the same as Γ and R respectively except that each component in the last n rows has the opposite sign as the corresponding component in Γ or R. (Hence $\tilde{\Gamma}=\tilde{R}+\tilde{A}$ and $\tilde{A}$ is symmetric.) Then, the linear equations of (7) can be written in matrix notation as

$$\mathcal{F}^T=-\Gamma^{-1}\cdot Z^T\quad\text{or}\quad Z^T=-\Gamma\cdot\mathcal{F}^T\quad\text{or}\quad \mathcal{Z}^T=-\tilde{\Gamma}\cdot\mathcal{F}^T.$$

Suitable differentiations yield (noting that $K(s,s)=\mathcal{F}(s)\cdot\mathcal{Z}^T(s)$)

$$(\mathcal{Z}^T)^{(1)}=\tilde{\Gamma}\cdot[-(\mathcal{F}^T)^{(1)}+K(s,s)\mathcal{F}^T],$$

$$(\mathcal{Z}^T)^{(2)}=\tilde{\Gamma}\cdot\Big[-(\mathcal{F}^T)^{(2)}-K(s,s)(-(\mathcal{F}^T)^{(1)}+K(s,s)\mathcal{F}^T)+\Big(\frac{d}{ds}K(s,s)+K_s(s,t)|_{t=s}\Big)\mathcal{F}^T\Big],$$

and hence that

$$2[K_{ss}-K_{tt}]|_{t=s}+AK(s,s)=2[\mathcal{F}^{(2)}\cdot\mathcal{Z}^T-\mathcal{F}\cdot(\mathcal{Z}^T)^{(2)}]-4[\mathcal{F}^{(1)}\cdot\mathcal{Z}^T+\mathcal{F}\cdot\mathcal{Z}^{T(1)}]\mathcal{F}\cdot\mathcal{Z}^T=0.$$

Having shown that all bracketed, $[\cdot]$, terms in (8) are zero, it can now be shown that y_{λ_i} satisfies the ordinary differential equation. This is done by first letting

$$\rho_iP_i=y_{\lambda_i}^{(4)}+(Ay_{\lambda_i}^{(1)})^{(1)}+By_{\lambda_i}-\lambda_iy_{\lambda_i},\qquad i=1,\cdots,n,$$

$$\rho_{i-n}^*P_i=y_{\lambda_{i-n}^*}+(Ay_{\lambda_{i-n}^*}^{(1)})^{(1)}+By_{\lambda_{i-n}^*}-\lambda_{i-n}^*y_{\lambda_{i-n}^*},\qquad i=n+1,\cdots,2n.$$

If P is the $1\times 2n$ column vector with ith component, P_i, then (8) together with the similar equation for $y_{\lambda_i^*}$ yields, after using (6) and the relations $y_{\lambda_i}=-\rho_iF_i$ and $y_{\lambda_i^*}=-\rho_i^*G_i$, $i=1,2,\cdots,n$, that $\Gamma\cdot P=0$. We have already shown (see proof of Theorem 3) that Γ is nonsingular for $0\leqq s\leqq 1$. Hence $P\equiv 0$ and y_{λ_i}, $i=1,2,\cdots,n$, satisfies the fourth order ordinary differential equation

$$y_{\lambda_i^*}^{(4)}+(Ay_{\lambda_i^*}^{(1)})^{(1)}+By_{\lambda_i}-\lambda_iy_{\lambda_i}=0;$$

similarly $y_{\lambda_i^*}$, $i=1,\cdots,n$, satisfies

$$y_{\lambda_i^*}^{(4)}+(Ay_{\lambda_i^*}^{(1)})^{(1)}+By_{\lambda_i^*}-\lambda_i^*y_{\lambda_i^*}=0.$$

This, in turn, implies, with the aid of $y_{\lambda_i}=-\rho_i F_i$ and $y_{\lambda_i^*}=-\rho_i^* G_i$, $i=1,2,\cdots,n$,

$$K_{ssss}(s,t)-K_{tttt}(s,t)+(A(s)K_s(s,t))_s+B(s)K(s,t)=0,$$

and hence, by (8),

$$y_{\lambda_i}^{(4)}(s)+(A(s)y_{\lambda_i}^{(1)}(s))^{(1)}+B(s)y_{\lambda_i}(s)-\lambda_i y_{\lambda_i}(s)=0 \quad \text{for } i=n+1,\cdots.$$

Remark 1. In contrast to § 2, use was made in this section, in particular to obtain expression (8), of the fact that Z_λ satisfies the differential equation $Z_\lambda^{(4)}(s)-\lambda Z_\lambda(s)=0$.

Remark 2. We have been requiring that for every $\lambda>0$,

$$Z_\lambda^{(4)}-\lambda Z_\lambda=0, \qquad 0\leqq s\leqq 1,$$

and

$$Z_\lambda(0)=Z_\lambda^{(1)}(0)=Z_\lambda(1)=0.$$

Further, for the particular values of λ, $\lambda=\lambda_i^*$, $i=1,2,\cdots$, we have $Z_{\lambda_i^*}^{(1)}(1)=0$. With this assumption, the generated functions, y_{λ_i}, $i=1,2,\cdots$, have been shown to satisfy the boundary conditions $y_{\lambda_i}(0)=y_{\lambda_i}^{(1)}(0)=y_{\lambda_i}(1)=y_{\lambda_i}^{(1)}(1)=0$.

These assumptions have made our analysis easier. It is possible, however, to carry out the analysis under more general circumstances. That is, the Z_λ's could be replaced by Y_λ's where for $\lambda>0$, each Y_λ satisfies the equation $Y_\lambda^{(4)}-\lambda Y_\lambda=0$, plus two boundary conditions at $s=0$ and one boundary condition at $s=1$. That is, there exist real α_i, β_i, γ_i, $i=1,2,3,4$, independent of λ and with

$$\sum_{i=1}^4 \alpha_i^2\neq 0, \quad \sum_{i=1}^4 \beta_i^2\neq 0, \quad \sum_{i=1}^4 \gamma_i^2\neq 0,$$

such that $\sum_{i=1}^4 \alpha_i Y_\lambda^{(i-1)}(0)=0=\sum_{i=1}^4 \beta_i Y_\lambda^{(i-1)}(0)$ and $\sum_{i=1}^4 \gamma_i Y_\lambda^{(i-1)}(1)=0$. Furthermore, ρ_i^*, λ_i^*, $i=1,2,3,\cdots$, are replaced by $\tilde\rho_i$, $\tilde\lambda_i$, $i=1,2,\cdots$, respectively, where $\tilde\lambda_i$, $i=1,2,\cdots$, is the entire set of eigenvalues (and $\tilde\rho_i$ the corresponding set of normalization constants) for the eigenvalue problem

$$Y^{(4)}-\lambda Y=0,$$

$$0=\sum_{i=1}^4 \alpha_i Y^{(i-1)}(0)=\sum_{i=1}^4 \beta_i Y^{(i-1)}(0)=\sum_{i=1}^4 \gamma_i Y^{(i-1)}(1)=\sum_{i=1}^4 \delta_i Y^{(i-1)}(1),$$

(here δ_i is real, independent of λ, $i=1,2,3,4$, and $\sum_{i=1}^4 \delta_i^2\neq 0$). Further, it is assumed that each λ_i is simple.

The boundary conditions given above are further restricted so that the above eigenvalue problem is self-adjoint and also so that for all μ, $\lambda>0$, Y_λ and Y_μ satisfy

$$[Y_\mu Y_\lambda^{(3)}-Y_\lambda Y_\mu^{(3)}-Y_\mu^{(1)}Y_\lambda^{(2)}+Y_\mu^{(2)}Y_\lambda^{(1)}]|_{s=0}=0.$$

These conditions insure that λ_i is real, $i=1,2,\cdots$, and that the analogies of Theorems 1, 2 and 3 of § 2 can be proved. Further, the boundary term in (8), with Z_λ being replaced by Y_λ (and λ_i^* being replaced by $\tilde\lambda_i$, $i=1,2,\cdots$), i.e.,

$$[-K(s,t)Y_{\lambda_i}^{(3)}(t)+K_t(s,t)Y_{\lambda_i}^{(2)}(t)-K_{tt}(s,t)Y_{\lambda_i}^{(1)}(t)+K_{ttt}(s,t)Y_{\lambda_i}(t)]|_{t=0}$$

would still be zero. Hence, the analogue of Theorem 4 could be proved; that is, if

$$x_{\lambda_i}(s)=Y_{\lambda_i}(s)+\int_0^s K(s,t)Y_{\lambda_i}(t)\,dt,$$

then there exists $A(s)$ and $B(s)$, independent of λ, such that each x_{λ_i} satisfies $x_{\lambda_i}^{(4)}+(Ax'_{\lambda_i})'+Bx_{\lambda_i}-\lambda_i x_{\lambda_i}=0$, $i=1,2,\cdots$.

The boundary conditions satisfied by x_{λ_i} would not in general be $x_{\lambda_i}(0)=x'_{\lambda_i}(0)=x_{\lambda_i}(1)=x'_{\lambda_i}(1)=0$ but would be related to the four boundary conditions satisfied by x_{λ_i}, $i=1,2,\cdots$. The proof that all x_{λ_i}, $i=1,2,\cdots$, do satisfy the same set of four, self-adjoint, boundary conditions is contained in § 4.

4. Other boundary conditions. It has already been shown that the set of functions $\{y_{\lambda_i}\}_{i=1}^{\infty}$, generated in § 2, form a complete orthogonal set on $0\leqq s\leqq 1$, and that each y_{λ_i} satisfies a differential equation of the form

$$y_{\lambda_i}^{(4)}+(Ay_{\lambda_i}^{(1)})^{(1)}+By_{\lambda_i}-\lambda_i y_{\lambda_i}=0,$$

where A and B are the same for each λ_i. In addition, it has been shown that

$$y_{\lambda_i}(0)=y_{\lambda_i}^{(1)}(0)=y_{\lambda_i}(1)=y_{\lambda_i}^{(1)}(1)=0,$$

that is, that the boundary conditions for y_{λ_i}, $i=1,2,\cdots$, are the same as those for $Z_{\lambda_i}^{*}$, $i=1,2,\cdots$. This was shown directly by examining the form given for y_{λ_i} and the solution $K(s,t)$ of the integral equation (4).

As mentioned at the end of § 3, if the $Z_{\lambda_i^*}$, $i=1,2,\cdots$, were chosen to be solutions of an eigenvalue problem with the same differential equation $Z^{(4)}-\lambda Z=0$ but with more general boundary conditions, and if the analysis of §§ 2 and 3 were repeated with this new set of $Z_{\lambda_i^*}$, $i=1,2,\cdots$, and (also) the appropriate new set of eigenvalues $\lambda_{n+1},\lambda_{n+2},\cdots$ and normalization constants ρ_{n+1}, $\rho_{n+2},\cdots$, then the boundary conditions satisfied by the resultant y_{λ_i}, $i=1,2,\cdots$, would not necessarily be the same as those satisfied by $Z_{\lambda_i^*}$.

One can, however, show that each y_{λ_i} satisfies the same set of self-adjoint boundary conditions. That is, we can prove the following theorem.

THEOREM 5. *If $\{y_{\lambda_i}\}_{i=1}^{\infty}$ is an orthogonal set of $C^{(4)}[0,1]$ functions, and if each y_{λ_i} satisfies a fourth order differential equation of the form*

$$y_{\lambda_i}^{(4)}+(Ay_{\lambda_i}^{(1)})^{(1)}+By_{\lambda_i}-\lambda_i y_{\lambda_i}=0,$$

where $A\in C^1[0,1]$ and $B\in C[0,1]$, A and B are the same real functions for each λ_i, $i=1,2,\cdots$, λ_i is real-valued for each $i=1,2,\cdots$, and $\lambda_1<\lambda_2<\cdots$, then each y_{λ_i} satisfies the same set of four linearly independent, self-adjoint boundary conditions.

Proof. Since $\{y_{\lambda_i}\}_{i=1}^{\infty}$ is an orthogonal set on $0\leqq s\leqq 1$ and since each y_{λ_i} satisfies $y_{\lambda_i}^{(4)}+(Ay_{\lambda_i}^{(1)})^{(1)}+By_{\lambda_i}-\lambda_i y_{\lambda_i}=0$, then for $i\neq j$ we have

$$\begin{aligned}0=\int_0^1(\lambda_i-\lambda_j)y_{\lambda_i}y_{\lambda_j}\,ds&=\int_0^1[y_{\lambda_j}\{y_{\lambda_i}^{(4)}+(Ay_{\lambda_i}^{(1)})^{(1)}+By_{\lambda_i}\}-y_{\lambda_i}\\&\qquad\cdot\{y_{\lambda_j}^{(4)}+(Ay_{\lambda_j}^{(1)})^{(1)}+By_{\lambda_j}\}]\,ds\\&=[y_{\lambda_j}y_{\lambda_i}^{(3)}-y_{\lambda_i}y_{\lambda_j}^{(3)}-y_{\lambda_j}^{(1)}y_{\lambda_i}^{(2)}+y_{\lambda_j}^{(2)}y_{\lambda_i}^{(1)}+A(y_{\lambda_j}y_{\lambda_i}^{(1)}-y_{\lambda_i}y_{\lambda_j}^{(1)})]\Big|_{s=0}^{s=1}\end{aligned}$$

Hence, if each y_{λ_i} satisfies the same set of boundary conditions, then these boundary conditions are self-adjoint [4, p. 189].

To show that the same set of four boundary conditions is satisfied by each y_{λ_i}, let λ_k, λ_1, λ_p, λ_q be arbitrary but distinct real numbers from the set $\{\lambda_i\}_{i=1}^{\infty}$. Then, for each $i = 1, 2, \cdots$, the following set of four linear equations in $y_{\lambda_i}(0)$, $y_{\lambda_i}^{(1)}(0)$, $y_{\lambda_i}^{(2)}(0)$, $y_{\lambda_i}^{(3)}(0)$, $y_{\lambda_i}(1)$, $y_{\lambda_i}^{(1)}(1)$, $y_{\lambda_i}^{(2)}(1)$, $y_{\lambda_i}^{(3)}(1)$, is satisfied:

$$[(-Ay_{\lambda_j}^{(1)} - y_{\lambda_j}^{(3)})y_{\lambda_i} + (Ay_{\lambda_j} + y_{\lambda_j}^{(2)})y_{\lambda_i}^{(1)} + (-y_{\lambda_j}^{(1)})y_{\lambda_i}^{(2)} + (y_{\lambda_j})y_{\lambda_i}^{(3)}]|_{s=0}^{s=1} = 0, \qquad j = k, l, p, q.$$

There are five cases to consider.

Case 1. Suppose for some distinct k, l, p, q that the coefficient matrix has rank 4. Then for these values of k, l, p, q the above four linear equations yield four linearly independent boundary conditions.

Case 2. Suppose that there is no set of distinct k, l, p, q such that the coefficient matrix above has rank 4. Suppose further that there exists some distinct k, l, p, q such that the coefficient matrix has rank 3. Then the above set of linear equations yields three linearly independent boundary conditions, say (without loss) with j replaced by k, l and p. The fourth linearly independent boundary condition comes from the coefficient matrix; we merely replace y_{λ_q} and its derivatives by y_{λ_i} and the corresponding derivatives and set the determinant of the resulting matrix equal to zero.

Case 3. Suppose that there is no distinct set of k, l, p, q such that the coefficient matrix has rank 3 or 4 but that there exists some distinct k, l, p, q such that the coefficient matrix has rank 2. The proof is similar to Case 3.

Case 4. Suppose that there is no distinct set of k, l, p, q such that the coefficient matrix has rank 2, 3 or 4 but there exists some distinct k, l, p, q such that the coefficient matrix has rank 1. The proof is similar to Case 3.

Case 5. Suppose for each distinct k, l, p, q the coefficient matrix has rank 0. This is impossible since then $y_{\lambda_i}^{(j)}(0) = 0, j = 0, 1, 2, 3$, while also $y_{\lambda_i} \neq 0$ in $0 < s < 1$.

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ESTIMATING THE SOLUTIONS OF SLOWLY VARYING RECURSIONS*

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Abstract. If the eigenvalues $\lambda_1, \cdots, \lambda_n$ of an $n \times n$ matrix A are distinct, then the recursion $x_{i+1} = Ax_i$ has n linearly independent solutions, one for each λ_j, such that the solution corresponding to λ_j grows by a factor λ_j per step. In this paper, similar results are obtained for the recursion $x_{i+1} = A_i x_i$ if the eigensystem of A_i changes slowly as a function of i. The conditions about the eigenvalues are relaxed and practicable error bounds are given.

CONTENTS

Quick reader. The reader who does not have the time to read all of the paper will get quite a good idea of what is going on by reading only:

§ 2; § 3;
§ 4 up to Theorem 4.6;
§ 5 up to Theorem 5.2;
§ 6 up to Remark 6.5;
§ 10 up to Theorem 10.4 and Theorem 10.13 (here L_2 is quite a specific set of solutions of (3.5), whose properties appear sufficiently from these theorems);
§ 11, Theorem 11.1;
§ 12, Theorem 12.1;
§ 17, Theorems 17.6 and 17.17;
§ 18.

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Part I. Introduction.

1. Aims. In this paper we study the behavior of the solutions of recursions

$$x_{i+1} = A_i x_i, \quad i \geqq 0, \tag{1.1}$$

where $\{x_i\}$ is a sequence of vectors and $\{A_i\}$ is a sequence of $n \times n$ real or complex matrices such that A_{i+1} is close to A_i for all i. As is well known, (1.1) contains as a special case the $(n+1)$-term linear homogeneous *scalar* recursions (see also Part V).

Often such recursions have solutions of very diverse growth characters. The solutions of fastest growth character are usually called the *dominant* solutions, whereas those of slowest growth character are called the *minimal* or dominated solutions.

This is very well known for the case of a 3-term scalar recursion like

$$u_{i+2} + (2i+2)u_{i+1} - u_i = 0, \qquad i \geqq 0, \tag{1.2}$$

which is satisfied by the rapidly *decreasing* sequence of coefficients of the Fourier–Chebyshev expansion of $\exp(x)$ on $[-1, 1]$, but which has rapidly *increasing* solutions as well.

Our main concern will be to give accurate estimates for the growth character of the different types of solutions. For instance, for (1.2) we will be able to estimate $|u_m/u_k|$ for any m and k and for both types of solutions with an error of only a few percentage points (see § 18, Example II), and this is not at all unusual.

This will be useful, for instance, for estimating *how many terms* of a series expansion (like the Fourier–Chebyshev expansion above) will be necessary in order to achieve a certain accuracy. But it will also be useful for estimating the truncation error of Miller's algorithm (cf. [9], [2]) for *determining* the dominated solutions of recursions like (1.2).[1]

In this paper we shall derive upper and lower bounds for the quotient $\|x_m\|/\|x_k\|$ for the various types of solutions of (1.1) and for any k and m. These bounds will have a ratio close to 1 for any k and m provided that A_i varies slowly enough as a function of i, but we shall also see that if for all i the moduli of the eigenvalues of A_i differ considerably, the sequence $\{A_i\}$ may vary quite rapidly and still be slowly varying in our sense.

Previous research in this area (for an account of which we may refer to [2], [3] and [13]), was mainly concerned with the asymptotic behavior of the various types of solutions of scalar recursions if the asymptotic behavior of the coefficients in the recursion is known, a line of research virtually started with Poincaré's paper [12], in which it is assumed that those coefficients have limits. Of a different nature are Olver's results for upper bounds of solutions of 3-term scalar recursions (cf. [10]

[1] This truncation error has been studied by various authors. Notably, Gautschi (cf. [2, (3.15)]) gave an explicit expression for it, but this depends on the values of u_m/u_k for both the dominant and the dominated solutions, and these were not so easily accessible. Olver (cf. [10]), got around this problem for a special case of Miller's algorithm (viz., normalization on the first term; cf. [10, (2.06) and (5.02)]) in a very elegant way, but his formulas may give considerable overestimates in the more general case (cf. [10, (9.01) and p. 126, top]), whereas Gautschi's formula then still is exact. Obviously, our present paper facilitates the application of Gautschi's formula; but it also has some meaning for Olver's formula (5.02). Finally, in [8] truncation and rounding errors are studied for a Miller-type algorithm for *matrix-vector recursions*, and there again expressions like $|u_m/u_k|$ play a part.

and [11]). And finally there is Schäfke's work [13] and [14], giving upper and lower bounds for the various types of solutions of matrix vector recursions (1.1) if all A_i are close together.

In our work we do not require $\{A_i\}$ to have a certain asymptotic character, nor do we require all A_i to be close together. Yet there is quite some parallelism between our work and Schäfke's, which we only discovered after our work was well under way. We believe, however, our approach to be more practical. Nevertheless we could have made use of a good deal of Schäfke's results. We refrained from this in the belief that a self-contained account in English might be useful; also Schäfke's framework of Abelian groups makes his work less suitable for straightforward application in a linear algebra environment. Nevertheless we realize that we may have been backtracking on Schäfke's steps. Also we have been inspired by his work on more than one occasion.

Similar results as have been obtained here for recursions can be obtained for differential equations. These results will be published separately (cf. [16]).

2. General ideas. In order to show the general idea of the paper, we first look at (1.1) for the case $n=2$ and $A_i=A$ with eigenvalues λ and μ, $|\lambda|>|\mu|$, and corresponding eigenvectors v and w.

Then we know from the well-known power method for determining eigenvalues and eigenvectors of matrices (cf. [18, Chap. 9, § 3]) that, as far as the direction is concerned, $\{x_i\}$ approaches v if x_0 is not a multiple of w. Moreover, in each step the ratio of the components of x_i with respect to w and v is reduced by a factor μ/λ, which therefore serves as a kind of *directional contraction factor*.

Therefore, if A_i does depend on i, with eigenvalues λ_i, μ_i, $|\lambda_i|>|\mu_i|$ and eigenvectors v_i and w_i, if moreover μ_i/λ_i is in some sense small in relation to the speed with which v_i and w_i move along as i increases, and if x_0 is close to v_0, then it is to be expected that $\{x_i\}$ will follow $\{v_i\}$ rather closely, and that $\|x_{i+1}\|/\|x_i\|\approx|\lambda_i|$.

It is more surprising, perhaps, that under similar circumstances there is also a solution $\{x_i\}$ following $\{w_i\}$ rather closely. Indeed, if A_i^{-1} exists for all i, and for given k we consider the sequence $y_i=A_i^{-1}y_{i+1}$, $i=k-1,\cdots,0$ with $y_k\approx w_k$, then according to what has just been said we may expect y_i to be directionally close to w_i (i.e., to have about the same direction) for $i<k$, since now the eigenvalue $1/\mu_i$ corresponding to w_i exceeds the eigenvalue $1/\lambda_i$ of v_i in modulus. Thus, in particular, y_0 will be directionally close to w_0. By doing this for $k=1,2,3,\cdots$, we get a sequence of $y_0(k)$, all directionally close to w_0, and a compactness argument then proves the existence of a vector y_0 which is a starting vector of a solution of (1.1) close to $\{w_i\}$.

Now turning to the case of arbitrary n, we assume that the moduli of the n_1 eigenvalues of A_i which are largest in modulus are well separated from the moduli of the other $n-n_1$ eigenvalues, n_1 independent of i, and that the corresponding invariant subspaces V_i and W_i vary slowly enough as functions of i. We shall then again expect solutions starting close to V_0 to be close to $\{V_i\}$, and we shall also expect the existence of an $(n-n_1)$-dimensional subspace of solutions close to $\{W_i\}$.

These observations set the pattern of our paper: we shall decompose each solution of (1.1) into components lying in $\{V_i\}$ and $\{W_i\}$, where V_i and W_i are reasonably close to invariant subspaces of A_i. Subsequently we shall estimate how close to $\{V_i\}$ or $\{W_i\}$ a solution will be if properly started, and this will then yield the growth character of this solution.

3. Definitions and conventions.

3.1. The space and its norm. For a given natural number n, R will be the n-dimensional real or complex Cartesian space and $\|\cdot\|$ will be a norm on R. We assume this norm to be *absolute*, i.e. $\| |x| \| = \|x\|$, or (equivalently) *monotonic*, i.e., $|x| \leqq |y| \Rightarrow \|x\| \leqq \|y\|$ (cf. [1]). This is particularly so if $\|\cdot\|$ is the linear, Euclidean or supremum norm.

3.2. Partitioning of the space. For given natural numbers n_1 and n_2, $n_1 + n_2 = n$, R_1 denotes the subspace of R consisting of vectors whose *last* n_2 coordinates are 0, and $R_2 \subset R$ consists of the vectors whose *first* n_1 coordinates are 0. Hence

$$R = R_1 \oplus R_2 \tag{3.1}$$

where $\oplus$ denotes the direct sum.

3.3. Partitioning of vectors and matrices. Correspondingly, we shall partition vectors and matrices as

$$x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \qquad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

This will induce norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the n_1- and n_2-dimensional Cartesian space as follows: $\|y\|_1 = \|\binom{y}{0}\|$ if y has n_1 coordinates; $\|z\|_2 = \|\binom{0}{z}\|$ if z has n_2 coordinates. Then, because of § 3.1, if $x \in R$, then $\|x^1\|_1 \leqq \|x\|$ and $\|x^2\|_2 \leqq \|x\|$. The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ induce in a natural way norms for A_{11}, A_{12}, A_{21} and A_{22}.

3.4. Notation of sequences and their elements. If we denote a sequence of vectors in R by x, then the elements of this sequence will be denoted by x_i; thus $x = \{x_i\}_{i \geqq 0}$. Similarly, if $x(k)$ is a sequence in R for any k, then its elements will be denoted by $x_i(k)$.

3.5. The recursion and its solutions. We shall consider the linear homogeneous recursion over R:

$$x_{i+1} = A_i x_i, \qquad i \geqq 0, \tag{3.2}$$

and any sequence x satisfying (3.2) will be called a *solution* of (3.2). The eigenvalues of A_i will be denoted as $\lambda_{1i}, \cdots, \lambda_{ni}$ in order of decreasing modulus.

3.6. The transformed recursion and its solutions. For any sequence $\{T_i\}$ of invertible matrices we write

$$\tilde{A}_i = (T_{i+1}^{-1} T_i)(T_i^{-1} A_i T_i) = \begin{pmatrix} B_i & C_i \\ D_i & E_i \end{pmatrix}, \tag{3.3}$$

where the right-hand side has been partitioned according to § 3.3.

For any solution x of (3.2) we define a sequence y by

$$y_i = T_i^{-1} x_i, \tag{3.4}$$

i.e., the coordinates of y_i are the coordinates of x_i on a basis consisting of columns of T_i (this is the representation referred to at the end of § 2). Then

$$y_{i+1} = \tilde{A}_i y_i. \tag{3.5}$$

By L we denote the n-dimensional linear space of all solutions of (3.5). L_1 will denote the n_1-dimensional linear subspace of L consisting of solutions with $y_0 \in R_1$. A subset L_2 of L will be defined in § 9.

3.7. Bounds for B_i, C_i, D_i and E_i.
In the following, b, $\tilde{b}$, c, d, e and $\tilde{e}$ will be nonnegative sequences such that

$$\begin{aligned} &b_i\|y\|_1 \leqq \|B_i y\|_1 \leqq \tilde{b}_i\|y\|_1, \qquad \|C_i y\|_1 \leqq c_i\|y\|_2, \\ &\|D_i y\|_2 \leqq d_i\|y\|_1, \qquad \tilde{e}_i\|y\|_2 \leqq \|E_i y\|_2 \leqq e_i\|y\|_2 \end{aligned} \tag{3.6}$$

for all vectors y of appropriate dimensions.

Therefore $b_i \leqq \text{g.l.b.}\,(B_i)(\overset{\text{def}}{=} \min_{v\neq 0}\|B_i v\|_1/\|v\|_1)$, $\tilde{b}_i \geqq \|B_i\|$, etc.

3.8. The SV case. As a consequence of the observations at the end of § 2, we shall be very interested in the case that for any i,

(a) T_i is a not-too-skew matrix whose first n_1 columns span approximately an invariant subspace V_i of A_i belonging to $\lambda_{1i}, \cdots, \lambda_{n_1,i}$ and whose last n_2 columns span (approximately) an invariant subspace W_i belonging to $\lambda_{n_1+1,i}, \cdots, \lambda_{ni}$;

(b) $T_{i+1}^{-1}T_i$ is close to the unit matrix, indicating that V_i and W_i vary slowly as functions of i.

On account of (a), (3.4) transforms solutions of (3.2) which are close to $\{V_i\}$ or $\{W_i\}$ into solutions of (3.5) which are close to R_1 or R_2.[2] Hence, in the setting of § 2 (solutions staying close to invariant subspaces), we should expect $\tilde{A}_i$ to have invariant subspaces close to R_1 and R_2.

Now $T_i^{-1}A_iT_i$ is (close to) the block diagonal matrix of A_i with respect to V_i and W_i, and on account of (b), the same will then apply to $\tilde{A}_i$. But this is not enough to guarantee that the invariant subspaces of $\tilde{A}_i$ are close to R_1 and R_2. We see this from the 2×2 matrix

$$\begin{pmatrix} b_i & c_i \\ -d_i & e_i \end{pmatrix}$$

(which in the 2×2 case would be a possibility for $\tilde{A}_i$) where, skipping the indices for the moment, $e \geqq 0$ is close to and less than b, whereas c and d are positive and less than $(b-e)/2$ (hence c and d are small with respect to b and e): then the eigenvalues λ and μ satisfy $e<\mu<(b+e)/2<\lambda<b$, and the corresponding eigenvectors $\binom{e-\lambda}{d}$ and $\binom{-c}{b-\mu}$ obviously are not close to R_1 and R_2 unless c and $d \ll b-e$.

However, if

(c) c_i and $d_i \ll b_i - e_i$ for all i, then it is easily seen, using Gershgorin circles, that all $\tilde{A}_i$ have invariant subspaces close to R_1 and R_2.

---

[2] By saying that a sequence is close to R_1 (or R_2) we obviously mean that any element of it is close to R_1 (or R_2).

The case that (a), (b) *and* (c) *hold will be referred to as the case of slowly varying* $\{A_i\}$, *or the SV case for short. In connection with* (3.5) *alone, we shall speak of the SV case if condition* (c) *holds.*

In order to appreciate what is going on, it will often be useful to think of b_i and e_i as being approximately $|\lambda_{n_1,i}|$ and $|\lambda_{n_1+1,i}|$ respectively, although even in the SV case this is not always realistic, e.g. if $\lambda_{n_1,i}$ and/or $\lambda_{n_1+1,i}$ has nonlinear elementary divisors.

Examples (cf. § 18) will show that the SV case is very common indeed, and even includes sequences $\{A_i\}$ for which $b_i - e_i \to 0$. On the other hand, it will be clear that for a given sequence $\{A_i\}$, the SV case may apply for one value of n_1 and may not for another value.

The reader is asked to bear in mind, however, that the SV case will only be a framework for checking the relevance of our results. The theorems are formulated independently of the SV assumptions, and will have a wider scope.

3.9. Dominant and dominated solutions. We shall say that a solution x of (3.2) or (3.5) *dominates* a solution $\tilde{x}$ if $\|\tilde{x}_i\|/\|x_i\| \to 0$ for $i \to \infty$.

In the case $n = 2$ it is then usual to call $\tilde{x}$ a *minimal* solution and x a dominant solution (cf. [2, p. 25]), and any solution which is not a multiple of $\tilde{x}$ is then automatically a dominant solution.

We shall, however, use the terms dominant and dominated solutions in a looser sense. By a *dominant* (*dominated*) solution of (3.5) we shall mean a solution y such that all y_i are close to R_1 (close to R_2) in some specified sense. Hence dominant solutions in this sense do not necessarily dominate the dominated solutions, although in the SV case we shall expect this to be the case.

The solutions of (3.2) corresponding via (3.4) to the dominant (dominated) solutions of (3.5) will then be referred to as dominant (dominated) solutions of (3.2) (we realize that this definition is not altogether nice since it depends on the choice of $\{T_i\}$).

Part II. The dominant solutions. (For the notion of dominance, see § 3.9.)

4. Dominant solutions of (3.5). Let y be a solution of (3.5). We partition y_i according to § 3.3 and introduce

$$r_i = \|y_i^2\|_2/\|y_i^1\|_1. \tag{4.1}$$

Then from (3.5), (3.3) and (3.6)

$$\begin{aligned} b_i\|y_i^1\|_1 - c_i\|y_i^2\|_2 \leqq \|y_{i+1}^1\|_1 \leqq \tilde{b}_i\|y_i^1\|_1 + c_i\|y_i^2\|_2, \\ \|y_{i+1}^2\|_2 \leqq d_i\|y_i^1\|_1 + e_i\|y_i^2\|_2, \end{aligned} \tag{4.2}$$

and hence, if $c_ir_i < b_i$,

$$r_{i+1} \leqq \frac{d_i + e_ir_i}{b_i - c_ir_i}, \tag{4.3}$$

$$b_i - c_ir_i \leqq \frac{\|y_{i+1}^1\|_1}{\|y_i^1\|_1} \leqq \tilde{b}_i + c_ir_i, \tag{4.4}$$

$$y_{i+1}^1 = (B_i + r_iC_iU_i)y_i^1 \tag{4.5}$$

for some $n_2 \times n_1$ matrix U_i with $\|U_i\|(=\max_{v\neq 0}\|U_i v\|_2/\|v\|_1)=1$. Indeed, for any vector $y \in R$ there exists a matrix U with $\|U\|=1$ such that $\|Uy^1\|=\|y^1\|$ and Uy^1 is a multiple of y^2 (cf. [15, Thm. 6.1]).

For later use, we note that in the SV case we shall expect the factor $b_i - c_i r_i$ to be close to b_i if r_0 is small, for the double reason that $c_i \ll b_i$ and $r_i \ll 1$.

Likewise $B_i + r_i C_i U_i$ will then be close to B_i, which means that y_i^1 is almost transformed by B_i.

We now have the following theorem.

THEOREM 4.6. *Let a nonnegative sequence $\{\tilde{\rho}_i\}$ exist that satisfies*

$$\tilde{\rho}_{i+1} = \frac{d_i + e_i\tilde{\rho}_i}{b_i - c_i\tilde{\rho}_i} \quad \textit{for all } i \geqq 0, \tag{4.7}$$

and let $\{\rho_i\}$ be a majorant of $\{\tilde{\rho}_i\}$ with $\rho_0 = \tilde{\rho}_0$.

Then all solutions y of (3.5) *satisfying $\|y_0^2\|_2 \leqq \rho_0\|y_0^1\|_1$ are close to R_1 in the sense that $\|y_i^2\|_2 \leqq \rho_i\|y_i^1\|_1$ for all i and they satisfy*

$$y_m^1 = \prod_{i=k}^{m-1} (B_i + \rho_i C_i U_i) y_k^1 \tag{4.8}$$

for some sequence of $n_2 \times n_1$ matrices $\{U_i\}$ with $\|U_i\| \leqq 1$.

Proof. After the foregoing, all we have to note is that any solution $\hat{\rho}$ of the recursion in (4.7) with $0 \leqq \hat{\rho}_0 \leqq \rho_0$ is also defined and satisfies $0 \leqq \hat{\rho}_i \leqq \rho_i$ for all $i \geqq 0$. □

COROLLARY 4.9.

$$\prod_{i=k}^{m-1} (b_i - c_i\rho_i) \leqq \frac{\|y_m^1\|_1}{\|y_k^1\|_1} \leqq \prod_{i=k}^{m-1} (\tilde{b}_i + c_i\rho_i) \tag{4.10}$$

and

$$\frac{1}{1+\rho_k} \prod_{i=k}^{m-1} (b_i - c_i\rho_i) \leqq \frac{\|y_m\|}{\|y_k\|} \leqq (1+\rho_m) \prod_{i=k}^{m-1} (\tilde{b}_i + c_i\rho_i) \tag{4.11}$$

unless $y_k = 0$, in which case also $y_{k+1} = y_{k+2} = \cdots = 0$. The estimates from below should be omitted when they contain negative factors (*but this will never happen if $\rho_i = \tilde{\rho}_i$*).

COROLLARY 4.12. *In the important case $n_1 = 1$, we may take $b_i = \tilde{b}_i = |(\tilde{A}_i)_{11}|$. Then* (4.10) *and* (4.11) *reduce to*

$$\frac{|y_m^1|}{|y_k^1|} = \prod_{i=k}^{m-1} (b_i + \theta_i c_i \rho_i) \quad \textit{with } |\theta_i| \leqq 1, \tag{4.10'}$$

$$\frac{\|y_m\|}{\|y_k\|} = \frac{1+\eta_m\rho_m}{1+\eta_k\rho_k} \prod_{i=k}^{m-1} (b_i + \theta_i c_i \rho_i) \quad \textit{with } |\theta_i| \leqq 1, \quad 0 \leqq \eta_i \leqq 1. \tag{4.11'}$$

Remark 4.13. If the conditions of Theorem 4.6 are satisfied for a certain choice of $\{b_i\}$ related to $\{A_i\}$ according to § 3.7 (i.e., $b_i \leqq |(\tilde{A}_i)_{11}|$ if $n_1 = 1$), then they are certainly satisfied for the choice in this corollary, since this means replacing the b_i by quantities which are not less.

Remark 4.14. For $\{\rho_i\}$ we may take any nonnegative solution of (4.7), if one exists.

Remark 4.15. For later use, we note that if the assumptions of Theorem 4.6 hold, then the estimates (4.10) and (4.11) apply in particular to the elements of the set L_1 defined in § 3.6, and the sharpest results are then obtained if $\{\rho_i\}=\{\tilde{\rho}_i\}$ with $\tilde{\rho}_0=0$ (which is obviously allowed for $y \in L_1$).

5. Determination of sequences $\{\rho_i\}$. In order to apply Theorem 4.6 in a quantitative sense, we must have sequences $\{\rho_i\}$ at our disposal. Such sequences will be derived in § 15, where it will become clear that the *characteristic equation*

$$c_i t^2-(b_i-e_i)t+d_i=0 \tag{5.1}$$

of (4.7) and its roots α_i, β_i, $0 \leqq \alpha_i \leqq \beta_i$, which will be assumed to exist, are of importance. In the SV case those roots wilt certainly exist, and $\alpha_i \approx d_i/(b_i-e_i)$, $\beta_i \approx (b_i-e_i)/c_i$. Hence in the SV case, α_i will be quite small and β_i will be quite large.

Therefore the following results are particularly useful in the SV case. For proofs, more results, more detail and comment, we refer to § 15.

THEOREM 5.2. *If $e_i<b_i$ for all i and* $\sup \alpha_j \leqq \inf \beta_j$, *then* $\rho_0=\alpha_0$, $\rho_{i+1}=\sup_{j \leqq i} \alpha_j$ *suffices in Theorem* 4.6. (cf. (15.12)).

COROLLARY 5.3. *In the SV case, $\tilde{\rho}_0=0$ allows all ρ_i to be small.*

THEOREM 5.4. *If $e_i<b_i$ for all i and* $\sup \alpha_j \leqq a \leqq \inf \beta_j$, *then $\rho_i=a$ suffices in Theorem* 4.6. *More generally, this is true for $\rho_0=a$,*

$$\rho_i=\sup \alpha_j+(a-\sup \alpha_j)\prod_{j=0}^{i-1}\left(\frac{e_j/b_j+\alpha_j/\beta_j}{1-a/\beta_j}\right) \tag{5.5}$$

(cf. Theorem 15.16).

THEOREM 5.6. *If we define $\alpha_{-1}=\alpha_0$, if $e_i<b_i$ and $\alpha_i/\alpha_{i-1}>e_i/b_i$ for all i, then $\rho_0=\tilde{a}\alpha_0$, $\rho_{i+1}=\tilde{a}\alpha_i$ suffices in Theorem* 4.6 *when*

$$2 \sup \frac{1-e_j/b_j}{1-(\alpha_{j-1}/\alpha_j)(e_j/b_j)} \leqq \tilde{a} \leqq \frac{1}{2} \inf \frac{\beta_j}{\alpha_j} \frac{\alpha_j/\alpha_{j-1}-e_j/b_j}{1-e_j/b_j} \qquad \text{(cf. (15.23))}. \tag{5.7}$$

Theorems 5.2 and 5.4 are useful if the sequence $\{\alpha_i\}$ does not tend to zero, and (5.5) says in fact how fast $\{\tilde{\rho}_i\}$ (cf. (4.7)) may approach $\sup \alpha_j$ if $\tilde{\rho}_0$ is not small at all. This shows that in the SV case, this approach is very fast if e_i/b_i is small for all i, since α_i/β_i is small, whereas if e_i/b_i is not small, then the approach is approximately as $\prod_0^{m-1}(e_i/b_i)$, as is to be expected from § 2 and § 3.8. As a corollary of Theorem 5.4, we have the following.

COROLLARY 5.8. *Let the conditions of Theorem* 5.4 *be satisfied. If all factors in $\prod$ in* (5.5) *are at most $s<1$, then with $\rho_0=a$,*

$$\|y_m\| \leqq [1+\sup \alpha_j+(\rho_0-\sup \alpha_j)s^m]e^{s^k/(1-s)}\prod_{i=k}^{m-1}(\tilde{b}_i+c_i \sup \alpha_j)\|y_k\|. \tag{5.9}$$

We note that the factor $[\cdot]$ will soon be close to $1+\sup \alpha_j$, and that, even for $k=0$, the factor $\exp(s^k/(1-s))$ will be very modest unless $s \approx 1$, whereas the important factor $\prod$ does not depend on ρ_0 at all.

Theorem 5.6 is useful if $\{\alpha_i\}$ does tend to zero or, indeed, has any behavior provided it doesn't vary too quickly. The reader should note that in the SV case $\tilde{a} \approx 2$ will be allowed if α_j/α_{j-1} is close enough to 1. This implies, e.g., that writing the right-hand side of (4.10′) as $\prod[b_i(1+\theta_i')]$ we have $|\theta_i'| \leqq c_i\rho_i/b_i \approx c_i\alpha_i/b_i \approx c_id_i/b_i(b_i-e_i)$ as the relative uncertainty in the growth factor per step.

6. Dominant solutions of (3.2). We now consider the solutions of the original equation (3.2) corresponding (via (3.4)) to the dominant solutions of (3.5). In order to obtain estimates for those solutions we partition all T_i:

$$T_i = (T_i^1 \vdots T_i^2), \tag{6.1}$$

where T_i^1 consists of n_1 columns, T_i^2 of n_2 columns.

Then

$$\frac{\|x_m\|}{\|x_k\|} = \frac{\|T_m^1 y_m^1 + T_m^2 y_m^2\|}{\|T_k^1 y_k^1 + T_k^2 y_k^2\|}. \tag{6.2}$$

The case $n_1 = 1$. Then T_i^1 just consists of the first column of T_i. We now have the following theorem immediately from (4.10′).

THEOREM 6.3. *Let $n_1 = 1$. Assume $b_i = \tilde{b}_i = |(\tilde{A}_i)_{11}|$ and let the assumptions of Theorem 4.6 be satisfied. Then all solutions of (3.2) satisfying $x_0 = uT_0^1 + T_0^2 v$ (u scalar) with $\|v\|_2 \leqq \rho_o|u|$ (i.e., in particular if $x_0 = uT_0^1$) satisfy*

$$\frac{\|x_m\|}{\|x_k\|} = \frac{1+\eta_m\rho_m}{1+\eta_k\rho_k} \frac{\|T_m^1\|}{\|T_k^1\|} \prod_{i=k}^{m-1} (b_i + \theta_i c_i \rho_i), \qquad |\theta_i| \leqq 1, \quad |\eta_i| \leqq \frac{\|T_i^2\|}{\|T_i^1\|} \tag{6.4}$$

unless $x_k = 0$. Here $\|T_i^2\| = \max_{v \neq 0} \|T_i^2 v\|/\|v\|_2$.

Remark 6.5. We note that in the SV case the terms $\theta_i c_i \rho_i$ are minor correction terms and the same holds for $\eta_i \rho_i$ provided that $\|T_i^2\|$ is not too much larger than $\|T_i^1\|$ (cf. the not-too-skew clause in § 3.8(a)). In this case, then, the uncertainty in estimating $\|x_m\|/\|x_k\|$ is only slightly greater than that in estimating $\|y_m\|/\|y_k\|$ (cf. (4.11′)), and the extent to which it is greater hardly depends on the number of iteration steps. This means that our theory is just as applicable to recursions whose solutions have limits or stay in some limited space angle as it is to recursions whose solutions wander freely about, if only not too fast.

Remark 6.6. We also note that in the case just considered,

$$(\|T_m^1\|/\|T_k^1\|) \prod_{i=k}^{m-1} b_i \tag{6.7}$$

will be a very good approximation of the growth factor of the iterated vector, and Theorem 6.3 enables us to estimate how good.

In § 2 we conjectured in fact that, λ_{1i} denoting the eigenvalue of A_i of largest modulus, $\prod_{i=k}^{m-1} |\lambda_{1i}|$ would be about the growth factor mentioned in the previous paragraph. Now, if T_i^1 is eigenvector to λ_{1i} and the columns of T_i^2 span an invariant subspace of A_i, and if we define

$$\tau_i = \text{first row of } T_i^{-1}, \tag{6.8}$$

then from (3.3):

$$b_i = |\tau_{i+1} T_i^1| |\lambda_{1i}|. \tag{6.9}$$

Hence (6.7) equals

$$\prod_{i=k}^{m-1} | \|T_{i+1}^1\| \tau_{i+1} T_i^1 / \|T_i^1\| | \prod_{i=k}^{m-1} |\lambda_{1i}| \tag{6.10}$$

and the factor preceding $\prod |\lambda_{1i}|$ clearly is independent of the special choice of T_i^1 (a scalar factor being the only degree of freedom since T_i^1 is eigenvector). Hence

$$(\|T_m^1\|/\|T_k^1\|)\prod_{i=k}^{m-1} b_i = \prod_{i=k}^{m-1} |f_i| \prod_{i=k}^{m-1} |\lambda_{1i}|, \tag{6.11}$$

where f_i denotes the element in the left upper corner of $T_{i+1}^{-1}T_i$ if the matrices T_i are so chosen that $\|T_i^1\|=1$ for all i.

This enables us to estimate how far (6.7) may be away from $\prod_{i=k}^{m-1}|\lambda_{1i}|$ if the T_i have been chosen as indicated, and it is clear that the f_i will be close to 1 if the eigensystem of $\{A_i\}$ does not vary too fast.

Examples. Numerical examples illustrating how remarkably well $\prod_{i=k}^{m-1}|\lambda_{1i}|$ approximates the growth of the iterated vector are given in Part V.

The case of arbitrary n_1. Here (4.6) and (4.10) yield the following.

THEOREM 6.12. *Under the assumptions of Theorem* 4.6, *we have: if* $x_0 = T_0^1 u + T_0^2 v$ *with* $\|v\|_2/\|u\|_1 \leqq \rho_0$, *then*

$$\begin{aligned}\frac{\text{g.l.b.}(T_m^1)-\|T_m^2\|\rho_m}{\|T_k^1\|+\|T_k^2\|\rho_k}\prod_{i=k}^{m-1}(b_i-c_i\rho_i) &\leqq \frac{\|x_m\|}{\|x_k\|} \\ &\leqq \frac{\|T_m^1\|+\|T_m^2\|\rho_m}{\text{g.l.b.}(T_k^1)-\|T_k^2\|\rho_k}\prod_{i=k}^{m-1}(\tilde{b}_i+c_i\rho_i).\end{aligned} \tag{6.13}$$

Here $\text{g.l.b.}(T_i^1)=\min_{u\neq 0}\|T_i^1 u\|/\|u\|_1$. *The estimate from below should be omitted when it contains negative factors.*

Remark 6.14. If the columns of all T_i have about unit norm, and if the columns of any T_i are not too nearly dependent, then in the SV case the factors in (6.13) preceding the expressions $\prod_{i=k}^{m-1}$ will be of the order of unity since the ρ_i are small, and therefore these expressions bound $\|x_m\|/\|x_k\|$ apart from a very modest factor which can be estimated.

7. Generalizations. If the moduli of the n_1 (in modulus) largest eigenvalues of A_i are well separated from the moduli of the other eigenvalues, and the corresponding invariant subspaces are slowly varying as functions of i, then the requirement $b_i > e_i$ will not be harsh if A_i is not too close to a defective matrix (cf. § 3.8, $b_i \approx \lambda_{n_1,i}$, $e_i \approx \lambda_{n_1+1,i}$). But if A_i is (close to) a defective matrix, then rather skew matrices T_i may be required to establish $b_i > e_i$ (cf. the reduction of a matrix to a kind of Jordan normal form with ε on the codiagonal instead of ones) and this may have annoying consequences for c_i and d_i.

For such cases, the theorems in this section may be useful. In these theorems, two or all three of the following assumptions will be invoked:

$$\left\|\left(\prod_{j=k}^{m} B_j\right)^{-1}\right\| \leqq f\left(\prod_{j=k}^{m} b_j\right)^{-1} \quad \text{for all } m \geqq k, \tag{7.1}$$

$$\left\|\prod_{j=k}^{m} B_j\right\| \leqq \tilde{f}\prod_{j=k}^{m} \tilde{b}_j \quad \text{for all } m \geqq k, \tag{7.2}$$

$$\left\|\prod_{j=k}^{m} E_j\right\| \leqq g\prod_{j=k}^{m} e_j \quad \text{for all } m \geqq k. \tag{7.3}$$

Here $\prod_{j=k}^{m} B_j$ denotes $B_m B_{m-1} \cdots B_k$, and similarly for $\prod_{j=k}^{m} E_j$; we assume the numbers $f \geqq 1$, $\tilde{f} \geqq 1$, $g \geqq 1$, $b_i > 0$, $\tilde{b}_i$ and e_i to be independent of k and m, and b_i, $\tilde{b}_i$ and e_i need not be as defined in § 3.7.

Then, generalizing Theorem 4.6, we have the following two theorems.

THEOREM 7.4. *Assume* (7.1) *and* (7.3). *Let a nonnegative sequence* $\{\tilde{\rho}_i\}$ *exist that satisfies*

$$\tilde{\rho}_{i+1} = \frac{gd_i + e_i\tilde{\rho}_i}{b_i - fc_i\tilde{\rho}_i} \quad \text{for all } i \geqq 0 \tag{7.5}$$

and let $\{\rho_i\}$ *be a majorant of* $\{\tilde{\rho}_i\}$ *with* $\rho_0 = \tilde{\rho}_0$.

Then all solutions y of (3.5) *with* $\|y_0^2\| \leqq \rho_0 \|y_0^1\|/g$ *satisfy*

$$\|y_i^2\| \leqq f\rho_i \|y_i^1\|, \tag{7.6}$$

$$y_m^1 = \prod_{i=k}^{m-1} (B_i + f\rho_i C_i U_i) y_k^1, \qquad \|U_i\| \leqq 1, \tag{7.7}$$

$$\frac{\|y_m^1\|}{\|y_k^1\|} \geqq \frac{1}{f} \prod_{i=k}^{m-1} (b_i - fc_i\rho_i), \tag{7.8}$$

the latter provided that $b_i - fc_i\rho_i \geqq 0$ *for all i.*

If (7.2) *is also satisfied, then*

$$\frac{\|y_m^1\|}{\|y_k^1\|} \leqq \tilde{f} \prod_{i=k}^{m-1} (\tilde{b}_i + f\tilde{f}c_i\rho_i). \tag{7.9}$$

THEOREM 7.10. *Assume* (7.2) *and* (7.3). *Define a sequence* $\{\tilde{\rho}_i\}$ *by*

$$\tilde{\rho}_{i+1} = \frac{gd_i + e_i\tilde{\rho}_i}{\tilde{b}_i + \tilde{f}c_i\tilde{\rho}_i}, \qquad i \geqq 0, \tag{7.11}$$

if $\tilde{f}gc_id_i \leqq \tilde{b}_ie_i$ *for all* $i \geqq 0$, *and by*

$$\tilde{\rho}_{i+1} = (gd_i + e_i\tilde{\rho}_i)/\tilde{b}_i, \qquad i \geqq 0, \tag{7.12}$$

otherwise, and let $\{\rho_i\}$ *be a majorant of* $\{\tilde{\rho}_i\}$ *with* $\rho_0 = \tilde{\rho}_0$.

Then all solutions y of (3.5) *with* $\|y_0^2\| \leqq \tilde{f}\rho_0 \|y_0^1\|/g$ *satisfy*

$$\|y_m^1\| \leqq \tilde{f} \prod_{i=0}^{m-1} (\tilde{b}_i + \tilde{f}c_i\rho_i) \|y_0^1\|, \tag{7.13}$$

$$\|y_m^2\| \leqq \rho_m \tilde{f} \prod_{i=0}^{m-1} (\tilde{b}_i + \tilde{f}c_i\rho_i) \|y_0^1\|. \tag{7.14}$$

Unfortunately, for $k = 0$, (7.9) is somewhat cruder than (7.13). We have been unable to remedy this.

Before giving the proofs, we note that (similar to Theorem 5.2), we have (cf. § 16) the following.

THEOREM 7.15. *If* $e_i \leqq \tilde{b}_i$ *and* $\tilde{f}gc_id_i \leqq b_ie_i$ *for all i, and* α_i *denotes the* (*always existing*) *positive root of* $c_it^2 + (b_i - e_i)t - d_i = 0$, *then* $\rho_0 = \alpha_0$, $\rho_{i+1} = \sup_{j \leqq i} \alpha_j$ *suffices in* (7.13) *and* (7.14). *The same holds for* $\rho_i = a$, *where* $a \geqq \sup \alpha_j$.

In preparation of the proofs of Theorems 7.4 and 7.10, we note the following three lemmas.

LEMMA 7.16. *Assume* (7.3). *Let, for any given* $p \geqq 0$, *quantities* $v_i (i \geqq p)$ *be defined by*

(7.17)
$$v_{i+1} = g\left\{d_i\|y_i^1\| + e_i d_{i-1}\|y_{i-1}^1\| + \cdots + \left(\prod_{p+1}^{i} e_j\right) d_p\|y_p^1\| + \left(\prod_{p}^{i} e_j\right)\|y_p^2\|\right\},$$
$$v_p = g\|y_p^2\|.$$

Then

(7.18) $$\|y_i^2\| \leqq v_i, \qquad i \geqq p$$

(7.19) $$v_{i+1} = g d_i \|y_i^1\| + e_i v_i, \qquad i \geqq p.$$

Proof.

$$y_{i+1}^2 = D_i y_i^1 + E_i D_{i-1} y_{i-1}^1 + \cdots + \left(\prod_{p+1}^{i} E_j\right) D_p y_p^1 + \left(\prod_{p}^{i} E_j\right) y_p^2. \quad \square$$

LEMMA 7.20. *Assume* (7.2) *and* (7.3). *Let, for any given* $p \geqq 0$, *quantities* $\tilde{u}_i (i \geqq p)$ *be defined by*

(7.21)
$$\tilde{u}_{i+1} = \tilde{f}\left\{c_i v_i + \tilde{b}_i c_{i-1} v_{i-1} + \cdots + \left(\prod_{p+1}^{i} \tilde{b}_j\right) c_p v_p + \left(\prod_{p}^{i} \tilde{b}_j\right)\|y_p^1\|\right\},$$
$$\tilde{u}_p = \tilde{f}\|y_p^1\|,$$

where the v_i *are according to* (7.17). *Then*

(7.22) $$\|y_i^1\| \leqq \tilde{u}_i, \qquad i \geqq p$$

(7.23) $$\tilde{u}_{i+1} = \tilde{b}_i \tilde{u}_i + \tilde{f} c_i v_i, \qquad i \geqq p.$$

Proof.

$$y_{i+1}^1 = C_i y_i^2 + B_i C_{i-1} y_{i-1}^2 + \cdots + \left(\prod_{p+1}^{i} B_j\right) C_p y_p^2 + \left(\prod_{p}^{i} B_j\right) y_p^1. \quad \square$$

LEMMA 7.24. *Assume* (7.1) *and* (7.3). *For any given* $p \geqq 0$ *and* $q > 0$, $p < q$, *let quantities* $u_i (p \leqq i \leqq q)$ *be defined by*

(7.25)
$$u_i = f\left\{b_i^{-1} c_i v_i + b_i^{-1} b_{i+1}^{-1} c_{i+1} v_{i+1} + \cdots + \left(\prod_{i}^{q-1} b_j\right)^{-1} c_{q-1} v_{q-1} + \left(\prod_{i}^{q-1} b_j\right)^{-1} \|y_q^1\|\right\},$$
$$u_q = f\|y_q^1\|,$$

where the v_i *are given by* (7.17). *Then*

(7.26) $$\|y_i^1\| \leqq u_i, \qquad p \leqq i \leqq q$$

(7.27) $$u_{i+1} = b_i u_i - f c_i v_i, \qquad p \leqq i \leqq q-1.$$

Proof.

$$y_i^1 = -B_i^{-1} C_i y_i^2 - B_i^{-1} B_{i+1}^{-1} C_{i+1} y_{i+1}^2 - \cdots - \left(\prod_{i}^{q-1} B_j\right)^{-1} C_{q-1} y_{q-1}^2 + \left(\prod_{i}^{q-1} B_j\right)^{-1} y_q^1$$

(this follows from the formula in the proof of Lemma 7.20 by the substitution $p \to i$, $i \to q-1$). $\square$

Proof of Theorem 7.4. We apply Lemma 7.16 with $p=0$ and Lemma 7.24 with $q=m$. Then (7.19) and (7.26) imply

$$v_{i+1} \leqq g d_i u_i + e_i v_i, \qquad 0 \leqq i \leqq m-1. \tag{7.28}$$

Together with (7.27) this yields

$$v_{i+1}(b_i u_i - f c_i v_i) \leqq u_{i+1}(g d_i u_i + e_i v_i). \tag{7.29}$$

Defining $t_i = v_i/u_i$, we obtain

$$t_{i+1} \leqq \frac{g d_i + e_i t_i}{b_i - f c_i t_i}, \qquad t_0 = \frac{v_0}{u_0} = \frac{g\|y_0^2\|}{u_0} \leqq g\frac{\|y_0^2\|}{\|y_0^1\|} \leqq \rho_0.$$

Hence $t_i \leqq \rho_i$, and since $u_{i+1} = (b_i - f c_i t_i)u_i$, we obtain

$$\|y_m^1\| = \frac{u_m}{f} = \frac{1}{f}\prod_k^{m-1}(b_i - f c_i t_i)u_k,$$

implying (7.8).

Also, $\|y_m^2\|/\|y_m^1\| \leqq f v_m/u_m \leqq f\rho_m$, implying (7.6) for $i=m$, but then for any i since m is arbitrary. Now (7.7) is obvious.

In order to prove the last assertion of Theorem 7.4, we expand the product in (7.7):

$$\prod_{i=k}^{m-1}(B_i + f\rho_i C_i U_i) = \prod_{i=k}^{m-1} B_i + \sum_{k \leqq i_1 \leqq m-1}\left(\prod_k^{i_1-1} B_i\right)(f\rho_{i_1} C_{i_1} U_{i_1})\left(\prod_{i_1+1}^{m-1} B_i\right) + \cdots,$$

which in norm is less than or equal to

$$\tilde f \prod_{i=k}^{m-1}\tilde b_i + \sum_{k \leqq i_1 \leqq m-1}\left(\tilde f \prod_k^{i_1-1}\tilde b_i\right)(f\rho_{i_1} c_{i_1})\left(\tilde f \prod_{i_1+1}^{m-1}\tilde b_i\right) + \cdots = \tilde f \prod_{i=k}^{m-1}(\tilde b_i + f\tilde f \rho_i c_i). \qquad \square$$

Proof of Theorem 7.10. We apply Lemmas 7.16 and 7.20 with $p=0$. Then from (7.19) and (7.22) we have

$$v_{i+1} \leqq g d_i \tilde u_i + e_i v_i \quad \text{for all } i \geqq 0. \tag{7.30}$$

Defining $t_i = v_i/\tilde u_i$, we obtain from (7.30) and (7.23)

$$t_{i+1} \leqq \frac{g d_i + e_i t_i}{\tilde b_i + \tilde f c_i t_i}, \qquad t_0 = \frac{g\|y_0^2\|}{\tilde f\|y_0^1\|} \leqq \rho_0, \tag{7.31}$$

hence

$$t_{i+1} \leqq (g d_i + e_i t_i)/\tilde b_i \tag{7.32}$$

and $t_i \leqq \rho_i \Rightarrow t_{i+1} \leqq \rho_{i+1}$ in either case, implying (7.13) and (7.14). $\square$

8. Inhomogeneous recursions. For an inhomogeneous recursion

$$y_{i+1} = \tilde A_i y_i + g_i, \tag{8.1}$$

$\tilde A_i$ as in (3.5), we have

$$y_{i+1} = \left(\prod_0^i \tilde A_j\right) y_0 + g_i + \tilde A_i g_{i-1} + \cdots + \left(\prod_1^i \tilde A_j\right) g_0. \tag{8.2}$$

The previous theory enables us to obtain quite realistic estimates for the terms $(\prod_{k+1}^{i} \tilde{A}_j)g_k$, even if the vectors g_k are not close to R_1. Just as in the latter case Theorem 5.4 and its Corollary 5.8 are useful.

In the particular case $n_1 = 1$ (if this is allowed), our theory actually says that in (8.2) all but the first few of the terms g_i, $\tilde{A}_i g_{i-1}, \cdots, \prod_1^i \tilde{A}_j)g_0$ are close to the first basis vector, and grow according to $\prod \lambda_{1j}$. In the inhomogeneous case corresponding to (1.1), we have similarly that the inhomogeneous term g_k at stage k of the recursion gives a contribution to x_i $(i > k)$ which has about the direction of the eigenvector corresponding to λ_{1i} and grows like $\prod_{k+1}^{i} \lambda_{1j}$.

This kind of argument may give rise, for instance, to a more straightforward and geometric proof of Henrici's result for the global discretization error in the numerical solution of differential equations (cf. [5]).

Another way to deal with vectors g_k which are not close to R_1 is to write $g_k = u_k + v_k$, where u_k is the starting vector of a dominating solution $\{u_i\}_{i \geqq k}$ and v_k is the starting vector of a dominated solution $\{v_i\}_{i \geqq k}$ (see the subsequent part of the theory). Then Theorems 4.6 and 10.4 give estimates for u_i and v_i and thus for $(\prod_{k+1}^{i} \tilde{A}_j)g_k$. We note that there is a close relationship with the decompositions used in [17].

Part III. The dominated solutions. (For the notion of dominance, see § 3.9.)

9. The set L_2 and its properties. Central to our theory is a set L_2 of solutions of (3.5) which is introduced and whose structure is studied in this section. In the next section we shall estimate the solutions belonging to this set.

DEFINITION 9.1. The sequence $z(0), z(1), z(2), \cdots$, where each $z(k)$ itself is a sequence $\{z_i(k)\}_{i \geqq 0}$, is said to converge to a sequence z if $\lim_{k\to\infty} z_i(k) = z_i$ for all i, and then z is called the limit. A sequence z will be called a limiting sequence of the sequence $z(0), z(1), z(2), \cdots$ if it is the limit of a subsequence of the latter sequence.

LEMMA 9.2. *If* $\lim_{k\to\infty} z(k) = z$ *and* $\lim_{k\to\infty} w(k) = w$, *then* $\lim_{k\to\infty} [z(k) + w(k)] = z + w$, *but if z and w are only limiting sequences of* $\{z(k)\}_{k \geqq 0}$ *and* $\{w(k)\}_{k \geqq 0}$, *respectively, then $z + w$ need not be a limiting sequence of* $\{z(k) + w(k)\}_{k \geqq 0}$.

LEMMA 9.3. *For reasons of continuity, any limiting sequence of a sequence of solutions of* (3.5) *is again a solution.*

LEMMA 9.4. *In order that a solution z of* (3.5) *is a limiting sequence of the sequence* $z(0), z(1), z(2), \cdots$ *of solutions, it is necessary and sufficient that z_0 is a limit point of the sequence* $\{z_0(k)\}_{k \geqq 0}$.

DEFINITION 9.5. Let M_k, k any natural number, denote the set of solutions z of (3.5) with $z_k^1 = 0$. Then L_2 is the set of all solutions of (3.5) occurring as limiting sequences of sequences $z(0), z(1), z(2), \cdots$ where $z(k) \in M_k$.

Remark 9.6. The mechanism used in defining L_2 clearly corresponds to the mechanism in § 2 for defining the solutions staying directionally close to $\{w_i\}$, but we have not assumed the existence of A_i^{-1}.

Remark 9.7. Although each M_k is a linear subspace of L (the space of all solutions of (3.5)), this does not necessarily hold for L_2. We illustrate this by the following example. Take $\tilde{A}_{2k} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\tilde{A}_{2k+1} = A_{2k}^{-1}$. With $z_0 = \begin{pmatrix} a \\ b \end{pmatrix}$, we get $z_{2k} =$

$\binom{a}{b}$, $z_{2k+1}=\binom{a+b}{b}$. Hence $z(2k)\in M_{2k}\Leftrightarrow a=0\Leftrightarrow z_0(2k)=\binom{0}{b}$; $z(2k+1)\in M_{2k+1}\Leftrightarrow a=-b\Leftrightarrow z_0(2k+1)=\binom{-b}{b}$. This implies that the limit points of $\{z_0(k)\}_{k\geqq 0}$ have only two possible types: $\binom{0}{b}$ and $\binom{-b}{b}$. Since a linear combination of two points of these types need not have one of these types, L_2 is not a linear space.

We summarize a number of important properties of L_2 in the following theorem, where the reader may note that condition (A) says, in fact, that all $z\in M_k$ have z_0 close to R_2 in a certain sense.

THEOREM 9.8. *Assume*

(A) *There exists a number σ_0 such that for any k and any $z\in M_k$ it is true that $\|z_0^1\|\leqq\sigma_0\|z_0^2\|$.*

Then

(a) *All solutions $z\in L_2$ satisfy $\|z_0^1\|\leqq\sigma_0\|z_0^2\|$.*

(b) *If there exists a sequence $\{\sigma_i\}$ such that for any k and any $z\in M_k$ it is true that $\|z_i^1\|\leqq\sigma_i\|z_i^2\|$ if $i\leqq k$, then all solutions $z\in L_2$ satisfy $\|z_i^1\|\leqq\sigma_i\|z_i^2\|$ for all i.*

(c) *Any solution $y\neq 0$ in L_1 has $y_i^1\neq 0$ for all i.*

(d) *L_2 contains at least one subset $\tilde{L}_2$ with the following properties* (e)–(h):

(e) *$\tilde{L}_2$ is an n_2-dimensional linear subspace of L.*

(f) *For any $a\in R$ there exists exactly one $z\in\tilde{L}_2$ such that $z_0^2=a^2$.*

(g) *The relation between a^2 and z in* (f) *is linear.*

(h) *$L=L_1\oplus\tilde{L}_2$ (direct sum).*

(k) *L_2 is the union of subsets $\tilde{L}_2$ having the properties mentioned in* (e)–(h).

If in addition to (A), *also* (B) *or* (B′):

(B) $\lim_{i\to\infty}\|z_i^1\|/\|y_i^1\|=0$ *for any $z\in L_2$ and any $y\in L_1$, $y\neq 0$,*

(B′) $\lim_{i\to\infty}\|z_i\|/\|y_i\|=0$ *for any $z\in L_2$ and any $y\in L_1$, $y\neq 0$,*

then

(l) *L_2 itself has the properties of $\tilde{L}_2$ in* (e)–(h).

(m) *The solution in* (f) *is the limit of the sequence $\{z(k)\}_{k\geqq 0}$, $z(k)\in M_k$, $z_0^2(k)=a^2$.*

Proof. By a limit argument, (a) and (b) follow trivially from (A) and the definition of L_2.

Assertion (c) follows from the observation that $y_i^1=0\Rightarrow y\in M_i\Rightarrow\|y_0^1\|\leqq\sigma_0\|y_0^2\|\Rightarrow y=0$ since $y_0^2=0$ in L_1.

In order to prove (d)–(h), we define $P_k=\tilde{A}_{k-1}\tilde{A}_{k-2}\cdots\tilde{A}_0$, and we partition P_k according to § 3.3. Then for any $y\in L_1$ we have $y_k^1=(P_k)_{11}y_0^1$; hence from (c), $(P_k)_{11}^{-1}$ exists. For any $z\in M_k$ we have, since $z_k^1=0$,

$$(P_k)_{11}z_0^1+(P_k)_{12}z_0^2=0. \tag{9.9}$$

Conversely, any solution of (3.5) satisfying (9.9) belongs to M_k. Therefore, if we define $Q_k=-(P_k)_{11}^{-1}(P_k)_{12}$, we have from (9.9), $z_0^1=Q_kz_0^2$ for $z\in M_k$. Because of (A), $\|Q_k\|\leqq\sigma_o$, hence $\{Q_k\}$ has at least a convergent subsequence, with limit Q, say. From the definition of L_2 it is then clear that any solution z with $z_0=\begin{pmatrix}Qa^2\\ a^2\end{pmatrix}$, a any vector in R, belongs to L_2, and clearly those solutions z constitute an n_2-dimensional subspace $\tilde{L}_2$ of L. This proves (d)–(g), whereas (h) is now trivial.

If z is a limiting sequence of a sequence $z(0), z(1), z(2)\cdots$, with $z(k)\in M_k$, then $z(k_i)\to z$ for a subsequence $\{k_i\}$, hence $z_0^2(k_i)\to z_0^2$, $Q_{k_i}z_0^2(k_i)\to z_0^1$. Then for a subsequence $\{k_i'\}$ of $\{k_i\}$, $\{Q_{k_i'}\}$ will converge to a limit Q, say, and obviously $z_0^1 = Qz_0^2$, which means that z belongs to a set $\tilde{L}_2$ as above and proves (k).

In order to prove (l), all we have to show is that for any $a \in R$ there exists only one $z \in L_2$ such that $z_0^2 = a^2$. Suppose that for $z, z' \in L_2$ we would have $z_0^2 = z_0'^2$. Then $y = z - z' \in L_1$ and if $z \neq z'$, then $y_i^1 \neq 0$ for all i (cf. (c)). Consequently $\|z_i^1 - z_i'^1\|/\|y_i^1\|$ is defined and equals 1 for all i. But (B) implies that this quantity tends to 0 as $i \to \infty$, which is a contradiction. We proceed similarly if (B′) is given.

Finally, if the sequence $\{z(k)\}$ in (m) does not converge, the sequence $\{z_0^1(k)\}$, which is bounded by $\sigma_0\|a^2\|$, would have at least two limit points, $\tilde{a}^1$ and $\hat{a}^1$, say, and both $\binom{\tilde{a}^1}{a^2}$ and $\binom{\hat{a}^1}{a^2}$ would serve as starting point for a solution in L_2, contradictory to what has just been proved. □.

Remark 9.10. Obviously condition (B′) states that all solutions in L_1 dominate those in L_2, whereas (B) also requires some kind of dominance. Therefore, in the SV case this condition will usually be satisfied, and therefore assertions (l) and (m) will hold frequently.

10. Dominated solutions of (3.5). Introducing $s_i = 1/r_i$:

$$s_i = \|y_i^1\|/\|y_i^2\|, \tag{10.1}$$

we may rewrite (4.3) as

$$s_i \leqq \frac{c_i + e_i s_{i+1}}{b_i - d_i s_{i+1}} \tag{10.2}$$

provided that $b_i - d_i s_{i+1} > 0$, and in the SV case we hope that the s_i are small for the solutions of L_2. Therefore the relevant relation for bounding the growth of these solutions is

$$\tilde{e}_i - d_i s_i \leqq \frac{\|y_{i+1}^2\|}{\|y_i^2\|} \leqq e_i + d_i s_i. \tag{10.3}$$

Hence, by Theorem 9.8(b) we have the following theorem.

THEOREM 10.4. *For all $k \geqq 1$, let a nonnegative sequence $\{\tilde{\sigma}_i\}_{0 \leqq i \leqq k}$ exist that satisfies*

$$\tilde{\sigma}_i = \frac{c_i + e_i \tilde{\sigma}_{i+1}}{b_i - d_i \tilde{\sigma}_{i+1}}, \qquad 0 \leqq i \leqq k-1, \quad \tilde{\sigma}_k = 0, \tag{10.5}$$

and let there exist a common majorant $\{\sigma_i\}$ of all sequences $\{\tilde{\sigma}_i\}$.

Then all solutions $y \in L_2$ of (3.5) are close to R_2 in the sense that $\|y_i^1\| \leqq \sigma_i \|y_i^2\|$ for all i, and they satisfy

$$y_m^2 = \prod_{i=k}^{m-1} (E_i + \sigma_i D_i U_i) y_k^2 \tag{10.6}$$

for some sequence of $n_1 \times n_2$ matrices $\{U_i\}$ with $\|U_i\| \leqq 1$.

COROLLARY 10.7.

$$\prod_{i=k}^{m-1} (\tilde{e}_i - d_i \sigma_i) \leqq \frac{\|y_m^2\|_2}{\|y_k^2\|_2} \leqq \prod_{i=k}^{m-1} (e_i + d_i \sigma_i) \tag{10.8}$$

and

$$(10.9)\qquad \frac{1}{1+\sigma_k}\prod_{i=k}^{m-1}(\tilde{e}_i-d_i\sigma_i)\leqq\frac{\|y_m\|}{\|y_k\|}\leqq(1+\sigma_m)\prod_{i=k}^{m-1}(e_i+d_i\sigma_i)$$

unless $y_k=0$, *in which case also* $y_{k+1}=y_{k+2}=\cdots=0$. *The estimates from below should be omitted when they contain negative factors.*

COROLLARY 10.10. *In the case* $n_2=1$, *we may take* $e_i=\tilde{e}_i=|(\tilde{A}_i)_{22}|$. *Then* (10.8) *and* (10.9) *reduce to*

$$(10.8')\qquad \frac{|y_m^2|}{|y_k^2|}=\prod_{i=k}^{m-1}(e_i+\theta_i d_i\sigma_i)\quad\text{with}\quad|\theta_i|\leqq 1,$$

$$(10.9')\qquad \frac{\|y_m\|}{\|y_k\|}=\frac{1+\eta_m\sigma_m}{1+\eta_k\sigma_k}\prod_{i=k}^{m-1}(e_i+\theta_i d_i\sigma_i)\quad\text{with}\,|\theta_i|\leqq 1,\quad 0\leqq\eta_i\leqq 1.$$

Remark 10.11. If the conditions of Theorem 10.4 are satisfied for a certain choice of $\{e_i\}$, then they are certainly satisfied for the choice in this corollary, since this means replacing the e_i by quantities which are not greater.

Remark 10.12. For $\{\sigma_i\}$ we may take any nonnegative solution of the recursion in (10.5) if one exists for all $i\geqq 0$ (and one certainly exists under the assumptions of Theorem 10.4 (cf. the proof of Theorem 13.1(b)).

Regarding the structure of L_2, we have directly the following theorem from Theorem 9.8:

THEOREM 10.13. *Let the conditions of Theorem* 10.4 *be satisfied. Then*:

(a) L_2 *contains at least one solution* y *for any given* y_0^2.

(b) *If, moreover, the conditions of Theorem* 4.6 *are satisfied, if* $b_i-c_i\rho_i>0$ *for all i, and*

$$(10.14)\qquad \lim_{m\to\infty}\sigma_m\prod_{i=0}^{m-1}(e_i+d_i\sigma_i)/(b_i-c_i\rho_i)=0,$$

then L_2 *is an* n_2-*dimensional subspace of* L *and* $L=L_1\oplus L_2$.

Again there is a generalization as in § 7.

THEOREM 10.15. *Assume* (7.1) *and* (7.3). *Let for all* $k\geqq 1$ *a nonnegative sequence* $\{\tilde{\sigma}_i\}_{0\leqq i\leqq k}$ *exist that satisfies*

$$(10.16)\qquad \tilde{\sigma}_i=\frac{fc_i+e_i\tilde{\sigma}_{i+1}}{b_i-gd_i\tilde{\sigma}_{i+1}},\qquad 0\leqq i\leqq k-1,\quad \tilde{\sigma}_k=0,$$

and let there exist a common majorant of all sequences $\{\tilde{\sigma}_i\}$.

Then all solutions $y\in L_2$ *of* (3.5) *satisfy*

$$(10.17)\qquad \|y_i^1\|\leqq g\sigma_i\|y_i^2\|,$$

$$(10.18)\qquad y_m^2=\prod_{i=k}^{m-1}(E_i+g\sigma_i D_i U_i)y_k^2,\qquad \|U_i\|\leqq 1,$$

$$(10.19)\qquad \frac{\|y_m^2\|}{\|y_k^2\|}\leqq g\prod_{i=k}^{m-1}(e_i+gd_i\sigma_i).$$

Moreover, L_2 contains at least one solution for any given y_0^2. Also, if in Theorem 7.4 *the condition about the sequence ρ is satisfied, if $b_i - c_i\rho_i > 0$ for all i, and*

$$\lim_{m\to\infty} \sigma_m \prod_{i=0}^{m-1} (e_i + gd_i\sigma_i)/(b_i - fc_i\rho_i) = 0, \tag{10.20}$$

then L_2 is an n_2-dimensional subspace of L and $L = L_1 \oplus L_2$.

Proof. Let k and m be given. We apply Lemma 7.24 with some $q > m$, and with $y_q^1 = 0$. We apply Lemma 7.16 with $p = k$. Then (7.29) holds for $k \leqq i \leqq q-1$ and yields

$$u_i(b_iv_{i+1} - gd_iu_{i+1}) \leqq v_i(e_iu_{i+1} + fc_iv_{i+1}). \tag{10.21}$$

Defining $t_i = u_i/v_i$, we obtain

$$t_i \leqq \frac{fc_i + e_it_{i+1}}{b_i - gd_it_{i+1}}, \qquad t_q = \frac{u_q}{v_q} = 0.$$

Hence $t_i \leqq \sigma_i$, and since $v_{i+1} \leqq (e_i + gd_it_i)v_i$, we obtain

$$\|y_m^2\| \leqq v_m \leqq \prod_{i=k}^{m-1} (e_i + gd_i\sigma_i)v_k \quad \text{with } v_k = g\|y_k^2\|,$$

implying (10.19) for $y \in M_q$, but then for $y \in L_2$.

Also, $\|y_k^1\|/\|y_k^2\| \leqq u_k/(v_k/g) \leqq g\sigma_k$, implying (10.17) for $i = k$ and $y \in M_q$, but then for any i and $y \in L_2$. Now (10.18) is obvious. □

11. Determination of sequences $\{\sigma_i\}$. For quantitative applications of Theorem 10.4, we have to establish sequences $\{\sigma_i\}$. Comparing (10.5) with (4.7), we note that the present recursion runs backward, and that the roles of c_i and d_i are interchanged. The latter fact implies inversion of the roots of the characteristic equation (cf. (5.1)).

Now applying Theorems 5.2 and 5.4, we get, with α_i and β_i defined as in § 5, the following theorem.

THEOREM 11.1. *If $e_i < b_i$ for all i and* $\sup \alpha_j \leqq \inf \beta_j$, *then* $\sigma_i = \sup_{j \geqq i} 1/\beta_j$ *suffices in Theorem* 10.4 *The same holds for $\sigma_i = a$, where a satisfies* $\sup 1/\beta_j \leqq a \leqq \inf 1/\alpha_j$.

COROLLARY 11.2. *In the SV case, all σ_i may be taken small.*

Similarly the analogues of the remainder of Theorem 5.4 and of Theorem 5.6 hold.

Although in the SV case β_i will be quite large (cf. § 5) and therefore σ_i quite small, we cannot state that then $\prod e_i$ and $\prod \tilde{e}_i$ are good upper and lower bounds for the growth of the solutions of L_2, since e_i might be very small with respect to d_i; actually e_i may be zero. If, however, d_i is at most of the order of e_i, then that statement could be made.

The additional condition in (10.14) is very weak in the SV case. It actually will then be satisfied if $\prod_0^\infty (e_i/b_i) = 0$, i.e., certainly if, for example, $e_i \leqq \theta b_i$, $\theta < 1$. This follows from the following theorem.

THEOREM 11.3. *If $d_i\sigma_i + c_i\rho_i \leqq \omega(b_i - e_i)$ for all i, $\omega < 1$, and $\{\sigma_i\}$ is bounded, then* (10.14) *is implied by* $\prod_{i=0}^\infty (e_i/b_i) = 0$.

Proof. Defining $\varepsilon_i = b_i - e_i$, we have from $\prod(1-\varepsilon_i/b_i)=0$ that $\sum \varepsilon_i/b_i = \infty$. Now $\prod_0^\infty (e_i + d_i\sigma_i)/(b_i - c_i\rho_i) = \prod_0^\infty (b_i - \varepsilon_i + d_i\sigma_i)/(b_i - c_i\rho_i) = \prod_0^\infty [1 + (c_i\rho_i + d_i\sigma_i - \varepsilon_i)/(b_i - c_i\rho_i)] \leqq \prod_0^\infty [1-(1-\omega)\varepsilon_i/b_i] = 0$ since $\sum \varepsilon_i/b_i = \infty$. □

Remark 11.4. Condition (10.14) goes back to condition (B) of Theorem 9.8, where $y \in L_1$ is required. Hence, on account of Remark 4.15, for the sequence $\{\rho_i\}$ we may take a solution $\{\tilde{\rho}_i\}$ of (4.7) with $\tilde{\rho}_0 = 0$. Then, if Theorem 5.6 and its analogue for $\{\sigma_i\}$ are applicable (as they often are in the SV case), we have $\sigma_i \approx \beta_i^{-1} \approx c_i/(b_i - e_i)$ and $\rho_i \approx \alpha_i \approx d_i/(b_i - e_i)$. In that case, the condition of Theorem 11.3 reads $2c_id_i \lesssim \omega(b_i - e_i)^2$, $\omega < 1$, and this condition is already satisfied as soon as (5.1) has two real roots.

12. Dominated solutions of (3.2). Again returning to the solutions of the original equation (3.2), we now state the analogues of Theorems 6.3 and 6.12, which could be provided with quite similar comment.

THEOREM 12.1. *Let* $n_2 = 1$. *Assume* $e_i = \tilde{e}_i = |(\tilde{A}_i)_{22}|$ *and let the assumptions of Theorem* 10.4 *be satisfied. Then for* $y \in L_2$ *and* $x_i = T_iy_i$,

$$(12.2)\qquad \frac{\|x_m\|}{\|x_k\|} = \frac{1+\eta_m\sigma_m}{1+\eta_k\sigma_k}\,\frac{\|T_m^2\|}{\|T_k^2\|}\prod_{i=k}^{m-1}(e_i+\theta_id_i\sigma_i), \qquad |\theta_i| \leqq 1, \quad |\eta_i| \leqq \frac{\|T_i^1\|}{\|T_i^2\|}$$

unless $x_k = 0$, *in which case also* $x_{k+1} = x_{k+2} = \cdots = 0$.

Numerical examples illustrating how remarkably well $\prod_{i=k}^{m-1} |\lambda_{ni}|$ approximates the growth of the solutions of (3.2) corresponding to L_2 in the case $n_2 = 1$ are again given in Part V.

THEOREM 12.3. *Under the assumptions of Theorem* 10.4, *we have, for* $y \in L_2$ *and* $x_i = T_iy_i$,

$$(12.4)\qquad \begin{aligned} &\frac{\text{g.l.b.}(T_m^2) - \|T_m^1\|\sigma_m}{\|T_k^1\| + \|T_k^2\|\sigma_k}\prod_{i=k}^{m-1}(\tilde{e}_i - d_i\sigma_i) \\ &\leqq \frac{\|x_m\|}{\|x_k\|} \leqq \frac{\|T_m^2\| + \|T_m^1\|\sigma_m}{\text{g.l.b.}(T_k^2) - \|T_k^1\|\sigma_k}\prod_{i=k}^{m-1}(e_i + d_i\sigma_i). \end{aligned}$$

The estimate from below should be omitted when it contains negative factors.

13. Relation between Theorems 4.6 and 10.4. The conditions of Theorems 4.6 and 10.4 are closely related as follows.

THEOREM 13.1. (a) *If the conditions of Theorem* 4.6 *are satisfied, and the sequence* $\{\tilde{\rho}_i\}$ *is positive, then the conditions of Theorem* 10.4 *are satisfied.*

(b) *If the conditions of Theorem* 10.4 *are satisfied and, moreover, the sequences* $\{\tilde{\sigma}_i\}$ *defined by* (10.5) *satisfy* $\tilde{\sigma}_i > 0$ *for any fixed i, if k is large enough, then the conditions of Theorem* 4.6 *can be satisfied.*

Proof. (a) We note that $b_i > 0$. We define $\sigma_i = 1/\tilde{\rho}_i$ and note that $(b_i - d_i\sigma_{i+1})\sigma_i = c_i + e_i\sigma_{i+1}$. Hence $b_i - d_i\sigma_{i+1} \geqq 0$.

We first assert that $b_i - d_iu > 0$ for all $u < \sigma_{i+1}$. This is obvious if $b_i - d_i\sigma_{i+1} > 0$; if $b_i - d_i\sigma_{i+1} = 0$, it follows from the observation that then $d_i > 0$.

Hence $\varphi(u) = (c_i + e_iu)/(b_i - d_iu)$ is defined for any $u < \sigma_{i+1}$, and we assert that $\varphi(u) < \sigma_i$ for $u < \sigma_{i+1}$. We first note that as a consequence of the positiveness of $\{\tilde{\rho}_i\}$, d_i and e_i cannot vanish simultaneously. If $e_i \neq 0$, then $\varphi(u)$ is monotonically increasing, proving our assertion. The same is true if $e_i = 0$ and $d_i \neq 0$ provided that $c_i \neq 0$. Finally, if $c_i = e_i = 0$, then $\varphi(u) = 0 < \sigma_i$.

Since $\tilde{\sigma}_k = 0 < \sigma_k$ it is now clear that $\tilde{\sigma}_k, \cdots, \tilde{\sigma}_0$ are defined, and $\tilde{\sigma}_i < \sigma_i$.

(b) In (10.5), $\tilde{\sigma}_i$ is a nondecreasing function of $\tilde{\sigma}_{i+1}$, and therefore if i is fixed and $k > i$, then $\tilde{\sigma}_i(k)$ is a nondecreasing function of k (here $\tilde{\sigma}(k)$ denotes the sequence defined by (10.5)). Hence $\lim_{k\to\infty} \tilde{\sigma}_i(k) = \text{l.u.b.}_k \tilde{\sigma}_i(k)$, and we denote this limit sequence by $\{\bar{\sigma}_i\}$, which is obviously positive. Then by a continuity argument, $\bar{\rho}_i = 1/\bar{\sigma}_i$ satisfies $(b_i - c_i\bar{\rho}_i)\bar{\rho}_{i+1} = d_i + e_i\bar{\rho}_i$. Now an exact analogue of the argument used in the proof of (a) shows that for any $\tilde{\rho}_0 < \bar{\rho}_0$, the solution of the recursion in (4.7) is defined and nonnegative. Hence $\rho_0 < \bar{\rho}_0$, $\rho_i = \bar{\rho}_i$ satisfies the requirements of Theorem 4.6. □

Remark 13.2. The additional condition in (a) that $\{\tilde{\rho}_i\}$ is positive is satisfied as soon as $\rho_0 > 0$ in Theorem 4.6, except in such cases where d_i and e_i vanish simultaneously, causing some kind of decomposition of (3.5). The additional condition of (b) is certainly satisfied if all $c_i \neq 0$, which is quite common. More precisely, this condition in (b) is equivalent to the property that e_i and c_i do not vanish simultaneously and that there is an infinite subsequence of nonvanishing c_i.

Remark 13.3. The observations at the end of either part of the proof of Theorem 13.1 should not be taken as an advice to take $\sigma_i = 1/\rho_i$ when applying Theorems 4.6 and 10.4; indeed, these theorems yield their sharpest results when the sequences $\{\rho_i\}$ and $\{\sigma_i\}$ are as small as possible, and in the SV case we may expect these smallest values to satisfy approximately $\rho_i \approx \alpha_i$, $\sigma_i \approx 1/\beta_i$, where $\alpha_i \ll \beta_i$, hence $\rho_i\sigma_i \ll 1$.

14. Further decomposition of *L*. Roughly speaking, L_1 and L_2 decompose L into solutions growing at least about a factor b_i per step and solutions growing at most about a factor e_i per step.

Now suppose that we apply a second partitioning to R, $\tilde{A}_i$, etc.: $R = R_1' \oplus R_2'$, R_1' of dimension $n_1' < n_1$, R_2' of dimension $n_2' = n - n_1'$. We shall provide all relevant quantities with a prime, e.g., b_i' and L_2', but we shall partition vectors into three parts:

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix},$$

and we write

$$\begin{pmatrix} y^1 \\ y^{2+3} \end{pmatrix}$$

for the $'$-partitioning and

$$\begin{pmatrix} y^{1+2} \\ y^3 \end{pmatrix}$$

for the original partitioning.

Now assuming that the conditions of Theorems 4.6 and 10.4 are satisfied for both partitionings, we find for $y \in L_1 \cap L_2'$:

$$\|y_m^{1+2}\| \geqq \prod_{i=k}^{m-1} (b_i - c_i r_i)\|y_k^{1+2}\| \quad \text{since } y \in L_1, \tag{14.1}$$

$$\|y_m^{2+3}\| \leqq \prod_{i=k}^{m-1} (e_i' + d_i' s_i') \|y_k^{2+3}\| \quad \text{since } y \in L_2'. \tag{14.2}$$

Also, $\|y_i^1\| = s_i'\|y_i^{2+3}\|$ and $\|y_i^3\| = r_i\|y_i^{1+2}\|$. Hence $\|y_i^1\| \leqq s_i'(\|y_i^2\| + \|y_i^3\|) \leqq s_i'(1+r_i)\|y_i^2\| + s_i' r_i \|y_i^1\|$, from which

$$\|y_i^1\| \leqq \frac{s_i'(1+r_i)}{1-r_i s_i'}\|y_i^2\| \quad \text{and similarly} \quad \|y_i^3\| \leqq \frac{r_i(1+s_i')}{1-r_i s_i'}\|y_i^2\|, \tag{14.3}$$

provided that $r_i s_i' < 1$, which will certainly be the case if we have the SV case for both partitionings. Consequently

$$\|y_i^{2+3}\| \leqq \|y_i^2\| + \|y_i^3\| \leqq \frac{1+r_i}{1-r_i s_i'}\|y_i^2\| \quad \text{and} \quad \|y_i^{1+2}\| \leqq \frac{1+s_i'}{1-r_i s_i'}\|y_i^2\|.$$

Hence we have the following theorem.

THEOREM 14.4. *If the conditions of Theorems* 10.4 *and* 10.13(b) *are satisfied for both partitionings, and* $\rho_i \sigma_i' < 1$ *for all i, then* $\dim (L_1 \cap L_2') = n_1 - n_1'$ *and all solutions y of* (3.5) *in* $L_1 \cap L_2'$ *satisfy*

$$\frac{1-\rho_m \sigma_m'}{1+\sigma_m'} \prod_{i=k}^{m-1} (b_i - c_i \rho_i) \leqq \frac{\|y_m^2\|}{\|y_k^2\|} \leqq \frac{1+\rho_k}{1-\rho_k \sigma_k'} \prod_{i=k}^{m-1} (e_i' + d_i' \sigma_i') \tag{14.5}$$

and

$$\frac{1}{1+\rho_k} \prod_{i=k}^{m-1} (b_i - c_i \rho_i) \leqq \frac{\|y_m\|}{\|y_k\|} \leqq (1+\sigma_m') \prod_{i=k}^{m-1} (e_i' + d_i' \sigma_i'). \tag{14.6}$$

A theorem for $\|x_m\|/\|x_k\|$ similar to Theorems 6.12 and 12.3 is now obvious.

In order to illustrate the implications of this theorem, we consider the case that the eigenvalues of A_i are different in modulus for all i : $|\lambda_{1i}| > |\lambda_{2i}| > \cdots > |\lambda_{ni}|$, and we suppose that we have the SV case for $n_1 = 1, 2, 3, \cdots, n-1$. Then $b_i \approx e_i' \approx \lambda_{n_1,i}$. Hence for each n_1 there is a solution of (3.2) for which $\|x_m\|/\|x_k\|$ is about $\prod_{i=k}^{m-1} |\lambda_{n_1,i}|$.

Part IV. The nonlinear recursions for the directions.

15. The nonlinear recursion $t_{i+1} = (d_i + e_i t_i)/(b_i - c_i t_i)$.

$$t_{i+1} = \frac{d_i + e_i t_i}{b_i - c_i t_i}. \tag{15.1}$$

We consider this recursion as a "time-dependent" (because the coefficients depend on i) successive substitution problem, of which Fig. 15.1 is the well-known graphic representation (cf. [4, p. 131] or [6, p. 7]), where

$$\varphi_i(t) = \frac{d_i + e_i t}{b_i - c_i t}. \tag{15.2}$$

We note that in the SV case the graph of φ_i will certainly intersect the line $y = t$. Indeed, the equality $t = (d_i + e_i t)/(b_i - c_i t)$ holds if the *characteristic equation* of (15.1),

$$c_i t^2 - (b_i - e_i)t + d_i = 0, \tag{15.3}$$

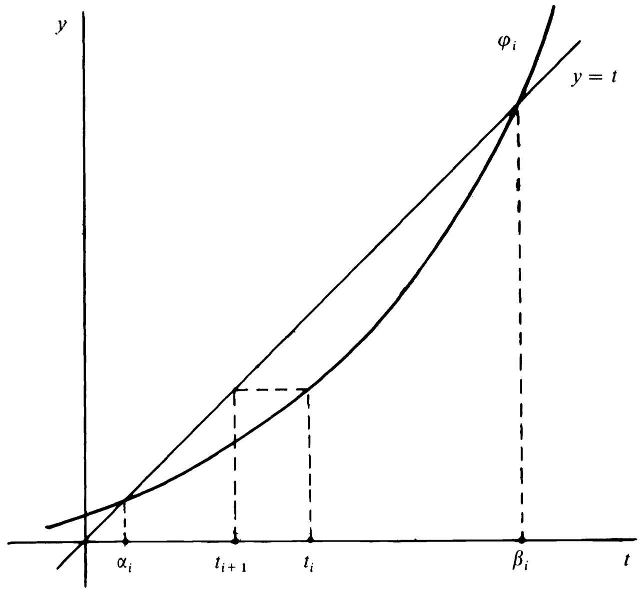

FIG. 15.1.

is satisfied, and since c_i, $d_i \ll b_i - e_i$, the discriminant of this equation is positive.

Because of our interest in the SV case, we shall always assume $b_i > e_i$ and the discriminant of (15.3) to be nonnegative, so that (15.3) has real roots α_i and β_i for which $0 \leqq \alpha_i \leqq \beta_i$. If $c_i = 0$ we shall take $\beta_i = \infty$ and still say that there are two real roots.

We have the following trivial but useful lemma estimating the roots of a quadratic equation.

LEMMA 15.4. *Let the equation*

$$pt^2 - qt + r = 0 \tag{15.5}$$

have nonnegative roots α and β, $0 \leqq \alpha \leqq \beta$. Then with $\eta = \alpha/\beta$,

$$\alpha = (1+\eta)\frac{r}{q}, \quad \beta = \frac{1}{1+\eta}\frac{q}{p}, \tag{15.6}$$

with $0 \leqq \eta \leqq 1$. More specifically, η is a monotonically decreasing function of $q^2/4pr$ satisfying

(15.7)

$q^2/(4pr)$	1	2	3	$\geqq 4$
η	1	0.16	0.1	$\approx pr/q^2$

Remark 15.8. We shall call $q^2/4pr$ the *discriminant quotient* (d.q.) of (15.5). The lemma is useful if we are only interested in estimates of the solutions which may have an error of several percentage points. In fact, we see that for d.q. $\geqq 2$, the roots are quite well approximated by r/q and q/p, respectively.

Hence the roots of (15.3) satisfy

$$\frac{d_i}{b_i-e_i} \leqq \alpha_i \leqq \frac{2d_i}{b_i-e_i}, \quad \frac{b_i-e_i}{2c_i} \leqq \beta_i \leqq \frac{b_i-e_i}{c_i}. \tag{15.9}$$

Since in the SV case the d.q. of (15.3) is large, the factors 2 in (15.9) may then be replaced by factors quite close to 1. In that case, also

$$\alpha_i \ll 1, \qquad \beta_i \gg 1. \tag{15.10}$$

In view of (15.9) and (15.10), the following theorem, whose proof is immediate from Fig. 15.1, is relevant.

THEOREM 15.11. *If*

(a) $b_i > e_i$ *and* $(b_i - e_i)^2 \geqq 4\, c_i d_i$ *for all i, so that* (15.3) *has real roots* α_i, β_i *with* $0 \leqq \alpha_i \leqq \beta_i$, *and*

(b) $\alpha_i \leqq u_i \leqq \beta_i$ *for all i for some nondecreasing sequence* $\{u_i\}$ *then* $t_{i+1} \leqq u_i$ *if* $t_0 \leqq u_0$.

Applications. (Here a is some suitable number):

(15.12) If (a) and $\alpha_i \leqq a \leqq \beta_i$ for all i, then $t_i \leqq a$ if $t_0 \leqq a$.

(15.13) If (a) and $2d_i/(b_i - e_i) \leqq a \leqq (b_i - e_i)/(2c_i)$ for all i, then $t_i \leqq a$ if $t_0 \leqq a$.

(15.14) If (a) and $\{\alpha_i\}$ is nondecreasing, then $t_{i+1} \leqq \alpha_i$ if $t_0 \leqq \alpha_0$.

(15.15) If (a) and $\{d_i/(b_i - e_i)\}$ is nondecreasing, then $t_{i+1} \leqq 2d_i/(b_i - e_i)$ if $t_0 \leqq 2d_0/(b_0 - e_0)$.

In the SV case, the factors 2 in (15.13) and (15.15) may obviously be replaced by a factor close to 1 in view of Remark 15.8.

In order to assess the sharpness of these results, we note that if b_i, c_i, d_i and e_i do not depend on i, then t_i approaches the constant value α_i, and hence we may expect (15.12) and (15.13) not to be too poor if all α_i are in the neighborhood of the number a ($\neq 0$).

If, on the other hand, $\{\alpha_i\}$ varies essentially and the graph in Fig. 15.1 "bulges" nicely, then t_{i+1} will be a good deal closer to α_i than t_i is, and consequently, if α_i does not move too fast as i increases, we may expect t_i to follow α_{i-1} rather closely. This settles to some extent the sharpness of (15.14) and (15.15), but at the same time points out the crudeness of Theorem 15.11 and its applications if $\{\alpha_i\}$ has a decreasing tendency.

Before exploring this more closely, we note that for the same reason, (15.12) and (15.13) are not very sharp either if t_0 is a good deal greater than $\sup \alpha_j$. The following result is better.

THEOREM 15.16. *If* $e_i < b_i$ *for all i and* $\sup \alpha_j \leqq t_0 \leqq \inf \beta_j$, *then*

$$\frac{t_{i+1} - \sup \alpha_j}{t_0 - \sup \alpha_j} \leqq \prod_{j=0}^{i} \frac{e_j/b_j + \alpha_j/\beta_j}{1 - t_0 c_j/b_j} \leqq \prod_{j=0}^{i} \frac{e_j/b_j + \alpha_j/\beta_j}{1 - t_0/\beta_j} \quad \textit{for all } i. \tag{15.17}$$

If $t_i \leqq \sup \alpha_j$ *for any* $i \geqq 0$, *then the same holds for all greater values of i.*

Proof. If $t_i \leqq \sup \alpha_j$, then $t_{i+1} \leqq \sup \alpha_j$, and (15.17) is trivial. Therefore let $t_i > \sup \alpha_j$. Since

$$\frac{t_{i+1}-\alpha_i}{t_i-\alpha_i}=\frac{d_i+e_it_i-\alpha_i(b_i-c_it_i)}{(t_i-\alpha_i)(b_i-c_it_i)}$$

and $d_i - b_i\alpha_i = -c_i\alpha_i^2 - e_i\alpha_i$, the numerator equals $e_it_i + \alpha_ic_it_i - c_i\alpha_i^2 - e_i\alpha_i = (t_i-\alpha_i)(e_i+\alpha_ic_i)$. With the observation that $c_i/b_i < 1/\beta_i$, the rest is easy. □

In order to deal with decreasing $\{\alpha_i\}$, we write

$$t_i = \theta_i\alpha'_{i-1}, \tag{15.18}$$

where $\{\alpha'_i\}_{i\geqq -1}$ is any positive sequence, which in applications will be assumed to resemble the behavior of $\{\alpha_i\}$. Then $\{\theta_i\}$ satisfies

$$\theta_{i+1}=\frac{d_i+e_i\alpha'_{i-1}\theta_i}{b_i\alpha'_i-c_i\alpha'_{i-1}\alpha'_i\theta_i} \tag{15.19}$$

with characteristic equation

$$c_i\alpha'_{i-1}\alpha'_i\tilde{t}^2-(b_i\alpha'_i-e_i\alpha'_{i-1})\tilde{t}+d_i=0 \tag{15.20}$$

or

$$c_i\frac{\alpha'_i}{\alpha'_{i-1}}(\alpha'_{i-1}\tilde{t})^2-\left(b_i\frac{\alpha'_i}{\alpha'_{i-1}}-e_i\right)(\alpha'_{i-1}\tilde{t})+d_i=0. \tag{15.21}$$

Hence we have the following theorem from Theorem 15.11.

THEOREM 15.22. *If $\{\alpha'_i\}_{i\geqq -1}$ is a positive sequence with $\alpha'_{-1}=\alpha'_0$ and*

(a) *$b_i(\alpha'_i/\alpha'_{i-1}) > e_i$ and $[b_i(\alpha'_i/\alpha'_{i-1})-e_i]^2 \geqq 4c_id_i(\alpha'_i/\alpha'_{i-1})$ for all i, so that* (15.20) *has real roots $\tilde{\alpha}_i$, $\tilde{\beta}_i$ satisfying $0\leqq\tilde{\alpha}_i\leqq\tilde{\beta}_i$, and*

(b) *$\tilde{\alpha}_i \leqq u_i \leqq \tilde{\beta}_i$ for all i for some nondecreasing sequence $\{u_i\}$,*

then $t_{i+1}\leqq u_i\alpha'_i$ if $t_0 \leqq u_0\alpha'_0$.

Although condition (a) does not look pleasant, we note that if α'_i/α'_{i-1} is close to 1, then the substitution $\alpha'_{i-1}\tilde{t}=t$ transforms (15.21) into something which is very similar to (15.3). In the SV case, Theorem 15.11(a) is generously satisfied, and therefore Theorem 15.22(a) will then also be generously satisfied provided that α'_i/α'_{i-1} is close enough to 1, and then $\tilde{\alpha}_i \approx \alpha_i/\alpha'_{i-1}$ and $\tilde{\beta}_i\approx\beta_i/\alpha'_{i-1}$, implying that $\tilde{\beta}_i$ and $\tilde{\alpha}_i$ have about the same (large) ratio as β_i and α_i.

Since the sequence $\{\alpha'_i\}$ is arbitrary, the condition that α'_i/α'_{i-1} is close to 1 can always be satisfied. If, however, one wishes $\{\alpha'_i\}$ to resemble $\{\alpha_i\}$, as has been observed above, it should be realized that, although α_i/α_{i-1} will often be close to 1 in the SV case, the SV assumptions do not really prevent $\{\alpha_i\}$ from tending to 0 quite rapidly.

In order to apply Theorem 15.22, we have to specify sequences $\{\alpha'_i\}$ and to find sequences $\{u_i\}$, and from what has been said before it will be clear that we hope that a constant sequence $\{u_i\}$ satisfies if $\{\alpha'_i\}$ resembles $\{\alpha_i\}$. Hence we have the following applications, which are all parallels of (15.13).

Applications. Define $\gamma_i = e_i/b_i$, and let any sequence $\{\alpha_i'\}$ to be defined below satisfy $\alpha_i'/\alpha_{i-1}' > \gamma_i$ $(\alpha_{-1}' = \alpha_0')$. Then $t_{i+1} \leqq \tilde{a}\alpha_i'$ if $t_0 \leqq \tilde{a}\alpha_0'$ whenever $\tilde{a}$ satisfies

$$(15.23)\quad 2\sup \frac{1-\gamma_i}{1-\gamma_i(\alpha_{i-1}'/\alpha_i')} \leqq \tilde{a} \leqq \frac{1}{2}\inf \frac{\beta_i'}{\alpha_i'}\frac{(\alpha_i'/\alpha_{i-1}')-\gamma_i}{1-\gamma_i} \quad \text{with } \alpha_i' = \alpha_i, \beta_i' = \beta_i,$$

or with

$$(15.24)\qquad \alpha_i' = \frac{d_i}{b_i - e_i}, \qquad \beta_i' = \frac{b_i - e_i}{c_i},$$

or

$$(15.25)\quad 2\sup \frac{1}{1-\gamma_i(\alpha_{i-1}'/\alpha_i')} \leqq \tilde{a} \leqq \frac{1}{2}\inf \frac{\beta_i'}{\alpha_i'}\left(\frac{\alpha_i'}{\alpha_{i-1}'}-\gamma_i\right) \quad \text{with } \alpha_i' = \frac{d_i}{b_i}, \quad \beta_i' = \frac{b_i}{c_i}$$

where again, using Remark 15.8, the factors 2 and $\frac{1}{2}$ might be replaced by factors close(r) to 1 if the discriminant quotient of (15.21) is large enough.

This shows that under rather liberal conditions (which can easily be made explicit), we have $t_i \leqq 2\alpha_{i-1}$ if $t_0 \leqq 2\alpha_0$. Among these conditions we obviously have that α_i/α_{i-1} should be fairly large with respect to $\gamma_i = e_i/b_i$. We conclude this section by showing that the latter condition is rather essential if one wishes t_i to follow α_{i-1} rather closely. Indeed, as is easily verified,

$$(15.26)\qquad \frac{t_{i+1}-\alpha_i}{t_i-\alpha_i} = \left(\frac{e_i}{b_i}+\frac{c_id_i}{b_i^2}\right)\frac{1}{(1-\alpha_ic_i/b_i)(1-t_ic_i/b_i)},$$

which shows that $t_{i+1}-\alpha_i > (e_i/b_i)(t_i-\alpha_i)$ if $t_i > \alpha_i$. Now assume that $t_i > \alpha_i$ for some i, that $e_j/b_j = \mu$ (const.) for $j \geqq i$, and that $\alpha_j = \lambda^j$, $0 < \lambda < \mu < 1$, for $j \geqq i$ (hence $\alpha_j/\alpha_{j-1} < e_j/b_j$). It is then easily verified that $t_{i+p} > \alpha_{i+p-1}(1-\mu)(1+\mu/\lambda+\cdots+(\mu/\lambda)^{p-1})$ and hence t_{i+p}/α_{i+p-1} increases rapidly for $p \to \infty$.

Finally, we refer to [7], where more refined work on the solutions of the recursion (15.1) is reported.

16. The nonlinear recursion $t_{i+1} = (d_i + e_it_i)/(b_i + c_it_i)$.

$$(16.1)\qquad t_{i+1} = \frac{d_i + e_it_i}{b_i + c_it_i}.$$

We shall be very brief about this recursion. The case which is of interest to us (cf. 7.11) is that

$$e_i < b_i, \qquad c_id_i \leqq b_ie_i.$$

Then the graph of $\varphi_i(t) = (d_i + e_it)/(b_i + c_it)$ is as shown in Fig. 16.1, where $\alpha_i \geqq 0$ and $\beta_i < 0$ are the roots of

$$(16.2)\qquad c_it^2 + (b_i - e_i)t - d_i = 0.$$

Obviously, $\alpha_i < d_i/(b_i - e_i)$.

THEOREM 16.3. *If*

(a) $b_i > e_i$ *and* $c_id_i \leqq b_ie_i$ *for all i, and*

(b) $\alpha_i \leqq u_i$ *for some nondecreasing sequence* $\{u_i\}$,

then $t_{i+1} \leqq u_i$ *if* $t_0 \leqq u_0$.

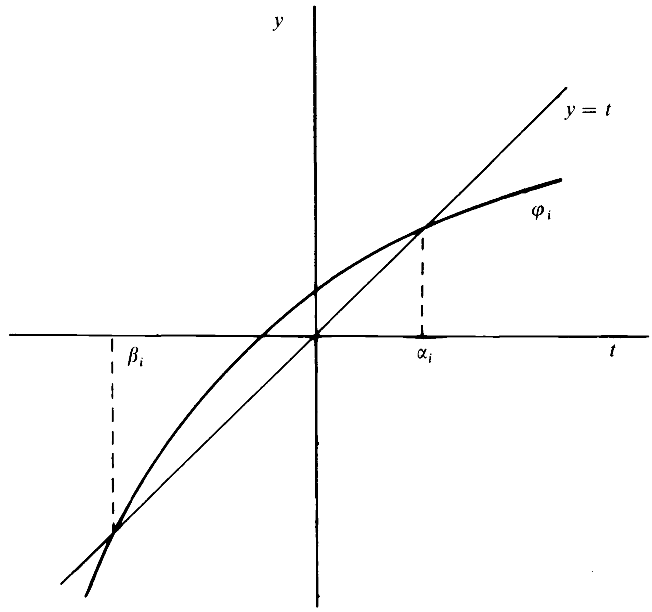

FIG. 16.1

Applications similar to (15.12)–(15.15) are now obvious; all factors 2 may be omitted, and in (15.12) $\alpha_i \leqq a$ suffices, whereas in (15.13) $d_i/(b_i - e_i) \leqq a$ suffices.

Similar to Theorem 15.22, we have the following theorem.

THEOREM 16.4. *If* $\{\alpha'_i\}_{i \geqq -1}$ *is a positive sequence with* $\alpha'_{-1} = \alpha'_0$ *and*

(a) $\alpha'_i/\alpha'_{i-1} > e_i/b_i$,

(b) $u_i \geqq \tilde{\alpha}_i$ *for some nondecreasing sequence* $\{u_i\}$, *where* $\tilde{\alpha}_i$ *denotes the positive root of*

$$c_i\alpha'_{i-1}\alpha'_i\tilde{t}^2 + (b_i\alpha'_i - e_i\alpha'_{i-1})\tilde{t} - d_i = 0, \tag{16.5}$$

then $t_{i+1} \leqq u_i\alpha'_i$ *if* $t_0 \leqq u_0\alpha'_0$.

Applications. With $\gamma_i = e_i/b_i$ and $\alpha'_i/\alpha'_{i-1} > \gamma_i$ $(\alpha'_{-1} = \alpha'_0)$ we have $t_{i+1} \leqq \tilde{a}\alpha'_i$ if $t_0 \leqq \tilde{a}\alpha'_0$ whenever

$$\sup \frac{1-\gamma_i}{1-\gamma_i(\alpha'_{i-1}/\alpha'_i)} \leqq \tilde{a} \quad \text{with } \alpha'_i = d_i/(b_i - e_i) \tag{16.6}$$

or

$$\sup \frac{1}{1-\gamma_i(\alpha'_{i-1}/\alpha'_i)} \leqq \tilde{a} \quad \text{with } \alpha'_i = d_i/b_i. \tag{16.7}$$

Part V. Applications.

17. Three-term linear homogeneous recursions. Because of the great practical interest of three-term linear homogeneous recursions, we now reformulate the preceding results for this case. Numerical results for this case will be given in § 18.

Let

$$u_{i+2}+p_iu_{i+1}+q_iu_i=0, \qquad i=0,1,2,\cdots \tag{17.1}$$

be such a recursion. A matrix-vector equivalent is

$$x_{i+1}=A_ix_i, \quad A_i=\begin{pmatrix}0 & 1\\ -q_i & -p_i\end{pmatrix}, \quad x_i=\begin{pmatrix}u_i\\ u_{i+1}\end{pmatrix}. \tag{17.2}$$

The eigenvalues λ_i and μ_i of A_i will also be called the *local characteristic roots* of (17.1). We assume $|\lambda_i|>|\mu_i|$ and define

$$T_i=\begin{pmatrix}1 & 1\\ \lambda_i & \mu_i\end{pmatrix}. \tag{17.3}$$

Obviously, if $|\lambda_i|\ll 1$ or $|\lambda_i|\gg 1$, then, no matter the ratio μ_i/λ_i, T_i will be very skew, thus violating one of the SV conditions. This could have been circumvented by a different choice of A_i(e.g., cf. [8, (4.10)]). However, this skewness will not matter here, since we are not going to use properties like Theorem 6.3, for which purpose we required the nonskewness of the T_i mainly.

Since

$$T_i^{-1}A_iT_i=\begin{pmatrix}\lambda_i & 0\\ 0 & \mu_i\end{pmatrix},$$

we get for $\tilde{A}_i$ as defined by (3.3):

$$\tilde{A}_i=\frac{1}{\lambda_{i+1}-\mu_{i+1}}\begin{pmatrix}\lambda_i(\lambda_i-\mu_{i+1}) & \mu_i(\mu_i-\mu_{i+1})\\ \lambda_i(\lambda_{i+1}-\lambda_i) & \mu_i(\lambda_{i+1}-\mu_i)\end{pmatrix}. \tag{17.4}$$

We now define

$$\begin{aligned} b_i'&=|\lambda_i(\lambda_i-\mu_{i+1})|, & c_i'&=|\mu_i(\mu_i-\mu_{i+1})|,\\ d_i'&=|\lambda_i(\lambda_{i+1}-\lambda_i)|, & e_i'&=|\mu_i(\lambda_{i+1}-\mu_i)|, \end{aligned} \tag{17.5}$$

and note that after division by $|\lambda_{i+1}-\mu_{i+1}|$, these quantities can be used as $b_i=\tilde{b}_i$, c_i, d_i and $e_i=\tilde{e}_i$, respectively, in (3.6).

We then have the following theorem for the *dominant* solutions.

THEOREM 17.6. *Let* $|\lambda_i|>|\mu_i|$. *Let for some* $\tilde{\rho}_0\geqq 0$ *the sequence* $\{\tilde{\rho}_i\}$,

$$\tilde{\rho}_{i+1}=(d_i'+e_i'\tilde{\rho}_i)/(b_i'-c_i'\tilde{\rho}_i), \tag{17.7}$$

be defined and nonnegative, and let $\{\rho_i\}$ *be a majorant of* $\{\tilde{\rho}_i\}$, $\rho_0=\tilde{\rho}_0$. *Then a solution* $\{u_i\}$ *of* (17.1) *with* $u_0=1$, $u_1=\lambda_0$ (*or more generally,* $u_0=a+b$, $u_1=a\lambda_0+b\mu_0$ *with* $|b|\leqq\rho_0|a|$) *satisfies*

$$\frac{|u_{m+1}|}{|u_{k+1}|}=\frac{1+\theta_m'\rho_m\mu_m/\lambda_m}{1+\theta_k'\rho_k\mu_k/\lambda_k}\pi_1\pi_2\prod_k^{m-1}|\lambda_i|, \tag{17.8}$$

where

$$\pi_1=\prod_k^{m-1}\left|\frac{1-\mu_{i+1}/\lambda_i}{1-\mu_{i+1}/\lambda_{i+1}}\right|$$

$$=\prod_k^{m-1}\left|1-\frac{\mu_{i+1}\Delta\lambda_i}{\lambda_i\lambda_{i+1}(1-\mu_{i+1}/\lambda_{i+1})}\right| \tag{17.9}$$

$$=\left|\frac{1-\mu_{k+1}/\lambda_k}{1-\mu_m/\lambda_m}\right|\prod_{k+1}^{m-1}\left|1-\frac{\Delta\mu_i}{\lambda_i-\mu_i}\right|,$$

(17.10)
$$\pi_2 = \prod_k^{m-1} \left[1 + \theta_i \rho_i \frac{\mu_i \Delta\mu_i}{\lambda_i(\lambda_i - \mu_{i+1})}\right],$$

with $\Delta\lambda_i = \lambda_{i+1} - \lambda_i$, $\Delta\mu_i = \mu_{i+1} - \mu_i$, $|\theta_i| \leqq 1$, $|\theta_i'| \leqq 1$.

This still applies for $k = -1$ *if one defines* $\lambda_{-1} = \lambda_0$, $\mu_{-1} = \mu_0$, $\rho_{-1} = \rho_0\lambda_0/\mu_0$.

Proof. From $x_i = T_i y_i$, we have

(17.11)
$$u_{i+1} = \lambda_i y_i^1 + \mu_i y_i^2 = \lambda_i y_i^1(1 + \theta_i' \rho_i \mu_i/\lambda_i), \qquad |\theta_i'| \leqq 1.$$

Hence, from (4.10′),

$$\frac{|u_{m+1}|}{|u_{k+1}|} = \frac{1 + \theta_m' \rho_m \mu_m/\lambda_m}{1 + \theta_k' \rho_k \mu_k/\lambda_k} \cdot \frac{|\lambda_m|}{|\lambda_k|} \prod_k^{m-1} \left\{\left|\lambda_i \frac{\lambda_i - \mu_{i+1}}{\lambda_{i+1} - \mu_{i+1}}\left[1 + \theta_i \rho_i \frac{\mu_i(\mu_i - \mu_{i+1})}{\lambda_i(\lambda_i - \mu_{i+1})}\right]\right|\right\}$$

and now (17.9) and (17.10) are clear.

For the case $k = -1$, we consider (17.2) for $i \geqq -1$ with $A_{-1} = A_0$. Since we then have $b_{-1}' = \lambda_0(\lambda_0 - \mu_0)$, $c_{-1}' = d_{-1}' = 0$, $e_{-1}' = \mu_0(\lambda_0 - \mu_0)$, we have $\tilde{\rho}_0 = \mu_0 \tilde{\rho}_{-1}/\lambda_0$.

Remark 17.12. In the important case $k = -1$, we have $\prod_k^{m-1} |\lambda_i| = |\lambda_0| \cdot \prod_0^{m-1} |\lambda_i|$ (cf. (17.8)). If, in particular, $\rho_0 = 0$ (i.e., $u_1/u_0 = \lambda_0$), then (17.8) reads

(17.13)
$$|u_{m+1}/u_0| = (1 + \theta_m' \rho_m \mu_m/\lambda_m)\pi_1 \pi_2 |\lambda_0| \prod_0^{m-1} |\lambda_i|.$$

Again, for $k = -1$ (but irrespective of ρ_0 being 0), we have $\prod_k^{m-1} = \prod_0^{m-1}$ everywhere in (17.9) and (17.10) since $\Delta\lambda_{-1} = \Delta\mu_{-1} = 0$.

Remark 17.14. In (17.8)–(17.10), we may replace λ_m, μ_m and ρ_m by λ_{m-1}, μ_{m-1} and $\rho_{m-1}\mu_{m-1}/\lambda_{m-1}$, respectively, and for (17.9) and (17.10) this means replacing $\prod^{m-1}$ by $\prod^{m-2}$. The replacement of λ_m and μ_m is a consequence of u_{m+1} being independent of λ_m and μ_m (cf. (17.1)), and then we get $e_{m-1}'/b_{m-1}' = |\mu_{m-1}/\lambda_{m-1}|$, $c_{m-1}' = d_{m-1}' = 0$.

Remark 17.15. Obviously, the factors π_1 and π_2 will be close to 1 in the SV case, and in Remark 17.16 we shall estimate how close, in order to see what error is made if these factors are skipped altogether. For the moment, we note that in this respect, the penultimate expression for π_1 in (17.9) may be advantageous if $\Delta\lambda_i$ is very small, whereas the final expression in (17.9) may be advantageous if $\Delta\mu_i$ is very small.

Remark 17.16. In any given case, the values of π_1 and π_2 (cf. (17.9) and (17.10)) may be estimated. In this remark we give (possibly quite conservative) estimates for some special, though rather common, cases. Proofs are contained in (c) and (h).

(a) If $\{\lambda_i\}$ has constant signs and is monotonic, then

$$1 - \varphi_1 \leqq \pi_1 \leqq \exp(\varphi_1) \quad \text{if } \varphi_1 = \left|\frac{1}{\lambda_m} - \frac{1}{\lambda_k}\right| \cdot \max_{k+1 \leqq i \leqq m} \frac{|\mu_i|}{1 - \mu_i/\lambda_i} \leqq 1.$$

(b) If $\{\mu_i\}$ is monotonic, then for $\pi_1' = \prod_{k+1}^{m-1} [1 - \Delta\mu_i/(\lambda_i - \mu_i)]$ (cf. (17.9)),

$$1 - \varphi_2 \leqq \pi_1' \leqq \exp(\varphi_2) \quad \text{if } \varphi_2 = |\mu_m - \mu_{k+1}| \max_{k+1 \leqq i \leqq m-1} 1/|\lambda_i - \mu_i| \leqq 1.$$

(c) The estimates in (a) and (b) follow immediately from the property

$$1-\sum|v_i|\leqq\prod(1+v_i)\leqq\exp\left(\sum|v_i|\right)\quad\text{if all } v_i>-1,$$

which may also be used for deriving some more detailed results.

(d) Let (S) denote the property that $\{\lambda_i\}$ and $\{\mu_i\}$ are slowly varying, that $\tilde{\rho}_i\leqq\omega d_i/(b_i-e_i)$ for some constant ω (which will often be of the order of 1 (cf. Theorem 15.22, applications, and (15.9)), and that the quotients $|\mu_i/\lambda_i|$ are not too close to 1.

(e) If (S) holds and $\{\lambda_i\}$ has constant signs and is monotonic, then

$$|\pi_2-1|\lesssim\varphi_3=\omega\varphi_1\cdot\max_{k\leqq i\leqq m}\frac{|\Delta\mu_i|}{|\lambda_i|(1-|\mu_i/\lambda_i|)^2}\quad\text{if }\varphi_3\ll 1;$$

hence π_2 may be expected to be a good deal closer to 1 than π_1.

(f) If (S) holds and $\{\lambda_i\}$ has constant signs and is monotonic, then

$$|\pi_2-1|\lesssim\varphi_4=\frac{\omega}{2}\left|\frac{1}{\lambda_m^2}-\frac{1}{\lambda_k^2}\right|\max_{k\leqq i\leqq m}\frac{|\mu_i\Delta\mu_i|}{(1-|\mu_i/\lambda_i|)^3}\text{ if }\varphi_4\ll 1.$$

(g) If (S) holds and $\{\mu_i\}$ has constant signs and is monotonic, then

$$|\pi_2-1|\lesssim\varphi_5=\frac{\omega}{2}|\mu_m^2-\mu_k^2|\max_{k\leqq i\leqq m}\frac{|\Delta\lambda_i|}{(|\lambda_i|-|\mu_i|)^3}\quad\text{if }\varphi_5\ll 1.$$

(h) In order to justify (e), (f) and (g) and to indicate under what circumstances they hold, we note that $\pi_2=\prod_k^{m-1}[1+w_i]$, with

$$\begin{aligned}|w_i|&\leqq\frac{\omega d_i'}{b_i'-e_i'}\,\frac{|\mu_i\Delta\mu_i|}{|\lambda_i(\lambda_i-\mu_{i+1})|}\\&\approx\frac{\omega|\lambda_i\Delta\lambda_i|}{|\lambda_i(\lambda_i-\mu_i)|-|\mu_i(\lambda_i-\mu_i)|}\cdot\frac{|\mu_i\Delta\mu_i|}{|\lambda_i(\lambda_i-\mu_i)|}\\&=\frac{\omega|\mu_i\Delta\mu_i\Delta\lambda_i|}{(|\lambda_i|-|\mu_i|)|\lambda_i-\mu_i|^2}\\&\leqq\frac{\omega|\mu_i\Delta\mu_i\Delta\lambda_i|}{(|\lambda_i|-|\mu_i|)^3}.\end{aligned}$$

Comparing the penultimate estimate for w_i with the penultimate expression for π_1 in (17.9) justifies (e). The observation that $\sum_k^{m-1}\mu_i\Delta\mu_i\approx\frac{1}{2}[\mu_m^2-\mu_k^2]$ and $\sum_k^{m-1}\Delta\lambda_i/\lambda_i^3\approx\frac{1}{2}[1/\lambda_k^2-1/\lambda_m^2]$ justifies (f) and (g).

For the *dominated* solutions, we have similarly from (10.8′), using $u_i=y_i^1+y_i^2=y_i^2(1+\theta_i'\sigma_i)$, $|\theta_i'|\leqq 1$:

THEOREM 17.17. *Let $|\lambda_i|>|\mu_i|$. Let for all $k\geqq 1$ the sequence $\{\tilde{\sigma}_i\}$,*

(17.18) $\quad\tilde{\sigma}_i=(c_i'+e_i'\tilde{\sigma}_{i+1})/(b_i'-d_i'\tilde{\sigma}_{i+1}),\qquad 0\leqq i\leqq k-1,\quad\tilde{\sigma}_k=0,$

be defined and nonnegative, and let $\{\sigma_i\}$ be a common majorant of all sequences $\{\tilde{\sigma}_i\}$. Then there are solutions $\{u_i\}$ of (17.1) (*dominated solutions*) *such that*

(17.19) $$\frac{|u_m|}{|u_k|}=\frac{1+\theta_m'\sigma_m}{1+\theta_k'\sigma_k}\pi_3\pi_4\prod_k^{m-1}|\mu_i|,$$

where

$$(17.20)\quad \pi_3 = \prod_{k}^{m-1}\left|1+\frac{\Delta\mu_i}{\lambda_{i+1}-\mu_{i+1}}\right| = \left|\frac{1-\mu_k/\lambda_{k+1}}{1-\mu_m/\lambda_m}\right| \prod_{k+1}^{m-1}\left|1+\frac{\mu_i\Delta\lambda_i}{\lambda_i\lambda_{i+1}(1-\mu_i/\lambda_i)}\right|,$$

$$(17.21)\quad \pi_4 = \prod_{k}^{m-1}\left[1+\theta_i\sigma_i\frac{\lambda_i\Delta\lambda_i}{\mu_i(\lambda_{i+1}-\mu_i)}\right],$$

with $\Delta\lambda_i=\lambda_{i+1}-\lambda_i$, $\Delta\mu_i=\mu_{i+1}-\mu_i$, $|\theta_i|\leqq 1$, $|\theta_i'|\leqq 1$, *and these solutions satisfy* $u_0=a+b$, $u_1=a\lambda_0+b\mu_0$ *with* $|a|\leqq\sigma_0|b|$.

Remark 17.22. (a) If $\{\lambda_i\}$ has constant signs and is monotonic, then for $\pi_3'=\prod_{k+1}^{m-1}|1+\mu_i\Delta\lambda_i/\lambda_i\lambda_{i+1}(1-\mu_i/\lambda_i)|$(cf. (17.20)),

$$1-\varphi_6\leqq\pi_3'\leqq\exp(\varphi_6)\quad\text{if } \varphi_6=\left|\frac{1}{\lambda_m}-\frac{1}{\lambda_{k+1}}\right|\max_{k+1\leqq i\leqq m-1}\frac{|\mu_i|}{1-\mu_i/\lambda_i}\leqq 1.$$

(b) If $\{\mu_i\}$ is monotonic, then

$$1-\varphi_7\leqq\pi_3\leqq\exp(\varphi_7)\quad\text{if } \varphi_7=|\mu_m-\mu_k|\cdot\max_{k+1\leqq i\leqq m}1/|\lambda_i-\mu_i|\leqq 1.$$

(c) Let (S′) denote the property that $\{\lambda_i\}$ and $\{\mu_i\}$ are slowly varying, that $\tilde{\sigma}_i\leqq\omega c_i/(b_i-e_i)$ for some constant ω (which will often be of the order of 1) and that the quotients $|\mu_i/\lambda_i|$ are not too close to 1.

(d) If (S′) holds and $\{\lambda_i\}$ has constant signs and is monotonic, then

$$|\pi_4-1|\lesssim\varphi_8=\omega\varphi_6\cdot\max_{k\leqq i\leqq m}\frac{|\Delta\mu_i|}{|\mu_i|(1-|\mu_i/\lambda_i|)^2}\quad\text{if } \varphi_8\ll 1.$$

(e) If (S′) holds and $\{\lambda_i\}$ has constant signs and is monotonic, then

$$|\pi_4-1|\lesssim\varphi_9=\omega\left|\frac{1}{\lambda_m}-\frac{1}{\lambda_k}\right|\max_{k\leqq i\leqq m}\frac{|\Delta\mu_i|}{(1-|\mu_i/\lambda_i|)^3}\quad\text{if } \varphi_9\ll 1.$$

(f) If (S′) holds and $\{\mu_i\}$ is monotonic, then

$$|\pi_4-1|\lesssim\varphi_{10}=\omega|\mu_m-\mu_k|\max_{k\leqq i\leqq m}\frac{|\lambda_i\Delta\lambda_i|}{(|\lambda_i|-|\mu_i|)^3}\quad\text{if } \varphi_{10}\ll 1.$$

18. Numerical examples. In these examples we consider four different cases for (17.1).

Example I. $\{\lambda_i\}$ and $\{\mu_i\}$ have finite limits of different modulus.

Example II. $\{\lambda_i\}\to\infty$, $\{\mu_i\}\to 0$.

Example III. $\{\lambda_i\}$ and $\{\mu_i\}$ have different finite limits of equal modulus.

Example IV. $\{\lambda_i\}$ and $\{\mu_i\}$ have equal limits.

In Examples I–III, the sequences of eigenvectors of $\{A_i\}$ automatically also have different limits as far as their directions are concerned; in Example IV they have equal limits.

Example I. We consider the recursion

$$(18.1)\qquad (2i+5)u_{i+2}+6(2i+3)u_{i+1}+(2i+1)u_i=0.$$

This (trivially solvable) recursion has as one solution the sequence of coefficients of odd index in the Chebyshev expansion of arctan (x) on $[-1, 1]$.

We find to reasonable accuracy for all $i \geqq 0$

$$\lambda_i \approx -6\frac{2i+3}{2i+5}, \qquad \mu_i \approx -\frac{1}{6}\frac{2i+1}{2i+3},$$

$$\Delta\lambda_i \approx -6\frac{1}{(i+3)^2}, \qquad \Delta\mu_i \approx -\frac{1}{6}\frac{1}{(i+2)^2},$$

$$b_i' \approx \lambda_i^2 \approx 35, \qquad c_i' \approx \frac{1}{9(2i+7)^2},$$

$$d_i' \approx \frac{36}{(i+4)^2}, \qquad e_i' \approx 1,$$

$$\alpha_i \approx \frac{4}{(2i+5)^2}, \qquad \beta_i \approx 300(2i+7)^2.$$

Theorem 5.2 yields $\rho_i \approx 0.16$. Theorem 5.6 yields $\tilde{a} \approx 2$, $\rho_i \approx 2\alpha_{i-1}$. Theorem 11.1 yields $\sigma_i \approx 0.0001$.

For the factors in (17.8) we find: the first factor differs from 1 by less than 0.01 (using $\rho_i = 0.16$), and by a much smaller amount if $\rho_0 = 0$, $k = -1$, $m > 2$ or if $k > 2$ (using $\rho_i < 8/(2i+5)^2$); $|\pi_1 - 1| \lesssim \varphi_1 \leqq 0.02$ (Remark 17.16(b)); $|\pi_2 - 1| \lesssim \varphi_3 \ll \varphi_1$ (Remark 17.16(e) with $\omega \approx 1$).

Hence $|u_{m+1}/u_{k+1}| \approx \prod_k^{m-1} |\lambda_i|$, with an error of only 3% or less (actual computation reveals no errors of over 1%).

For the factors in (17.19) we find: the first factor differs from 1 by at most 0.0002; $|\pi_3 - 1| \lesssim \varphi_7 \leqq 0.025$; $|\pi_4 - 1| \lesssim \varphi_8 \ll \varphi_6 \leqq 0.02$.

Hence $|u_{m+1}/u_{k+1}| \approx \prod_k^{m-1} |\mu_i|$, with an error of only $2\frac{1}{2}$% (actual computation shows errors of this size).

Thus our theory gives very accurate estimates for the dominant and dominated solutions.

Also, by Theorem 11.3, we have $L = L_1 \oplus L_2$.

Example II. We consider the recursion

$$u_{i+2} + (2i+2)u_{i+1} - u_i = 0. \tag{18.2}$$

This recursion has as one solution the sequence of coefficients in the Chebyshev expansion of exp (x) on $[-1, 1]$.

We find to reasonable accuracy for all $i \geqq 0$

$$\lambda_i \approx -(2i+2), \qquad \mu_i \approx \frac{1}{2i+2},$$

$$\Delta\lambda_i \approx -2, \qquad \Delta\mu_i \approx \frac{-2}{(2i+2)(2i+4)},$$

$$b_i' \approx (2i+2)^2, \qquad c_i' \approx \frac{2}{(2i+3)^3},$$

$$d_i' \approx 4i+4, \qquad e_i' \approx 1,$$

$$\alpha_i \approx \frac{1}{i+1}, \qquad \beta_i \approx \frac{(2i+2.7)^3}{2}.$$

Theorem 5.2 yields $\rho_i \approx 1$. Theorem 5.6 yields $\tilde{a} \approx 2$, but since the discriminant quotient (cf. Remark 15.8) of (15.21) is >4, we actually have $\tilde{a} \approx 1$. Hence we may take $\rho_0 \approx \alpha_0 \approx 1$ (or $\rho_0 = 0$), $\rho_i \approx \alpha_{i-1} \approx 1/i$, $i \geqq 1$. Similarly, $\sigma_i \approx 2/(2i+2.4)^5$, which is very small for all i.

Now remarks similar to those in Example I can be made, though the error bounds for $k=-1$ are now somewhat wider.

Example III. We consider the recursion ($p>0$)

$$u_{i+2} - \frac{2}{(i+1)^p} u_{i+1} - u_i = 0. \tag{18.3}$$

Then

$$\lambda_{i-1} \approx 1 + i^{-p}, \qquad \mu_{i-1} \approx -1 + i^{-p},$$

$$\Delta\lambda_{i-1} \approx -pi^{-p-1}, \qquad \Delta\mu_{i-1} \approx -pi^{-p-1},$$

$$b_{i-1}' \approx 2(1+i^{-p}), \qquad c_{i-1}' \approx pi^{-p-1},$$

$$d_{i-1}' \approx pi^{-p-1}, \qquad e_{i-1}' \approx 2(1-i^{-p}).$$

Hence $(b_i'-e_i')^2/4c_i'd_i' \approx 4p^{-2}(i+1)^2 > 1$ if $2(i+1) > p$. Hence, for $p<2$, the discriminant of (15.3) is positive for all values of $i \geqq 0$, whereas for $p \leqq 1$, the discriminant quotient is "large" and hence $\alpha_i \approx d_i'/(b_i'-e_i')$ and $\beta_i \approx (b_i'-e_i')/c_i'$. *We shall assume this henceforth.* Hence

$$\alpha_{i-1} \approx \tfrac{1}{4}pi^{-1}, \qquad \beta_{i-1} \approx 4p^{-1}i.$$

Note, however, that for larger values of i, this is true for much larger values of p. Theorem 5.2 yields $\rho_i \approx \frac{1}{4}p$. Theorem 5.6 is applicable only if $p \leqq 1$, since otherwise $\alpha_i/\alpha_{i-1} < e_i'/b_i'$ and, indeed, it can be shown that in that case $\{t_i\}$ (cf. (15.1)) does not tend to zero. If $p \leqq 1$, then $\tilde{a} \approx 4$ satisfies (and, indeed, since the discriminant quotient of (15.21) might again be called large, even $\tilde{a} \approx 2$ would be allowed). Hence, in this case $\rho_i \approx 4\alpha_{i-1}$ (or even $2\alpha_{i-1}$) is allowed. (This corresponds nicely with reality since $p=1$, $t_0=0$ yields $t_1 = \frac{1}{4}$, $t_2 \approx \frac{1}{6}$, $t_3 \approx \frac{1}{8}$, whereas $\alpha_0 \approx \frac{1}{4}$, $\alpha_1 \approx \frac{1}{8}$, $\alpha_2 \approx \frac{1}{12}, \cdots$).

From the foregoing, it is clear that the first factor at the right of (17.8) differs from 1 by less than 0.3 if $p \leqq 1$ and m is not too small, and this factor becomes arbitrarily small if $k=-1$ and m is large.

For the factor π_1 in (17.8), we have $|\pi_1 - 1| \lesssim \varphi_1 = \frac{1}{2}(\lambda_m^{-1} - \lambda_k^{-1}) \approx \frac{1}{2}(k^{-p} - m^{-p})$, and again $|\pi_2 - 1| \ll |\pi_1 - 1|$ (cf. Remark 17.16(e)) if, e.g., $k>3$.

In the case of this example, therefore, $\prod |\lambda_i|$ still represents the growth of dominant solutions very well, and similar things can be said about dominated solutions.

This shows that for $p \leqq 1$ the difference in growth character of the two kinds of solutions is quite marked, since then $\prod^m \mu_i / \prod^m \lambda_i \to 0$ for $m \to \infty$, which implies condition (B) in Theorem 9.8, hence $L = L_1 \oplus L_2$.

For $p > 1$, this is no longer the case.

Example IV. We consider the recursion

$$u_{i+2} - 2u_{i+1} + (1-(i+1)^{-2p})u_i = 0, \qquad 0 < p \leqq 1. \tag{18.4}$$

Then

$$\begin{aligned} \lambda_{i-1} &= 1 + i^{-p}, & \mu_{i-1} &= 1 - i^{-p}, \\ \Delta\lambda_{i-1} &\approx -pi^{-p-1}, & \Delta\mu_{i-1} &\approx pi^{-p-1}, \\ b'_{i-1} &\approx 2i^{-p}\lambda_{i-1}, & c'_{i-1} &\approx pi^{-p-1}\lambda_{i-1}, \\ d'_{i-1} &\approx pi^{-p-1}\mu_{i-1}, & e'_{i-1} &\approx 2i^{-p}\mu_{i-1}. \end{aligned}$$

Hence $(b'_{i-1} - e'_{i-1})^2 / 4c'_{i-1}d'_{i-1} \approx 4i^{2-2p}/p^2 \geqq 4$ (this causes the restriction $p \leqq 1$, since otherwise the discriminant would become negative for increasing i). Thus the discriminant quotient is again "large", implying

$$\alpha_{i-1} \approx \tfrac{1}{4}pi^{p-1}, \qquad \beta_{i-1} \approx 4i^{1-p}/p.$$

Theorem 5.2 yields $\rho_i \approx \frac{1}{2}p$. Theorem 5.6 yields $\tilde{a} \approx 2$, but since in (15.21) the discriminant is again large, $\tilde{a} \approx 1$ is more appropriate. Hence $\rho_i \approx \alpha_{i-1}$.

The factor π_1 in (17.8) is now unbounded; it is approximately equal to $(m/k)^{p/2}$. Hence the factor $\prod |\lambda_i|$ is now no longer a good estimate of the growth, although obviously $(m/k)^{p/2}$ is increasing only very slowly as m increases. Anyway, this factor π_1 allows a good estimate from above and from below.

The latter cannot be said of the factor π_2 in (17.8) if $p = 1$. This factor is then about $\prod_k^{m-1} (1 + \theta_i/8i) \approx (m/k)^{\theta/8}$, $|\theta| < 1$, and this means, therefore, an unbounded uncertainty factor. If, on the other hand, $p < 1$, this factor is about $1 + [p^2/8(1-p)](k^{p-1} - m^{p-1})$, bounded therefore.

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A SOLVABILITY THEOREM FOR HOMOGENEOUS FUNCTIONS*

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Abstract. Extensions of the solvability theorem of Farkas to include particular homogeneous functions have been proved by a number of authors and have been used to derive duality theorems for particular programming problems. In this paper, the subject of solvability theorems is approached from the viewpoint of the theory of convex sets and a fairly general theorem is derived. By working in an arbitrary real space rather than R^n, we get as special cases complex as well as real solvability theorems.

Introduction. Starting with the celebrated theorem of Farkas, a number of solvability or transposition theorems have been published and subsequently used to derive duality theorems. In many cases the solvability theorem is of the form: $\langle u, x\rangle \leqq h_0(x)$ for all x in some cone K if and only if u satisfies some stated condition. Here h_0 is some given positively homogeneous convex function. In the original Farkas lemma, h_0 is identically zero. In the theorem of Eisenberg [1], [2], generalized by Kaul [4] and Mond [5], h_0 is the square root of a positive semidefinite quadratic form and in the theorem of Smiley [7], h_0 is the support function of a compact convex set. In this paper, we derive a theorem for a more general homogeneous function h_0 which includes, for example, the results cited above. Furthermore, we use set theoretic rather than matrix methods so that our results show the relationship between solvability theorems and convexity theory and is not heavily dependent on polyhedrality of the cone K. Also by working in an arbitrary real space rather than R^n, we can get complex solvability theorems as special cases.

1. The polar of an intersection. Let X be a real finite-dimensional inner product space with the inner product denoted by $\langle \cdot , \cdot \rangle$. For any set S in X, we define the *polar of* S, denoted by S^0, by

$$S^0 = \{y \mid \langle x, y\rangle \leqq 1 \text{ for all } x \in S\}. \tag{1}$$

We will make use of the result, which will be referred to as the *bipolar theorem*, that if S is a closed convex set containing the origin then $S^{00} = S$ (see [6, Thm. 14.5].) Note that if K is a cone, then

$$K^0 = \{y \mid \langle x, y\rangle \leqq 0 \text{ for all } x \text{ in } K\}. \tag{2}$$

THEOREM 1.1. *Let K be a closed convex cone and let C be a closed convex set containing the origin. Then*

$$(K \cap C)^0 = \text{cl}\,(K^0 + C^0).$$

Proof. It is immediate from (2) that $K^0 + C^0 \subseteq (K \cap C)^0$ and since the polar of any set is closed we have $\text{cl}\,(K^0 + C^0) \subseteq (K \cap C)^0$. To get the reverse inclusion it

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will suffice to show that

$$(3) \qquad [\mathrm{cl}\,(K^0+C^0)]^0 \subseteq K \cap C$$

because cl (K^0+C^0) is a closed convex set containing the origin and taking polars reverses inclusion; hence if (3) is satisfied, then the bipolar theorem gives the desired reverse inclusion. Using the bipolar theorem and the fact that C^0 contains the origin, we have $(K^0+C^0)^0 \subseteq K^{00}=K$. Similarly we get $(K^0+C^0)^0 \subseteq C$. Finally, since a set and its closure have the same polar, (3) is verified and the theorem proved.

2. Support and gauge functions. In this section, we outline some properties of the gauge and support functions of a convex set. The properties we list are well known (e.g., see [6]) but we briefly outline proofs.

Let R'_+ denote the nonnegative real numbers with ∞ adjoined. Let X be a real finite-dimensional inner product space with the inner product denoted by $\langle \cdot , \cdot \rangle$. With each each set C in X we associate two functions: the *gauge function* $g(\cdot\,|C)$ and the *support function* $s(\cdot\,|C)$ defined as follows:

$$g(x|C)=\inf\{t>0|x\in tC\},$$
$$s(y|C)=\sup\{\langle x, y\rangle, x\in C\}.$$

We take the inf of the empty set to be $+\infty$ and we suppose that C contains the origin. Then both s and g map X into R'_+. It is clear that these functions are both *positively homogeneous*; i.e., they both satisfy $f(\tau x)=\tau f(x)$ for $\tau>0$. We cannot say the same for $\tau=0$ because either function may assume the value ∞.

For the rest of this section let C be a closed convex set containing the origin, let $g=g(\cdot\,|C)$ and let $s=s(\cdot\,|C)$.

LEMMA 2.1. *Both s and g are positively homogeneous and convex functions. (Therefore they are sub-additive.) Furthermore, s is lower semicontinuous and*

$$C=\{x|g(x)\leqq 1\}=\{x|\langle x, y\rangle \leqq s(y)\forall y\}.$$

The proof of this result is elementary. We only comment that the lower semicontinuity of s follows from the fact that it is the supremum of a family of linear, therefore continuous, functions. Also note that if $\langle x, y\rangle \leqq s(y)$ for all y, then the strict separating hyperplane theorem implies, by contradiction, that x lies in C.

The following simple observation, the proof of which we leave to the reader, allows us to relate the gauge and support functions of a set and its polar and is also generally useful.

LEMMA 2.2. *Let $h: x\to R'_+$ be a positively homogeneous function. Then*

$$\{y|\langle x, y\rangle \leqq h(x)\forall x\}=\{x|h(x)\leqq 1\}^0.$$

LEMMA 2.3.

$$g=g(\cdot\,|C)=s(\cdot\,|C^0),$$
$$s=s(\cdot\,|C)=g(\cdot\,|C^0).$$

Proof. It suffices to prove the first assertion, the second then following by the bipolar theorem. By Lemma 2.1,

$$C=(x|g(x)\leqq 1).$$

By applying Lemma 2.1 and Lemma 2.2, we get

$$C^0 = \{y|\langle x, y\rangle \leqq s(x|C^0)\forall x\} = \{x|s(x|C^0)\leqq 1\}^0.$$

The lower semicontinuity of a support function implies that $\{x|s(x|C^0)\leqq 1\}$ is closed, hence the bipolar theorem gives $C = C^{00} = \{x\,|\,s(x\,|\,C^0)\leqq 1\}$. Comparing our two representations for C we deduce that $g(x)\leqq 1$ if and only if $s(x|C^0)\leqq 1$. From the fact that both g and $s(\cdot\,|C^0)$ are positively homogeneous, it follows easily that the functions are equal.

LEMMA 2.4. *Let h be a convex, positively homogeneous and lower semicontinuous function from x into R'_+. Then h is the gauge function of some closed convex set containing the origin and it is also the support function of some such set.*

Proof. Let $C = \{x|h(x)\leqq 1\}$. Then C has all the stated properties. From Lemma 2.1 we have $h(x)\leqq 1$ if and only if $g(x|C)\leqq 1$, and by positive homogeneity this implies $h(x) = g(x|C)$ for all x. Finally, $h(x) = s(x|C^0)$.

3. The solvability theorem. Let a closed K and a closed convex set C containing the origin be given. Let the function h be defined by $h(x) = s(x\,|\,C)$ if $x\in K$, $h(x) = \infty$ otherwise. Then

$$\{x|h(x)\leqq 1\} = K\cap\{x|s(x|C)\leqq 1\} = K\cap C^0,$$

where we have used Lemmas 2.3 and 2.1. By Lemma 1.1,

$$\{x|h(x)\leqq 1\}^0 = (K\cap C^0)^0 = \mathrm{cl}\,[K^0 + C].$$

But by Lemma 2.2,

$$\{x|h(x)\leqq 1\}^0 = \{y|\langle x, y\rangle\leqq h(x)\forall x\} = \{y|\langle x, y\rangle \leqq s(x|C)\forall x\in K\}.$$

Thus we have proved that $\langle x, y\rangle\leqq s(x|C)$ for all x in K if and only if y is in $\mathrm{cl}\,(K^0 + C)$. Now we show that this result holds even without assuming that the origin lies in C. Suppose x_0 is a point of C. Then $C - x_0$ is a closed convex set containing the origin, and therefore from what has already been proved, we may write that

$$\langle x, y\rangle \leqq s(x|C - x_0) = s(x|C) - \langle x, x_0\rangle \quad \text{for all } x \text{ in } K$$

if and only if y is in $\mathrm{cl}\,(K^0 + C - x_0) = \mathrm{cl}\,(K^0 + C) - x_0$. Replacing y by $y + x_0$ we have therefore proved the following theorem.

THEOREM 3.1. *Let K be a closed convex cone and C a closed convex nonempty set. Then $\langle x, y\rangle\leqq s(x|C)$ for all x in K, if and only if $y\in\mathrm{cl}\,(K^0 + C)$.*

COROLLARY 3.1. *Suppose C and K are as above and $K^0 + C$ is closed. Then*

$$\langle x, y\rangle \leqq s(x|C) \quad \forall x\in K$$

if and only if there exist u and v satisfying

$$y = u + v, \qquad u\in K^0, \quad g(v|C)\leqq 1.$$

Note that if we are given an arbitrary positively homogeneous, convex lower semicontinuous function h from X to R_+, not identically ∞, we are able to apply the last two results with h in place of $s(x|C)$ by defining $C = \{x|h(x)\leqq 1\}^0$.

4. Special cases. In this section, we will show how the result of the preceding section includes a number of previously obtained alternative theorems and we will obtain a new specific result.

First, in Theorem 3.1, let $C=\{0\}$. Then $s(x|C)=0$ for all x and our theorem therefore says that $\langle x, y\rangle \leqq 0$ for all x in K if and only if y is in K^0; i.e., we get the Farkas lemma.

Now, suppose C is compact. Then K^0+C is closed and

$$s(x|C)=\sup\{\langle x, y\rangle, y\in C\}=\max\{\langle x, y\rangle, y\in C\}$$

and hence $\langle x; y\rangle \leqq s(x|C)$ if and only if there exists y_0 in C such that $\langle x, y\rangle \leqq \langle x, y_0\rangle$. Then Theorem 3.1 takes the following form:

$x\in K$ implies $\langle x, y-y_0\rangle \leqq 0$ for some y_0 in C if and only if $y\in K^0+C$.

Now let $X=C^m=m$-tuples of complex numbers. Note that X is a real space (as well as a complex space). Write the elements of X as columns and let superscript H denote conjugate transpose. Define $\langle x, y\rangle = \text{Re } y^H x$. It is easily verified that this is a legitimate inner product. Let $\mathscr{A}$ be an $n\times n$ matrix and S a closed convex cone in C^n. Define the cone K in X by $K=\{y|A^H\in S^0\}$. K is closed since a linear transformation is continuous. Furthermore, it is easily verified that $K^0=\text{cl } A(S)$. Hence Theorem 3.1 gives us almost immediately the following, which reduces exactly to the theorem of Smiley [7] in case S is a polyhedral cone.

THEOREM 4.1. *Let S be a closed convex cone in C^n, A an $m\times n$ matrix and C a closed convex set in C^m.*

$A^H x\in S^0$ *implies* $\text{Re } x^H(y-y_0)\leqq 0$ *for some* y_0 *in* C *if and only if* $y\in \text{cl } A(S)+C$.

As another special case we can get a result of Mond [5] which is an extension of a result of Kaul [4] which is itself an extension of a result of Eisenberg [1], [2]. To do this we need a lemma about positive semidefinite matrices.

LEMMA 4.1. *Let B be a positive semidefinite linear transformation on X. Let $H=\{x|\langle x, Bx\rangle \leqq 1\}$. Then BH is compact and $H^0=BH$.*

Proof. Let L be the nullspace of B. Restricted to $L^\perp$, B is nonsingular so there exists a positive constant δ_1 such that $\|Bx\|\geqq \delta_1\|x\|$ for all x in $L^\perp$. Also it is easily proved that $H=L\oplus(L^\perp\cap H)$. To show that BH is closed, let Z be in cl BH. There is a sequence $\{x_n\}$ in H such that $Bx_n\to Z$ and because of the above decomposition of H, we may suppose x_n lies in $L^\perp\cap H$. Then $\|x_n\|\leqq\|Bx_n\|/\delta_1\to \|Z\|/\delta_1$ and hence $\{x_n\}$ is a bounded sequence. The closure of BH follows from the fact that it has a convergent subsequence. To show that BH is bounded, we note that $BH=B(H\cap L^\perp)$. Since B restricted to $L\cap H$ is nonsingular, there exists a positive constant δ_2 such that $\langle Bx, x\rangle \geqq \delta_2\|x\|^2$ for x in $L^\perp\cap H$; hence for any x in $H\cap L^\perp$,

$$\|Bx\|^2\leqq\|B\|^2\cdot\|x\|^2\leqq\|B\|^2\cdot\langle Bx, x\rangle/\delta_2\leqq\|B\|^2/\delta_2.$$

Therefore BH is compact.

Now let us show that $BH=H^0$. For any y and x in H,

$$\langle By, x\rangle \leqq [\langle x, Bx\rangle\langle y, By\rangle]^{1/2} \leqq 1.$$

Therefore By lies in H^0. Now suppose u is in H^0 but not in BH. By the preceding part, BH is closed and it is surely convex, so u and BH may be strictly separated by a hyperplane; i.e., there exists a d such that

$$\langle d, u\rangle > 1 > \sup\{\langle d, x\rangle, x \in BH\} = \langle d, Bd\rangle^{1/2},$$

where we have used the Schwarz inequality to evaluate the supremum. But this is a contradiction because the last inequality implies that d lies in H and yet $\langle d, u\rangle > 1$; therefore $H^0 = BH$.

Now to use this lemma to get the desired solvability theorem let $C=\{x|\langle x, Bx\rangle \leqq 1\}^0$, where B is as in the lemma. Then the last lemma says $C = BH$.

$$s(x|C) = \sup\{\langle x, Bz\rangle | \langle z, Bz\rangle \leqq 1\} = \langle x, Bx\rangle^{1/2}.$$

With this choice of C, Theorem 3.1 gives

$$x \in K \quad \text{implies } \langle x, y\rangle \leqq \langle x, Bx\rangle^{1/2} \quad \text{if and only if} \quad y \in K^0 + BH,$$

where we have used the compactness of BH to get the closure of this sum. Choosing the space X and the inner product as in the preceding theorem, we get the following, which is exactly the theorem Mond gets in [5] in the case of polyhedral S. In order to get better notational agreement with [5] we put $-b$ in place of our y.

THEOREM 4.2. *Let S be a closed convex cone in C^m, A an $m \times n$ matrix and B an $n \times n$ positive semidefinite matrix. Then*

$$Ax \in S \quad \textit{implies } [x^H Bx]^{1/2} + \operatorname{Re} b^t \geqq 0$$

if and only if

$$-(b + Bv) \in \operatorname{cl}(A^H S^0) \quad \textit{for some } v \textit{ such that } v^t Bv \leqq 1.$$

To conclude this section, we give an example of a new concrete solvability theorem derived from Theorem 3.1. We get such a theorem for p norms, $p \geqq 1$. With $X = C^n$ let

$$\|x\|_p = \left[\sum_1^n |x_i|^p\right]^{1/p} \quad \text{for } 1 \leqq p < \infty,$$

$$\|x\|_\infty = \max\{|x_i|, i = 1, 2, \cdots, n\},$$

and let q be the *conjugate exponent*, i.e., let q be the unique solution to $p^{-1} + q^{-1} = 1$. When $p = 1$, we take $q = +\infty$. Let $C = \{x | \|x\|_q \leqq 1\}$. It is well known that $C^0 = \{y | \|y\|_p \leqq 1\}$. Also C is compact. Then taking, for the sake of familiarity of form, the case of a cone defined as in Theorem 4.2, we get the following from Theorem 3.1 and the results of § 2.

THEOREM 4.3. *Let S be a closed convex cone in C^m and A an $n \times m$ matrix. Let p and q be conjugate exponents. Then*

$$Ax \in S \quad \textit{implies } \operatorname{Re} y^H x \leqq \|x\|_p$$

if and only if

$$y - v \in \operatorname{cl} AS^0 \quad \textit{for some } v \textit{ satisfying } \|v\|_q \leqq 1.$$

5. More general spaces. The setting for all the results obtained so far has been a real finite-dimensional inner product space. Finite-dimensionality has been used explicitly only in Lemma 4.1 and implicitly it has been used by invoking the bipolar theorem and separating hyperplane theorem. Both these work in a Hilbert space so the results of this paper, except for those involving the positive semidefinite transformation, are valid for a Hilbert space. More generally, let X and Y be a dual pair of locally convex spaces. Now $\langle x, y\rangle$ does not mean the inner product of x and y but the continuous linear functional y evaluated at x or the continuous linear functional x evaluated at y. Then the separating hyperplane and bipolar theorems still hold ([3, p. 182 and p. 195]). The proof of Lemma 2.1 may easily be modified to work in a locally convex space, hence the main results obtained above hold in this more general setting.

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ON EXPANSION PROBLEMS: NEW CLASSES OF FORMULAS FOR THE CLASSICAL FUNCTIONS*

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Abstract. Using operators, a theorem is initially proved which generalizes a pair of known Lagrange expansions. These expansions are subsequently employed in the derivation of generating and multiplication theorems, giving new families of formulas involving the Gegenbauer, Laguerre, Hermite polynomials, as well as the Bessel and other functions. The salient feature of these results is that the degree of the polynomial is incorporated in the argument through an arbitrary parameter l. The known classical results present themselves by letting l go to zero. The second half of the paper deals with the more usual generating and multiplication theorems. With the aid of another pair of Lagrange expansions, we generalize known formulas of Fields and Wimp [7], Verma [19], and others.

1. Lagrange's generalization of Taylor's theorem is a powerful tool when trying to find interesting generating functions. For example, Jacobi [11] found his well-known generating function for Jacobi polynomials by use of Lagrange's theorem

$$\sum_{n=0}^{\infty} r^n P_n^{(\alpha,\beta)}(x) = 2^{\alpha+\beta}\rho^{-1}(1 - r + \rho)^{-\alpha}(1 + r + \rho)^{-\beta}, \tag{1.1}$$

where $\rho = (1 - 2xr + r^2)^{1/2}$, and

$$(1 - x)^{\alpha}(1 + x)^{\beta}P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1 - x)^{n+\alpha}(1 + x)^{n+\beta}]. \tag{1.2}$$

Elementary proofs of Lagrange's theorem have been given by Schur [16] and Carlitz [4]. These proofs, while elementary in a technical sense, are reasonably complicated. However, it is possible to give simple proofs for some of the key results which are usually proved by use of Lagrange's theorem. For example, Pólya and Szegö [14, pp. 301–302, problems 210 and 214] give the expansions

$$e^{-z} = \sum_{n=0}^{\infty} \frac{(\omega)^n(ln + 1)^{n-1}}{n!}, \qquad \omega = -ze^{zl}, \tag{1.3}$$

$$\frac{e^{-z}}{1 + zl} = \sum_{n=0}^{\infty} \frac{(\omega)^n(ln + 1)^n}{n!}. \tag{1.4}$$

Theorem 1 in this paper is a generalization of (1.3) and (1.4), obtained with the aid of operators. The subsequent theorems are then proved using (1.3) and (1.4). These theorems give new families of generating functions and expansions for the classical functions. The degree of the polynomial is incorporated in the argument through an arbitrary parameter l. Letting $l \to 0$ gives the well-known classical results.

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The second half of the paper is devoted to employing the problems 212 and 216 of Pólya and Szegö [14, pp. 301–302]. Again these Lagrange expansions aid in proving formulas that generalize equations (1.9) and (1.10) of Fields and Wimp [7], equation (11) of Srivastava [17], equation (3.1) of Verma [19]. Earlier workers in this area have been Al-Salam [1], Brafman [2], Brown [3], Carlitz [5], Chaundy [6], Gould and Hopper [9], Feldheim [8], Niblett [13], Toscano [18], Zeitlin [21], and others. The generalization of some of the classical polynomials have ramifications of interest. For example, the generalized Hermite polynomial treated by Gupta et al. [10] and others is a useful tool in probability distribution problems.

THEOREM 1. *Let β and l be real or complex. Then*

$$\sum_{k=0}^{\infty} \frac{(\beta)[-ze^{zl}]^k(lk+1)^k}{(k+\beta)k!} = \exp(-z)\,{}_1F_1[1;\beta+1;z(1-\beta l)], \tag{1.5}$$

where $|zl\exp(1-zl)| < 1$, and β is not a negative integer.

Proof. For the case l a nonnegative integer,

$$(Dx)\{(1-x^l)^n\} = \sum_{k=0}^{n} \frac{(-n)_k(lk+1)}{k!}x^{lk}, \tag{1.6}$$

$$(Dx)^n\{(1-x^l)^n\} = \sum_{k=0}^{n} \frac{(-n)_k(lk+1)^n}{k!}x^{lk}, \tag{1.7}$$

where $D \equiv d/dx$. Now

$$\int_0^1 x^{\beta l-1}(Dx)^n\{(1-x^l)^n\}\,dx = \int_0^1 x^{\beta l-1}\sum_{k=0}^{n}\frac{(-n)_k(lk+1)^n x^{lk}}{k!}\,dx. \tag{1.8}$$

Integrating the left-hand side by parts n times, and evaluating the right-hand side directly, one has

$$\sum_{k=0}^{n}\frac{(-n)_k(lk+1)^n}{k!(k+\beta)} = \frac{(1-\beta l)^n n!}{(\beta+1)_n}. \tag{1.9}$$

This equation holds for l a nonnegative integer. Now both sides are rational functions of l, hence the relation is true for complex values of l.

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(lk+1)^n}{k!(k+\beta)} = \sum_{n=0}^{\infty}\frac{(1-\beta l)^n z^n}{(\beta)(1+\beta)_n}. \tag{1.10}$$

By applying the transformation

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/s]} f(n,k) = \sum_{n,k=0}^{\infty} f(n+sk,k) \tag{1.11}$$

to the left-hand side of (1.10) gives the required theorem.

Taking $\beta = 1/l$ in Theorem 1 gives the result (1.3), and letting $\beta \to \infty$ gives (1.4).

2.

THEOREM 2. *Let $\{c_k\}$ be a sequence of arbitrary complex numbers, and let l be complex. Then for any positive integer s,*

$$\text{(2.1)}\quad \text{(a)}\qquad \sum_{n=0}^{\infty} \frac{t^n(ln+1)^n}{n!} F_n(x) = \frac{\exp(z)}{1-lz} \sum_{k=0}^{\infty} \frac{\{c_k\} x^k z^{sk}}{k!},$$

$$\text{(2.2)}\quad \text{(b)}\qquad \sum_{n=0}^{\infty} \frac{t^n(ln+1)^{n-1}}{n!} F_n(x) = \exp(z) \sum_{k=0}^{\infty} \frac{\{c_k\} x^k z^{sk}}{k!\,(1+lsk)},$$

where

$$\text{(2.3)}\qquad F_n(x) = \sum_{k=0}^{[n/s]} \frac{(-1)^{sk}(-n)_{sk}\{c_k\} x^k}{k!\,(ln+1)^{sk}},$$

$$t = z\exp(-zl), \quad |zl\exp(1-zl)| < 1,$$

$[n/s]$ is the greatest integer notation.

$$\text{(2.4)}\quad \text{(c)}\qquad \sum_{n=0}^{\infty} \frac{\omega^n G_n(x)}{n!} = \frac{\exp(z)}{1-slz} \sum_{n=0}^{\infty} \frac{\{c_n\} x^n z^{n/s}}{n!},$$

$$\text{(2.5)}\quad \text{(d)}\qquad \sum_{n=0}^{\infty} \frac{\omega^n G_n(x)}{n!\,(ln+1)} = \exp(z) \sum_{n=0}^{\infty} \frac{\{c_n\} x^n z^{n/s}}{n!\,(ln+1)},$$

where

$$\text{(2.6)}\qquad G_n(x) = \sum_{k=0}^{[n/s]} \frac{(-1)^{sk}(-n)_{sk}(ln+1)^k \{c_{n-sk}\} x^{n-sk}}{k!},$$

$\omega = z^{1/s}\exp(-lz)$, and $|zls\exp(1-zls)| < 1$.

On the right-hand side of each of the above equations, the infinite series is convergent.

Proof of (a).

$$\text{(2.7)}\qquad \sum_{n=0}^{\infty} \frac{(ze^{-zl})^n}{n!}(ln+1)^n \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}(-1)^{sk}\{c_k\} x^k}{k!\,(ln+1)^{sk}}$$

$$\text{(2.8)}\qquad = \sum_{n,k=0}^{\infty} \frac{\{c_k\}(z\,e^{-zl})^{sk} x^k (ln+lsk+1)^n (z\,e^{-zl})^n}{k!\,n!}.$$

Now

$$\text{(2.9)}\qquad \sum_{n=0}^{\infty} \frac{(ln+lsk+1)^n z^n \exp(-zln)}{n!} = \frac{\exp[z(lsk+1)]}{1-lz}.$$

Hence (2.8) simplifies to give the right-hand side of the theorem. The transformation

$$\text{(2.10)}\qquad \sum_{n=0}^{\infty} \sum_{k=0}^{[n/s]} f(n,k) = \sum_{n,k=0}^{\infty} f(n+sk,k)$$

is used in obtaining (2.8) from (2.7).

Proof of (b). For the proof of this equation, the same procedure is followed as above. The essential difference lies in using

$$\sum_{n=0}^{\infty} \frac{(ln + lsk + 1)^{n-1} z^n \exp(-zln)}{n!} = \frac{\exp[z(lsk + 1)]}{lsk + 1} \tag{2.11}$$

instead of (2.9).

Proof of (c).

$$\sum_{n=0}^{\infty} \frac{z^{n/s} e^{-lnz} x^n}{n!} \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}(-1)^{sk}(ln + 1)^k \{c_{n-sk}\}}{k!\, x^{sk}} \tag{2.12}$$

$$= \sum_{n,k=0}^{\infty} \frac{(ln + lsk + 1)^k x^n z^{n/s} e^{-lnz} z^k e^{-lskz} \{c_n\}}{n!\, k!}. \tag{2.13}$$

The summation over k is simplified with a suitable modification of (2.9) to give the right-hand side of (2.4).

Proof of (d). For the proof of this equation, a similar procedure to the proof of (c) is adopted. Instead of using (2.9) one employs (2.11) in the simplification process.

The following special cases of Theorem 2 appear to be new. We deduce four families of generating functions for the Hermite polynomial, two families for the Laguerre, and three families for the Gegenbauer polynomial.

Hermite polynomial. The representation [15, p. 191] is used in the following four cases, in Theorem 2.

1. From 2a, one obtains

$$\sum_{n=0}^{\infty} \frac{(\xi_1)^n}{n!} H_n[y(ln + 1)] = \frac{\exp[z - z^2/4y^2]}{1 - lz}, \tag{2.14}$$

where $2y\xi_1 = z \exp(-zl)$.

2. From 2b,

$$\sum_{n=0}^{\infty} \frac{(\xi_1)^n}{n!\,(ln + 1)} H_n[y(ln + 1)] = \exp(z)\, {}_1F_1\left[\begin{matrix} 1/2l; & -z^2 \\ 1/2l + 1; & \overline{4y^2} \end{matrix}\right]. \tag{2.15}$$

3. From 2c,

$$\sum_{n=0}^{\infty} \frac{(\xi_2)^n (ln + 1)^{(1/2)n}}{n!} H_n\left[\frac{y}{(ln + 1)^{1/2}}\right] = \frac{\exp(2yz^{1/2} - z)}{1 + 2lz} \tag{2.16}$$

where $\xi_2 = z^{1/2} \exp(lz)$.

4. From 2d,

$$\sum_{n=0}^{\infty} \frac{(\xi_2)^n (ln + 1)^{((1/2)n - 1)}}{n!} H_n\left[\frac{y}{(ln + 1)^{1/2}}\right] = \exp(-z)\, {}_1F_1\left[\begin{matrix} 1/l; & \\ 1/l + 1; & 2yz^{1/2} \end{matrix}\right].$$

(2.17)

For equations (2.14) and (2.15), $|zl \exp(1 - zl)| < 1$. For (2.16) and (2.17) $|2zl \exp(1 - 2zl)| < 1$.

Letting $l \to 0$ in all the above special cases gives the known generating function [15, p. 190].

Laguerre polynomial. Using the representation [15, p. 203] in 2a and 2b, respectively, gives the two special cases below.

1.

$$\sum_{n=0}^{\infty} \frac{t^n(ln+1)^n}{(1+a)_n} L_n^a\left[\frac{x}{ln+1}\right] = \frac{\Gamma(a+1)\,e^z}{(1-lz)(xz)^{(1/2)a}} J_a([4xz]^{1/2}), \tag{2.18}$$

where $t = z\exp(-zl)$.

2.

$$\sum_{n=0}^{\infty} \frac{t^n(ln+1)^{n-1}}{(1+a)_n} L_n^a\left[\frac{x}{ln+1}\right] = \exp(z)\, {}_1F_2\left[\begin{matrix}1/l; \\ 1+a, 1/l+1;\end{matrix} \quad -xz\right]. \tag{2.19}$$

For (2.18) and (2.19), $|zl\exp(1-zl)| < 1$.

When $l \to 0$ the known result [15, p. 201] is obtained in both cases.

Gegenbauer polynomial.

1. Using [15, p. 280], in 2a,

$$\begin{aligned}\sum_{n=0}^{\infty} & \frac{t^n[(ln+1)^2+y^2]^{(1/2)n}}{(2v)_n} C_n^v\left[\frac{(ln+1)}{[(ln+1)^2+y^2]^{1/2}}\right] \\ &= \frac{e^z}{1-lz}\, {}_0F_1[-;\ v+\tfrac{1}{2};\ -y^2z^2/4] \\ &= \frac{\Gamma(v+\frac{1}{2})\,e^z 2^{v-(1/2)}}{(1-lz)(yz)^{v-(1/2)}} J_{v-(1/2)}(yz),\end{aligned} \tag{2.20}$$

where $t = z\exp(-zl)$. For $l = 0$, the known result [15, p. 278] is obtained.

2. Using [15, p. 280], in 2b,

$$\begin{aligned}\sum_{n=0}^{\infty} & \frac{t^n[(ln+1)^2-4x]^{(1/2)n}}{(2v)_n(ln+1)} C_n^v\left[\frac{(ln+1)}{[(ln+1)^2-4x)]^{1/2}}\right] \\ &= \exp(z)\, {}_1F_2\left[\begin{matrix}1/2l; \\ v+\frac{1}{2}, 1/2l+1;\end{matrix} \quad xz^2\right].\end{aligned} \tag{2.21}$$

$l \to 0$ gives the known Bessel function generator [15, p. 278].

3. Using [15, p. 282], in 2b,

$$\begin{aligned}\sum_{n=0}^{\infty} & \frac{t^n(ln+1)^{(1/2)n-1}(ln+1+x)^{(1/2)n}}{(2v)_n} C_n^v\left[\frac{2ln+2+x}{2(ln+1)^{1/2}(ln+1+x)^{1/2}}\right] \\ &= \exp(z)\, {}_2F_2\left[\begin{matrix}v, 1/l; \\ 2v, 1/l+1;\end{matrix} \quad xz\right].\end{aligned} \tag{2.22}$$

Letting $l \to 0$ gives [15, p. 278]. For equations (2.20), (2.21), and (2.22), $|zl\exp(1-zl)| < 1$.

3.

THEOREM 3. *Let $\{c_k\}$ and $\{d_k\}$ be sequences of arbitrary complex numbers, and let l be complex. Then for any positive integer s and nonnegative integer r,*

$$\text{(a)} \qquad \sum_{n=0}^{\infty} \frac{(ln+1)^{n-1}}{n!} F_n(x)A_n(z) = \sum_{k=0}^{\infty} \frac{\{c_k\}\{d_{sk}\}x^k z^{sk}}{k!\,(1+lsk)}, \tag{3.1}$$

where $F_n(x)$ *is defined through* (2.3), *and*

(3.2) $$A_n(z) = \sum_{p=0}^{\infty} \frac{(-1)^p(ln+1)^p\{d_{n+p}\}z^{n+p}}{p!},$$

(3.3) (b) $$x^r = \frac{(-1)^{rs}(1+lrs)r!}{(rs)!\{c_r\}} \sum_{n=0}^{rs} \frac{(ln+1)^{rs-1}(-rs)_n}{n!} F_n(x).$$

(3.4) (c) $$\sum_{n=0}^{\infty} \frac{1}{(ln+1)n!} G_n(x) B_n(z) = \sum_{n=0}^{\infty} \frac{\{c_n\}\{d_n\}(xz)^n}{n!(ln+1)},$$

where $G_n(x)$ *is defined through* (2.6), *and*

(3.5) $$B_n(z) = \sum_{p=0}^{\infty} \frac{(-1)^p(ln+1)^p\{d_{n+sp}\}z^{n+sp}}{p!}.$$

(3.6) (d) $$x^r = \frac{(lr+1)r!}{\{c_r\}} \sum_{k=0}^{[r/s]} \frac{(-1)^k(lr-lsk+1)^{k-1}}{k!(r-sk)!} G_{r-sk}(x),$$

where $A_n(z)$ *and* $B_n(z)$ *are convergent series, and the right-hand side of* (a) *and* (c) *are convergent series. The domain of convergence is as in Theorem* 2.

Proof of (a). From Theorem 2b,

(3.7) $$\sum_{n=0}^{\infty} \frac{(ln+1)^{n-1}z^n \exp\{-z-zln\}}{n!} \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}\{c_k\}x^k}{k!(ln+1)^{sk}} = \sum_{k=0}^{\infty} \frac{x^k z^{sk}\{c_k\}}{k!(1+lsk)}.$$

On the left-hand side, let

(3.8) $$\exp\{-z-zln\} = \sum_{p=0}^{\infty} \frac{(-1)^p(ln+1)^p z^p}{p!}.$$

With this change, and taking transforms with respect to z on both sides of the equation, introduces the arbitrary sequence $\{d_n\}$.

Proof of (b). The expansion formula presents itself with the aid of

(3.9) $$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f(n,k) = \sum_{k=0}^{\infty} \sum_{n=0}^{k} f(n,k-n),$$

and equating coefficients of z on both sides of the resulting equations.

Proof of (c). Using Theorem 2d as we have used Theorem 2b in the proof of part (a), gives equation (3.4).

Proof of (d). Following the same procedure as in the proof of (b), but employing instead

(3.10) $$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f(n,k) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/s]} f(n-sk,k),$$

proves (3.6).

For $l=0$, $s=1$, Theorem 3 reduces to essentially equation (1.10) of Fields and Wimp [7].

All the following special cases of Theorem 3 appear to be new. Other new results for the classical functions may also be derived, as required.

1. From 3b, the Hermite polynomial representation yields

$$\frac{1}{y^{2p}} = \frac{(-1)^p(1+2lp)}{(\frac{1}{2})_p} \sum_{n=0}^{2p} \frac{(ln+1)^{2p-n-1}(-2p)_n}{n!\,(2y)^n} H_n[(ln+1)y]. \tag{3.11}$$

For $l = 0$, a known result is obtained.

2. From 3d,

$$y^p = \frac{(lp+1)p!}{2^p} \sum_{n=0}^{[p/2]} \frac{(lp-2ln-1)^{(1/2)p-1}}{n!(p-2n)!} H_{p-2n}\left[\frac{y}{(lp-2ln+1)^{1/2}}\right]. \tag{3.12}$$

For $l = 0$, [15, p. 194] presents itself.

3. From 3b, the Laguerre polynomial representation gives

$$y^p = (1+a)_p(1+lp) \sum_{n=0}^{p} \frac{(ln+1)^{p-1}(-p)_n}{(1+a)_n} L_n^a\left[\frac{y}{ln+1}\right]. \tag{3.13}$$

The known result [15, p. 207] is obtained for $l = 0$.

4. Using [15, p. 280], and 3b,

$$\begin{aligned} y^p = {} & \frac{(v+\frac{1}{2})_p(1+2lp)}{2^{2p}(\frac{1}{2})_p} \sum_{n=0}^{2p} \frac{(ln+1)^{2p-n-1}(-2p)_n[(ln+1)^2-4y]^{(1/2)n}}{(2v)_n} \\ & \cdot C_n^v\left[\frac{ln+1}{[(ln+1)^2-4y]^{1/2}}\right], \end{aligned} \tag{3.14}$$

for the Gegenbauer polynomial.

5. Using the representation [15, p. 108] in 3a gives a type of generating function for the Bessel function

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(t+ln+1)^n y^n}{n!\,2^n(ln+1)^{(1/2)(n+a+2)}} J_{n+a}[y(ln+1)^{1/2}] \\ & = \frac{2^{-a}y^a}{\Gamma(a+1)} {}_1F_2[1/l;\, a+1,\, 1/l+1;\, ty^2/4]. \end{aligned}$$

$l \to 0$ gives the known result [20, p. 141].

4.

THEOREM 4. *Let $\{e_k\}$ be a sequence of arbitrary complex numbers, and let α, l, and m be complex. Then for any positive integer s,*

$$\text{(a)} \quad \sum_{n=0}^{\infty} \frac{v^n}{n!} \frac{(\alpha)_{mn}}{(\alpha)_{(m-1)n}} C_n(x) = \frac{(1-z)^\alpha}{[1+z(m-1)]} \Phi(x,z). \tag{4.1}$$

$$\text{(b)} \quad \sum_{n=0}^{\infty} \frac{v^n}{n!} \frac{(\alpha)_{mn}}{(\alpha+1)_{(m-1)n}} D_n(x) = (1-z)^\alpha \Phi(x,z), \tag{4.2}$$

where

$$C_n(x) = \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}(\alpha + mn)_{lk}\{e_k\}x^k}{k!(\alpha + 1 + mn - n)_{(s+l)k}}, \tag{4.3}$$

$$D_n(x) = \sum_{k=0}^{[n/s]} \frac{((\alpha + msk + lk)/\alpha)(-n)_{sk}(\alpha + mn)_{lk}\{e_k\}x^k}{k!(mn - n + \alpha + 1)_{(s+l)k}} \tag{4.4}$$

$$\Phi(x, z) = \sum_{k=0}^{\infty} \frac{\{e_k\}x^k z^{sk}(1 - z)^{lk}}{k!} \tag{4.5}$$

and $v(1 - z)^m = (-z)$, $v(0) = 0$, *and* $\Phi(x, z)$ *is convergent.* z *lies in the connected component of the origin of the set which satisfies*

$$|z| < 1 \quad and \quad \left| \frac{m^m z}{(m - 1)^{m-1}(1 - z)^m} \right| < 1.$$

Proof of (a).

$$\sum_{n=0}^{\infty} \frac{(\alpha)_{mn}(-z)^n}{n!\,(\alpha)_{(m-1)n}(1 - z)^{mn}} \sum_{k=0}^{[n/s]} \frac{(-n)_{sk}(mn + \alpha)_{lk}\{e_k\}x^k}{k!\,(mn - n + \alpha)_{(s+l)k}} \tag{4.6}$$

$$= \sum_{k=0}^{\infty} \frac{z^{sk}\{e_k\}x^k}{k!\,(1 - z)^{msk}} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + msk + lk + mn)(-z)^n}{n!\,\Gamma(\alpha + msk + lk + mn - n)(1 - z)^{mn}}. \tag{4.7}$$

Using

$$\sum_{n=0}^{\infty} \frac{\Gamma(\alpha + msk + lk + mn)(-z)^n}{n!\,\Gamma(\alpha + msk + lk + mn - n)(1 - z)^{mn}} = \frac{(1 - z)^{\alpha + msk + lk}}{[1 + z(m - 1)]} \tag{4.8}$$

gives the right-hand side of the theorem.

The transformations

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/s]} f(n, k) = \sum_{k,n=0}^{\infty} f(n + sk, k) \quad \text{and} \quad (-n)_{sk} = \frac{(-1)^{sk} n!}{(n - sk)!} \tag{4.9}$$

are used in going from (4.6) to (4.7).

Proof of (b). The same procedure as above is followed except the transformation

$$\sum_{n=0}^{\infty} \frac{(msk + \alpha + lk)_{mn}(-z)^n}{n!\,(msk + lk + 1)_{(m-1)n}(1 - z)^{mn}} = (1 - z)^{msk + \alpha + lk} \tag{4.10}$$

is used.

For $l = 0$, Theorem 4a reduces to the generating function of Srivastava [17, p. 32]. For Theorem 4b, there is no corresponding known result for $l = 0$. However, Brown [3, eq. 8, p. 265] is a special case with $l = 0$, $s = 1$, and $m = \frac{1}{2}$.

Special cases. From 4b, we obtain as a special case a hypergeometric transformation

$$(4.11)\quad {}_{m+1}F_m\left[\begin{matrix}\alpha/m, \Delta(\alpha - a + 1, m); \\ \alpha/m + 1, \Delta(\alpha - a + 1, m - 1);\end{matrix}\ \frac{-zm^m}{(1 - z)^m(m - 1)^{m-1}}\right] = (1 - z)^\alpha\, {}_2F_1\left[\begin{matrix}\alpha/m, a; \\ \alpha/m + 1;\end{matrix}\ z\right],$$

where $\Delta(c, m) \equiv c/m, (c + 1)/m, \cdots, (c + m - 1)/m$. Equation (4.11) is deduced by letting $l = 0$, $s = 1$, $x = 1$, $e_k = (a)_k/(\alpha + mk)$ in (4.2). z lies in the connected component of the origin of the set which satisfies

$$|z| < 1 \quad \text{and} \quad \left|\frac{m^m z}{(m - 1)^{m-1}(1 - z)^m}\right| < 1.$$

From 4a

$$(4.12)\quad \sum_{n=0}^{\infty} \frac{(\alpha)_{mn} v^n}{n!\,(\alpha)_{mn-n}}\, {}_{s+l}F_{s+l}\left[\begin{matrix}\Delta(-n, s), \Delta(\alpha + mn, l); \\ \Delta(\alpha + mn - n, s + l);\end{matrix}\ \frac{xs^s l^l}{(s + l)^{s+l}}\right] = \frac{(1 - z)^\alpha}{(1 + zm - z)} \exp\,[xz^s(1 - z)^l],$$

where $v(1 - z)^m = (-z)$, $v(0) = 0$.

The above polynomial is a type of generalized Laguerre polynomial. $s = 1$, $l \to 0$ gives a result of Carlitz [5, eq. 8].

From 4b,

$$(4.13)\quad \sum_{n=0}^{\infty} \frac{(\alpha)_{mn} v^n}{n!\,(\alpha + 1)_{mn-n}}\, {}_{s+l}F_{s+l}\left[\begin{matrix}\Delta(-n, s), \Delta(\alpha + mn, l); \\ \Delta(\alpha + 1 + mn - n, s + l);\end{matrix}\ \frac{xs^s l^l}{(s + l)^{s+l}}\right] = (1 - z)^\alpha\, {}_1F_1\left[\begin{matrix}\dfrac{\alpha}{ms + l}; \\ \dfrac{\alpha}{ms + l} + 1;\end{matrix}\ xz^s(1 - z)^l\right].$$

From 4a,

$$(4.14)\quad \sum_{n=0}^{\infty} \frac{(\alpha)_{mn} v^n}{n!\,(\alpha)_{mn-n}}\, {}_{s+l+1}F_{s+l}\left[\begin{matrix}\Delta(-n, s), \Delta(\alpha + mn, l), c; \\ \Delta(\alpha + mn - n, s + l);\end{matrix}\ \frac{xs^s l^l}{(s + l)^{s+l}}\right] = \frac{(1 - z)^\alpha}{(1 + zm - z)}\{1 - xz^s(1 - z)^l\}^{-c}.$$

A corresponding expression to (4.14) may be deduced from Theorem 4b. A known generating function for the Jacobi polynomial [15, p. 254] and two known generating functions for the Gegenbauer polynomial [15, p. 277, p. 279] are special cases of Theorem 4. A generating function for the generalized Hermite polynomial [10, (2.1)] may also be deduced. The above examples of Theorem 4 are not among those obtained by other workers.

5.

THEOREM 5. *Let* $\{e_k\}$ *and* $\{f_k\}$ *be sequences of arbitrary complex numbers, and let* α, l, *and* m *be complex. Then for any positive integer* s,

$$\text{(5.1)}\quad \text{(a)}\qquad \sum_{n=0}^{\infty} \frac{(-1)^n(\alpha)_{mn}}{n!(\alpha+1)_{mn-n}} D_n(x)E_n(y) = \theta(x, y),$$

where

$$\text{(5.2)}\qquad E_n(y) = \sum_{p=0}^{\infty} \frac{(mn+\alpha+ln/s)_{(l/s+1)p}\{f_{n+p}\}\, y^{n+p}}{p!\,(mn+\alpha+1+ln/s)_{lp/s}},$$

$$\text{(5.3)}\qquad \theta(x, y) = \sum_{k=0}^{\infty} \frac{\{e_k\}\{f_{sk}\}x^k y^{sk}}{k!}$$

and $D_n(x)$ *is defined by* (4.4).

$$\text{(5.4)}\quad \text{(b)}\qquad x^r = \sum_{n=0}^{rs} \mu_n D_n(x),$$

where

$$\text{(5.5)}\qquad \mu_n = \frac{(-1)^n(\alpha+msr+lr)(\alpha+mn+ln/s)(\alpha+mn-n)_{lr+sr}}{(\alpha+mn)(\alpha+mn-n)n!\,(rs-n)!\,(\alpha+mn+1)_{lr}\{e_r\}}.$$

$E_n(y)$ and $\theta(x, y)$ are convergent series. The domain of convergence is as in Theorem 4.

Proof of (a). In Theorem 4b, let

$$\text{(5.6)}\qquad (1-z)^{-mn-\alpha-ln/s} = \sum_{p=0}^{\infty} \frac{(mn+\alpha+ln/s)_{(l/s+1)p} z^p (1-z)^{lp/s}}{(mn+\alpha+1+ln)_{lp/s}\, p!}.$$

Making the change of variable

$$y = z(1-z)^{l/s}$$

and taking transforms of both sides of the resulting equation creates the function $E_n(y)$.

Proof of (b). (5.4) is proved by comparing coefficients of y on both sides of (5.1), suitably transformed.

In Theorem 5a for $s = 1$, $l = 0$, one obtains essentially equation (3.1) [19] of a recent result by Verma. Note that there appears to be a misprint in his equation. The factor $[(e_u) + k]_{s+n-k}$ has been absorbed in c_k and d_{s+n} and should be omitted. The Theorem 5 is also a generalization of the two sets of equations (1.9) and (1.10) given by Fields and Wimp [7]. Letting $m = 1$, $l = 1$, $s = 1$, one obtains essentially (1.9), and $m = 1$, $l = 0$, $s = 1$, gives equation (1.10).

The generalized Hermite polynomial is defined by Gupta and Jain [10] as

$$\text{(5.7)}\qquad H_{n,s}(\omega, \lambda) = \sum_{k=0}^{[n/s]} \frac{n!\,\omega^{n-sk}\lambda^k}{(n-sk)!\,k!}.$$

From 5a and 5b, respectively, one obtains

$$\text{(5.8)}\qquad \sum_{n=0}^{\infty} \frac{(z/2\omega)^n H_{n,s}(\omega, \lambda) J_{a+n}(z)}{n!} = \frac{(\tfrac{1}{2}z)^{a+1}}{\Gamma(a+1)}\, {}_0F_s\left[-; \frac{a+1}{s}, \ldots, \frac{a+s}{s}; \frac{\lambda z^{2s}}{(4\omega)^s}\right]$$

and

$$\lambda^r = \sum_{n=0}^{rs} \frac{(-\omega)^{rs-n} r!}{n!\,(rs-n)!} H_{n,s}(\omega, \lambda). \tag{5.9}$$

The above special cases are among those results not derivable from either Fields and Wimp, or Verma. For known special cases of the multiplication theorem, which are many and varied, see Chapter 9 of Luke [12] for an excellent exposition.

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HALF-PLANE REPRESENTATIONS AND HARMONIC CONTINUATION*

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Abstract. Representations of boundary-integral type are presented for solutions of Laplace's equation on the half-plane. These are derived using related Lions–Magenes inspired results of Saylor [9], [11], and conformal mapping. Applications to numerical harmonic continuation are briefly discussed. Results carry over to R^n_+.

1. Introduction. The object of this paper is to present two representations of boundary-integral type for solutions of Laplace's equation on the half-plane. Also, we will discuss briefly how such representations can be used to numerically continue harmonic functions to all or part of the half-plane given approximate function values on particular finite subsets of points.

For the sake of simplicity, the functions involved in this paper will be taken as real-valued, and results will be discussed only for the plane R^2. However, the discussion can be carried over to complex-valued functions and to finite-dimensional spaces of higher dimension (also to solutions of more general elliptic type equations).

The main representation comes from using Lions–Magenes [6] inspired results of Saylor [9], [11] along with conformal mapping. It has the form

$$u(x)=\sum_{q=0}^{\infty}\int_{-\infty}^{\infty}\tilde{\Delta}_y^{(q)}\tilde{P}(x_1,x_2;y)\,d\tilde{\mu}_q(y), \tag{1.1}$$

where $\tilde{P}$ is a function related to the Poisson kernel, $\tilde{\Delta}_y^{(q)}$ is the qth iterate of an operator related to the Laplacian, the $\{\tilde{\mu}_q\}$ are Borel measures whose total variations satisfy a boundedness condition, $x=(x_1,x_2)$, and u has boundary values in a (distributional) space to be specified later. The second representation to be presented will be an offshoot of the one above.

As concerns the applications segment of the paper and continuation, it is important to point out that the process of harmonic continuation is unstable. That this is so can be seen by considering the standard example which involves the harmonic functions

$$u_k(z)=\operatorname{Re}\left(\frac{z}{R}\right)^k,\qquad z=x_1+ix_2, \tag{1.2}$$

which converge to zero on $\{|z|<R\}$ and diverge otherwise. The imposition of global a priori bounds eliminates the difficulties arising from instability.

Related boundary-integral results for both elliptic and parabolic problems are found in [5], [7], [9], [11]. Of particular interest with respect to continuation procedures here are the papers [1], [2], [3], [4], [8], [10].

2. A unit disk result. Before treating the half-plane situation, we present a representation for (distributional) solutions of Laplace's equation on the unit disk.

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For the sake of completeness, its proof will be indicated briefly. This result, in more generality, is due to Saylor [9], [11], and the reader is referred to his work for more detail since the discussion here is only sketchy. In the next section, we will use the unit disk representation and conformal mapping to obtain the half-plane results.

First we need some notation and some spaces. Let $x=(x_1, x_2)$ be a typical point in Euclidean R^2 space with $|x|=\sqrt{x_1^2+x_2^2}$. Take $U=\{|x|<1\}$ to be the unit open disk, $\bar{U}$ its closure, and $\partial U=\{|x|=1\}$ as the boundary. Also let $\Delta\equiv\partial^2/\partial x_1^2+\partial^2/\partial x_2^2$ be the Laplace operator.

We designate by $\mathscr{E}(U)$ the space of all infinitely differentiable functions on U and by $\mathscr{D}(U)$ Those functions in $\mathscr{E}(U)$ with compact support. Both of these are equipped with the usual Schwartz topologies. $\mathscr{E}'(U)$ and $\mathscr{D}'(U)$ represent the strong duals of $\mathscr{E}(U)$ and $\mathscr{D}(U)$ and are the standard distribution spaces of Schwarz.

For K any compact subset, let

$$\|\phi\|_{C(K)}\equiv\sup_{x\in K}|\phi(x)| \tag{2.1}$$

Then, $C^\infty(K)$ is the space of all infinitely differentiable functions on K provided with the norm (2.1).

The next spaces, as will be seen shortly, are connected with the traces of the harmonic distributions. Define $H(\partial U)$ to be the space of all real analytic functions ϕ on ∂U. It can be described equivalently as the space of all functions $\phi\in C^\infty(\partial U)$ for which positive constants A and B exist such that

$$\|\Delta^{(q)}\phi\|_{C(\partial U)}\leqq A(2q)!B^q, \qquad q=0, 1, \cdots. \tag{2.2}$$

In (2.2), Δ is the Laplace–Beltrami operator and the superscript (q) indicates a qth iterate. If (x_1, x_2) is replaced by polar coordinates (r, θ), then $\Delta=\partial^2/\partial\theta^2$ on ∂U. Occasionally, a subscript will be appended to operators such as Δ to denote the particular variable with respect to which the operator acts.

$H(\partial U)$ is topologized by writing it as

$$H(\partial U)=\bigcup_B H_B$$

and with respect to this giving it the inductive limit topology. In the above,

$$H_B=\left\{\phi\in C^\infty(\partial U)\,\middle|\, \|\phi\|_B\equiv\sup_q\frac{\|\Delta^{(q)}\phi\|_{C(\partial U)}}{(2q)!B^q}<\infty\right\}. \tag{2.3}$$

Each H_B is equipped with the topology provided by the norm $\|\cdot\|_B$ and is a Banach space.

Define $H'(\partial U)$ to be the dual of $H(\partial U)$. Elements h in $H'(\partial U)$ can be represented as

$$h(\phi)=\sum_{q=0}^{\infty}\int_{\partial U}\Delta^{(q)}\phi(x)\,d\mu_q(x) \tag{2.4}$$

for all $\phi \in H(\partial U)$, where the $\{\mu_q\}$ are Borel measures on ∂U satisfying

$$\text{var}\,(\mu_q) \leqq \frac{A(\varepsilon)\varepsilon^q}{(2q)!}, \qquad q = 0, 1, \cdots, \tag{2.5}$$

for any $\varepsilon > 0$. In (2.5), $A(\varepsilon)$ is a positive function which in general tends to infinity as ε approaches zero; var (μ_q) is the total variation of μ_q.

The unit disk representation will hold for distributions in the space

$$S = \{u \in \mathscr{D}'(U) | \Delta u = 0\}, \tag{2.6}$$

where Δu is taken in the distributional sense and S is provided with the induced $\mathscr{D}'(U)$ topology. Note that members of S can also be regarded as classical solutions since the distributional solutions can be corrected on a set of measure zero to make this so.

For elements in S, it is possible to define a linear trace operator γ which maps S onto $H'(\partial U)$ and which for functions $u \in C^\infty(\bar{U}) \cap S$ just gives the restriction of u to ∂U. Thus the functions in S can be considered as that class of harmonic functions whose boundary traces fall in $H'(\partial U)$.

Note that it can be shown that the problem

$$\begin{aligned} \Delta u &= 0, \qquad u \in \mathscr{D}'(U), \\ \gamma(u) &= u_0, \qquad u_0 \in H'(\partial U), \end{aligned} \tag{2.7}$$

has a unique solution. Also note that γ defines a topological and algebraic isomorphism between S and $H'(\partial U)$. The topologies involved are the induced weak * of $\mathscr{D}'(U)$ for S and the weak * for $H'(\partial U)$.

We are now ready for the unit disk representation.

THEOREM 2.1. *Let u be a harmonic solution belonging to S. Then it has representation*

$$u(x) = \sum_{q=0}^{\infty} \int_0^{2\pi} \Delta_t^{(q)} P(r, \theta; t)\, d\mu_q(t), \tag{2.8}$$

where the $\{\mu_q\}$ are Borel measures on $[0, 2\pi]$ satisfying

$$\text{var}\,(\mu_q) \leqq \frac{A(\varepsilon)\varepsilon^q}{(2q)!}, \qquad q = 0, 1, \cdots, \tag{2.9}$$

for any $\varepsilon > 0$. In (2.8), $x = re^{i\theta}$,

$$P(r, \theta; t) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}, \tag{2.10}$$

and $A(\varepsilon)$ is a positive function which depends on the choice of u.

Sketch of Proof. It can be shown that with respect to the $\mathscr{E}(U)$ topology, $S \cap C^\infty(\bar{U})$ is dense in S. Thus, given any $u \in S$, it is possible to pick a sequence of harmonic functions $\{u_k\} \subset S \cap C^\infty(\bar{U})$ such that $u_k \to u$ in $\mathscr{E}(U)$. Since the Poisson integral formula holds for functions in $S \cap C^\infty(\bar{U})$, we can write

$$u_k(x) = \int_0^{2\pi} P(r, \theta; t) u_k(e^{it})\, dt = \langle P(r, \theta; \cdot), \gamma(u_k) \rangle, \tag{2.11}$$

where $x=re^{i\theta}\in U$ and $\langle\ ,\ \rangle$ represents the duality between $H(\partial U)$ and $H'(\partial U)$. The bracketed part of (2.11) is legitimate since it is possible to show that $P(r,\theta;t)\in H(\partial U)$ when it is considered a function of $y=e^{it}$.

If we now let $k\to\infty$ and recall the continuity properties of γ, we have that

$$u(x)=\langle P(r,\theta;\cdot),\gamma(u)\rangle. \tag{2.12}$$

Recalling (2.4), reconsidered here in terms of the polar coordinate variable t instead of x, we find that (2.8) then follows.

3. Half-plane representations. We turn now to a study of the half-plane results. Let $v=(v_1,v_2)$ represent a point in the Euclidean plane R_v^2 and $x=(x_1,x_2)$ a point in the plane R_x^2. Polar coordinates for v will be (r,θ). U, $\bar{U}$ and ∂U will be the same as before and are taken as subsets of R_v^2. Let $R_+^2=\{x\in R_x^2|x_2>0\}$ with $\bar{R}_+^2$ its closure and $\partial R_+^2=\{x\in R_+^2|x_2=0\}$ its boundary.

We will be using the transformation

$$x_1=\frac{2v_2}{(1+v_1)^2+v_2^2},\qquad x_2=\frac{1-(v_1^2+v_2^2)}{(1+v_1)^2+v_2^2}, \tag{3.1}$$

with inverse

$$v_1=\frac{1-(x_1^2+x_2^2)}{x_1^2+(1+x_2)^2},\qquad v_2=\frac{2x_1}{x_1^2+(1+x_2)^2}, \tag{3.2}$$

which maps $\bar{U}$ 1–1 onto $\bar{R}_+^2$. It takes $\{|v|=1\}$ onto ∂R_+^2 and maps concentric circles $\{|v|=r<1\}$ onto circles C_r in R_+^2. The circles C_r are symmetric with respect to the x_2-axis and C_{r_1} falls inside C_{r_2} if $r_1<r_2$.

The derivatives

$$\tilde{D}_1\equiv -x_1(1+x_2)\frac{\partial}{\partial x_1}+\frac{1}{2}[x_1^2-(1+x_2)^2]\frac{\partial}{\partial x_2}, \tag{3.3}$$

$$\tilde{D}_2\equiv -\frac{1}{2}[x_1^2-(1+x_2)^2]\frac{\partial}{\partial x_1}-x_1(1+x_2)\frac{\partial}{\partial x_2}, \tag{3.4}$$

$$\tilde{D}_3\equiv (1+x_1^2)\frac{\partial}{\partial x_1}\qquad \text{(operating on } \partial R_+^2\text{)}, \tag{3.5}$$

$$\tilde{\Delta}\equiv \tilde{D}_3^2=\left[(1+x_1^2)\frac{\partial}{\partial x_1}\right]^2\qquad \text{(operating on } \partial R_+^2\text{)} \tag{3.6}$$

will be relevant to our discussion. They come from writing $\partial/\partial v_1$, $\partial/\partial v_2$, $\partial/\partial\theta|_{r=1}$, and $\Delta|_{r=1}$, respectively, in terms of R_x^2 variables. Taking $\tilde{D}=(\tilde{D}_1,\tilde{D}_2)$, differentiation involving components of $\tilde{D}$ will be referred to as $\tilde{D}$-differentiation.

Using the above, we can define spaces which are analogues of the spaces encountered in the last section. Let $\tilde{\mathscr{D}}(R^2_+)$ be the space of infinitely $\tilde{D}$-differential functions having compact support on R^2_+. To topologize it, let K be any compact subset of R^2_+ and let $\tilde{C}^\infty_0(K)$ consist of those functions in $\tilde{\mathscr{D}}(R^2_+)$ having their support in K. Topologize $\tilde{C}^\infty_0(K)$ by means of the seminorms

$$\|\phi\|_{m,K} \equiv \sum_{|l|\leqq m} \sup_{x\in K} |\tilde{D}^l\phi(x)|, \tag{3.7}$$

where $l=(l_1, l_2)$ is a multi-index, l_1 and l_2 are nonnegative integers, $|l|=l_1+l_2$, and $\tilde{D}^l=\tilde{D}_1^{l_1}\tilde{D}_2^{l_2}$. Then represent $\tilde{\mathscr{D}}(R^2_+)$ as $\bigcup_K \tilde{C}^\infty_0(K)$ (the sets K becoming increasingly larger) and equip it with the inductive limit topology. Let $\tilde{\mathscr{D}}'(R^2_+)$ be the strong dual of $\tilde{\mathscr{D}}(R^2_+)$.

Next, let $\tilde{H}(\partial R^2_+)$ be the space of all infinitely $\tilde{D}_3$-differentiable functions ϕ on ∂R^2_+ for which positive constants A and B exist such that

$$\|\tilde{\Delta}^{(q)}\phi\|_{C(\partial R^2_+)} \leqq A(2q)!B^q, \qquad q=0, 1, \cdots. \tag{3.8}$$

In (3.8), $\|\cdot\|_{C(\partial R^2_+)}$ is defined the same as $\|\cdot\|_{C(K)}$. If

$$\tilde{H}_B=\left\{\phi\in\tilde{H}(\partial R^2_+) \,\middle|\, |||\phi|||_B \equiv \sup_q \frac{\|\tilde{\Delta}^{(q)}\phi\|_{C(\partial R^2_+)}}{(2q)!B^q} < \infty\right\}, \tag{3.9}$$

then $|||\cdot|||_B$ is a norm, $\tilde{H}_B$ is a Banach space, and we specify the inductive limit topology for $\tilde{H}(\partial R^2_+)=\bigcup_B \tilde{H}_B(\partial R^2_+)$. Let $\tilde{H}'(\partial R^2_+)$ be the strong dual of $\tilde{H}(\partial R^2_+)$.

Paralleling the space S, the half-plane representations will hold for distributions in the space

$$\tilde{S}=\{u\in\tilde{\mathscr{D}}'(R^2_+) \mid \Delta u=0\}, \tag{3.10}$$

where, just as in the last section, these can be considered as classical solutions. $\tilde{S}$ is equipped with the induced $\tilde{\mathscr{D}}'(R^2_+)$ topology.

Again, it is possible to define a linear trace operator. Transferring γ from R^2_v to R^2_x, we obtain an operator $\tilde{\gamma}$ which maps $\tilde{S}$ onto $\tilde{H}'(\partial R^2_+)$ and which gives as its map the restriction of u to ∂R^2_+ for functions $u\in\tilde{C}^\infty(\bar{R}^2_+)\cap\tilde{S}$. Functions in $\tilde{S}$ can be considered as those harmonic functions whose traces fall in $\tilde{H}'(\partial R^2_+)$.

Because of (2.7) and the correspondence set up between U and R^2_+, the problem

$$\begin{aligned} \Delta u&=0, & u&\in\tilde{\mathscr{D}}'(R^2_+),\\ \tilde{\gamma}(u)&=u_0, & u_0&\in\tilde{H}'(\partial R^2_+) \end{aligned} \tag{3.11}$$

has a unique solution. Also, $\tilde{\gamma}$ defines a topological and algebraic isomorphism between $\tilde{S}$ and $\tilde{H}'(\partial R^2_+)$, where the topologies are the induced weak * of $\tilde{\mathscr{D}}'(R^2_+)$ for $\tilde{S}$ and the weak * for $\tilde{H}'(\partial R^2_+)$.

With the background above, we now can present the main representation.

THEOREM 3.1. *Let u be harmonic on R^2_+ and belong to $\tilde{S}$ (with boundary trace in $\tilde{H}'(\partial R^2_+)$). Then it has representation*

$$u(x)=\sum_{q=0}^{\infty}\int_{-\infty}^{\infty} \tilde{\Delta}_y^{(q)}\tilde{P}(x, y)\, d\tilde{\mu}_q(y), \tag{3.12}$$

where the $\{\tilde{\mu}_q\}$ *are Borel measures on* $(-\infty, \infty)$ *satisfying*

$$\text{var}\,(\tilde{\mu}_q) \leqq \frac{A(\varepsilon)\varepsilon^q}{(2q)!}, \qquad q=0,1,\cdots, \tag{3.13}$$

for any $\varepsilon > 0$. *In* (3.13),

$$\tilde{P}(x, y) = \frac{1}{\pi}\frac{x_2(1+y^2)}{(x_1-y)^2+x_2^2}, \tag{3.14}$$

and $A(\varepsilon)$ *is a positive function which depends on the choice of* u.

Proof. This follows easily considering the discussion above and transformation (3.1). $\tilde{P}$ corresponds to P, $\tilde{\Delta}_y^{(q)}$ to $\Delta_t^{(q)}$, and the $\{\tilde{\mu}_q\}$ on $(-\infty, \infty)$ to $\{\mu_q\}$ on $[0, 2\pi]$, so that applying (3.1) to the representation given by Theorem 2.1 gives our result.

Consider now $P(r, \theta; t)$ as given by (2.10). We have that $\Delta_t P = \partial^2 P/\partial t^2 = \partial^2 P/\partial \theta^2$. Further, for $r<1$, $\partial/\partial\theta$ transforms into

$$\tilde{D}_4 \equiv \left(\frac{1-x_1^2-x_2^2}{2}\right)\frac{\partial}{\partial x_1} + x_1x_2\frac{\partial}{\partial x_2}. \tag{3.15}$$

Define

$$\hat{\Delta}_x \equiv \tilde{D}_4^2 = \left[\left(\frac{1-x_1^2-x_2^2}{2}\right)\frac{\partial}{\partial x_1} + x_1x_2\frac{\partial}{\partial x_2}\right]^2. \tag{3.16}$$

If we replace Δ_t in (2.8) with $\partial^2 P/\partial\theta^2$ and map this over to R_+^2, we have

$$\hat{\Delta}_x^{(q)} u(x) = \sum_{q=0}^{\infty} \int_{-\infty}^{\infty} \hat{\Delta}_x^{(q)} \tilde{P}(x, y)\, d\tilde{\mu}_q. \tag{3.17}$$

If we next replace each of the measures $\{\tilde{\mu}_q\}$ by measures $\{\tilde{\nu}_q\}$ such that $d\nu_q = (1+y^2)\, d\tilde{\mu}_q$, we have the following corollary to Theorem 3.1 in which $\tilde{P}$ is replaced by the standard half-plane Poisson kernel.

COROLLARY 3.1. *Let* u *be harmonic on* R_+^2 *and belong to* $\tilde{S}$ (*with boundary trace in* $\tilde{H}'(\partial R_+^2)$). *Then it has representation*

$$u(x) = \sum_{q=0}^{\infty} \int_{-\infty}^{\infty} \hat{\Delta}_x^{(q)} \hat{P}(x, y)\, d\tilde{\nu}_q(y), \tag{3.18}$$

where the $\{\tilde{\nu}_q\}$ *are Borel measures on* $(-\infty, \infty)$ *such that* $d\tilde{\nu}_q = (1+y^2)\, d\tilde{\mu}_q$, $q=0, 1, \cdots$, *where the* $\{\tilde{\mu}_q)$ *are Borel measures satisfying* (3.13). *In* (3.18),

$$\hat{P}(x, y) = \frac{1}{\pi}\frac{x_2}{(x_1-y)^2+x_2^2}. \tag{3.19}$$

Before closing this section, it should be noted that solutions given by the usual Poisson integral formula clearly fall in the class of solutions covered by (3.12) and (3.18).

Also note that representation (2.8) extends to spheres in higher dimensional Euclidean R_v^n spaces. Thus the representations given above carry over to n

dimensions if one uses the transformation

$$(3.20)\qquad \begin{aligned} x_i &= \frac{2v_{i+1}}{(1+v_1)^2+v_2^2+\cdots+v_n^2}, \qquad i=1,\cdots,n-1,\\ x_n &= \frac{1-(v_1^2+\cdots+v_n^2)}{(1+v_1)^2+v_2^2+\cdots+v_n^2}, \end{aligned}$$

which takes a sphere in R_v^n onto the upper half-plane $\bar{R}_+^n=\{x_n \geqq 0\}$ of Euclidean R_x^n space. The applications in the next section also will carry over for the n-dimensional case.

4. Applications. We conclude by discussing the application of the representations just derived to numerical harmonic continuation. Douglas, Cannon, Saylor, Meyer [1], [2], [3], [4], [8], [10] and others, as mentioned previously, have studied continuation procedures of a related nature.

Consider first the approximation of harmonic functions u on all of the half-plane assuming the following:

1. $u(x) \in \tilde{S}$;
2. $|u(x^j)-F(x^j)|<\varepsilon$, where F is known on a finite set of points $\{x^j\}\subset C_\rho$, $0<\rho<1$ (the $\{x^j\}$ correspond to points $\{v^j\}$ on $\{|v|=\rho\}$);
3. $A(\varepsilon)$ (appearing in (3.13)) is known.

Note that $A(\varepsilon)$ serves as the global bound mentioned in the Introduction as needed.

Using (3.12) and picking positive integer parameters Q and N, we have

$$(4.1)\qquad u(x) \doteq \sum_{q=0}^{Q}\sum_{k=1}^{N} \tilde{\Delta}_y^{(q)}\tilde{P}(x, y^k)\tilde{\mu}_q([y^{k-1/2}, y^{k+1/2}]),$$

where points $(y^s, 0)$ are images of the points

$$(4.2)\qquad v^s = e^{i(2\pi s/N)}, \qquad s=0, 1/2, \cdots, N, N+1/2,$$

which are equally spaced on the unit circle in R_v^2. Explicitly, the $\{y^s\}$ are given by

$$(4.3)\qquad y^s = \frac{\cos(2\pi s/N)}{1+\sin(2\pi s/N)}, \qquad s=0, 1/2, \cdots, N, N+1/2.$$

Each of the measures $\tilde{\mu}_q$ can be decomposed into positive and negative parts

$$(4.4)\qquad \tilde{\mu}_q = \tilde{\mu}_q^+ - \tilde{\mu}_q^-,$$

where

$$(4.5)\qquad \tilde{\mu}_q^+([y^{k-1/2}, y^{k+1/2}]),\ \tilde{\mu}_q^-([y^{k-1/2}, y^{k+1/2}]) \geqq 0,$$

$$(4.6)\qquad \operatorname{var}(\tilde{\mu}_q^+),\ \operatorname{var}(\tilde{\mu}_q^-) \leqq \frac{A(\varepsilon)\varepsilon^q}{(2q)!}.$$

Let us substitute the sequence of parameters $\{a_{k,q}, b_{k,q}\}$ for the parameters $\{\tilde{\mu}_q^+([y^{k-1/2}, y^{k+1/2}]), \tilde{\mu}_q^-([y^{k-1/2}, y^{k+1/2}])\}$. Corresponding to (4.5) and (4.6), we require

$$a_{k,q},\ b_{k,q} \geqq 0, \tag{4.7}$$

$$\sum_{k=1}^{N} a_{k,q},\ \sum_{k=1}^{N} b_{k,q} \leqq A_q, \qquad q = 0, \cdots, Q, \tag{4.8}$$

$$A_q \equiv A\left(\frac{1-R}{2}\right)\left(\frac{1-R}{2}\right)^q \Big/ (2q)!, \tag{4.9}$$

where R is selected so that $0 < \rho < R < 1$.

The above and (4.1) then suggest the following form for our approximation:

$$U_{Q,N}(x, \{a_{k,q}, b_{k,q}\}) = \sum_{q=0}^{Q} \sum_{k=1}^{N} \tilde{\Delta}_y^{(q)} \tilde{P}(x, y^k)(a_{k,q} - b_{k,q}). \tag{4.10}$$

The actual approximation $U_{Q,N}(x)$ now comes from picking a set of $\{a_{k,q}, b_{k,q}\}$, not necessarily unique, such that

$$\max_j |F(x^j) - U_{Q,N}(x^j, \{a_{k,q}, b_{k,q}\})| \tag{4.11}$$

is minimized subject to the constraints (4.7)–(4.9). Determining the $\{a_{k,q}, b_{k,q}\}$ amounts to treating a standard linear programming problem.

Note that in evaluating the terms $\tilde{\Delta}_y^{(q)} \tilde{P}(x, y^k)$ used in the approximation it is easier to evaluate instead the equivalent versions of these terms in R_v^2.

The procedure presented just above, by virtue of (3.1), matches up with a corresponding unit disk harmonic continuation procedure based on (2.8). This latter procedure is essentially the same as still another procedure for the unit disk discussed by Douglas [4].

Douglas bounds the error in his approximation by estimating the error on $\{|v| = \rho\}$ and $\{|v| = R\}$ and then applying Hadamard's three-circle theorem. The estimate on $\{|v| = \rho\}$ comes from knowing $A((1-R)/2)$ and that for $\{|v| = R\}$ comes from the data.

It is an easy matter to modify Douglas' error results so that they hold for our unit disk case. If these results, in turn, are transferred to the half-plane, it is then easily seen that we have the following error estimate.

THEOREM 4.1. *Let u satisfy* 1–3 *and $U_{Q,N}(x)$ be determined as discussed above. Let $0 < \rho < R < 1$. Then*

$$\begin{aligned} |u(x) - U_{Q,N}(x)| \leqq 2^{1/2}\left[\left(1 - \frac{\rho^2}{r^2}\right)^{-1/2} + \left(1 - \frac{r^2}{R^2}\right)^{-1/2}\right] \\ \cdot M_R^{1-[\ln(r/R)/\ln(\rho/R)]}[2\varepsilon \end{aligned} \tag{4.12}$$

for x on C_r, $\rho < r < R$, where the constants M_R and C_R depend on the choice of R and $\delta \equiv \sup_{|v|=\rho} \{\max|v - v^j|\}$.

From (4.12), we see that $U_{Q,N}(x)$ converges to $u(x)$ as ε and δ tend to zero and N and Q to infinity. It should be pointed out that the constants in (4.12) become large as R tends to one. It also should be pointed out that a bound for $|u-U_{Q,N}|$ inside C_ρ is given by the estimate

$$|u-U_{Q,N}|\leqq 2\varepsilon + C_R(2^{-Q}+N^{-1}+\delta^2). \tag{4.13}$$

This estimate bounds $|u-U_{Q,N}|$ on C_ρ (it is obtained in the course of deriving (4.12)) and holds for u inside C_ρ by virtue of the maximum principle.

A second minimax approximation based on (3.18) can be set up in a fashion paralleling that for $U_{Q,N}(x)$. The error estimate will be similar.

Note further, that our hypothesis that data be given on some C_ρ is not unreasonable. For many distributions of data points in R^2_+, it is possible to take the information from these and bound the error on some C_ρ. When this is so, we are in essence treating the same situation as above as far as the error estimate is concerned. When data is given this way, the approximation is determined by proceeding as before only using the new points $\{x^j\}$ in (4.11).

Observe, also, that the continuation problem 1–3 is not a practical one for values of ρ close to one. In such instances, one would be measuring data at points tending toward infinity.

Before closing, we will briefly discuss continuation assuming data is given in a second way. We assume

1′. $u(x)\in\tilde{S}$,

2′. $|u(\zeta^j, Y)-F(\zeta^j, Y)|<\varepsilon$, where F is known on a finite set of points $\{(\zeta^j, Y)\}$ on the line $\{x_2=Y\}$,

3′. $A(\varepsilon)$ is known,

and we wish to continue u to some region $\mathscr{X}=\{|x_1|\leqq X,\ \eta\leqq x_2\leqq Y\}\subset R^2_+$.

This is done by picking an approximation of form

$$U_{Q,N}(x,\{a_{k,q}, b_{k,q}\})=\sum_{q=0}^{Q}\sum_{|\xi^k-x_1|\leqq\tilde{X}}\tilde{\Delta}_y^{(q)}\tilde{P}(x, y^k)(a_{k,q}-b_{k,q}), \tag{4.14}$$

which is required to be a best minimax fit to the data, subject to the constraints (4.7)–(4.9). In (4.14), $y^k=(\xi_k, 0)$. We pick the ξ^k such that $\xi^k=k\Delta x$ and $\tilde{X}$ is an appropriately chosen parameter. The R in (4.9) must be picked sufficiently close to one in the range $0<R<1$.

Error bounds can be found using techniques similar to those employed by Cannon and Douglas in [2]. An alternative approximation procedure based on (3.18) can also be formulated.

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SELF-INVERSE SHEFFER SEQUENCES*

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Abstract. The sequence of Laguerre polynomials is known to be self-inverse in the group of Sheffer sequences, and our main goal here is to furnish a generating function characterization of *all* the self-inverse Sheffer sequences. We present our result in the broader context of generalized Appell sequences of arbitrary order and obtain it by solving a system of functional equations.

1. Let $P = \{P_n(x)\}_{n=0}^{\infty}$ be a polynomial sequence generated by a relation of the form

$$\sum_{n=0}^{\infty} \phi_n P_n(x) t^n = G(t)\Phi(xH(t)), \tag{1}$$

where

$$\Phi(t) = \sum_{n=0}^{\infty} \phi_n t^n, \qquad \phi_n \neq 0, \quad n = 0, 1, 2, \cdots. \tag{2}$$

It is understood, moreover, that $H(t)$ and $G(t)$ represent power series of the types

$$H(t) = \sum_{k=1}^{\infty} h_k t^k, \qquad h_1 \neq 0, \quad \text{and} \quad G(t) = \sum_{k=0}^{\infty} g_k t^k, \qquad g_0 \neq 0, \tag{3}$$

respectively, our entire discussion being in the context of *formal* power series with complex coefficients. Such sequences were first treated in their full generality by Boas and Buck [3], [2] who called them generalized Appell sequences since they reduce to Appell [1] sequences when $\Phi(t) = \exp t$ and $H(t) = t$.

As was done in [5] and [4], we use (Φ) to denote the class of all generalized Appell sequences generated by relations of the form (1) when a fixed $\Phi(t)$ is taken, the most important of those classes being the class (exp) of Sheffer [8] sequences which occurs when $\Phi(t) = \exp t$. Note that for any given sequence P in (Φ), the pair $H(t)$ and $G(t)$ appearing in (1) is uniquely determined since the identity

$$G^*(t)\Phi(xH^*(t)) = G(t)\Phi(xH(t)), \tag{4}$$

where the pair $H^*(t)$ and $G^*(t)$ is also of type (3), implies that

$$H^*(t) = H(t) \quad \text{and} \quad G^*(t) = G(t). \tag{5}$$

To see this, simply set $x = 0$ in (4) to get the second of identities (5). Equation (4) then becomes

$$\sum_{n=0}^{\infty} \phi_n (H^*(t))^n x^n = \sum_{n=0}^{\infty} \phi_n (H(t))^n x^n,$$

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and the first identity in (5) is obtained from this by equating the coefficients of powers of x on each side.

Boas and Buck pointed out that each $P_n(x)$ is of degree exactly n; and in references [5] and [4], mentioned above, the class (Φ) was exhibited as a group where if

$$P_n(x) = \sum_{k=0}^{n} p_{n,k}x^k, \qquad n = 0, 1, 2, \cdots, \tag{6}$$

the product $P\tilde{P}$ of P and a sequence $\tilde{P} = \{\tilde{P}_n(x)\}_{n=0}^{\infty}$ in (Φ) is the sequence of polynomials

$$(P\tilde{P})_n(x) = \sum_{k=0}^{n} p_{n,k}\tilde{P}_k(x), \qquad n = 0, 1, 2, \cdots.$$

Also, it was shown there that if

$$\sum_{n=0}^{\infty} \phi_n \tilde{P}_n(x)t^n = \tilde{G}(t)\Phi(x\tilde{H}(t))$$

is the generating relation placing $\tilde{P}$ in (Φ), then

$$\sum_{n=0}^{\infty} \phi_n (P\tilde{P})_n(x)t^n = G(t)\tilde{G}(H(t))\Phi(x\tilde{H}(H(t))) \tag{7}$$

is the corresponding one for $P\tilde{P}$. The identity element I is evidently the sequence of polynomials $I_n(x) = x^n$, $n = 0, 1, 2, \cdots$, generated by

$$\sum_{n=0}^{\infty} \phi_n I_n(x)t^n = \Phi(xt).$$

We plan here to establish necessary and sufficient conditions on the pair $H(t)$ and $G(t)$ in (1) such that $P^N = I$ for any given positive integer N. Our chief aim is, however, to provide a generating function characterization of those sequences in (exp) such that $P^2 = I$, or, in terms of the polynomials themselves,

$$\sum_{k=0}^{n} p_{n,k}P_k(x) = x^n, \qquad n = 0, 1, 2, \cdots.$$

The problem is suggested to us by the fact that the sequence of Laguerre polynomials $L_n^{(\alpha)}(x)$, $n = 0, 1, 2, \cdots$, generated by

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\frac{t^n}{n!} = (1 - t)^{-\alpha-1} \exp\left(\frac{-xt}{1 - t}\right), \tag{8}$$

has this "remarkable" self-inverse property, so described by Rota, Kahaner and Odlyzko [7, p. 729] in their recent and exhaustive study of Sheffer sequences. While it is not usual to include the $n!$ on the left side of (8), we follow the aforementioned authors in doing so. As they pointed out, their notation also has precedent in the literature.

The general problem is treated in § 2, and its solution in the special case of (exp) when $N = 2$ is given in § 3.

2. Our result in the general case is as follows.

THEOREM. *Let $P = \{P_n(x)\}_{n=0}^{\infty}$ be a generalized Appell sequence generated by* (1), *and let N be a positive integer. A necessary and sufficient condition that $P^N = I$ within the group* (Φ) *is that the pair $H(t)$ and $G(t)$ appearing in* (1) *be of the form*

$$H(t) = V^{-1}(\tau V(t)), \quad G(t) = \sigma \exp(U(V(t))), \tag{9}$$

where τ and σ are Nth roots of unity, $V(t)$ is a power series of the type

$$V(t) = \sum_{k=1}^{\infty} v_k t^k, \qquad v_1 \neq 0, \tag{10}$$

$V^{-1}(t)$ is its power series inverse defined by $V(V^{-1}(t)) = V^{-1}(V(t)) = t$, and $U(t)$ is a power series of the type

$$U(t) = \sum_{k=1}^{\infty} u_k t^k \tag{11}$$

satisfying the identity

$$\sum_{n=0}^{N-1} U(\tau^n t) = 0. \tag{12}$$

Proof. As remarked upon in § 1, the pair $H(t)$ and $G(t)$ is unique for any given P in (Φ). Hence it is immediate from (7) that $P^N = I$ if and only if that pair is a solution of type (3) to the system of functional equations

$$H_N(t) = t, \tag{13}$$

$$\prod_{n=0}^{N-1} G(H_n(t)) = 1, \tag{14}$$

where

$$H_0(t) = t \quad \text{and} \quad H_n(t) = H_{n-1}(H(t)) \qquad n = 1, 2, \cdots, N$$

(cf. [6, (0.3) and (15.1)]). It is a simple matter to verify that (9) is actually such a solution, once it has been observed that the first expression there generalizes to

$$H_n(t) = V^{-1}(\tau^n V(t)), \qquad n = 0, 1, \cdots, N. \tag{15}$$

The sufficiency part of the theorem is therefore evident.

To prove the necessity part, we let $H(t)$ and $G(t)$ be any pair of power series of type (3) satisfying (13)–(14) and show that it must be of the form (9).

We note from (13) that $h_1^N = 1$, where h_1 is as indicated in (3), and consider first the possibility that $h_1 = 1$:

$$H(t) = t + \sum_{k=2}^{\infty} h_k t^k.$$

The fact that (13) is satisfied implies that $h_k = 0$ for all $k \geqq 2$. For, supposing the contrary, let h_m be the first of those coefficients which is nonzero and write

$$H(t) = t + h_m t^m + \cdots, \qquad h_m \neq 0, \quad m \geqq 2.$$

It is then easy to see that

$$H_N(t) = t + N h_m t^m + \cdots;$$

and, because of (13), it follows that $h_m = 0$. But this contradicts the fact that $h_m \neq 0$. Hence $H(t) = t$ and, in view of (14), $G(t) = \sigma$ where σ is an Nth root of unity. The pair $H(t)$ and $G(t)$ is therefore of the form (9) with $\tau = 1$, that choice of τ forcing (12) to become $U(t) \equiv 0$.

If $h_1 \neq 1$, it must be some root τ of unity other than unity itself; and we use τ to define the power series

$$V(t) = \sum_{n=0}^{N-1} H_n(t)/\tau^n$$

(cf. [6, (6.47)]). With (13), it is straightforward to show that

$$V(H(t)) = \tau V(t). \tag{16}$$

Furthermore, the series $V(t)$ is as stated in (10), where $v_1 = N \neq 0$; and it is consequently invertible. Hence (16) is equivalent to the first of expressions (9). Turning now to $G(t)$, put $\sigma = G(0) = g_0$ and set $t = 0$ in (14) to see that σ must be an Nth root of unity, not necessarily the same as τ. Evidently then,

$$G(t)/\sigma = 1 + \sum_{k=1}^{\infty} (g_k/\sigma)t^k;$$

and so $\log[G(t)/\sigma]$ represents a well-defined power series vanishing at $t = 0$. Putting

$$U(t) = \log[G(V^{-1}(t))/\sigma], \tag{17}$$

we are thus assured that $U(t)$ also represents such a power series, as indicated in (11); and (17) is readily inverted into the explicit expression for $G(t)$ given in (9). It remains only to show that (12) is satisfied. In view of (14), this can be accomplished by referring to definition (17) of $U(t)$ and then generalization (15) of the first expression in (9), that expression having just been established. To be precise,

$$\sum_{n=0}^{N-1} U(\tau^n V(t)) = \sum_{n=0}^{N-1} \log[G(V^{-1}(\tau^n V(t)))/\sigma]$$
$$= \log\left[\prod_{n=0}^{N-1} G(H_n(t))\right] = 0;$$

and (12) follows if we replace t by $V^{-1}(t)$ in this result. The proof of the theorem is now complete.

Observe that in the necessity part of the proof we did not actually need to treat the case $h_1 = 1$ separately. That case does, however, deserve special attention, not only because it can be handled in an especially simple manner, but because it occurs when the sequence P is *monic*, or when $p_{n,n} = 1$ in (6). Indeed, the formula [2, (6.4)]

$$p_{n,n} = g_0 h_1^n, \qquad n = 0, 1, 2, \cdots,$$

reveals that both h_1 and g_0 are unity for such sequences. According to our theorem, then, the identity sequence is the only monic sequence P in (Φ) such that $P^N = I$.

Finally, we note that the sequence in the theorem may, in fact, be a group element of order N. This occurs, for example, when either τ or σ is a *primitive* Nth root of unity.

3. When $\Phi(t) = \exp t$ and $N = 2$, our theorem yields the result that is of particular interest to us, a generating function characterization of self-inverse Sheffer sequences. The reader will note that the characterization remains valid even within the context of formal power series with *real* coefficients. For convenience, we shall exclude what we call the *trivial* self-inverse sequences $\{x^n\}_{n=0}^{\infty}$ and $\{-x^n\}_{n=0}^{\infty}$, occurring when $\tau = 1$.

COROLLARY. *A necessary and sufficient condition for a Sheffer sequence* $P = \{P_n(x)\}_{n=0}^{\infty}$ *to be a nontrivial self-inverse element of the group (exp) is that it be generated by a relation of the form*

$$\sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!} = s \exp\left[U(V(t)) + xV^{-1}(-V(t))\right], \tag{18}$$

where $s = \pm 1$, $V(t)$ *is a power series of the type* $V(t) = \sum_{k=1}^{\infty} v_k t^k$, $v_1 \neq 0$, $V^{-1}(t)$ *is its power series inverse, and* $U(t)$ *is a power series of the type* $U(t) = \sum_{k=1}^{\infty} u_k t^{2k-1}$.

Observe that

$$s \exp(U(V(t))) = (1 - t)^{-\alpha - 1}, \qquad V^{-1}(-V(t)) = \frac{-t}{1 - t}$$

when

$$s = 1, \quad V(t) = \log(1 - t) = \sum_{k=1}^{\infty} (-1/k)t^k, \quad U(t) = -(1 + \alpha)t.$$

Thus the sequence of Laguerre polynomials, generated by (8), is indeed a self-inverse element of (exp).

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ADJOINT SEMIGROUP THEORY FOR A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS*

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Abstract. We consider the semigroup and adjoint semigroup for a class of linear functional differential equations with infinite delays, which includes certain linear Volterra integro-differential equations. In particular, we show that by an appropriate choice of the state space the semigroup constructed by Miller [13] can be considered the adjoint semigroup of the semigroup constructed by Barbu and Grossman [2]. This provides a useful characterization of Miller's semigroup which can be applied to obtain additional information about the semigroup defined by Barbu and Grossman.

1. Introduction. In recent years, considerable attention has been given to the semigroup theory for a class of functional differential equations with infinite delays. In order to construct a theory which extends that for finite delays, much of the work has been done in spaces with fading memory (see [4], [5], [10], [11], [15]). However, it has been noted that for the case of finite delays, the L_p spaces are sometimes more suitable for certain applications (see [1], [3], [6], [7]).

In this paper we consider a linear functional differential equation with infinite delays in a product space of the form $C^n \times L_p(-\infty, 0)$ (compare with [6], [7]). Using $C^n \times L_p(-\infty, 0)$ as the state space, we construct a C_0 semigroup. In § 2 the infinitesimal generator and its adjoint are computed, and this is applied to compute the "adjoint" semigroup in § 3.

Section 4 is devoted to an application of the semigroup theory to a class of Volterra integro-differential equations. In particular, we show that by an appropriate choice of the state space the semigroup constructed by Miller [13] can be considered the adjoint semigroup of the semigroup defined below, which in turn corresponds to the semigroup constructed by Barbu and Grossman in [2]. This provides a useful characterization of Miller's semigroup.

Let $1 \leqq p \leqq +\infty$ and n be fixed. The usual Lebesgue space of C^n-valued functions defined on an interval with endpoints a and b $(-\infty \leqq a < b \leqq +\infty)$ will be denoted by $L_p(a, b)$. The space of bounded continuous functions on an interval such as $(a, b]$ will be denoted by $BC(a, b]$. If $x : (-\infty, \infty) \to C^n$ is given, then $x_t : (-\infty, 0] \to C^n$ is defined by $x_t(s) = x(t + s)$ for $t \geqq 0$ and $s \leqq 0$.

We assume that L is a linear function with domain in the linear space of C^n-valued Lebesgue measurable functions defined on $(-\infty, 0]$ such that L restricted to $BC(-\infty, 0]$ is a bounded linear operator. Moreover, we require that the following hypothesis hold:

($\mathscr{H}$) If $t_1 > 0$, $1 \leqq p < +\infty$, then:

(i) For each $x \in L_p(-\infty, t_1)$, $t \to L(x_t)$ defines a function almost everywhere (a.e.) on $[0, t_1]$, and depends only on the equivalence class of x.

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(ii) There is a continuous function Γ such that if $x \in L_p(-\infty, t_1)$, then $g(t) = L(x_t)$ belongs to $L_1(0, t_1)$ and

$$\int_0^t |L(x_s)|\, ds \leqq \Gamma(t) \left[\int_{-\infty}^t |x(s)|^p\, ds \right]^{1/p},$$

for all $t \in [0, t_1]$.

(iii) If $x \in L_p(-\infty, t_1) \cap BC(-\infty, t_1]$, then $g(t) = L(x_t)$ is continuous at 0 from the right.

The system is defined by the linear retarded functional differential equation with infinite delays

$$x'(t) = L(x_t) + f(t), \qquad t \geqq 0, \tag{1.1}$$

and the initial data

$$x(0) = \eta, \quad x_0 = \varphi, \tag{1.2}$$

where f is measurable and bounded on finite intervals, $\eta \in C^n$, and $\varphi \in L_p(-\infty, 0)$.

A solution to system (1.1)–(1.2) is a function $x \in L_p(-\infty, t_1)$ for each $t_1 > 0$, such that x is absolutely continuous (a.c.) on compact subsets of $[0, +\infty)$, x satisfies (1.1) a.e. on $[0, +\infty)$, $x(0) = \eta$, and $x_0(s) = \varphi(s)$ a.e. on $(-\infty, 0]$.

The above formulation is analogous to the formulation for finite delays found in Borisovič and Turbabin [3] and Banks and Burns [1].

Throughout the remainder of this paper, the state space (i.e., the space of initial data) will be the product space $Z_p = C^n \times L_p(-\infty, 0)$, with norm defined by

$$\|(\eta, \varphi)\|^p = \|\eta\|^p + \|\varphi\|^p.$$

It should be noted that Z_p is not a hereditary space in the sense of Coleman and Mizel [4], [5]; however, Z_p is equivalent to the state space (denoted by M^p) used in some of Delfour's and Mitter's work (see [6], [7]). It will become clear that the choice of Z_p and the explicit representation of Z_p as a product space has many advantages.

The following result guarantees the existence, uniqueness, and continuation of solutions to (1.1)–(1.2). It is not obvious that the stated conditions imply existence. However, standard techniques (i.e., the contraction mapping principle) are used to prove the following theorem. For completeness, we give a sketch of the proof. The interested reader can easily fill in the details.

THEOREM 1.1. *If* $(\mathscr{H})$ *holds*, $(\eta, \varphi) \in Z_p$, *and* $f \in L_1(0, t_2)$ *for all* $t_2 > 0$, *then there is a unique solution of* (1.1)–(1.2) *defined on* $(-\infty, +\infty)$.

Sketch of proof. Let $\alpha > 0$ be such that $K = \int_0^\alpha (\Gamma(t))^p\, dt < 1$, and define the operator $T: L_p(-\infty, \alpha) \to L_p(-\infty, \alpha)$ by

$$[T\psi](t) = \begin{cases} \eta + \int_0^t L(\psi_s)\, ds + \int_0^t f(s)\, ds, & 0 \leqq t \leqq \alpha, \\ \varphi(t), & t < 0. \end{cases}$$

It follows that

$$\begin{aligned}\|T\psi - T\chi\|^p &= \int_{-\infty}^{\alpha} \|[T\psi](t) - [T\chi](t)\|^p\,dt \\ &= \int_{-\infty}^{0} \|\varphi(t) - \varphi(t)\|^p\,dt + \int_0^{\alpha}\left\|\int_0^t L(\psi_s - \chi_s)\,ds\right\|^p dt \\ &\leqq \int_0^{\alpha}\left(\int_0^t \|L(\psi_s - \chi_s)\|\,ds\right)^p dt \\ &\leqq \int_0^{\alpha}\left(\Gamma(t)\left[\int_{-\infty}^{t} \|\psi(s) - \chi(s)\|^p\,ds\right]^{1/p}\right)^p dt \\ &\leqq \int_0^{\alpha} [\Gamma(t)]^p\,dt\left[\int_{-\infty}^{\alpha} \|\psi(s) - \chi(s)\|^p\,ds\right] \\ &= K\|\psi - \chi\|^p,\end{aligned}$$

and since $K < 1$, the operator T is a contraction. Therefore, the contraction mapping principle implies that there is a unique fixed point for T, say x^1. Clearly x^1 is a solution to (1.1)–(1.2) defined on $(-\infty, \alpha]$.

Let $\eta^1 = x^1(\alpha)$ and $\varphi^1(t) = x^1_\alpha(t)$. The above procedure yields a solution $x^2(t)$ defined on $(-\infty, \alpha]$ with initial data (η^1, φ^1). Define x on $(-\infty, 2\alpha]$ by

$$x(t) = \begin{cases} x^1(t), & -\infty < t \leqq \alpha, \\ x^2(t - \alpha), & \alpha \leqq t \leqq 2\alpha, \end{cases}$$

and note that x is a solution of (1.1)–(1.2) with initial data (η, φ). By continuing this process it is clear that x can be continued to $+\infty$.

An application of Gronwall's inequality yields the uniqueness of solutions.

If $(\mathscr{H})$ holds, then there exists a unique solution to (1.1)–(1.2) for each $(\eta, \varphi) \in Z_p$. Also, suppose x is a solution to (1.1)–(1.2) with $(\eta, \varphi) = (\eta^1, \varphi^1)$ and y is a solution to (1.1)–(1.2) with $(\eta, \varphi) = (\eta^2, \varphi^2)$. Let $\mathscr{M}(t)$ be defined by

$$\mathscr{M}(t) = \sup_{0 \leqq s \leqq t} \{2^p \Gamma(s)^p\} + 2^p,$$

and $B(t)$ be defined by

$$B(t) = 2^p\left(1 + \mathscr{M}(t)\,e^{t\mathscr{M}(t)} + \int_0^t \mathscr{M}(s)\,e^{s\mathscr{M}(s)}\,ds\right).$$

An application of Gronwall's inequality yields the following inequalities. If $t \geqq 0$, then

$$\|x(t) - y(t)\|^p \leqq (\|\eta^1 - \eta^2\|^p + \|\varphi^1 - \varphi^2\|^p)\mathscr{M}(t)\,e^{t\mathscr{M}(t)}, \tag{1.3}$$

$$\|x_t - y_t\|^p \leqq \|\eta^1 - \eta^2\|^p + \left(1 + \int_0^t \mathscr{M}(s)\,e^{s\mathscr{M}(s)}\,ds\right)\|\varphi^1 - \varphi^2\|^p, \tag{1.4}$$

and hence we have that

$$\|(x(t), x_t) - (y(t), y_t)\|^p \leqq B(t)(\|(\eta^1, \varphi^1) - (\eta^2, \varphi^2)\|^p). \tag{1.5}$$

If x is the solution to (1.1)–(1.2), then

$$\lim_{t \to 0^+} \int_{-\infty}^{0} \|x_t(s) - \varphi(s)\|^p \, ds = 0.$$

Consequently, we have that

$$\lim_{t \to 0^+} \|(x(t), x_t) - (\eta, \varphi)\|^p = 0, \tag{1.6}$$

and this now allows us to define a semigroup on Z_p.

For $t \geqq 0$, $S(t): Z_p \to Z_p$ is defined by

$$S(t)(\eta, \varphi) = (x(t), x_t), \tag{1.7}$$

where x is the unique solution to (1.1)–(1.2) with $f(t) \equiv 0$. In view of (1.5) and (1.6), it follows that $S(t)$ is a C_0-semigroup defined on Z_p.

For $1 < p < +\infty$, p' is defined by $(1/p) + (1/p') = 1$ and if $p = 1$, then $p' = +\infty$. Therefore, it follows that the adjoint space of Z_p is $Z_{p'}$ for $1 \leqq p < +\infty$. If $(\eta, \varphi) \in Z_p$ and $(\xi, \Psi) \in Z_p^*$, then we shall use the symbol $\langle(\eta, \varphi), (\xi, \Psi)\rangle$ to denote the "product" $(\xi, \Psi)((\eta, \varphi))$. Thus, by definition we have that

$$\langle(\eta, \varphi), (\xi, \Psi)\rangle = \Psi(\varphi) + \xi(\eta) = \langle\varphi, \Psi\rangle + \langle\eta, \xi\rangle.$$

Example 1.1. Let A and B be constant $n \times n$ matrices and $K(s)$ an $n \times n$ measurable matrix function such that $\int_{-\infty}^{0} \|K(s)\| \, ds < +\infty$. Given a real number $r > 0$ let L be the operator defined by

$$L(\varphi) = A\varphi(0) + B\varphi(-r) + \int_{-\infty}^{0} K(s)\varphi(s) \, ds. \tag{1.8}$$

It follows that $L(\varphi)$ is defined for all $\varphi \in L_p(-\infty, 0) \cap BC(-\infty, 0]$. Moreover, if $x \in L_p(-\infty, t_1)$, then

$$g(t) = L(x_t) = Ax(t) + Bx(t - r) + \int_{-\infty}^{0} K(s)x(t + s) \, ds$$

is defined a.e. on $[0, t_1]$ and depends only on the equivalence class of x. Therefore, hypothesis ($\mathscr{H}$i) is satisfied.

Define $h(t) = \int_{-\infty}^{0} K(s)x(t + s) \, ds$ for $t \in [0, t_1]$, where $x \in L_p(-\infty, t_1)$. By the convolution theorem the function h is in $L_p(0, t_1)$; thus it follows that $g(t) \in L_1(0, t_1)$. Applying elementary inequalities to (1.8) we have that

$$\int_0^t \|L(x_s)\| \, ds \leqq \|A\| \int_0^t \|x(s)\| \, ds + \|B\| \int_0^t \|x(s - r)\| \, ds$$
$$+ \int_0^t \int_{-\infty}^{0} \|K(s)\| \, \|x(s + u)\| \, ds \, du$$

$$\leqq t^{(p-1)/p}(\|A\| + \|B\|)\left[\int_{-\infty}^{t} \|x(s)\|^p \, ds\right]^{1/p}$$

$$+ t^{(p-1)/p}\int_{-\infty}^{0} \|K(s)\| \, ds\left[\int_{-\infty}^{t} \|x(v)\|^{p} dv\right]^{1/p},$$

for $t \in [0, t_1]$. Consequently,

$$\int_0^t \|g(s)\| \, ds = \int_0^t \|L(x_s)\| \, ds \leqq \Gamma(t)\left[\int_{-\infty}^{t} \|x(s)\|^p \, ds\right]^{1/p},$$

for $0 \leqq t \leqq t_1$, where Γ is the continuous function defined on $[0, t_1]$ by

$$\Gamma(t) = t^{(p-1)/p}\left(\|A\| + \|B\| + \int_{-\infty}^{0} \|K(s)\| \, ds\right).$$

We now have that ($\mathscr{H}$ii) is satisfied.

For $x \in BC(-\infty, t_1]$ it follows that $h(t) = \int_{-\infty}^{0} K(s)x(t+s)\,ds$ is continuous on $[0, t_1]$; therefore, $g(t)$ is continuous at 0 from the right. In particular, hypothesis ($\mathscr{H}$iii) is satisfied.

We shall need the following three lemmas.

LEMMA 1.1. *If $f \in L_p(-\infty, t_1)$ is a.c. on compact subsets of $(-\infty, t_1]$ and $f' \in L_p(-\infty, t_1)$, then $f \in BC(-\infty, t_1]$.*

LEMMA 1.2. *Given $r \geqq 0$, if φ is a.c. on compact subsets of $(-\infty, -r]$, $\varphi \in L_1(-\infty, -r)$ and $\varphi' \in L_1(-\infty, -r)$, then $\lim_{t \to -\infty} \|\varphi(t)\| = 0$.*

The above Lemmas are easily obtained by using elementary arguments.

Remark 1.1. It is to be noted that Lemma 1.1 together with ($\mathscr{H}$iii) implies that $g(t) = L(x_t)$ is continuous at 0 from the right whenever $x \in L_p(-\infty, t_1)$ and x is a.c. on compact subsets of $(-\infty, t_1]$ with $x' \in L_p(-\infty, t_1)$.

We shall use this result throughout the remainder of this paper.

LEMMA 1.3. *Let $q = p'$ and suppose that g is locally integrable on $(-\infty, -r]$, $r \geqq 0$, and $f \in L_q(-\infty, -r]$. If*

$$\int_{-\infty}^{-r} [\langle \varphi'(s), f(s)\rangle + \langle \varphi(s), g(s)\rangle] \, ds = 0$$

for all $\varphi \in L_p(-\infty, -r)$ such that φ is a.c. on compact subsets of $(-\infty, -r]$, $\varphi' \in L_p(-\infty, -r)$, $\varphi(-r) = 0$, and φ has compact support, then f is a.c. on compact subsets of $(-\infty, -r]$ and $f'(s) = g(s)$ a.e. on $(-\infty, -r]$. Moreover, if $\varphi \in L\ (-\infty, -r)$ is a.c. on compact subsets of $(-\infty, -r]$, $\varphi' \in L_p(-\infty, -r)$, and $\langle \varphi, g\rangle \in L_1(-\infty, -r)$, then $[\langle \varphi, f\rangle]'$ belongs to $L_1(-\infty, -r)$, $\lim_{t \to -\infty} |\langle \varphi(t), f(t)\rangle| = 0$ and

$$\int_{-\infty}^{-r} [\langle \varphi'(s), f(s)\rangle + \langle \varphi(s), g(s)\rangle] \, ds = \int_{-\infty}^{-r} [\langle \varphi(s), f(s)\rangle]' \, ds = \langle \varphi(-r), f(-r)\rangle.$$

Proof. Let $t < -r$ and suppose that φ is a.c. on $[t, -r]$ with $\varphi(t) = \varphi(-r) = 0$. It follows that

$$\int_{t}^{-r} [\langle \varphi'(s), f(s)\rangle + \langle \varphi(s), g(s)\rangle] \, ds = \int_{-\infty}^{-r} [\langle \bar{\varphi}'(s), f(s)\rangle + \langle \bar{\varphi}(s), g(s)\rangle] \, ds = 0,$$

where

$$\bar{\varphi}(s) = \begin{cases} \varphi(s), & \text{if } t \leqq s \leqq -r, \\ 0, & \text{if } s \leqq t. \end{cases}$$

By the fundamental lemma of the calculus of variations, (see [17, p. 112]), we have that f is a.c. on $[t, -r]$ and $f(s) = f(-r) + \int_{-r}^{s} g(u)\,du$. In particular,

$$f(t) = f(-r) + \int_{-r}^{t} g(u)\,du$$

and since $t < -r$ is arbitrary, it follows that f is a.c. on compact subsets of $(-\infty, -r]$ and $f'(s) = g(s)$ a.e.

Suppose $\varphi \in L_p(-\infty, -r)$ is a.c., $\varphi' \in L_p(-\infty, -r)$, and $\langle \varphi, g\rangle \in L_1(-\infty, -r)$. Since $f \in L_q(-\infty, -r)$, it follows that $\langle \varphi, f\rangle$ and $\langle \varphi', f\rangle$ belong to $L_1(-\infty, -r)$. Consequently, $[\langle \varphi, f\rangle]' = \langle \varphi', f\rangle + \langle \varphi, f'\rangle = \langle \varphi', f\rangle + \langle \varphi, g\rangle$ and $[\langle \varphi, f\rangle]'$ is also in $L_1(-\infty, -r)$. An application of Lemma 1.2 yields that $\lim_{t\to-\infty} |\langle \varphi(t), f(t)\rangle|$ exists and equals zero. Moreover,

$$\begin{aligned} \int_{-\infty}^{-r} [\langle \varphi(s), f(s)\rangle]'\,ds &= \lim_{t\to-\infty} \int_{t}^{-r} [\langle \varphi(s), f(s)\rangle]'\,ds \\ &= \lim_{t\to-\infty} \langle \varphi(-r), f(-r)\rangle - \langle \varphi(t), f(t)\rangle \\ &= \langle \varphi(-r), f(-r)\rangle, \end{aligned}$$

and this completes the proof.

2. The infinitesimal generator and its adjoint. Since $S(t): Z_p \to Z_p$ is a C_0-semigroup, it follows that there exist constants $M > 0$ and $\gamma > 0$ such that $\|S(t)\| \leqq Me^{\gamma t}$. Also, $S(t)$ has a closed densely defined generator, $\mathscr{A}$.

THEOREM 2.1. *If $\mathscr{A}$ is the infinitesimal generator of $S(t)$, then:*

(i) *The domain of $\mathscr{A}$ is given by $\mathscr{D}(\mathscr{A}) = \{(\eta, \varphi) | \varphi$ is a.c. on compact subsets of $(-\infty, 0]$, $\varphi' \in L_p(-\infty, 0)$, and $\eta = \varphi(0)\}$.*

(ii) *If $(\eta, \varphi) \in \mathscr{D}(\mathscr{A})$, then $\mathscr{A}(\eta, \varphi) = (L(\varphi), \varphi')$.*

(iii) *If* $\operatorname{Re} \lambda \leqq 0$, *then λ belongs to the spectrum of $\mathscr{A}$.*

(iv) *If* $\operatorname{Re} \lambda > 0$, *then λ belongs to the point spectrum, $P_\sigma(\mathscr{A})$, or λ belongs to the resolvant of $\mathscr{A}$, $\rho(\mathscr{A})$. Also, if* $\operatorname{Re} \lambda > 0$, *then $\lambda \in P_\sigma(\mathscr{A})$ if and only if* $\det \Delta(\lambda) = 0$, *where*

$$\Delta(\lambda) = L(e^{\lambda I}) - \lambda I.$$

Proof. Recall that $(\eta, \varphi) \in \mathscr{D}(\mathscr{A})$ if and only if $\lim_{t\to0^+} t^{-1}[S(t)(\eta, \varphi) - (\eta, \varphi)]$ exists.

Suppose $(\eta, \varphi) \in Z_p$ is such that φ is a.c. on compact subsets of $(-\infty, 0]$, $\varphi' = \Psi \in L_p(-\infty, 0)$ and $\eta = \varphi(0)$. As a consequence of Lemma 1.1 we have that

φ is bounded on $(-\infty, 0]$. Consider

$$\begin{aligned}
&\|t^{-1}[S(t)(\eta, \varphi) - (\eta, \varphi)] - (L(\varphi), \Psi)\|^p \\
&= \|t^{-1}[x(t) - \eta] - L(\varphi)\|^p + \|t^{-1}[x_t - \varphi] - \Psi\|^p \\
&= \|t^{-1}[x(t) - x(0)] - L(\varphi)\|^p + \int_{-\infty}^{0} \|t^{-1}[x(t+s) - x(s)] - \varphi'(s)\|^p \, ds,
\end{aligned}$$

where x is the solution to (1.1)–(1.2) with $f(t) \equiv 0$.

Since x is a.c. on compact subsets of $[0, +\infty)$ and $x_0 = \varphi$, we have that $x \in L_p(-\infty, t_1) \cap BC(-\infty, t_1]$ for $t_1 > 0$. Consequently, $g(t) = L(x_t)$ is continuous from the right at 0 and hence $\lim_{t\to 0^+} t^{-1} \int_0^t L(x_s)\, ds = L(x_0) = L(\varphi)$. Since $t^{-1}[x(t) - x(0)] = t^{-1} \int_0^t L(x_s)\, ds$, it follows that

$$\|t^{-1}[x(t) - x(0)] - L(\varphi)\|^p \to 0.$$

Let $x^t(s) = t^{-1} \int_s^{s+t} x'(u)\, du$, and note that $x^t(s) = t^{-1}[x(t+s) - x(s)]$. It is well known that x^t converges to x' in mean of order p on $(-\infty, 0]$ as $t \to 0^+$ (see Graves [9, pp. 254–259]). Hence

$$\begin{aligned}
&\lim_{t\to 0^+} \int_{-\infty}^{0} \|t^{-1}[x(t+s) - x(s)] - \varphi'(s)\|^p \, ds \\
&= \lim_{t\to 0^+} \int_{-\infty}^{0} \|x^t(s) - \varphi'(s)\|^p \, ds \\
&= \lim_{t\to 0^+} \int_{-\infty}^{0} \|x^t(s) - x'(s)\|^p \, ds = 0,
\end{aligned}$$

and this proves that $(\eta, \varphi) \in \mathscr{D}(\mathscr{A})$.

Conversely, suppose $(\eta, \varphi) \in \mathscr{D}(\mathscr{A})$. There exists a $(\xi, \Psi) \in Z_p$ such that $\lim_{t\to 0^+} t^{-1}[S(t)(\eta, \varphi) - (\eta, \varphi)] = (\xi, \Psi)$ and by definition, $\mathscr{A}(\eta, \varphi) = (\xi, \Psi)$. Let x be the solution to (1.1)–(1.2) with $f(t) \equiv 0$, and note that $x \in L_p(-\infty, 1)$. Define F_N on $(-\infty, 0]$ by

$$F_N(s) = N \int_0^{1/N} x(s+u)\, du = N \int_s^{s+1/N} x(u)\, du.$$

The following facts concerning F_N are well known (see Graves [9, pp. 254–259]).

(α) F_N is a.c. on compact subsets of $(-\infty, 0]$ and $F_N'(s) = N[x(s + 1/N) - x(s)]$ a.e. on $(-\infty, 0]$.

(β) $F_N \in L_p(-\infty, 0]$ and F_N converges to x in mean of order p. Because $\lim_{t\to 0^+} t^{-1}[S(t)(\eta, \varphi) - (\eta, \varphi)] = (\xi, \Psi)$, we have that

$$\lim_{t\to 0^+} t^{-1}[x(t) - x(0)] = \xi$$

and

$$\lim_{t\to 0^+} t^{-1}[x_t - x] = \Psi.$$

Consequently, it follows that

$$\int_{-\infty}^{0} \|N[x(s+1/N) - x(s)] - \Psi(s)\|^p \, ds \to 0$$

as $N \to \infty$. But $N[x(s+1/N) - x(s)] = F_N'(s)$, and this implies that F_N' converges to Ψ in mean of order p. Since the differential operator on $L_p(-\infty, 0)$ is closed, and $F_N \to \varphi$ and $F_N' \to \Psi$, we have that φ is a.c. on compact subsets of $(-\infty, 0]$ and $\varphi' = \Psi \in L_p(-\infty, 0)$. The solution x is continuous from the right at zero, thus $F^N(0) = N\int_0^{1/N} x(u)\,du \to x(0) = \eta$, and since $x_0 = \varphi$ is a.c. it follows that $\eta = x(0) = x_0(0) = \varphi(0)$. Now we have that x is a.c. on compact subsets of $(-\infty, 1]$, $x \in L_p(-\infty, 1)$, and $x' \in L_p(-\infty, 1)$. Again by the Remark 1.1, the function $g(t) = L(x_t)$ is continuous from the right and

$$\xi = \lim_{t \to 0^+} t^{-1}[x(t) - x(0)] = \lim_{t \to 0^+} t^{-1} \int_0^t L(x_s)\,ds = L(\varphi).$$

This completes the proof of parts (i) and (ii).

Suppose that $(\eta, \varphi) \in \mathscr{D}(\mathscr{A})$ and $(\mathscr{A} - \lambda I)(\eta, \varphi) = (\xi, \Psi)$. This is equivalent to the equations

$$\varphi(s) = \eta\, e^{\lambda s} + \int_0^s e^{\lambda(s-t)}\Psi(t)\,dt$$

and

$$\begin{aligned}
\xi &= L(\eta\, e^{\lambda(\cdot)}) - \lambda\eta + L\left(\int_0^{(\cdot)} e^{\lambda((\cdot)-t)}\Psi(t)\,dt\right) \\
&= (L(e^{\lambda(\cdot)}I_n) - \lambda I_n)\eta + L\left(\int_0^{(\cdot)} e^{\lambda((\cdot)-t)}\Psi(t)\,dt\right) \\
&= \Delta(\lambda)\eta + L\left(\int_0^{(\cdot)} e^{\lambda((\cdot)-t)}\Psi(t)\,dt\right).
\end{aligned}$$

However, it is clear that if φ is to belong to $L_p(-\infty, 0)$, then $\operatorname{Re}\lambda$ must be positive. In particular, if $(\xi, \Psi) = (\xi, \theta)$, then $\varphi(s) = \eta\, e^{\lambda s} \in L_p(-\infty, 0)$ and $\operatorname{Re}\lambda \leqq 0$ implies that $\eta = 0$. Consequently, $\xi = L(\varphi) - \lambda\eta = 0$ and nothing of the form (ξ, θ), $\xi \neq 0$ can belong to the range of $\mathscr{A} - \lambda I$, if $\operatorname{Re}\lambda \leqq 0$. This proves that $\{\lambda | \operatorname{Re}\lambda \leqq 0\} \subseteq \sigma(\mathscr{A})$.

On the other hand, suppose $\operatorname{Re}\lambda > 0$. Define $K_\lambda : L_p(-\infty, 0) \to C^n$ by

$$K_\lambda(\Psi) = L\left(\int_0^{(\cdot)} e^{\lambda((\cdot)-t)}\Psi(t)\,dt\right),$$

and observe that $(\mathscr{A} - \lambda I)(\eta, \varphi) = (\xi, \Psi)$ is equivalent to

$$\begin{aligned}
&\Delta(\lambda)\eta = \xi - K_\lambda(\Psi), \\
(2.1)\qquad &\varphi(s) = \eta\, e^{\lambda s} + \int_0^s e^{\lambda(s-t)}\Psi(t)\,dt.
\end{aligned}$$

If $\det \Delta(\lambda) \neq 0$, then we have that (2.1) is equivalent to

$$\eta = [\Delta(\lambda)]^{-1}(\xi - K_\lambda(\Psi)),$$
$$\varphi(s) = \eta\, e^{\lambda s} + \int_0^s e^{\lambda(s-t)}\Psi(t)\,dt. \tag{2.2}$$

If $(\mathscr{A} - \lambda I)(\eta, \varphi) = 0$, then in view of (2.1) it follows that $\Delta(\lambda)\eta = 0$ and $\varphi(s) = \eta\, e^{\lambda s}$. However, $\Delta(\lambda)\eta = 0$ has a nonzero solution if $\det(\Delta(\lambda)) = 0$, and for this case λ belongs to the point spectrum of $\mathscr{A}$, $P_\sigma(\mathscr{A})$. On the other hand, if $\det \Delta(\lambda) \neq 0$, the only solution to $(\mathscr{A} - \lambda I)(\eta, \varphi) = 0$ is $(\eta, \varphi) = (0, \theta)$ and $(\mathscr{A} - \lambda I)^{-1}$ exists. Also in view of (2.2),

$$(\mathscr{A} - \lambda I)^{-1}(\xi, \Psi) = \left(E_\lambda(\xi, \Psi), E_\lambda(\xi, \Psi)\, e^{\lambda(\cdot)} + \int_0^{(\cdot)} e^{\lambda((\cdot)-t)}\Psi(t)\,dt\right),$$

where

$$E_\lambda(\xi, \Psi) = [\Delta(\lambda)]^{-1}[\xi - K_\lambda(\Psi)]. \tag{2.3}$$

Moreover, $(\mathscr{A} - \lambda I)^{-1}$ is bounded since $K_\lambda(\Psi) = L(f)$, where

$$f(s) = \int_0^s e^{\lambda(s-t)}\Psi(t)\,dt$$

is in $L_p(-\infty, 0) \cap BC(-\infty, 0)$, L is a bounded operator on $BC(-\infty, 0]$, and

$$\|K_\lambda(\Psi)\| = \|L(f)\| \leqq \|L\|_\infty \|f\|_\infty \leqq \|L\|_\infty \|\Psi\|.$$

Therefore, it follows that $\lambda \in \rho(\mathscr{A})$ and the resolvent $R_\lambda = (\mathscr{A} - \lambda I)^{-1}$ is given by (2.3).

Remark 2.1. Working in a space with fading memory, Hale [10], [11] considered equation (1.1). In [10] Hale applied the theory of α-contractions to obtain exponential estimates on the solution operator. As pointed out in that paper, these estimates are applicable to the semigroup $S(t)$ defined on Z_p. In particular, if

$$\{\lambda | \operatorname{Re} \lambda > 0 \text{ and } \det \Delta(\lambda) = 0\}$$

is empty, then for any $\varepsilon > 0$, there is an $M_\varepsilon > 0$ such that

$$\|S(t)(\eta, \varphi)\| \leqq M_\varepsilon\, e^{\varepsilon t} \|(\eta, \varphi)\|,$$

for $t \geqq 0$.

However, our prime interest is in the study of the adjoint semigroup, and some of its stability properties. For this reason, the state space Z_p is a useful space in that the computation of the adjoint is almost trivial.

Also, it should be noted that results similar to Theorem 2.1 may be found in Naito's paper [15]. However, again it should be noted that Naito was working in a space with fading memory.

In order to compute the adjoint semigroup we shall need the adjoint operator $\mathscr{A}^*$. However, as we shall see below, the choice of Z_p as the state space allows us to compute $\mathscr{A}^*$ very easily for a large class of operators L. In particular, we shall restrict attention to the case where L is defined by

$$L(\varphi) = M\varphi(0) + N\varphi(-r) + \int_{-\infty}^{0} K(s)\varphi(s)\,ds, \tag{2.4}$$

where M and N are $n \times n$ constant matrices and K is an $n \times n$ matrix function such that $\int_{-\infty}^{0} \|K(s)\|\,ds < +\infty$, and $r \geqq 0$. As shown in Example 1.1, the operator L defined by (2.4) satisfies hypothesis $\mathscr{H}$. Moreover, in this case $\mathscr{A}$ is given by

$$\mathscr{A}(\eta, \varphi) = \left(M\eta + N\varphi(-r) + \int_{-\infty}^{0} K(s)\varphi(s)\,ds, \varphi'(\cdot)\right).$$

In order to simplify notation, we define the functions Ψ^- and Ψ^+ as follows. If Ψ is continuous on $(-\infty, 0]$ except for a jump discontinuity at $-r$, then $\Psi^- \in \mathscr{C}(-\infty, -r]$ and $\Psi^+ \in \mathscr{C}[-r, 0]$ are defined by

$$\Psi^-(s) = \begin{cases} \Psi(s), & s < -r, \\ \Psi(-r^-), & s = -r, \end{cases}$$

and

$$\Psi^+(s) = \begin{cases} \Psi(-r^+), & s = -r, \\ \Psi(s), & -r < s \leqq 0. \end{cases}$$

THEOREM 2.2. *If $\mathscr{A}^*$ denotes the adjoint of $\mathscr{A}$, then:*

(i) *The domain of $\mathscr{A}^*$ is given by $\mathscr{D}(\mathscr{A}^*) = \{(\xi, \Psi) \in Z_p^* | \Psi^-$ is a.c. on compact subsets of $(-\infty, -r]$, Ψ^+ is a.c. on $[-r, 0]$, $K^*(\cdot)\xi - \Psi'(\cdot) \in L_q(-\infty, 0)$, and $N^*\xi = [\Psi(-r^+) - \Psi(-r^-)]\}$, where $q = p'$.*

(ii) *If $(\xi, \Psi) \in \mathscr{D}(\mathscr{A}^*)$, then $\mathscr{A}^*(\xi, \Psi) = (M^*\xi + \Psi(0), K^*(\cdot)\xi - \Psi'(\cdot))$.*

(iii) *If $1 < p < +\infty$, then $\mathscr{A}^*$ is densely defined, i.e., $\overline{\mathscr{D}(\mathscr{A}^*)} = Z_p^* = Z_q$.*

(iv) *If $p = 1$, then $\mathscr{A}^*$ is not densely defined and $\overline{\mathscr{D}(\mathscr{A}^*)} = Z_1^+$, where Z_1^+ is the closed subspace of $C^n \times L_\infty(-\infty, 0)$ given by*

$$Z_1^+ = \{(\xi, \Psi) | \Psi^- \in \mathrm{BUC}(-\infty, -r], \Psi^+ \in C[-r, 0],$$

$$\text{and} \quad N^*\xi = \Psi^+(-r) - \Psi^-(-r)\},$$

where BUC $(-\infty, -r]$ *is the space of bounded uniformly continuous functions on* $(-\infty, -r]$.

Proof. Recall that $(\xi, \Psi) \in \mathscr{D}(\mathscr{A}^*)$ if and only if there is a $(\hat{\xi}, \hat{\Psi}) \in Z_p^*$ such that

$$\langle \mathscr{A}(\eta, \varphi), (\xi, \Psi) \rangle - \langle (\eta, \varphi), (\hat{\xi}, \hat{\Psi}) \rangle = 0 \tag{2.5}$$

for all $(\eta, \varphi) \in \mathscr{D}(\mathscr{A})$, and in this case, $\mathscr{A}^*(\xi, \Psi) = (\hat{\xi}, \hat{\Psi})$.

Concentrating on the left side of (2.5) we obtain

$$\begin{aligned}
&\langle \mathscr{A}(\eta,\varphi),(\xi,\Psi)\rangle - \langle(\eta,\varphi),(\hat{\xi},\hat{\Psi})\rangle \\
&= \langle L(\varphi),\xi\rangle + \int_{-\infty}^{0}\langle\varphi'(s),\Psi(s)\rangle\,ds - \langle\eta,\hat{\xi}\rangle - \int_{-\infty}^{0}\langle\varphi(s),\hat{\Psi}(s)\rangle\,ds \\
&= \langle M\varphi(0),\xi\rangle + \langle N\varphi(-r),\xi\rangle - \langle\eta,\hat{\xi}\rangle + \int_{-\infty}^{0}\langle K(s)\varphi(s),\xi\rangle\,ds \\
&\quad + \int_{-\infty}^{0}[\langle\varphi'(s),\Psi(s)\rangle + \langle\varphi(s),-\hat{\Psi}(s)\rangle]\,ds \\
&= \langle\varphi(0),M^*\xi - \hat{\xi}\rangle + \langle\varphi(-r),N^*\xi\rangle \\
&\quad + \int_{-\infty}^{0}[\langle\varphi'(s),\Psi(s)\rangle + \langle\varphi(s),K^*(s)\xi - \hat{\Psi}(s)\rangle]\,ds.
\end{aligned}$$

If $(\eta,\varphi)\in\mathscr{D}(\mathscr{A})$, $\eta = \varphi(0) = 0$, and for $s \leqq -r$, $\varphi(s) = \varphi(-r) = 0$, then (2.5) implies that

$$\int_{-r}^{0}[\langle\varphi'(s),\Psi(s)\rangle + \langle\varphi(s),K^*(s)\xi - \hat{\Psi}(s)\rangle]\,ds = 0.$$

Consequently, the fundamental lemma of the calculus of variations implies that Ψ is a.c. on $[-r,0]$ and $\Psi'(s) = K^*(s)\xi - \hat{\Psi}(s)$ a.e. on $[-r,0]$. In particular, $\hat{\Psi}(s) = K^*(s)\xi - \Psi'(s)$ and $K^*(\cdot)\xi - \Psi'(\cdot)\in L_q(-r,0)$.

Also, if $(\eta,\varphi)\in\mathscr{D}(\mathscr{A})$, $\varphi(s) = 0$ for $-r \leqq s \leqq 0$, and φ has compact support on $(-\infty,-r)$, then (2.5) implies that

$$\int_{-\infty}^{-r}[\langle\varphi'(s),\Psi(s)\rangle + \langle\varphi(s),K^*(s)\xi - \hat{\Psi}(s)\rangle]\,ds = 0.$$

From Lemma 1.3, it follows that Ψ is a.c. on compact subsets of $(-\infty,-r]$ and $\Psi'(s) = K^*(s)\xi - \hat{\Psi}(s)$ a.e. on $(-\infty,-r]$. Therefore, we have that $\hat{\Psi}(s) = K^*(s)\xi - \Psi'(s)$ and $K^*(\cdot)\xi - \Psi'(\cdot)\in L_q(-\infty,-r)$. Again applying Lemma 1.3, we have that

$$\begin{aligned}
&\int_{-\infty}^{0}[\langle\varphi'(s),\Psi(s)\rangle + \langle\varphi(s),K^*(s)\xi - \hat{\Psi}(s)\rangle]\,ds \\
&= \int_{-r}^{0}[\langle\varphi(s),\Psi(s)\rangle]'\,ds + \int_{-\infty}^{-r}[\langle\varphi(s),\Psi(s)\rangle]'\,ds \\
&= \langle\varphi(0),\Psi(0)\rangle - \langle\varphi(-r),\Psi(-r^+)\rangle + \langle\varphi(-r),\Psi(-r^-)\rangle,
\end{aligned}$$

and hence if $(\eta,\varphi)\in\mathscr{D}(\mathscr{A})$, then (2.5) reduces to

$$0 = \langle\varphi(-r),M^*\xi + \Psi(0) - \hat{\xi}\rangle + \langle\varphi(-r),N^*\xi - [\Psi(-r^+) - \Psi(-r^-)]\rangle.$$

Thus, it follows that

$$\hat{\xi} = M^*\xi + \Psi(0)$$

and

$$N^*\xi = (\Psi(-r^+) - \Psi(-r^-)),$$

and we have shown that if $(\xi, \Psi) \in \mathscr{D}(\mathscr{A}^*)$, then Ψ is a.c. on $[-r, 0]$, and on compact subsets of $(-\infty, -r]$, $K^*(\cdot)\xi - \Psi'(\cdot) \in L_q(-\infty, 0)$, and $N^*\xi = (\Psi(-r^+) - \Psi(-r^-))$. Moreover, for $(\xi, \Psi) \in \mathscr{D}(\mathscr{A}^*)$, $\mathscr{A}^*$ is given by $\mathscr{A}^*(\xi, \Psi) = (M^*\xi + \Psi(0), K^*(\cdot)\xi - \Psi'(\cdot))$. To complete the proof of parts (i) and (ii) one need only reverse the above equalities.

The proof of (iii) is well known since Z_p is reflexive (see [16, p. 270]). For the case where $p = 1$, it is clear that $\mathscr{D}(\mathscr{A}^*)$ is not dense in Z_∞. If $(\xi, \Psi) \in \mathscr{D}(\mathscr{A}^*)$, then Ψ^- is a.c. on compact subsets of $(-\infty, -r]$ and $K^*(\cdot)\xi - [\Psi^-(\cdot)]' = \hat{\Psi}(\cdot) \in L_\infty(-\infty, -r)$. In particular, we have that $\Psi^-(t) = \Psi^-(-r) + \int_{-r}^{t} [K^*(s)\xi - \hat{\Psi}(s)]\, ds$ and hence $\Psi^-(t)$ is a bounded uniformly continuous function on $(-\infty, -r]$. Consequently, $\mathscr{D}(\mathscr{A}^*) \subseteq Z_1^+$. Let K be defined by $K = \{F | F^-$ and $[F^-]' \in \mathrm{BUC}(-\infty, -r]$, F^+ and $[F^+]' \in C[-r, 0]\}$, and for each $\xi \in C^n$ define the set Y_ξ as follows:

$$Y_\xi = \{F | F \in K \quad \text{and} \quad F^+(-r) - F^-(-r) = N^*\xi\}.$$

For each $(\xi, \Psi) \in Z_1^+$ and $\varepsilon > 0$, it follows that there exists a $F \in Y_\xi$ such that $\|(\xi, F) - (\xi, \Psi)\|_\infty < \varepsilon$. Let $\mathscr{D} = \{(\xi, \Psi) | \Psi^-(s) = \int_{-\infty}^{s} K^*(t)\xi\, dt + F^-(s),\ \Psi^+(s) = \int_{-\infty}^{s} K(t)\xi\, dt + F^+(s)$, where $F \in Y_\xi\}$, and note that $\mathscr{D}$ is dense in Z_1^+. Moreover, $\mathscr{D} \subseteq \mathscr{D}(\mathscr{A}^*) \subseteq Z_1^+$ implies that $\overline{\mathscr{D}(\mathscr{A}^*)} = Z_1^+$.

Remark 2.2. It should be noted that if $r = 0$ and $N = 0$, then system (1.1)–(1.2) becomes the Volterra integro-differential system

$$x'(t) = Mx(t) + \int_{-\infty}^{t} K(s - t)x(s)\, ds, \qquad t \geqq 0,$$

$$x(0) = \eta, \qquad x_0 = \varphi.$$

In this case, $\mathscr{D}(\mathscr{A})$ is given by $\mathscr{D}(\mathscr{A}) = \{(\eta, \varphi) | \varphi$ is a.c. on compact subsets of $(-\infty, 0]$, $\varphi(\cdot) \in L_p(-\infty, 0)$, $\varphi'(\cdot) \in L_p(-\infty, 0)$, and $\eta = \varphi(0)\}$, and $\mathscr{A}(\eta, \varphi) = (M\eta + \int_{-\infty}^{0} K(s)\varphi(s)\, ds, \varphi'(\cdot))$. Also, $\mathscr{D}(\mathscr{A}^*)$ is given by $\mathscr{D}(\mathscr{A}^*) = \{(\xi, \Psi) | \Psi \in L_q(-\infty, 0)$, Ψ is a.c. on compact subsets of $(-\infty, 0]$, and $K^*(\cdot)\xi - \Psi'(\cdot) \in L_q(-\infty, 0)\}$, and $\mathscr{A}^*(\xi, \Psi) = (M^*\xi + \Psi(0), K^*(\cdot)\xi - \Psi'(\cdot))$. In particular, if $p = 1$ and $q = \infty$, then

$$Z_1^+ = C^n \times \mathrm{BUC}(-\infty, 0].$$

Note that for this case there is no restriction on Ψ at 0.

Also, it is clear that the above arguments can be made for more general operators L. For example, L could be defined by

$$L(\varphi) = \sum_{i=0}^{N} M_i \varphi(-r_i) + \sum_{i=1}^{Q} \int_{-\infty}^{0} K_i(s)\varphi(s)\, ds,$$

where $0 = r_0 < r_1 < \cdots < r_n$. In this case, one would have to take into account the jumps of Ψ at $r_1, r_2, \cdots, r_n$.

3. The adjoint semigroup. In [16], R. S. Phillips developed a general theory for the "adjoint" semigroup of operators. Earlier, Feller [8] had formally obtained an adjoint semigroup and applied his formal theory to parabolic differential equations.

In this section, we shall present a brief description of the construction of the "adjoint" semigroup as defined by Phillips in [16]. Also, we shall restrict our attention to C_0-semigroups.

Let X be a Banach space and $T(t): X \to X$ a C_0-semigroup with infinitesimal generator C. It is clear that $T^*(t): X^* \to X^*$ is a semigroup on X^*. If X is reflexive, then it follows that $T^*(t)$ is a C_0-semigroup with infinitesimal generator C^* (see [16, p. 277]). However, if X is not reflexive, then $T^*(t)$ need not be a C_0-semigroup; in fact, C^* may not be densely defined. One is able to avoid some of these difficulties by taking the "adjoint" semigroup to be the restriction of the adjoints $T^*(t)$ to a properly chosen "adjoint" Banach space X^+, which in general will be a proper subspace of X^*. Moreover, $X^+ \subseteq X^*$ shall be the largest domain for which the ordinary adjoint $T^*(t)$ is a C_0-semigroup and the generator of the semigroup, C^+, turns out to be the maximal restriction of C^* with domain and range in X^+.

To be more precise, let $X^+ = \overline{\mathscr{D}(C^*)}$ and $T^+(t)$ be the restriction of $T^*(t)$ to X^+. The following results may be found in [12, p. 429] or [16].

THEOREM 3.1. *If $T(t)$ is a C_0-semigroup, then $T^+(t)$ is a C_0-semigroup on X^+. Moreover, if C^+ denotes the infinitesimal generator of $T^+(t)$, then $\mathscr{D}(C^+) = \{x^* \in \mathscr{D}(C^*) | C^*x^* \in X^+\}$ and C^+ is C^* restricted to $\mathscr{D}(C^+)$.*

It is clear from the above theorem that one may be able to obtain certain properties (i.e., such as stability) for the semigroup $T(t)$ from known properties of $T^+(t)$. For example, if X is reflexive, then $X^+ = X^*$ and if $T^*(t)y^* = T^+(t)y^* \to 0$ for all $y^* \in X^*$, we can conclude that $T(t)x \to 0$ weakly for all $x \in X$.

THEOREM 3.2. *If $T(t)$ is a C_0-semigroup, then $\rho(C) = \rho(C^+)$.*

We shall not devote time to the general theory of the adjoint semigroup for the system (1.1)–(1.2), but rather to a specific application to a Volterra integro-differential system.

4. A Volterra integro-differential system. In this section we shall consider the linear Volterra integro-differential equation

$$x'(t) = Mx(t) + \int_{-\infty}^{t} K(s-t)x(s)\,ds, \qquad t \geqq 0, \tag{4.1}$$

with initial data

$$x(0) = \eta, \quad x_0(\cdot) = \varphi(\cdot). \tag{4.2}$$

System (4.1)–(4.2) was studied by Barbu and Grossman [2] via semigroup methods. In [2], the state space was $BC_l(-\infty, 0]$, i.e., the space of bounded continuous functions on $(-\infty, 0]$ with finite limit at $-\infty$. Also, η was identified with

$\varphi(0)$. Their method of constructing the semigroup is direct and analogous to the construction of the semigroup $S(t)$, presented in § 2.

In [13], R. K. Miller also studied system (4.1)–(4.2) by semigroup methods, under very mild assumptions on the kernel function K. (However, we shall always assume that $K(\cdot) \in L_1(-\infty, 0)$.) The method employed by Miller is indirect and involves embedding system (4.1)–(4.2) in a larger class of problems. As indicated by Miller, the indirect approach has certain advantages. In particular, the semigroup constructed by Barbu and Grossman can be uniformly bounded, but cannot tend strongly to zero. On the other hand, the semigroup constructed by the indirect method can have both of these properties.

As we shall now show, by appropriately choosing the state space, one may consider Miller's semigroup as a restriction of the adjoint of the semigroup constructed by Barbu and Grossman.

Remark 4.1. Note that if $f \in L_p(0, +\infty)$, then $\hat{f} \in L_p(-\infty, 0)$ defined by $\hat{f}(s) = f(-s)$ provides an obvious identification of $C^n \times L_p(0, +\infty)$ with $C^n \times L_p(-\infty, 0)$. Also, note that under this identification, $[\hat{f}(s)]' = -[\hat{f'}](s)$. We point this out because Miller's semigroup is defined on a subspace of $C^n \times L_p(0, +\infty)$, whereas the adjoint semigroup mentioned above will be defined on a subspace of $C^n \times L_p(-\infty, 0)$. With the identification given above, it will be clear that the two semigroups are the same. We continue now assuming that the reader has Miller's paper [13] at hand.

Let the state space for the system (4.1)–(4.2) be $Z_1 = C^n \times L_1(-\infty, 0)$. In view of Remark 2.2 and previous theorems we have the following facts:

(i) If $S(t): Z_1 \to Z_1$ is defined by (1.7), then the infinitesimal generator $\mathscr{A}$ is given by

$$\mathscr{D}(\mathscr{A}) = \{(\eta, \varphi) | \varphi \in L_1(-\infty, 0), \varphi \text{ a.c.}, \varphi' \in L_1(-\infty, 0) \text{ and } \eta = \varphi(0)\}.$$

Moreover, if $(\eta, \varphi) \in \mathscr{D}(\mathscr{A})$, then $\mathscr{A}(\eta, \varphi) = (M\eta + \int_{-\infty}^{0} K(s)\varphi(s)\, ds, \varphi'(\cdot))$.

(ii) By Theorem 2.2, $Z_1^+ = \overline{\mathscr{D}(\mathscr{A}^*)}$ is given by $Z_1^+ = C^n \times \mathrm{BUC}(-\infty, 0]$, where $\mathscr{D}(\mathscr{A}^*) = \{(\xi, \Psi) \in Z_\infty | \Psi(\cdot)$ is a.c. and $K^*(\cdot)\xi - \Psi'(\cdot) \in L_\infty(-\infty, 0)\}$, and if $(\xi, \Psi) \in \mathscr{D}(\mathscr{A}^*)$, then $\mathscr{A}^*(\xi, \Psi) = (M^*\xi + \Psi(0), K^*(\cdot)\xi - \Psi'(\cdot))$.

(iii) From Theorem 3.1, we have that $S^+(t)$ is $S^*(t)$ restricted to Z_1^+ and $\mathscr{A}^+$ is $\mathscr{A}^*$ restricted to

$$\begin{aligned}\mathscr{D}(\mathscr{A}^+) &= \{(\xi, \Psi) \in \mathscr{D}(\mathscr{A}^*) | \mathscr{A}^*(\xi, \Psi) \in Z_1^+\} \\ &= \{(\xi, \Psi) | \Psi(\cdot) \in \mathrm{BUC}(-\infty, 0] \text{ and } K^*(\cdot)\xi - \Psi'(\cdot) \in \mathrm{BUC}(-\infty, 0]\}.\end{aligned}$$

(iv) The resolvent set of $\mathscr{A}$ and $\mathscr{A}^+$ is given by

$$\rho(\mathscr{A}) = \rho(\mathscr{A}^+) = \{\lambda | \operatorname{Re} \lambda > 0 \text{ and } \det \Delta(\lambda) \neq 0\},$$

where

$$\Delta(\lambda) = L(e^{\lambda s} I) - \lambda I = \left[M + \int_{-\infty}^{0} K(s)\, e^{\lambda s}\, ds - \lambda I \right].$$

THEOREM 4.1. *If* $(\xi, \Psi) \in Z_1^+$, *then the semigroup* $S^+(t)$ *is defined by*

$$S^+(t)(\xi, \Psi) = (y(t), y^t(\cdot)), \tag{4.3}$$

where y *is the solution to*

$$y'(t) = M^*y(t) + \int_0^t K^*(s-t)y(s)\,ds + \Psi(-t) \tag{4.4}$$

with initial value $y(0) = \xi$, *and* $y^t(\cdot)$ *is defined by*

$$y^t(s) = \Psi(s-t) + \int_0^t K^*(v+s-t)y(v)\,dv. \tag{4.5}$$

Proof. It suffices to show that (4.3) holds for $(\xi, \Psi) \in \mathscr{D}(\mathscr{A}^+)$ since $\overline{\mathscr{D}(\mathscr{A}^+)} = Z_1^+$. However, if $(\xi, \Psi) \in \mathscr{D}(\mathscr{A}^+)$, then $z(t) = S^+(t)(\xi, \Psi) = (x(t), y(t;\cdot))$ belongs to $\mathscr{D}(\mathscr{A}^+)$ and satisfies the system

$$\frac{d}{dt}z(t) = A^+z(t), \qquad t \geqq 0,$$

and $z(0) = (x(0), y(0;\cdot)) = (\xi, \Psi(\cdot))$. If we define $g(u)$ by $g(u) = y(u; u+s-t)$ for $t \geqq 0$ and $s \leqq 0$, then in view of the definition of $\mathscr{A}^+$ we have that ($\partial/\partial x_j$ means derivative with respect to the jth variable, $j = 1, 2$)

$$\frac{d}{du}g(u) = \frac{\partial}{\partial x_1}y(u; u+s-t) + \frac{\partial}{\partial x_2}y(u; u+s-t) = K^*(u+s-t)x(u).$$

Therefore, it follows that

$$g(t) = g(0) + \int_0^t K^*(u+s-t)x(u)\,du,$$

or

$$\begin{aligned} y(t,s) &= y(0; s-t) + \int_0^t K^*(u+s-t)x(u)\,du. \\ &= \Psi(s-t) + \int_0^t K^*(u+s-t)x(u)\,du. \end{aligned}$$

Moreover, $x(\cdot)$ satisfies

$$\begin{aligned} \frac{d}{dt}x(t) &= M^*x(t) + y(t;0) \\ &= M^*x(t) + \int_0^t K^*(u-t)x(u)\,du + \Psi(-t), \end{aligned}$$

with $x(0) = \xi$. Consequently, $x(\cdot)$ satisfies (4.4) and $y(t;s) = x^t(s)$ which completes the proof.

We now direct attention to the construction of the semigroup presented in [13].

If $F \in \mathrm{BC}(-\infty, 0]$, then $\mathscr{L}(F) = f$ is the function in $\mathrm{BUC}(-\infty, 0]$ defined by

$$[\mathscr{L}(F)](s) = f(s) = \int_{-\infty}^{0} K^*(u - s)F(u)\,du,$$

(compare with 2.1 in [13]), and $Y(K^*)$ will be the subspace of Z_1^+ given by

$$Y(K^*) = \{(x, f) | f = \mathscr{L}(F), x = F(0), F \in \mathrm{BC}(-\infty, 0]\}.$$

Moreover, the closure of $Y(K^*)$ will be denoted by Y.

Let $U(t)$ be the restriction of $S^+(t)$ to Y. In particular, if $(x,f) \in Y$, then $U(t)(x,f) = S^+(t)(x,f) = (y(t), y^t(\cdot))$, where y satisfies (4.4) with $y(0) = x$ and $y^t(\cdot)$ is defined by (4.5) with $f(\cdot) = \Psi(\cdot)$.

The following results may be found in [13].

THEOREM 4.2. *Let $\mathscr{L}$, Y and $U(t)$ be as above. Then*:

(i) *The space Y contains all pairs of the form $(x, \mathscr{L}(F)) \in Z_1^+$, where $x \in C^n$ and F is uniformly bounded and piecewise continuous on each interval of the form $[-T, 0]$.*

(ii) *The operator $U(t)$ maps Y into Y and is a C_0-semigroup on Y.*

(iii) *The following statements are equivalent*; (a) $\det \Delta(\lambda) \neq 0$ *for* $\operatorname{Re} \lambda \geqq 0$. (b) $\sup \{\|U(t)\| \,|t \geqq 0\} < +\infty$ *and for each* $(x,f) \in Y$, $U(t)(x,f) \to 0$ *as* $t \to +\infty$.

As Miller indicated in [13], the semigroup he constructed can be embedded in a larger semigroup. Moreover, we have shown above that $U(t)$ is actually the restriction of $S^+(t)$, the adjoint semigroup of $S(t)$. Although stability of the semigroup $U(t)$ (or even of $S^+(t)$) does not imply the corresponding stability for $S(t)$, one is still able to obtain some information about $S(t)$ from $S^+(t)$ and $U(t)$. For example, consider the following obvious result.

THEOREM 4.3. *Let $(\eta, \varphi) \in Z_1$ and suppose that $x(\cdot)$ is the solution to system* (4.1)–(4.2). *If* $\det \Delta(\lambda) \neq 0$ *for* $\operatorname{Re} \lambda \geqq 0$, *then* $x(t) \to 0$ *as* $t \to \infty$.

Proof. Let $\xi \in C^n$ and $\theta(\cdot)$ denote the zero function. Since $(\xi, \theta) \in Y$, we have that

$$\begin{aligned} \langle x(t), \xi \rangle &= \langle (x(t), x_t(\cdot)), (\xi, \theta) \rangle \\ &= \langle S(t)(\eta, \varphi), (\xi, \theta) \rangle \\ &= \langle (\eta, \varphi), S^*(t)(\xi, \theta) \rangle \\ &= \langle (\eta, \varphi), U(t)(\xi, \theta) \rangle \to 0. \end{aligned}$$

Since $\xi \in C^n$ is arbitrary, we have that $x(t) \to 0$.

In conclusion, we should also note that Miller in [14] has considered the Volterra integro-differential system (4.1)–(4.2) in a Banach space X. In this paper, Miller constructed the semigroup $U(t)$ and indicated that $U(t)$ could be extended to a semigroup on the space of bounded and uniformly continuous X-valued functions. It is clear that under certain assumptions that again this extension may be considered as the adjoint semigroup, $S^+(t)$, where $S(t)$ is the semigroup defined in § 2 for the X-valued system (1.1)–(1.2). (For the finite delay system in a Banach space see Borisovič and Turbabin [3].)

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AN EXPLICIT A PRIORI ESTIMATE FOR PARABOLIC EQUATIONS WITH APPLICATIONS TO SEMILINEAR EQUATIONS*

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Abstract. A coercivity inequality for the first initial-boundary value problem for a second order parabolic equation is established. This result is applied to a semilinear problem.

Introduction. We establish here an a priori estimate for second order parabolic operators with bounded coefficients. The spirit of this work is similar to that of recent work of one of the authors [1] in which explicit estimates were given for elliptic operators satisfying analogous hypotheses. The estimates provide a bound for the L_2-norm of the second spatial derivatives and the time derivative of a function u in terms of the L_2-norm of Lu, where L is the operator in question, with an explicitly given constant. If the number of space variables is less than four, the Sobolev embedding theorem implies a bound for the maximum norm of u and a corresponding a posteriori estimate for the approximation of the solution of a linear parabolic equation by linear combinations of functions which satisfy the boundary conditions (cf. [2]). (We impose the Dirichlet boundary condition on the lateral boundary, but it is anticipated that analogous results can be derived for mixed boundary conditions.)

In the second part of the paper, we apply the above result to obtain a constructive existence theorem for a semilinear equation of a type that occurs in nonlinear heat conduction. An interesting feature of this is an a priori estimate for the solution of a nonlinear equation which followed from a version of the strong maximum principle for weak solutions of parabolic inequalities [3].

1. An a priori estimate. We will study the parabolic operator

$$Lu = a_{ij}u_{,ij} + b_i u_{,i} - au - cu_t$$

acting on functions in $D = \Omega \times [0, T]$ which belong to the Hilbert space $W_2^{2,1}(D)$ of functions with finite norm

$$(\|u\|_{2,1})^2 = \int_D (u^2 + |\nabla u|^2 + |D^2 u|^2 + u_t^2)\, dx\, dt,$$

where the gradient is with respect to space variables, and $|D^2u|^2$ represents the sum of the squares of all the second derivatives with respect to space variables. We denote by $\|\cdot\|_0$ the norm in $L_2(D)$. We assume immediately that the n-dimensional domain Ω has a piecewise smooth boundary with nonnegative mean

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curvature and that

$$a_{ij}\xi_i\xi_j \geqq v^2|\xi|^2$$

for some positive constant v, all n-vectors ξ and all (x, t) in D. This assumption about the mean curvature of $\partial\Omega$ can be weakened somewhat (as indicated in [1]), but we will not pursue that further here. Further, we will use the notation $_{,i}$ to denote $\partial/\partial x_i$; t used as a subscript denotes $\partial/\partial t$, and the summation convention is used so that repeated indices are to be summed over the spatial coordinates, i.e., from 1 to n.

Our goal is to establish an inequality of the form

$$\|u\|_{2,1} \leqq C\|Lu\|_0$$

for functions $u \in W_2^{2,1}(D)$ with zero trace on the parabolic boundary of D and to obtain specific information about the dependence of C on the coefficients of L.

The assumptions about Ω imply that for each t, $u(x, t) \in W_{2,0}^2(\Omega) = W_2^2(\Omega) \cap \dot{W}_2^1(\Omega)$ and that any function in this class is the norm limit of functions in $C_2(\bar{\Omega})$ which vanish on $\partial\Omega$. We will denote the subspace of $W_2^{2,1}(D)$ whose elements vanish on the parabolic boundary of D by W_0.

The results obtained here utilize similar estimates for elliptic operators which were given by one of the authors in a recent paper [1], and these are summarized below following our assumptions about the coefficients of L:

(A_2) (Regularity of coefficients) $a_{ij}(x, t) \in W_\infty^1(D)$, b_i, $a \in L_\infty(D)$, $c = c(x) \in W_\infty^1(\Omega)$.

(A_2) If we define $S = \sup|b_i - (a_{ij})_{,j}|$ and $a_0 = \inf_D a$, then $S < \sqrt{\lambda}\, v^2$ and $a_0 > \sqrt{\lambda}\, S - \lambda v^2$. Also, we assume $\inf_\Omega c > 0$.

In the above, λ denotes the lowest eigenvalue of $-\Delta$ in Ω. See [4] and [5] for methods of estimating λ.

The constant S may be taken to be a measure of formal self-adjointness of the operator L_0, that is, if $S = 0$, $L_0 u = (a_{ij}u_{,i})_{,j} - au$. In any case, the Faber–Krahn inequality [5] implies that the restrictions $S < \sqrt{\lambda}\, v^2$ and $a_0 > \sqrt{\lambda}\, S - \lambda v^2$ are automatically satisfied if the volume of Ω is sufficiently small.

In order to state the previous results in a convenient form, we denote by

$$L_0 u = a_{ij}u_{,ij} + b_i u_{,i} - au$$

the elliptic operator obtained from L by deleting the term $-cu_t$ and holding t fixed.

LEMMA 1. *The inequality*

$$\int_\Omega (L_0 u)^2\, dx \geqq (a_0 + \lambda v^2 - S\sqrt{\lambda})^2 \int_\Omega u^2\, dx$$

holds for u in $W_{2,0}^2(\Omega)$.

LEMMA 2. *The inequality*

$$\int_\Omega |\nabla u|^2\, dx \leqq C_1 \int_\Omega (L_0 u)^2\, dx$$

holds for u in $W^2_{2,0}(\Omega)$, *where*

$$\nu^2 C_1 = \begin{cases} a_0 + \lambda\nu^2 - S\sqrt{\lambda} & \text{if } a_0 \geqq \lambda\nu^2 - S\sqrt{\lambda} + \dfrac{S^2}{2\nu^2}, \\ 2(a_0 + \lambda\nu^2 - S\sqrt{\lambda}) + \left(\dfrac{S^2}{\nu^2} - 2a_0\right) & (a_0 + \lambda\nu^2 - S\sqrt{\lambda})^2 \\ & \text{if } a_0 < \lambda\nu^2 - S\sqrt{\lambda} + \dfrac{S^2}{2\nu^2}. \end{cases}$$

Now we define

$$Pu = a_{ij}u_{,ij} \quad \text{and} \quad B = \sup_D |(a_{ij}a_{kl} - a_{ik}a_{jl})_{,k}|;$$

then we have the next lemma.

LEMMA 3. *The inequality*

$$\frac{\nu^2}{2}\int_\Omega |D^2 u|^2\, dx \leqq \int_\Omega (Pu)^2\, dx + \frac{n^4 B^2}{2\nu^2}\int_\Omega |\nabla u|^2\, dx$$

holds for u in $W^2_{2,0}(\Omega)$.

These results are derived in [1]. It is remarked there that if $n = 2$, Lemma 3 may be replaced by an inequality which does not involve the gradient on the right-hand side.

The basic identity used in proving Lemmas 1 and 2 is

$$-\int_\Omega uL_0 u\, dx = \int_\Omega a_{ij}u_{,i}u_{,j}\, dx + \int_\Omega au^2\, dx + \int_\Omega [(a_{ij})_{,j} - b_i]u_{,i}u\, dx,$$

and its analogue in the present situation is

$$\begin{aligned} -\int_D uLu\, dx\, dt &= \int_D a_{ij}u_{,i}u_{,j}\, dx\, dt + \int_\Omega au^2\, dx\, dt \\ &\quad + \int_D [(a_{ij})_{,j} - b_i]u_{,i}u\, dx\, dt + \int cuu_t\, dx\, dt \quad \text{for } u \in W_0. \end{aligned} \tag{1}$$

If we also observe that

$$\int_D cuu_t\, dx\, dt = \frac{1}{2}\int_D c(u^2)_t\, dx\, dt = \frac{1}{2}\int_\Omega cu^2(x, T)\, dx \geqq 0 \tag{2}$$

and that, by integrating the usual inequality,

$$\lambda\int_D u^2\, dx\, dt \leqq \int_D |\nabla u|^2\, dx\, dt, \tag{3}$$

results analogous to Lemmas 1 and 2 follow immediately. In particular, we have the following results.

LEMMA 1′. *The inequality*

$$\int_D (Lu)^2\,dx\,dt \geqq (a_0 + \lambda\nu^2 - S\sqrt{\lambda})^2 \int_D u^2\,dx\,dt$$

holds for $u \in W_0$.

LEMMA 2′. *The inequality*

$$\int_D |\nabla u|^2\,dx\,dt \leqq C_1 \int_D (Lu)^2\,dx\,dt$$

holds for $u \in W_0$, *where* C_1 *is as defined in Lemma* 2.

LEMMA 3′.

$$\frac{\nu^2}{2}\int_D |D^2u|^2\,dx\,dt \leqq \int_D (Pu)^2\,dx\,dt + \frac{n^4B^2}{2\nu^2}\int_D |\nabla u|^2\,dx\,dt \quad \text{for } u \in W_0. \tag{4}$$

We are now ready to give our main estimate. We will define the norm by

$$(\|u\|_{2,1})^2 = \nu^2 \int_D |D^2u|^2\,dx\,dt + \int_D (cu_t)^2\,dx\,dt$$

in W_0.

THEOREM 1. *Assume* A_1 *and* A_2. *Then there is a constant C such that*

$$\|u\|_{2,1} \leqq C\|Lu\|_0$$

for $u \in W_0$.

Proof. By repeated applications of the weighted arithmetic-geometric mean inequality to (1), we obtain

$$\begin{aligned}\int_D (Lu)^2\,dx\,dt \geqq \frac{1}{4}\int_D (Pu)^2\,dx\,dt - 5\int_D (b_iu_{,i})^2\,dx\,dt \\ - 5\int_D a^2u^2\,dx\,dt - 5\int_D c^2u_t^2\,dx\,dt.\end{aligned} \tag{5}$$

In order to estimate the last term on the right, we observe that

$$\int_D c^2u_t^2\,dx\,dt = -\int_D (cu_t)Lu\,dx\,dt + \int_D (cu_t)L_0u\,dx\,dt,$$

so that, for positive ε,

$$\begin{aligned}\left(1 - \frac{\varepsilon}{2}\right)\int_D c^2u_t^2\,dx\,dt \leqq \frac{1}{2\varepsilon}\int_D (Lu)^2\,dx\,dt + \int_D cu_tPu\,dx\,dt \\ + \int_D cu_tb_iu_{,i}\,dx\,dt - \int_D cu_tau\,dx\,dt.\end{aligned}$$

The second term on the right can be written as

$$\int_D (cu_t a_{ij} u_{,i})_{,j}\, dx\, dt - \int_D cu_t (a_{ij})_{,j} u_{,i}\, dx\, dt - \int_D cu_{,jt} a_{ij} u_{,i}\, dx\, dt - \int_D c_{,j} u_t a_{ij} u_{,i}\, dx\, dt.$$

For the first term in this expression, we have

$$\int_D (cu_t a_{ij} u_{,i})_{,j}\, dx\, dt = \int_0^T \int_{\partial\Omega} cu_t (\partial u/\partial \nu)\, dS\, dt = 0,$$

where ν denotes the conormal direction associated with a_{ij} on $\partial\Omega$, and the third term can be written as

$$-\frac{1}{2}\int_D (ca_{ij}u_{,i}u_{,j})_t\, dx\, dt + \frac{1}{2}\int_D c(a_{ij})_t u_{,i} u_{,j}\, dx\, dt$$

$$= -\frac{1}{2}\int_\Omega (ca_{ij}u_{,i}u_{,j})\Big|_{t=T} dx + \frac{1}{2}\int_D c(a_{ij})_t u_{,i} u_{,j}\, dx\, dt.$$

Since the first term in this last expression is nonpositive, we obtain

$$\left(1 - \frac{\varepsilon}{2}\right)\int_D c^2 u_t^2\, dx\, dt \leqq \frac{1}{2\varepsilon}\int_D (Lu)^2\, dx\, dt + \int_D cu_t[b_i - (a_{ij})_{,j}]u_{,i}\, dx\, dt$$
$$+ \int_D cu_t a u\, dx\, dt - \int_D \frac{c_{,j}}{c} a_{ij} u_{,i} c u_t\, dx\, dt$$
$$+ \frac{1}{2}\int_D c(a_{ij})_t u_{,i} u_{,j}\, dx\, dt,$$

and by further applications of the arithmetic-geometric mean inequality, we obtain

$$\frac{1}{2}\int_D (cu_t)^2\, dx\, dt \leqq 4\int_D (Lu)^2\, dx\, dt + 2a_1^2 \int_D u^2\, dx\, dt$$
$$+ (2S + 2\mu\gamma_1 + \gamma_2)\int_D |\nabla u|^2\, dx\, dt, \tag{6}$$

where $a_1 = \sup_D |a|$, $\mu = \sup_D |a_{ij}|$, and γ_1, γ_2 are the suprema of $|\nabla c/c|$, $|c(a_{ij})_t|$, respectively, over D. The inequalities (4) and (5) imply that

$$\nu^2 \int_D |D^2 u|^2\, dx\, dt \leqq 8\int_D (Lu)^2\, dx\, dt + (n^4 B^2 \nu^{-2} + 40\beta^2)\int_D |\nabla u|^2\, dx\, dt$$
$$+ 40a_1^2 \int_D u^2\, dx\, dt + 40\int_D (cu_t)^2\, dx\, dt,$$

where β denotes the supremum over D of $|b_i|$. This inequality, together with Lemmas 1′ and 2′ and (6) above, implies the conclusion of the theorem.

The above discussion yields the expression

$$C^2 = 336 + 204a_1^2(a_0 + \lambda v^2 - S\sqrt{\lambda})^{-2} + C_1[n^4B^2v^{-2} + 40\beta^2 + 40\beta^2 + 82(2S + 2\mu\gamma_1 + \gamma_2)]$$

for the constant C. It is interesting to note that C grows linearly with a_1.

Remarks. (i) If the coefficients of L are smooth, Theorem 1 implies a weak maximum principle for L. In fact, if $Lu \geqq 0$ for some $u \in W_0$, define $f = Lu$, approximate f in $L_2(D)$ by smooth functions f_n with $f_n \geqq 0$ and observe that, if $Lu_n = f_n$, $u_n \in W_0$, we have

$$\|u - u_n\|_{2,1} \leqq C\|f - f_n\|_0.$$

Since u_n is smooth, the classical maximum principle implies that $u_n \leqq 0$ in D, and we have shown that u is the limit of W_0 of smooth, nonpositive functions.

(ii) It appears that much of the above can be carried through for mixed boundary conditions on $\partial\Omega \times (0, T)$. The authors plan to return to this question in a subsequent paper.

2. An application to nonlinear equations. The results of the previous paragraph can be used to study the initial-boundary value problem

$$\begin{aligned} Mu &\equiv a_{ij}u_{,ij} + b_iu_{,i} - cu_t = f(x, t, u) \quad \text{in } D, \\ u &= 0 \quad \text{for } t = 0 \quad \text{and for } (x, t) \in \partial\Omega \times (0, T). \end{aligned} \tag{7}$$

We assume that the coefficients of M satisfy A_1 and that $S < \sqrt{\lambda}\, v^2$. We also assume that $n \leqq 3$, so that $W_2^2(\Omega)$ may be imbedded in $C(\bar{\Omega})$, and that $\partial\Omega$ is sufficiently regular to imply that $M(W_0)$ is dense in $L_2(D)$.

We also need to assume that f has certain properties which are given below. First we recall that a function $h(x, t, u)$ defined on $D \times R$ is called an *N-function* if it is continuous in u for almost all (x, t) and measurable in (x, t) for all u. We denote by $N(D)$ the class of N-functions h for which $h(D \times [-A, A])$ is a bounded set of all A. The operator F is defined on W_0 by $F(u)(x, t) = f(x, t, u(x, t))$. We assume

(F): $f \in N(D)$, and, for almost all (x, t), f has three derivatives with respect to u, and these functions belong to $N(D)$.

The problem (7) may be formulated as an operator equation $P(u) = 0$, where $P(u) = Mu - f(x, t, u)$ is thought of as a mapping of W_0 into $L_2(D)$, and this is the approach we will take.

Our results are based on the convergence of certain "hybrid" iterative methods which arise from modifying the first N steps of Newton's method. Suppose that P is a twice continuously differentiable mapping of X into Y, where X and Y are Hilbert spaces, that $P'(x)$ is invertible for all x, and that

$$\|[P'(x)]^{-1}\| \leqq C.$$

Further, assume that P'' is locally bounded, that is,

$$\|P''(x)\| \leqq K(r)$$

for x in $S(0, r)$. Then it is known [6] that the unique solution of $P(x) = 0$ is obtained as the limit of the iterative processes:

$$(8) \qquad \begin{aligned} x_{n+1} &= \Phi(x_n, N), \qquad n = 1, \cdots, N-1, \\ x_{n+1} &= x_n - [P'(x_n)]^{-1}P(x_n), \qquad n = N, \cdots, \end{aligned}$$

independently of the initial guess x_0. The first N steps arise from replacing the differential equation

$$(9) \qquad \dot{x}(t) = -[P'(x)]^{-1}P(x), \qquad t \in [0, 1],$$

with a difference equation with discretization error h^p ($p \geqq 1$), where $h = 1/N$. The size of N is determined by C and the constant in the discretization error bound $\|x(1) - x_n\| \leqq \text{const.} \cdot h^p$. If Euler's method is used to discretize (9),

$$\Phi(x_n, N) = x_n - N^{-1}[P'(x_n)]^{-1}P(x_n).$$

Our existence theorem uses the following result, which is the analogue of a result for elliptic equations [1], [7, p. 426].

THEOREM 2. *Suppose that u is a solution of* (7) *and that* $uf(x, t, u) \geqq 0$ *if* $|u| > m$. *Then* $|u| \leqq m$.

Proof. It suffices to show that if $f(x, t, u) \geqq 0$ for $u > m$, then $u \leqq m$.

We note that u is continuous on $\bar{D}$ and vanishes on the parabolic boundary of D. Let $v = u - m$. If v is positive anywhere in D, it follows that there is a point $P = (x_0, t_0) \in \Omega \times (0, T]$ with

$$v(P) = \max_{\bar{D}} v \equiv k > 0.$$

Furthermore, we can find a P that also has a "backward neighborhood"

$$S = \{|x - x_0| < \varepsilon\} \times (t_0 - \delta, t_0]$$

on which v is positive, but not identically equal to k. Since $f(x, t, u(x, t))$ is non-negative on S, we have

$$Mv = Mu \geqq 0$$

there. At this point, we make use of a generalization of the strong maximum principle [3] which applies to weak solutions of parabolic inequalities, and in particular to v. This maximum principle implies that v is identically equal to k in S, and we have obtained the required contradiction.

The proof of the following is essentially given in [1].

LEMMA. *The assumption* (F) *implies that F maps W_0 into $L_2(D)$ continuously, F has two continuous Fréchet derivatives, and F'' is locally bounded.*

We are now ready to state our main theorem.

THEOREM 3. *Assume that the coefficients of M satisfy A_1, that $S < \sqrt{\lambda}\, v^2$, and that $M(W_0)$ is dense in $L_2(D)$. Further, assume that f satisfies* (F), *that*

$$uf(x, t, u) \geqq 0$$

if $|u| > m$, *and that*

$$\inf f_u > \sqrt{\lambda}\, S - \lambda v^2 \quad on\ \bar{D} \times [-m, m].$$

Then the iteration (8) *converges to the unique solution of* (7).

Proof. Since Theorem 2 provides the a priori estimate $|u| \leqq m$, we may use a familiar device (see [8, Chap. 4]) to replace (7) with an equivalent problem with nonlinear term $\tilde{f}(x, t, u)$, where $\tilde{f}$ coincides with f for $|u| \leqq m$ and $\tilde{f}$ and its derivatives with respect to u are bounded. Then we need only observe that

$$P'(u) = M - \tilde{f}_u,$$

make use of our hypotheses on M and f, and invoke Theorem 1.

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NEW IDENTITIES FOR LEGENDRE ASSOCIATED FUNCTIONS OF INTEGRAL ORDER AND DEGREE. II: EXTENSION TO OTHER POLYNOMIALS*

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Abstract. The concept of an orthogonal polynomial $p_n(x)$ is extended by defining a generalized orthogonal function $p_\nu(x)$ of nonintegral degree ν. A sufficiency condition is then found under which a generalized orthogonal function which satisfies an identity of the type known as Dougall's identity (for a generalized orthogonal function of nonintegral degree), will also satisfy the related S-type identity (for a generalized orthogonal function of integral degree).

The sufficiency condition is applied to Legendre associated functions, generalized Legendre associated functions, and to Jacobi functions.

1. In a previous paper [5] we started with Dougall's identity for Legendre associated functions $P_\nu^m(x)$ of nonintegral degree [1, 3.10(9)] which is of the form

$$D_\nu P_\nu^m(x)P_\nu^m(y) = \{\sin(\nu\pi)/\pi\} \sum_{n=0}^{\infty} (-1)^n D_n P_n^m(x)P_n^m(y) \cdot \{1/(\nu - n) - 1/(\nu + n + 1)\}$$

with D_ν independent of x and y, and we derived a set of new identities for Legendre associated functions $P_l^m(x)$ of integral order and degree of the type

$$S_l P_l^m(x) = \sum_n S_n P_n^m(x)\{1/(l - n) - 1/(l + n + 1)\},$$

where $\sum_n$ denotes either $\sum_{n=0}^{\infty}$ or $\sum_{\substack{n=0\\ n\neq l}}^{\infty}$ and where the parameters S_l, S_n are functions of $P_l^m(x)$ and $P_n^m(x)$ respectively and their respective first derivatives evaluated at the origin (and are therefore independent of x).

We now attempt to generalize these results in two ways. First, in place of Legendre associated functions we study certain functions $p_\nu(x)$ defined on the interval $(a, b) \subset R$, which satisfy Dougall-like identities, namely

$$D_\nu p_\nu(x)p_\nu(y) = \{\sin(\nu\pi)/\pi\} \sum_{n=0}^{\infty} (-1)^n D_n p_n(x)p_n(y) \cdot \{1/(\nu - n) - 1/(\nu + n + 1)\}$$

for all $\nu \in R \sim I$, for all $x, y \in Y \subseteq (a, b)$, with D_ν independent of x and y. We will refer to the above as a D-type identity.

Second, instead of evaluating our parameters S_l, S_n at the origin as in [5], we investigate whether points $x_0 \in (a, b)$ in addition to the origin exist for which identities similar to those derived in [5] can be found. More specifically, we obtain

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a sufficiency condition under which a function $p_\nu(x)$ which satisfies a D-type identity will also satisfy an identity of the type

$$p_l(x) = \sum_n S_n^l(x_0)p_n(x)\{1/(l-n) - 1/(l+n+1)\}$$

for $l \in N$, for some $x_0 \in Y \subseteq (a, b)$, where we have written $S_n^l(x_0)$ for $S_n(x_0)/S_l(x_0)$. N is the set of nonnegative integers.

Again $\sum_n$ denotes either $\sum_{n=0}^{\infty}$ or $\sum_{\substack{n=0\\ n\neq l}}^{\infty}$. As we will see the parameters $S_n^l(x_0)$ are very similar in form whether or not the term for $n = l$ is included in the summation, and we will thus refer to both types of series as S-type identities.

Meulenbeld and van de Wetering [4] have derived a D-type identity for the generalized Legendre associated function (or GLAF for short) $P_\nu^{m,n}(x)$. Hence those functions which can be expressed in terms of GLAF's (e.g., the Jacobi function $P_\nu^{(\alpha,\beta)}$) will satisfy a D-type identity for nonintegral ν, as will those which can be expressed in terms of Legendre associated functions (such as Gegenbauer or ultraspherical functions $C_\nu^{(\lambda)}$). Thus the set $\{p_\nu(x)\}$ is certainly nontrivial.

In searching for such functions we will consider only those functions which are generalizations of the classical orthogonal polynomials $p_n(x)$ to the case of nonintegral n by expressing $p_n(x)$ in terms of the hypergeometric function and replacing n by ν, or by considering the original differential equation defining $p_n(x)$ and making the same replacement. We shall refer to such functions as generalized orthogonal functions. Before following this program we must recall a few basic properties of orthogonal polynomials in order to generalize them later.

2. Properties of orthogonal polynomials. (The results of this and the following subsection are taken from [1, Chap. 10] and [6]).

Let $\{p_n(x) | n = 0,1, \cdots\}$ be a sequence of polynomials of exact degree n, defined on the interval (a, b). Further let $w(x)$, the weight function, be a nonnegative function (measurable in the Lebesgue sense) for which

$$\int_a^b dx\, w(x) > 0.$$

Then if $\int_a^b dx\, w(x)p_i(x)p_j(x)$ exists for all $i, j \in N$ (in Lebesgue's sense) we may define the scalar product

$$(p_i, p_j) = \int_a^b dx\, w(x)p_i(x)p_j(x).$$

If

$$(p_i, p_j) = 0 \quad \text{for} \quad i \neq j,$$

then the sequence $\{p_n(x)\}$ is said to be a system of orthogonal polynomials.

We can show that every orthogonal polynomial system is complete on (a, b) if the interval is finite.

The classical polynomials (the Hermite, Laguerre and Jacobi polynomials, and their special cases) are characterized by three major properties:

(I) $\{p_n'(x)\}$ is a system of orthogonal polynomials,

(II) $y = p_n(x)$ satisfies a differential equation of the form

$$A(x)y'' + B(x)y' + \lambda_n y, \tag{1}$$

where $A(x)$ and $B(x)$ are independent of n, and λ_n is independent of x.

(III) There is a generalized Rodrigues formula

$$p_n(x) = \frac{1}{K_n w(x)} \frac{d^n}{dx^n}(w(x)X^n), \tag{2}$$

where K_n is a constant and X is a polynomial of degree at most 2 in x whose coefficients are independent of n.

Conversely, any one of these properties characterizes the classical orthogonal polynomial in the sense that any system of orthogonal polynomials which has one of these properties can be reduced to a classical system. Thus by considering only the classical orthogonal polynomials we nevertheless are including a wide range of functions.

From (2) we can deduce that the differential equation for $y = p_n(x)$ has the form

$$X\frac{d^2y}{dx^2} + K_1 p_1(x)\frac{dy}{dx} + \lambda_n y = 0, \tag{3}$$

where

$$\lambda_n = -n\{k_1 K_1 + \tfrac{1}{2}(n-1)X''\} \tag{4}$$

with k_1 the coefficient of x in $p_1(x)$, K_1 and X as defined in (2).

Since X is at most quadratic in x, and $p_1(x)$ is a linear function of x, (3) can be reduced to the hypergeometric equation or one of its special or limiting cases.

3. The recurrence and differentiation formulas. Let k_n, $\tilde{k}_n$ be the coefficients of x^n, x^{n-1} respectively in $p_n(x)$; $r_n = \tilde{k}_n/k_n$ and $h_n = (p_n, p_n)$. Then [6, (3.2.1)] $p_n(x)$ satisfies the recurrence relation

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x), \qquad n = 0, 1, 2, \cdots, \tag{5}$$

with $p_{-1}(x) = 0$ where

$$\begin{aligned} A_n &= k_{n+1}/k_n, \\ B_n &= A_n(r_{n+1} - r_n), \\ C_n &= A_n h_n/(A_{n-1}h_{n-1}) = k_{n+1}k_{n-1}h_n/(k_n^2 h_{n-1}). \end{aligned} \tag{6}$$

The differentiation formula [1, 10.7(4)] obeyed by $p_n(x)$ is

$$X\frac{dp_n(x)}{dx} = (\alpha_n + \tfrac{1}{2}nX''x)p_n(x) + \beta_n p_{n-1}(x), \tag{7}$$

where

$$\alpha_n = nX'(0) - \tfrac{1}{2}X''r_n, \tag{8}$$
$$A_n\beta_n = -C_n\{k_1K_1 + (n - \tfrac{1}{2})X''\},$$

where X, K_1 are defined in (2), r_n and k_1 in (5) and A_n, C_n in (6).

4. Generalized orthogonal functions. Since $p_n(x)$, the solution of (3), is a hypergeometric function or one of its special or limiting cases, let us define the generalized orthogonal function $z = p_v(x)$ to be the solution of the equation (cf. (3), (4))

$$X\frac{d^2x}{dx^2} + K_1p_1(x)\frac{dz}{dx} + \lambda_v z = 0, \tag{9}$$

where

$$\lambda_v = -v\{k_1K_1 + \tfrac{1}{2}(v - 1)X''\}.$$

Further, replacing n by v in (5) and (7) we obtain

$$p_{v+1}(x) = (A_vx + B_v)p_v(x) - C_vp_{v-1}(x), \qquad p_{-1}(x) = 0, \tag{10}$$

$$X\frac{dp_v(x)}{dx} = (\alpha_v + \tfrac{1}{2}vX''x)p_v(x) + \beta_vp_{v-1}(x). \tag{11}$$

This substitution of n by v can be justified rigorously by considering the appropriate hypergeometric function for each classical system and using Gauss' 15 relations between contiguous hypergeometric functions [1, 2.1.2] to deduce (10) and the differentiation formula for $F(a, b; c; z)$ [1, 2.8.20] to obtain (11).

5. The sufficiency conditions. Armed with the above results we can now find a sufficient condition for a generalized orthogonal function $p_v(x)$ which satisfies a D-type identity to satisfy an S-type identity. We allow $p_v(x)$ to be a function of an integer parameter m. However, to avoid over-cumbersome notation, the m-dependence is not explicitly indicated except where essential.

Our main result follows from Lemmas 1 and 2.

LEMMA 1. *Let $p_v(x) = p_v(x, m), m \in I$ be a generalized orthogonal function defined on $(a, b) \subset R$ which satisfies a D-type identity, viz.,*

$$D_vp_v(x)p_v(y) = \{\sin(v\pi)/\pi\} \sum_{n=0}^{\infty} (-1)^n D_np_n(x)p_n(y) \cdot \{(1/v - n) - 1/(v + n + 1)\} \tag{12}$$

for all $v \in R$, for all $x, y \in Y \subseteq (a, b)$, with D_v independent of x and y.

Then a sufficient condition that $p_v(x)$ satisfy an S-type identity, namely

$$p_l(x) = \sum_{\substack{n=0\\ n\neq l}}^{\infty} S_n^l(x_0)p_n(x)\{1/(l - n) - 1/(l + n + 1)\} \tag{13}$$

for all $x \in Y$, *for* $l \in N$, *for some* $x_0 \in Y$, $S_n^l(x_0)$ *independent of* x, *is that the series*

$$D_\nu p_\nu(x)p'_\nu(y) = \{\sin(\nu\pi)/\pi\} \sum_{n=0}^{\infty} (-1)^n D_n p_n(x) p'_n(y) \cdot \{1/(\nu - n) - 1/(\nu + n + 1)\} \tag{14}$$

is uniformly convergent for all $x, y \in Y$, *and that there exists an* $x_0 \in Y$, *such that for all* $\nu \in R$, $p_\nu(x_0)$ *can be expressed as*

$$p_\nu(x_0) = M_\nu(x_0)\cos\{\tfrac{1}{2}(\nu + m)\pi\}, \tag{15}$$

where $M_l(x_0) = \lim_{\nu \to l \in N} p_\nu(x_0)/\cos\{\tfrac{1}{2}(\nu + m)\pi\}$ *exists with* $M_l(x_0) \neq 0$ *for all* $l \in N$. *The parameters* $S_n^l(x_0)$ *are given by*
$l + m$ *even*:

$$S_n^l(x_0) = -(2/\pi)\cos\{\tfrac{1}{2}(l + m)\pi\}(D_n/D_l)\Big(\beta_n M_{n-1}(x_0) \cdot \sin\{\tfrac{1}{2}(n + m)\pi\} - M_n(x_0)\{(\alpha_n - \alpha_l) + \tfrac{1}{2}(n - l)X''x_0\} \cdot \cos\{\tfrac{1}{2}(n + m)\pi\}\Big) \Big/ \Big(M_{l-1}(x_0)\beta_l + (2/\pi)M_l(x_0)\{\alpha'_l + \tfrac{1}{2}X''x_0\}\Big), \tag{16}$$

where

$$\alpha'_l = \left(\frac{d\alpha_\nu}{d\nu}\right)_{\nu = l}.$$

$l + m$ *odd*:

$$S_n^l(x_0) = \frac{2}{\pi}\sin\{\tfrac{1}{2}(l + m)\pi\}\frac{D_n M_n(x_0)}{D_l M_l(x_0)}\cos\{\tfrac{1}{2}(n + m)\pi\}. \tag{17}$$

Proof. Differentiate (12) term-by-term with respect to y, and set $y = x_0$; this step is valid by virtue of the uniform convergence condition (14) (which in turn implies uniform convergence of (12)). Multiplying both sides by $X(x_0)$ we obtain

$$D_\nu X(x_0)p_\nu(x)p'_\nu(x_0) = \{\sin(\nu\pi)/\pi\} \sum_{n=0}^{\infty} (-1)^n \cdot D_n p_n(x) X(x_0) p'_n(x_0)\{1/(\nu - n) - 1/(\nu + n + 1)\}. \tag{18}$$

Multiply both sides of (12) by $\{\alpha_\nu + \tfrac{1}{2}\nu X''y\}$ and set $y = x_0$ to give

$$D_\nu\{\alpha_\nu + \tfrac{1}{2}\nu X''x_0\}p_\nu(x)p_\nu(x_0) = \{\sin(\nu\pi)/\pi\} \sum_{n=0}^{\infty}(-1)^n D_n p_n(x)\{\alpha_\nu + \tfrac{1}{2}\nu X''x_0\}p_n(x_0) \cdot \{1/(\nu - n) - 1/(\nu + n + 1)\}. \tag{19}$$

Subtract (19) from (18). Using (11) we obtain the result

$$\begin{aligned} D_\nu p_\nu(x)\beta_\nu p_{\nu-1}(x_0) = {} & \{\sin(\nu\pi)/\pi\} \sum_{n=0}^{\infty} (-1)^n D_n p_n(x) \\ & \cdot \Big(\beta_n p_{n-1}(x_0) + \{\alpha_n + \tfrac{1}{2}nX''x_0\}p_n(x_0) \\ & - \{\alpha_\nu + \tfrac{1}{2}\nu X''x_0\}p_n(x_0)\Big)\{1/(\nu-n) - 1/(\nu+n+1)\}. \end{aligned} \tag{20}$$

Now if condition (15) holds, then

$$p_\nu(x_0)p_{\nu-1}(x_0) = \tfrac{1}{2}M_\nu(x_0)M_{\nu-1}(x_0)\sin(\nu\pi)(-1)^m. \tag{21}$$

Thus multiplying both sides of (20) by $p_\nu(x_0)$ gives

$$\begin{aligned} & (-1)^m D_\nu\beta_\nu M_\nu(x_0)M_{\nu-1}(x_0)p_\nu(x) \\ & = (2/\pi)p_\nu(x_0)\sum_{n=0}^{\infty}(-1)^n D_n p_n(x)\Big(\beta_n p_{n-1}(x_0) + \{\alpha_n + \tfrac{1}{2}nX''x_0\} \\ & \quad \cdot p_n(x_0) - \{\alpha_\nu + \tfrac{1}{2}\nu X''x_0\}p_n(x_0)\Big)\{1/(\nu-n) - 1/(\nu+n+1)\}. \end{aligned}$$

As in [5] we separate out the term for $n = l \geqq 0$. We deduce

$$\begin{aligned} & (-1)^m D_\nu\beta_\nu M_\nu(x_0)M_{\nu-1}(x_0)p_\nu(x) \\ & = (2/\pi)p_\nu(x_0)(-1)^l D_l p_l(x)\Big(\beta_l p_{l-1}(x_0) + \{\alpha_l + \tfrac{1}{2}lX''x_0\} \\ & \quad \cdot p_l(x_0) - \{\alpha_\nu + \tfrac{1}{2}\nu X''x_0\}p_l(x_0)\Big)\{1/(\nu-l) - 1/(\nu+l+1)\} \\ & \quad + (2/\pi)p_\nu(x_0)\sum_{\substack{n=0 \\ n\neq l}}^{\infty}(-1)^n D_n p_n(x)\Big(\beta_n p_{n-1}(x_0) \\ & \quad + \{\alpha_n + \tfrac{1}{2}nX''x_0\}p_n(x_0) - \{\alpha_\nu + \tfrac{1}{2}\nu X''x_0\}p_n(x_0)\Big) \\ & \quad \cdot \{1/(\nu-n) - 1/(\nu+n+1)\}. \end{aligned} \tag{22}$$

Now take $\lim_{\nu\to l\in N}$ of (22). The only nonsmooth term is (using condition (15))

$$\lim_{\nu\to l} p_\nu(x_0)p_{l-1}(x_0)/(\nu-l) = -(\tfrac{1}{2}\pi)M_l(x_0)M_{l-1}(x_0)\sin^2\{\tfrac{1}{2}(l+m)\pi\}. \tag{23}$$

We substitute this term into $\lim_{\nu\to l}$ of (22) to obtain

$$\begin{aligned} & M_l(x_0)M_{l-1}(x_0)\beta_l p_l(x)D_l(-1)^m(1 + (-1)^{l+m}\sin^2\{\tfrac{1}{2}(l+m)\pi\}) \\ & = -(2/\pi)(-1)^l M_l^2(x_0)p_l(x)D_l\{\alpha_l' + \tfrac{1}{2}X''x_0\}\cos^2\{\tfrac{1}{2}(l+m)\} \\ & \quad + (2/\pi)M_l(x_0)\cos\{\tfrac{1}{2}(l+m)\pi\}\sum_{\substack{n=0 \\ n\neq l}}^{\infty}(-1)^n D_n p_n(x) \\ & \quad \cdot \Big(\beta_n M_{n-1}(x_0)\sin\{\tfrac{1}{2}(n+m)\pi\} \\ & \quad + M_n(x_0)\{(\alpha_n - \alpha_l) + \tfrac{1}{2}(n-l)X''x_0\}\cos\{\tfrac{1}{2}(n+m)\pi\}\Big) \\ & \quad \cdot \{1/(l-n) - 1/(l+n+1)\}. \end{aligned} \tag{24}$$

The identity becomes trivial unless we assume

$$M_l(x_0) \neq 0 \quad \text{for all} \quad l \in N. \tag{25}$$

Further, from elementary trigonometry we know that

$$(1 + (-1)^{l+m} \sin^2 \{\tfrac{1}{2}(l + m)\}) = \begin{cases} 1 & \text{for } (l + m) \text{ even}, \\ 0 & \text{for } (l + m) \text{ odd}, \end{cases}$$

and

$$\cos \{\tfrac{1}{2}(l + m)\pi\} = \begin{cases} (-1)^{(1/2)(l+m)} & \text{for } (l + m) \text{ even}, \\ 0 & \text{for } (l + m) \text{ odd}. \end{cases}$$

We must therefore assume that $(l + m)$ is even, and (24) then gives

$$\begin{aligned} p_l(x) = (2/\pi) \cos \{\tfrac{1}{2}(l + m)\pi\} \sum_{\substack{n=0 \\ n \neq l}}^{\infty} (-1)^{n+m}(D_n/D_l) \\ \cdot \Big(\beta_n M_{n-1}(x_0) \sin \{\tfrac{1}{2}(n + m)\pi\} + M_n(x_0)\{(\alpha_n - \alpha_l) \\ + \tfrac{1}{2}(n - l)X''x_0\} \cos \{\tfrac{1}{2}(n + m)\pi\}\Big) p_n(x)\{1/(l - n) - 1/(l + n + 1)\} \\ (M_{l-1}(x_0)\beta_l + (2/\pi)M_l(x_0)\{\alpha_l' + \tfrac{1}{2}X''x_0\}). \end{aligned} \tag{26}$$

Now, $\sin(\tfrac{1}{2}k\pi) = 0$ unless k is odd and $\cos(\tfrac{1}{2}k\pi) = 0$ unless k is even. Therefore defining $S_m^l(x_0)$ for $(l + m)$ even by (16) we obtain (13) as required.

The corresponding identity for $(l + m)$ odd is found by setting $y = x_0$ in (12) and multiplying both sides by $p_{\nu-1}(x_0)$; we obtain (using (21))

$$\begin{aligned} &(-1)^m \tfrac{1}{2} D_\nu M_\nu(x_0) M_{\nu-1}(x_0) \sin(\nu\pi) p_\nu(x) \\ &\quad = \{\sin(\nu\pi)/\pi\} \sum_{n=0}^{\infty} (-1)^n D_n p_n(x) p_n(x_0) p_{\nu-1}(x_0) \{1/(\nu - n) - 1/(\nu + n + 1)\}. \end{aligned}$$

As before we separate out the $n = l \geqq 0$ term and take the limit $\nu \to l \in N$. We find

$$\begin{aligned} &(-1)^m D_l M_l(x_0) M_{l-1}(x_0) p_l(x)(1 - (-1)^{l+m} \cos^2 \{\tfrac{1}{2}(l + m)\pi\}) \\ &\quad = (2/\pi) M_{l-1}(x_0) \sin \{\tfrac{1}{2}(l + m)\pi\} \sum_{\substack{n=0 \\ n \neq l}}^{\infty} (-1)^n D_n p_n(x) \\ &\qquad \cdot M_n(x_0) \cos \{\tfrac{1}{2}(n + m)\pi\} \{1/(l - n) - 1/(l + n + 1)\}. \end{aligned} \tag{27}$$

Again (27) reduces to a trivial identity unless we assume condition (25); further $(l + m)$ must be odd, in which case

$$(1 - (-1)^{l+m} \cos^2 \{\tfrac{1}{2}(l + m)\pi\}) = 1$$

and (27) reduces to

$$p_l(x) = (2/\pi)\sin\{\tfrac{1}{2}(l+m)\pi\} \sum_{\substack{n=0\\ n\neq l}}^{\infty} (-1)^{n+m} \frac{D_n M_n(x_0)}{D_l M_l(x_0)}$$
$$\cdot \cos\{\tfrac{1}{2}(n+m)\pi\} p_n(x)\{1/(l-n) - 1/(l+n+1)\}.$$

Defining $S_n^l(x_0)$ by (17), we obtain (13) as required. Q.E.D.

It would appear from Lemma 1 that there are many possible values of $x_0 \in Y$ for which S-type identities exist.

However, this is not so; in the following lemma we show that x_0 must be zero. In addition, the parameter $r_n = \tilde{k}_n/k_n$ must be a constant (independent of n) for condition (15) of Lemma 1 to be possible.

LEMMA 2. *Let $p_\nu(x) = p_\nu(x, m)$, $m \in I$ be a generalized orthogonal function defined on $(a, b) \subset R$. If there exists an $x_0 \in Y$ such that for all $\nu \in R$, $p_\nu(x_0)$ can be expressed as*

$$p_\nu(x_0) = M_\nu(x_0)\cos\{\tfrac{1}{2}(\nu+m)\pi\},$$

where $M_l(x_0) = \lim_{\nu\to l\in N} p_\nu(x_0)/\cos\{\tfrac{1}{2}(\nu+m)\pi\}$ exists, with $M_l(x_0) \neq 0$ for all $l \in N$, then

$$x_0 = 0$$

and

$$r_n = \tilde{k}_n/k_n$$
$$= r, \quad \textit{independent of } n.$$

Proof.

Case A: *m even.* Assume

$$p_s(x_0) = M_s(x_0)\cos\{\tfrac{1}{2}(s+m)\pi\} \quad \text{for } s \leqq N$$
$$= (-1)^{(1/2)m} M_s(x_0)\cos(\tfrac{1}{2}s\pi).$$

Choose N even. Then by (5) since m is even,

$$p_{N+1}(x_0) = (-1)^{(1/2)m}(A_N x_0 + B_N) M_N \cos(\tfrac{1}{2}N\pi)$$

This is zero as required if

$$M_N = 0 \quad \text{(impossible by (15))}$$

or

$$A_N x_0 + B_N = 0.$$

Since A_N and B_N are independent of x this implies

$$B_N = 0 \quad \text{and} \quad A_N x_0 = 0. \tag{28}$$

Now $B_N = A_N(r_{N+1} - r_N)$, and (28) is equivalent to

$$x_0 = 0 \quad \text{and} \quad r_{N+1} - r_N = 0 \tag{29}$$

or

$$A_N = 0.$$

But $A_N = k_{N+1}/k_N$ by (6), and $A_N = 0$ means that k_{N+1} (the coefficient of x^{N+1} in p_{N+1}) is zero, which is impossible because $p_{N+1}(x)$ is defined to be a polynomial of exact degree $(N + 1)$.

Thus (29) gives

$$x_0 = 0, \quad \text{and} \quad r_n = \text{const. for all } n.$$

Case B: *m odd*. Choosing N odd leads to an analogous proof.

Combining Lemmas 1 and 2 we deduce (again not explicitly indicating m-dependence except where essential) the following.

THEOREM 3. *Let $p_\nu(x) = p_\nu(x, m)$, $m \in I$ be a generalized orthogonal function defined on $(a, b) \subset R$ for which $r_n = r$ (a constant, independent of n), and which satisfies a D-type identity, viz.,*

$$D_\nu p_\nu(x)p_\nu(y) = \{\sin(\nu\pi)/\pi\} \sum_{n=0}^{\infty} (-1)^n D_n p_n(x) p_n(y)\{1/(\nu - n) - 1/(\nu + n + 1)\}$$

for all $\nu \in R$, for all $x, y \in Y \subset (a, b)$, where $0 \in Y$, with D_ν independent of x and y.

Then a sufficient condition that $p_n(x)$ satisfies an S-type identity

$$p_l(x) = \sum_{\substack{n=0 \\ n \neq l}}^{\infty} S_n^l p_n(x)\{1/(l - n) - 1/(l + n + 1)\}$$

for all $l \in N$ and for all $x \in Y$, is that

$$D_\nu p_\nu(x)p'_\nu(y) = \{\sin(\nu\pi)/\pi\} \sum_{n=0}^{\infty} (-1)^n D_n p_n(x) p'_n(y)\{1/(\nu - n) - 1/(\nu + n + 1)\} \tag{30}$$

is uniformly convergent for all $x, y \in Y$, and that for all $\nu \in R$, $p_\nu(0)$ can be expressed as

$$p_\nu(0) = M_\nu \cos\{\tfrac{1}{2}(\nu + m)\pi\}, \tag{31}$$

where $M_l = \lim_{\nu \to l \in N} p_\nu(0)/\cos\{\frac{1}{2}(\nu + m)\pi\}$ exists with $M_l \neq 0$ for all $l \in N$.

The parameters S_n^l are given by

$(l + m)$ even

$$\begin{aligned} S_n^l = {} & -(2/\pi)\cos\{\tfrac{1}{2}(l + m)\pi\}(D_n/D_l)\Big(\beta_n M_{n-1} \sin\{\tfrac{1}{2}(n + m)\pi\} \\ & - M_n(\alpha_n - \alpha_l)\cos\{\tfrac{1}{2}(n + m)\pi\}\Big)/(M_{l-1}\beta_l + (2/\pi)M_l\alpha'_l) \end{aligned} \tag{32}$$

$(l + m)$ odd

$$S_n^l = (2/\pi)\sin\{\tfrac{1}{2}(l + m)\pi\}\frac{D_n M_n}{D_l M_l}\cos\{\tfrac{1}{2}(n + m)\pi\}. \tag{33}$$

Proof. The proof follows from Lemmas 1 and 2. Q.E.D.

In equations (22) and (27) we removed the term for $n = l \geqq 0$, thereby giving in Theorem 3 an S-type identity with infinite sum $\sum_{n=0, n\neq l}^{\infty}$.

As in [5] we now replace the $n = l$ term to obtain an S-type identity summed over all values of n, since we have shown in Lemma 1 that the apparently diverging term exists. (In fact, for $(l + m)$ odd, the $n = l$ term is zero, as can be seen from (27). This result agrees with equations (I.45) and (I.47) of [5]).

We immediately deduce the following.

COROLLARY 4. *If the conditions of Theorem 3 hold, then an S-type identity for $p_n(x)$ of the form*

$$p_l(x) = \sum_{n=0}^{\infty} S_n^l p_n(x)\{1/(l-n) - 1/(l+n+1)\}$$

exists. The parameters S_n^l are given by (33) for $(l + m)$ odd, and for $(l + m)$ even we find

$$S_n^l = -(2/\pi)\cos\{\tfrac{1}{2}(l+m)\pi\}(D_n/D_l)\Big(\beta_n M_{n-1}\sin\{\tfrac{1}{2}(n+m)\pi\} - M_n(\alpha_n - \alpha_l)\cos\{\tfrac{1}{2}(n+m)\pi\}\Big)/M_{l-1}\beta_l.$$

6. Application to Legendre associated functions. Let us apply Theorem 3 to Legendre associated functions $P_n^m(x)$ with m fixed. Since $\{P_n^m(x)\}$ satisfy a differential equation of the type (1) they can be reduced to a classical system.

We are going to show that S_n^l of equations (32) and (33) are in agreement with equations (I.19) and (I.20) of [5].

Using Lemma 1 of [5] we can show that

$$\gamma_{\nu,m}P_\nu^m(x)P_\nu'^m(y) = \{\sin(\nu\pi)/\pi\}\sum_{n=0}^{\infty}(-1)^n\gamma_{n,m} \cdot P_n^m(x)P_n'^m(y)\{1/(\nu-n) - 1/(\nu+n+1)\} \tag{34}$$

is uniformly convergent for

$$-\pi < \theta \pm \xi < \pi, \quad \text{where } x = \cos\theta, \quad y = \cos\xi,$$

$$\gamma_{\nu,m} = \Gamma(\nu - m + 1)/\Gamma(\nu + m + 1).$$

We see that the range of validity of (34) is not in the form required by Theorem 3, but following the methods of [5] the extension of Theorem 3 to include limits of this nature is straightforward.

Comparing the differentiation formula for Legendre associated functions [1, 3.8(19)] with (7) we deduce that

$$\alpha_n = 0, \quad \beta_n = n + m, \quad X = 1 - x^2,$$

whence from (8) we obtain

$$r_n = 0.$$

By uniform convergence of (34) condition (30) is satisfied with

$$D_\nu = \gamma_{\nu,m}.$$

Finally, from (I.15) we know that

$$P_\nu^m(0) = 2^m\pi^{-1/2}\cos\{\tfrac{1}{2}(\nu + m)\pi\}\Gamma(\tfrac{1}{2} + \tfrac{1}{2}\nu + \tfrac{1}{2}m)/\Gamma(1 + \tfrac{1}{2}\nu - \tfrac{1}{2}m)$$

whence (cf. (31))

$$M_n = 2^m\pi^{-1/2}\Gamma(\tfrac{1}{2} + \tfrac{1}{2}\nu + \tfrac{1}{2}m)/\Gamma(1 + \tfrac{1}{2}\nu - \tfrac{1}{2}m).$$

Thus $\{P_n^m(x)\}$ satisfies the conditions of Theorem 3, and on substituting the above values in (32) and (33) we find:

$$(l \times m)\ even: S_n^l = -\gamma_{n,m}P_l^m(0)P_n'^m(0),$$

$$(l + m)\ odd: S_n^l = \gamma_{n,m}P_n^m(0)P_l'^m(0);$$

these results agree with Theorem 3 of [5] (equations (I.19) and (I.20)).

7. Application to generalized Legendre associated functions (GLAFs) and to Jacobi functions. We have noted that Meulenbeld and van de Wetering [4] have derived a D-type identity for the GLAF $P_\nu^{m,n}(x)$. Starting from this identity and making use of the fact that the Jacobi function $P_\nu^{(\alpha,\beta)}$ can be expressed in terms of $P_\nu^{m,n}$, we derive a D-type identity for the Jacobi function and using Theorem 3 investigate whether an S-type identity can be derived for $P_n^{(\alpha,\beta)}$. We work with Jacobi polynomials rather than GLAF's, because the former are a set of classical orthogonal polynomials.

We define the GLAF $w = P_\nu^{m,n}(z)$ to be the solution of the differential equation [3, (1)]

$$(1 - z)^2\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \{\nu(\nu + 1) - \tfrac{1}{2}m^2/(1 - z) - \tfrac{1}{2}n^2/(1 + z)\}.$$

In [4, (6)] we find the following D-type identity for GLAF's with ν nonintegral, with m and n constants such that $m, \frac{1}{2}(n - m)$ and $\frac{1}{2}(n + m)$ are nonnegative integers:

$$\gamma_{\nu,(1/2)(m-n)}P_\nu^{-n,-m}(\cos\theta)P_\nu^{m,n}(\cos\xi)$$

$$(35)\qquad = \{\sin(\nu\pi)/\pi\}\sum_{q=1/2(m+n)}^{\infty}(-1)^q\gamma_{q,1/2(m-n)}P_q^{-n,-m}(\cos\theta)$$

$$\cdot P_q^{m,n}(\cos\xi)\{1/(\nu - q) - 1/(\nu + q + 1)\}, \qquad -\pi < \theta \pm \xi < \pi.$$

The Jacobi function $P_\nu^{(\alpha,\beta)}$, the generalization of the Jacobi polynomial, satisfies the differential equation [1, 10.8(14)]

$$(1 - x^2)y' + (\beta - \alpha - (\alpha + \beta + 2)x)y' + \nu(\nu + \alpha + \beta + 1)y = 0.$$

It can be shown that [2, (4)]

$$P_\nu^{m,n}(x) = 2^{-m}(1 + x)^{(1/2)n}(1 - x)^{(1/2)m}\frac{\Gamma(\nu + \frac{1}{2}[m + n] + 1)}{\Gamma(\nu - \frac{1}{2}[m - n] + 1)}P^{(m,n)}_{\nu-(1/2)[m+n]}(x),$$

whence (35) may be shown to reduce to

$$\begin{aligned}
&\frac{\Gamma(\nu - \frac{1}{2}[m + n] + 1)\Gamma(\nu + \frac{1}{2}[m + n] + 1)}{\Gamma(\nu - \frac{1}{2}[n - m] + 1)\Gamma(\nu + \frac{1}{2}[n - m] + 1)}P^{(n,m)}_{\nu-1/2[m+n]}(x)P^{(m,n)}_{\nu-1/2[m+n]}(y)\\
&\quad = \{\sin(\nu\pi)/\pi\} \sum_{q=1/2(m+n)}^{\infty} (-1)^q\\
&\quad \cdot\frac{\Gamma(q - \frac{1}{2}[m + n] + 1)\Gamma(q + \frac{1}{2}[m + n] + 1)}{\Gamma(q - \frac{1}{2}[n - m] + 1)\Gamma(q + \frac{1}{2}[n - m] + 1)}P^{(n,m)}_{q-1/2[m+n]}(x)\\
&\quad \cdot P^{(m,n)}_{q-1/2[m+n]}(y)\{1/(\nu - q) - 1/(\nu + q + 1)\},
\end{aligned} \tag{36}$$

with $m, \frac{1}{2}(n \pm m)$ nonnegative integers, which is a D-type identity for the Jacobi function.

For the Jacobi polynomial $P_n^{(\alpha,\beta)}$ [1, 10.8(5)]

$$r_n = n(\alpha - \beta)/(2n + \alpha + \beta).$$

Thus r_n is a constant if $\alpha = \pm\beta$, and for these two cases an S-type identity exists assuming that the uniform convergence condition (30) holds and that $P_\nu^{(m,\pm m)}(0)$ has the required form. However, we need not check this explicitly because we can show that

$$P^{(m,m)}_{\nu-m}(x) = 2^m(1 - x^2)^{-(1/2)m}\frac{\Gamma(\nu + 1)(-1)^m}{\Gamma(\nu + m + 1)}P_\nu^m(x)$$

and

$$P_\nu^{(m,-m)}(x) = \left(\frac{1 + x}{1 - x}\right)^{(1/2)m}\frac{\Gamma(\nu - m + 1)(-1)^m}{\Gamma(\nu + 1)}P_\nu^m(x). \tag{37}$$

Uniform convergence of (36) then follows in both cases from uniform convergence of (34).

Consider first the case $\alpha = +\beta = m$. From (37) we see that

$$P^{(m,m)}_{n-m}(0) = 2^{2m}\pi^{-1/2}(-1)^m\frac{\Gamma(n + 1)\Gamma(\frac{1}{2} + \frac{1}{2}n + \frac{1}{2}m)}{\Gamma(n + m + 1)\Gamma(1 + \frac{1}{2}n - \frac{1}{2}m)}\cdot\cos\{\tfrac{1}{2}(n + m)\pi\}.$$

Theorem 3 is satisfied; the constants S_n^l are found to be

$$l + m\ \textit{even}:\quad S_n^l = -\gamma_{n,m}2^{-2m}P^{(m,m)}_{l-m}(0)P'^{(m,m)}_{n-m}(0)\frac{\Gamma^2(n + m + 1)}{\Gamma^2(n + 1)},$$

$$l + m\ \textit{odd}:\quad S_n^l = \gamma_{n,m}2^{-2m}P^{(m,m)}_{n-m}(0)P'^{(m,m)}_{l-m}(0)\frac{\Gamma^2(n + m + 1)}{\Gamma^2(n + 1)}.$$

The coefficients for the S-type identity for $P_n^{(m,-m)}(x)$ can be found similarly.

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REMARKS ON A PAPER OF A. ERDÉLYI*

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Abstract. An alternative asymptotic expansion is given for an integral, which was recently considered by Erdélyi by means of fractional derivatives. The new expansion is simpler and the bounds of the remainder terms are of the same kind.

1. Introduction. In a recent paper [3], Professor Erdélyi considered integrals of the form

$$F(z,a) = \int_a^\infty e^{-z(t-a)} t^{\lambda-1} g(t)\,dt, \tag{1.1}$$

where $a \geqq 0$, $0 < \lambda < 1$, and z is a large parameter. In order to obtain an asymptotic expansion for $z \to \infty$, uniformly valid for $a \geqq 0$, he replaced the function $t^{\lambda-1}g(t)$ by a fractional integral $I^{\lambda-1}f(t)$, the operator I^α being defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds.$$

By an integration by parts procedure, Erdélyi obtained the uniform expansion

$$F(z,a) = Q \sum_{k=0}^{n-1} \Gamma(k+\lambda) g^k(0) z^{-k}/k! + \sum_{k=1}^{n-1} z^{-k} I^\lambda f^{(k)}(a) + R_n, \tag{1.2}$$

where Q is related to the incomplete gamma function and is given by

$$Q = z^{-\lambda} e^{az} \Gamma(\lambda, az)/\Gamma(\lambda). \tag{1.3}$$

The remainder R_n is estimated uniformly in a for $a \geqq 0$. The expression $I^\lambda f^{(k)}(a)$ is explicitly given in terms of derivatives of the function $g(t)$ at $t = 0$ and $t = a$ as

$$\begin{aligned} I^\lambda f^{(k)}(a) = \sum_{m=1}^{k} \frac{a^{\lambda-m}}{(k-m)!} \Bigg[& (-1)^{m-1} \frac{\Gamma(k)\Gamma(m-\lambda)}{\Gamma(m)\Gamma(1-\lambda)} g^{(k-m)}(a) \\ & - \frac{\Gamma(k+\lambda-m)}{\Gamma(\lambda-m+1)} g^{(k-m)}(0) \Bigg], \qquad k = 1, 2, \cdots. \end{aligned} \tag{1.4}$$

As remarked by Erdélyi, the expansion (1.2) could have been obtained via integration by parts of (1.1), but the explicit form (1.4) in (1.2) is not easily obtained in that way.

In this note we give an alternative expansion of $F(z, a)$, which is simpler than (1.2), and in which the bounds of the remainder terms are of the same kind. Both expansions may be derived from each other by formal rearrangement of infinite series.

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2. From a numerical point of view, (1.2) is not attractive because of the $I^\lambda f^{(k)}(a)$ in the second series. Recurrence relations for these factors based on (cf. [3, (2.3)])

$$I^{\lambda-1}f(t) = \frac{d}{dt}I^\lambda f(t) = \frac{f(0)}{\Gamma(\lambda)}t^{\lambda-1} + I^\lambda f'(t)$$

are not suitable for numerical evaluation of a sequence of $I^\lambda f^{(k)}(a)$, $k = 0, 1, \cdots, n$.

Furthermore, the terms $g^{(k)}(0)$ in (1.2) are somewhat surprising. Of course, the singularity at $t = 0$ due to $t^{\lambda-1}$ gives a hint that this point may significantly contribute to the asymptotic expansion, especially when a is small. But for moderate and large values of a, we cannot expect relevant information from the function values at $t = 0$.

In our opinion, the expansion (1.2) can be considerably simplified. Let us suppose that g and its first n derivatives are continuous and bounded on $[0, \infty)$. We write

$$g(t) = \sum_{k=0}^{n-1} c_k(t-a)^k + r_n(t), \qquad c_k = g^{(k)}(a)/k!.$$

Then we have

$$F(z,a) = \sum_{k=0}^{n-1} c_k F_k + R_n \tag{2.1}$$

with

$$F_k = \int_a^\infty e^{-z(t-a)}t^{\lambda-1}(t-a)^k\,dt, \tag{2.2}$$

$$R_n = \int_a^\infty e^{-z(t-a)}t^{\lambda-1}r_n(t)\,dt. \tag{2.3}$$

The first few functions F_k are easily computed. It turns out that

$$F_0 = \Gamma(\lambda)Q, \qquad F_1 = (\lambda z^{-1} - a)F_0 + a^\lambda z^{-1}, \tag{2.4}$$

where Q is essentially an incomplete gamma function and is defined in (1.3). By partial integration of (2.2) we obtain

$$F_{k+1} = z^{-1}[(k+\lambda-az)F_k + akF_{k-1}], \qquad k \geqq 1. \tag{2.5}$$

Hence, if F_0 is computed, the remaining F_k can be generated by (2.5).

The functions F_k are confluent hypergeometric functions. In the notation of [1], we have

$$\begin{aligned} F_k &= k!a^{k+\lambda}U(k+1, k+1+\lambda, az) \\ &= k!z^{-k-\lambda}U(1-\lambda, 1-\lambda-k, az). \end{aligned} \tag{2.6}$$

The second representation enables us to write for $0 < \lambda < 1$,

$$F_k = \frac{k!\,z^{-k-\lambda}}{\Gamma(1-\lambda)}\int_0^\infty e^{-azt}t^{-\lambda}(1+t)^{-k-1}\,dt, \tag{2.7}$$

from which follows, by majorizing the exponential function in the integrand by 1,

$$F_k \leqq z^{-k-\lambda}\Gamma(k+\lambda). \tag{2.8}$$

As follows from (2.2), this bound is also valid for $\lambda = 1$.

If on $[0, \infty)$ an estimate is known for $g^{(k)}$, say $|g^{(k)}(t)| \leqq a_k$, and a, λ and z are real, then R_n in (2.3) may be majorized by $|R_n| \leqq a_n F_n/n!$. Using (2.8), we obtain uniformly in a for $a \geqq 0$,

$$|R_n| \leqq a_n z^{-n-\lambda}\Gamma(n+\lambda)/n!.$$

Consequently, in the notation of [2], we have

$$F(z,a) \sim \sum c_k F_k \quad \{z^{-k-\lambda}\} \quad \text{as } z \to \infty.$$

This shows that (2.1) is an asymptotic expansion, holding uniformly in a for $a \geqq 0$, with respect to the asymptotic sequence $\{z^{-n-\lambda}\}$, which does not depend on a.

From a practical point of view, the expansion in (2.1) is more suitable than (1.2), since the coefficients c_k are simply expressed in terms of $g^{(k)}(a)$. Both expansions have the same bounds for the remainders. As a minor improvement, our expansion is also uniformly valid with respect to λ on compact subintervals of $(0, 1]$.

3. The numerical analyst may wonder if the sequence $\{F_k\}$ can be generated in a stable way by using (2.5). The answer is affirmative, as one easily deduces from the qualitative behavior of the linearly independent solutions of the second order difference equation (2.5). With

$$G_k = \int_0^a e^{-zt}(t-a)^k t^{\lambda-1}\,dt = a^{\lambda+k}(-1)^k \frac{\Gamma(\lambda)\Gamma(k+1)}{\Gamma(k+\lambda+1)} M(\lambda, k+\lambda+1, -az), \tag{3.1}$$

the functions F_n, G_n constitute a linearly independent pair of solutions of (2.5), as follows from the asymptotic behavior

$$F_n \sim n!\, z^{-n-\lambda}(1+a/n)^{\lambda+1} n^{\lambda-1}, \quad n \to \infty, \quad \text{uniformly in } a \geqq 0, \tag{3.2}$$

and from the inequality,

$$|G_n| \leqq a^{n+\lambda}\Gamma(\lambda)\Gamma(n+1)/\Gamma(n+\lambda+1), \qquad n = 0, 1, \cdots. \tag{3.3}$$

Formula (3.2) is easily derived with saddle point techniques from (2.7), and (3.3) follows from (3.1) by majorizing the exponential function by 1.

The relations (3.2) and (3.3) show that, in the sense of [4], the solution G_n is a minimal solution of (2.5) and F_n a dominant solution.

4. The relation between Erdélyi's expansion (1.2) and our expansion (2.1) can be illustrated by writing

$$F_k = P_k F_0 + Q_k a^\lambda z^{-1}, \qquad k = 0, 1, \cdots.$$

P_k and Q_k are polynomials in z^{-1} satisfying (2.5) with initial values $P_0 = 1$, $Q_0 = 0$, $P_1 = \lambda z^{-1} - a$, $Q_1 = 1$. By using the recurrence relation it can be proved that

$$P_k = z^{-k}\sum_{j=0}^{k} (-az)^{k-j}\binom{k}{j}\Gamma(\lambda+j)/\Gamma(\lambda), \qquad k = 0, 1, \cdots. \tag{4.1}$$

Hence, in a formal way, our expansion (2.1) can be written as

$$F(z, a) \sim F_0 \sum c_k P_k + a^\lambda z^{-1} \sum c_k Q_k. \tag{4.2}$$

With the substitution of (4.1) and using the (formal) expansion

$$g^{(j)}(t) = \sum_{k=j}^{\infty} c_k \frac{k!}{(k-j)!}(t-a)^{k-j}$$

at $t = 0$, we obtain, by interchanging the order of summation,

$$F(z, a) \sim Q \sum z^{-k}\Gamma(k + \lambda)g^{(k)}(0)/k! + a^\lambda z^{-1} \sum c_k Q_k.$$

The first series in this expression is exactly the first series of Erdélyi in (1.2). The second series is much more complicated, but probably it can be identified with the corresponding series of Erdélyi.

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SOME CLASSES OF WATSON TRANSFORMS AND RELATED INTEGRAL EQUATIONS FOR GENERALIZED FUNCTIONS*

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Abstract. Spaces of testing functions which are isomorphically mapped onto one another by the Mellin and the inverse Mellin transform are used to prove that certain spaces are also mapped isomorphically onto one another by the so-called Watson transform. Then Watson transforms for generalized functions are defined. Applications on Hankel transforms, fractional integrals and integral equations of Love involving hypergeometric functions and of Fox involving H-functions are given. Furthermore, dual integral equations for generalized functions with Hankel transforms and H-functions are treated.

Introduction. In this paper we define Watson transforms and other convolution transforms for generalized functions. To this end we introduce spaces of testing functions which are mapped isomorphically onto each other by means of the Mellin transform (§ 1). Using the connection of Watson transforms and Mellin transforms (cf. Titchmarsh [13]) we show that Watson transforms map these function spaces continuously into function spaces of the same type (§ 2). Then these transforms are generalized to generalized functions in the dual spaces. Also the inverses of these transforms are considered. In § 3 the same analysis is done on certain subspaces of the spaces of testing functions of § 1. Examples including Hankel transforms are given in § 4.

Another type of product convolutions is treated in § 5. In particular, operators of fractional integration are considered including the so-called cut fractional integrals. Using these fractional integrals we extend the definition of the Hankel transform in § 6. Here also the cut Hankel transform appears which is useful for the inversion of Hankel transforms. Furthermore relations between Hankel transforms and fractional integrals of generalized functions are given. In §§ 7 and 8 we give applications to dual integral equations for generalized functions involving Hankel transforms and, more generally, transforms with H-functions of Fox which contain many special integral transforms (cf. Fox [6]). Here we use a method of Erdélyi and Sneddon [5]. We give precise conditions for the existence of solutions of the dual integral equations, which were obtained formally by Fox. In § 9 we consider a special case of product convolutions involving hypergeometric functions and related integral equations, which have been studied among others by Love [11a] and [11b]. Some of the results of Love are also extended for ordinary functions.

Other applications to differential equations may be given analogous to those in Zemanian's study of generalized integral transformations [14]. Our approach to Mellin and Hankel transforms is different from Fung Kang's [7] and from Zemanian's approach. Fractional integrals for distributions have been studied recently by Erdélyi and McBride [4] and Erdélyi [3]. Our treatment is similar to theirs, though we do not assume that the testing functions have compact support. Watson transforms for generalized functions have been considered also by

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Hsing-Yuan Hsu [8], starting from testing function spaces closely related to those of Zemanian.

1. The spaces $T(\lambda, \mu)$ and $S(\lambda, \mu)$. Throughout this paper $\mathbb{R}$ denotes the set of the real numbers and $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$. $\mathbb{C}$ is the set of complex numbers. $\mathbb{N} = \{0, 1, 2, \cdots\}$.

Let $\lambda, \mu \in \mathbb{R}^*$, $\lambda < \mu$. Let $(\lambda_n)_{n=0}^{\infty}$ and $(\mu_n)_{n=0}^{\infty}$ be sequences of real numbers with $\lambda_n \downarrow \lambda$, $\mu_n \uparrow \mu$ and $\lambda_n < \mu_n$ for all $n \in \mathbb{N}$. $T(\lambda, \mu)$ is the space of all functions $\phi \in C^{\infty}(0, \infty)$ with the property

$$\tau_n(\phi) = \sup_{\substack{t>0 \\ p=0,1,\cdots,n \\ \lambda_n \leqq c \leqq \mu_n}} |t^{c+p}\phi^{(p)}(t)| < \infty \quad \text{for all } n \in \mathbb{N}. \tag{1.1}$$

$T(\lambda, \mu)$ is a locally convex vector space with the topology generated by the sequence of norms (τ_n). Related spaces have been considered by Zemanian [14, § 4.2].

Let λ, μ, λ_n and μ_n be as above. $S(\lambda, \mu)$ is the space of all functions Φ, analytic on $\lambda < \operatorname{Re} s < \mu$, with the property

$$\sigma_n(\Phi) = \sup_{\substack{\lambda_n \leqq \operatorname{Re} s \leqq \mu_n \\ p=0,1,\cdots,n}} |s^p\Phi(s)| < \infty \quad \text{for all } n \in \mathbb{N}. \tag{1.2}$$

With the topology generated by the sequence of norms (σ_n), $S(\lambda, \mu)$ is a locally convex vector space.

The topologies of $T(\lambda, \mu)$ and $S(\lambda, \mu)$ are independent of the particular choice of the sequences (λ_n) and (μ_n). Using standard arguments it may be shown that both spaces are Fréchet spaces. In the following, isomorphisms and automorphisms between spaces are interpreted as linear continuous mappings onto with continuous inverses.

If ϕ is some function, we denote its Mellin transform by $\mathcal{M}\phi$,

$$(\mathcal{M}\phi)(s) = \int_0^{\infty} t^{s-1}\phi(t)\, dt. \tag{1.3}$$

If Φ is some function we denote its inverse Mellin transform by $\mathcal{M}^{-1}\Phi$,

$$(\mathcal{M}^{-1}\Phi)(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Phi(s)t^{-s}\, ds.$$

We prove the following theorem.

THEOREM 1. *The Mellin transform $\mathcal{M}$ defines an isomorphism of $T(\lambda, \mu)$ onto $S(\lambda, \mu)$. The adjoint Mellin transform $\mathcal{M}'$ defines an isomorphism of $S'(\lambda, \mu)$ onto $T'(\lambda, \mu)$.*

Proof. If $\phi \in T(\lambda, \mu)$ and $\lambda < \operatorname{Re} s < \mu$, $p \in \mathbb{N}$, then

$$(\mathcal{M}\phi)(s) = \Phi(s) = \int_0^{\infty} t^{s-1}\phi(t)\, dt = \frac{(-1)^p}{(s)_p}\int_0^{\infty} t^{s+p-1}\phi^{(p)}(t)\, dt$$

by virtue of (1.1). (Notation: $(s)_0 = 1$, $(s)_p = (s+p-1)(s)_{p-1}$, $p \geqq 1$). Note that $\int_0^{\infty} t^{s+p-1}\phi^{(p)}(t)\, dt$ has a zero in $s = -h$, $h \in \mathbb{N}$, if $\lambda < -h < \mu$. It follows that $\Phi \in S(\lambda, \mu)$.

If $\Phi \in S(\lambda, \mu)$, $\lambda < c < \mu$, $t > 0$ and $p \in \mathbb{N}$, then

$$(\mathscr{M}^{-1}\Phi)(t) = \phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Phi(s)t^{-s}\,ds,$$

$$\phi^{(p)}(t) = \frac{(-1)^p}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-s-p}(s)_p\Phi(s)\,ds,$$

where the integrals are absolutely convergent. It follows that $\phi \in T(\lambda, \mu)$. From the well-known inversion theorem for Mellin transforms it follows that $\mathscr{M} \circ \mathscr{M}^{-1}$ and $\mathscr{M}^{-1} \circ \mathscr{M}$ are the identity maps on $S(\lambda, \mu)$ and $T(\lambda, \mu)$. It remains to prove the continuity.

We may assume that the sequences (λ_n) and (μ_n) are chosen in such a way that $\lambda_n, \mu_n \neq 0, -1, -2, \cdots$. Consider the strip $\lambda_n \leqq \operatorname{Re} s \leqq \mu_n$. For each integer $h \leqq 0$ with $\lambda_n < h < \mu_n$, let D_h be the interior of a disc with center h and which lies entirely in the strip. We omit all the sets D_h from the strip and denote the remaining "reduced" strip by S. Let $\phi \in T(\lambda, \mu)$ and $\Phi = \mathscr{M}\phi$. Then

$$\sigma_n(\Phi) = \sup_{\substack{0 \leqq p \leqq n \\ \lambda_n \leqq \operatorname{Re} s \leqq \mu_n}} |s^p\Phi(s)| = \sup_{\substack{0 \leqq p \leqq n \\ s \in S}} \left| \frac{s^p}{(s)_p}\int_0^\infty t^{s+p-1}\phi^{(p)}(t)\,dt \right|.$$

Now

$$K_0 = \sup_{\substack{0 \leqq p \leqq n \\ s \in S}} |s^p/(s)_p| < \infty$$

and with

$$\varepsilon = \min\{\lambda_n - \lambda_{n+1}, \mu_{n+1} - \mu_n\}, \qquad c = \operatorname{Re} s,$$

we have

$$\sigma_n(\Phi) \leqq K_0 \sup_{\substack{0 \leqq p \leqq n \\ \lambda_n \leqq c \leqq \mu_n}} \left\{ \int_0^1 |t^{c+p-\varepsilon}\phi^{(p)}(t)|t^{-1+\varepsilon}\,dt + \int_1^\infty |t^{c+p+\varepsilon}\phi^{(p)}(t)|t^{-1-\varepsilon}\,dt \right\} \leqq \frac{2K_0}{\varepsilon}\tau_{n+1}(\phi).$$

This proves the continuity of $\mathscr{M}$.

Let $\Phi \in S(\lambda, \mu)$ and let $\phi = \mathscr{M}^{-1}\Phi$. Then

$$\begin{aligned}
\tau_n(\phi) &= \sup_{\substack{0 \leqq p \leqq n \\ \lambda_n \leqq c \leqq \mu_n \\ t > 0}} |t^{c+p}\phi^{(p)}(t)| \\
&= \sup_{t,c,p} \left| t^{c+p}\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} (s)_p\Phi(s)t^{-s-p}\,ds \right| \\
&\leqq \sup_{t,c,p} \frac{1}{2\pi}\left\{ \left(\int_{c-i\infty}^{c-i} + \int_{c+i}^{c+i\infty} \right) |s^{p+2}\Phi(s)| \cdot \left| \frac{(s)_p}{s^{p+2}} \right| \cdot |ds| + \left| \int_{c-i}^{c+i} (s)_p\Phi(s)t^{c-s}\,ds \right| \right\} \\
&\leqq K\sigma_{n+2}(\Phi),
\end{aligned}$$

where K depends only on n. Thus $\mathcal{M}^{-1}$ is continuous. The second assertion of the theorem follows at once.

2. Watson transforms on $T(\lambda, \mu)$. In this section we will consider a Watson transformation between two spaces of type $T(\lambda, \mu)$. Formally such a transformation is described by a pair of reciprocal formulas

$$\psi(x) = \int_0^\infty k(xt)\phi(t)\,dt, \qquad \phi(x) = \int_0^\infty h(xt)\psi(t)\,dt.$$

By applying the Mellin transform to these formulas we may formally show that the Mellin transforms $K(s)$ and $H(s)$ of $k(t)$ and $h(t)$ satisfy $K(s)H(1-s) = 1$, (cf. Titchmarsh [13]). We prove two theorems on these transforms in spaces $T(\lambda, \mu)$.

THEOREM 2. *Let $\lambda, \mu \in \mathbb{R}^*$, $\lambda < \mu$. Let $K(s)$ be an analytic function on $\lambda < \operatorname{Re} s < \mu$ such that $K(c + it) \in L(-\infty, \infty)$ for some c with $\lambda < c < \mu$. Assume moreover that for every pair (a, b) such that $\lambda < a \leqq b < \mu$ there exists a real number γ such that*

$$K(s) = O(s^\gamma) \quad \text{as } s \to \infty, \text{ uniformly on } a \leqq \operatorname{Re} s \leqq b. \tag{2.1}$$

Let

$$k(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K(s)t^{-s}\,ds, \qquad t > 0. \tag{2.2}$$

Then the map $A: T(1-\mu, 1-\lambda) \to T(\lambda, \mu)$, defined by

$$\psi(x) = (A\phi)(x) = \int_0^\infty k(xt)\phi(t)\,dt \tag{2.3}$$

is linear and continuous. The adjoint operator A' is continuous from $T'(\lambda, \mu)$ into $T'(1-\mu, 1-\lambda)$.

Proof. The integral in (2.2) is absolutely convergent, hence $k(t)$ exists for $t > 0$. It follows from the definition that if $\phi \in T(1-\mu, 1-\lambda)$, then $t^{-c}\phi(t) \in L(0, \infty)$. Then the reversion of the order of integration in the following computation is allowed:

$$\begin{aligned} \psi(x) &= (A\phi)(x) = \frac{1}{2\pi i}\int_0^\infty dt\,\phi(t)\int_{c-i\infty}^{c+i\infty} K(s)x^{-s}t^{-s}\,ds, \\ \psi(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K(s)\Phi(1-s)x^{-s}\,ds, \end{aligned} \tag{2.4}$$

where $\Phi = \mathcal{M}\phi$. Since $\Phi \in S(1-\mu, 1-\lambda)$, we have $\Phi(1-s) \in S(\lambda, \mu)$. Moreover, from (2.1) we obtain

$$s^q K(s)\Phi(1-s) = O(s^{q+\gamma-p}) \quad \text{as } s \to \infty, \text{ on } a \leqq \operatorname{Re} s \leqq b \text{ if } p, q \in \mathbb{N},$$

and we see that $K(s)\Phi(1-s) \in S(\lambda, \mu)$. Define the map

$$\mathcal{K}: S(1-\mu, 1-\lambda) \to S(\lambda, \mu)$$

by

$$(\mathscr{K}\Phi)(s) = K(s)\Phi(1 - s).$$

It is clear that $\mathscr{K}$ is linear and continuous. Now (2.4) reads as

$$(A\phi)(x) = (\mathscr{M}^{-1} \circ \mathscr{K} \circ \mathscr{M}\phi)(x) \tag{2.5}$$

and the desired properties of A follow from the corresponding ones of the factors.

If we impose further conditions on $K(s)$ in Theorem 2, then the map A is even an isomorphism. From Fig. 1 it is seen that we have to choose $K(s)$ in such a way that $\mathscr{K}$ is an isomorphism. The following theorem gives the precise conditions.

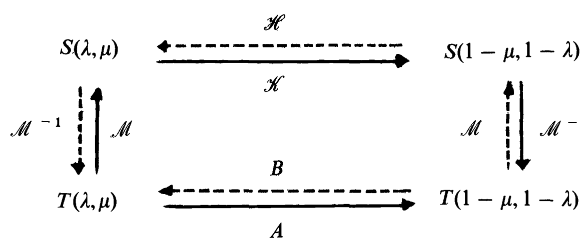

FIG. 1.

THEOREM 3. *Let λ, μ and $K(s)$ be as in Theorem 2 and let $K(s)$ have no zeros in $\lambda < \operatorname{Re} s < \mu$. Define $H(s) = K^{-1}(1 - s)$, $1 - \mu < \operatorname{Re} s < 1 - \lambda$. Assume $H(c_1 + it) \in L(-\infty, \infty)$ for some c_1 with $1 - \mu < c_1 < 1 - \lambda$. Moreover, assume that to every pair (a_1, b_1), $1 - \mu < a_1 \leqq b_1 < 1 - \lambda$, there exists a constant γ_1 such that*

$$H(s) = O(s^{\gamma_1}) \quad \text{as } s \to \infty, \text{uniformly on } a_1 \leqq \operatorname{Re} s \leqq b_1. \tag{2.6}$$

Then the map A in Theorem 2 is an isomorphism on $T(1 - \mu, 1 - \lambda)$ onto $T(\lambda, \mu)$ and the inverse B of A is given by

$$\phi(x) = (B\psi)(x) = \int_0^\infty h(xt)\psi(t)\,dt, \qquad \psi \in T(\lambda, \mu), \tag{2.7}$$

where

$$h(t) = \frac{1}{2\pi i}\int_{c_1 - i\infty}^{c_1 + i\infty} H(s)t^{-s}\,ds. \tag{2.8}$$

The adjoint operator A' is an isomorphism from $T'(\lambda, \mu)$ onto $T'(1 - \mu, 1 - \lambda)$ with $(A')^{-1} = B'$.

Proof. Define the map $\mathscr{H} : S(\lambda, \mu) \to S(1 - \mu, 1 - \lambda)$ by

$$(\mathscr{H}\Psi)(s) = H(s)\Psi(1 - s), \qquad \Psi \in S(\lambda, \mu).$$

It is easy to see that $\mathscr{H}$ is the continuous inverse of $\mathscr{K}$. If B is defined by (2.7), then

$$B = \mathscr{M}^{-1} \circ \mathscr{H} \circ \mathscr{M}. \tag{2.9}$$

This may be proved in the same way as (2.5). Combining (2.5) and (2.9) we see that A and B are inverses of one another.

Remark 1. The conditions $K(c + it) \in L(-\infty, \infty)$ and $H(c_1 + it) \in L(-\infty, \infty)$ in Theorems 2 and 3 may be omitted provided (2.2), (2.3), (2.7) and (2.8) are modified as follows.

From the assumptions on $K(s)$ we deduce that there are numbers $d \in \mathbb{R}$, $\varepsilon > 0$ and a positive integer n such that

$$K(s) = O(s^{n-1-\varepsilon}) \quad \text{as } s \to \infty \text{ on } \operatorname{Re} s = d, \quad \lambda < d < \mu, \quad d \neq 1, 2, \cdots, n.$$

Then define

$$k_n(t) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \frac{K(s)}{(1-s)_n} t^{n-s}\, ds$$

and

$$(A\phi)(x) = \frac{d^n}{dx^n}\int_0^\infty k_n(xt)\phi(t)t^{-n}\, dt, \quad \text{if } \phi \in T(1-\mu, 1-\lambda). \tag{2.10}$$

Now

$$\begin{aligned}(A\phi)(x) &= \frac{d^n}{dx^n}\int_0^\infty dt\, \phi(t)t^{-n} \cdot \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \frac{K(s)}{(1-s)_n}(xt)^{n-s}\, ds \\ &= \frac{1}{2\pi i}\frac{d^n}{dx^n}\int_{d-i\infty}^{d+i\infty} ds \frac{K(s)}{(1-s)_n} x^{n-s}\int_0^\infty \phi(t)t^{-s}\, dt \\ &= \frac{1}{2\pi i}\frac{d^n}{dx^n}\int_{d-i\infty}^{d+i\infty} \frac{K(s)}{(1-s)_n}(\mathcal{M}\phi)(1-s)x^{n-s}\, ds \\ &= \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} K(s)(\mathcal{M}\phi)(1-s)x^{-s}\, ds.\end{aligned}$$

Similarly, $h_m(t)$ and B are defined. Fig. 1 remains valid.

3. Watson transforms on the subspaces T_m and S_m. In this section we shall take $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{R}^*$, $\operatorname{Re} \lambda < \mu$. We want to define subspaces of $T(\operatorname{Re} \lambda, \mu)$ and $S(\operatorname{Re} \lambda, \mu)$ which are mapped onto one another by the maps A and B of § 2. The motivation will become clear in the next section.

Let m be a positive number and $\operatorname{Re} \lambda < \mu$. Then $T_m(\lambda, \mu)$ is the linear space of functions $\phi \in T(\operatorname{Re} \lambda, \mu)$ such that

$$\phi(t) = t^{-\lambda}\tilde{\phi}(t^m), \qquad t > 0, \quad \tilde{\phi} \in C^\infty[0, \infty).$$

We choose a topology on $T_m(\lambda, \mu)$ which is finer than the induced topology of $T(\operatorname{Re} \lambda, \mu)$. If μ_n tends monotonically to μ from below we define

$$\tilde{\tau}_n(\phi) = \sup_{\substack{t \geqq 0 \\ p = 0, 1, \cdots, n}} (1 + t^{((\mu_n - \operatorname{Re} \lambda)/m) + p})|\tilde{\phi}^{(p)}(t)|,$$

and we take the topology generated by the norms $\tilde{\tau}_n$, $n \in \mathbb{N}$ on $T_m(\lambda, \mu)$. Then $T_m(\lambda, \mu)$ is a Fréchet space.

Furthermore $S_m(\lambda, \mu)$ is the linear space of elements $\Phi \in S(\mathrm{Re}\,\lambda, \mu)$ such that

(i) $\Phi(s)$ is analytic if $\mathrm{Re}\, s < \mu$ except for at most simple poles in the points $s = \lambda - mj, j \in \mathbb{N}$.

(ii) $\Phi(s) = O(s^{-p})$ as $s \to \infty$ uniformly on any strip $a \leqq \mathrm{Re}\, s \leqq b < \mu$ for any $p \in \mathbb{N}$.

We choose on $S_m(\lambda, \mu)$ the topology generated by the norms $\tilde{\sigma}_n$, $n \in \mathbb{N}$, where

$$\tilde{\sigma}_n(\Phi) = \sup_{s \in G_n} |\Phi(s)| \prod_{j=0}^{n} |s - \lambda + mj|,$$

$$G_n = \{s \in \mathbb{C} : \mathrm{Re}\,\lambda - mn + m \leqq \mathrm{Re}\, s \leqq \mu_n\}.$$

It is very easy to prove that $S_m(\lambda, \mu)$ is a Fréchet space.

THEOREM 4. *The Mellin transform* $\mathscr{M}$ *is an isomorphism from* $T_m(\lambda, \mu)$ *onto* $S_m(\lambda, \mu)$. *Its adjoint* $\mathscr{M}'$ *is an isomorphism from* $S'_m(\lambda, \mu)$ *onto* $T'_m(\lambda, \mu)$.

Proof. If $\phi \in T_m(\lambda, \mu)$, $\mathrm{Re}\,\lambda < \mathrm{Re}\, s < \mu$ and $p \in \mathbb{N}$, then

$$\begin{aligned}\mathscr{M}\phi(s) &= \frac{1}{m}\int_0^\infty \tau^{((s-\lambda)/m)-1}\tilde{\phi}(\tau)\,d\tau \\ &= \frac{(-1)^p}{m\left(\dfrac{s-\lambda}{m}\right)_p}\int_0^\infty \tau^{((s-\lambda)/m)+p-1}\tilde{\phi}^{(p)}(\tau)\,d\tau.\end{aligned}$$

The last integral is analytic in s if $\mathrm{Re}\,\lambda - mp < \mathrm{Re}\, s < \mu$. Hence $\mathscr{M}\phi \in S_m(\lambda, \mu)$ and it easily follows that $\mathscr{M}$ is continuous.

If $\Phi \in S_m(\lambda, \mu)$, $\phi = \mathscr{M}^{-1}\Phi$, $\mathrm{Re}\,\lambda < c < \mu$, then

$$\tilde{\phi}(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Phi(s)t^{(\lambda-s)/m}\,ds, \qquad t > 0.$$

Consequently,

$$\begin{aligned}\tilde{\phi}^{(p)}(t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Phi(s)\left(\frac{\lambda-s}{m}\right)\cdots\left(\frac{\lambda-s}{m} - p + 1\right)t^{((\lambda-s)/m)-p}\,ds \\ &= p!\,\operatorname*{Res}_{s=\lambda-mp}\Phi(s) + \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\Phi(s)\left(\frac{\lambda-s}{m}\right) \\ &\quad\cdots\left(\frac{\lambda-s}{m} - p + 1\right)t^{((\lambda-s)/m)-p}\,ds,\end{aligned}$$

where $\mathrm{Res}_{s=\lambda-mp}$ denotes "residue at $s = \lambda - mp$ of"; if $\mathrm{Re}\,\lambda - m(p+1) < c_1 < \mathrm{Re}\,\lambda - mp$, $t > 0$. Therefore $\tilde{\phi} \in C^p\,[0, \infty)$. Further it is easily seen that $\phi \in T_m(\lambda, \mu)$ and that $\mathscr{M}^{-1}$ is continuous.

We now follow the method of § 2 to derive some further theorems.

THEOREM 5. *Let* $\lambda \in \mathbb{C}$, $\mu \in \mathbb{R}^*$, $\mathrm{Re}\,\lambda < \mu$ *and* m *be a positive number. Assume that* $K(s)$ *is analytic for* $\mathrm{Re}\, s < \mu$ *except for simple poles at* $s = \lambda - jm$, $j \in \mathbb{N}$. *Assume moreover that for each pair* (a, b), $a \leqq b < \mu$, *there exists a constant* γ *such that* (2.1) *holds. Let* $K(c + it) \in L(-\infty, \infty)$ *for some* c *with* $\mathrm{Re}\,\lambda < c < \mu$. *Then the* *map A of Theorem* 2 *maps* $T(1-\mu,\infty)$ *linearly and continuously into* $T_m(\lambda,\mu)$. *The adjoint map* A' *is a continuous operator on* $T'_m(\lambda,\mu)$ *into* $T'(1-\mu,\infty)$.

Proof. The map $\mathscr{K}$ used in the proof of Theorem 2 is a continuous map on $S(1-\mu,\infty)$ into $S_m(\lambda,\mu)$.

In the same way we have the following.

THEOREM 6. *Let* λ, μ *and* m *be as in Theorem* 5. *Assume* $H(s)$ *is analytic for* $\operatorname{Re} s > 1-\mu$ *and* $H(s) = 0$ *if* $s = 1-\lambda+jm$, $j = 0, 1, 2, \cdots$. *Assume that for each pair* (a_1, b_1) *such that* $1-\mu < a_1 \leqq b_1$ *there exists a constant* γ_1 *such that* (2.6) *holds. Moreover let* $H(c_1+it) \in L(-\infty,\infty)$ *for some* c_1 *with* $1-\mu < c_1 < 1-\operatorname{Re}\lambda$. *Then the map* B *defined by* (2.7) *maps* $T_m(\lambda,\mu)$ *linearly and continuously into* $T(1-\mu,\infty)$ *and* B' *is a continuous operator on* $T'(1-\mu,\infty)$ *into* $T'_m(\lambda,\mu)$.

If $H(s)K(1-s) = 1$ *and* $\operatorname{Re} s > 1-\mu$, *then* A *is an isomorphism of* $T(1-\mu,\infty)$ *onto* $T_m(\lambda,\mu)$ *with inverse* B.

Remark 2. Here also we may omit the conditions $K(c+it) \in L(-\infty,\infty)$ and $H(c+it) \in L(-\infty,\infty)$ as in Remark 1 of § 2.

4. Examples.

Example 1. Let m be a positive number, $\lambda \in \mathbb{C}$, λ_0, $\mu \in \mathbb{R}$, $\operatorname{Re}\lambda \leqq \lambda_0 < \mu \leqq 1-\operatorname{Re}\lambda$ and let $K_1(s)$ be analytic on $\operatorname{Re} s < \mu$ and on $\operatorname{Re} s > 1-\mu$, whereas $K_1(s) = K_1^{-1}(1-s)$. Assume that (2.1) holds for $K_1(s)$ on any set $\lambda' \leqq \operatorname{Re} s \leqq \mu' < \mu$ and any set $1-\mu < 1-\mu' \leqq \operatorname{Re} s \leqq 1-\lambda'$. Assume

$$K_1(c+it) = O(t^{-((2c-1)/m)-1-\varepsilon}) \quad \text{as } t \to \infty \tag{4.1}$$

for some c with $\lambda_0 < c < \mu$ and for some c with $1-\mu < c < 1-\lambda_0$, and some $\varepsilon > 0$. Define

$$K(s) = \frac{\Gamma((s-\lambda)/m)}{\Gamma((1-\lambda-s)/m)} K_1(s).$$

Then $K(s) = K^{-1}(1-s)$ and Theorems 2 and 6 imply that A is a homeomorphism from $T(1-\mu, 1-\lambda_0)$ onto $T(\lambda_0,\mu)$ and from $T(1-\mu,\infty)$ onto $T_m(\lambda,\mu)$, whereas $A = A^{-1}$. Condition (4.1) may be omitted if A is interpreted as in Remark 1.

Example 2. A special case of Example 2 is the following. Let $K_1(s) = 2^{s-(1/2)}$ and

$$K(s) = \frac{\Gamma((\nu+\frac{1}{2}+s)/2)}{\Gamma((\nu+\frac{3}{2}-s)/2)} 2^{s-(1/2)}.$$

Now $K(s) = K^{-1}(1-s)$, $K_1(s) = O(1)$, $K(s) = O(s^{\operatorname{Re} s-(1/2)})$ as $s \to \infty$ on any strip $a \leqq \operatorname{Re} s \leqq b$, $m = 2$, $c < \frac{1}{2}$ and

$$k(t) = t^{1/2} J_\nu(t)$$

(cf. [13, p. 214]). Suppose $\operatorname{Re}\nu > -1$ and choose λ and μ such that $-\operatorname{Re}\nu - \frac{1}{2} \leqq \lambda < \mu \leqq \operatorname{Re}\nu + \frac{3}{2}$. If $\operatorname{Re}\nu > 0$, $\lambda < -\frac{1}{2}$, $\frac{3}{2} < \mu$ we may choose c and c_1 such that $-\operatorname{Re}\nu - \frac{1}{2} < c < -\frac{1}{2}$, $c < \mu$, $1-\mu < c_1 < -\frac{1}{2}$, $c_1 < 1-\lambda$. Then $K(c+it)$, $K(c_1+it) \in L(-\infty,\infty)$. Hence, if $\operatorname{Re}\nu > 0$, $-\operatorname{Re}\nu - \frac{1}{2} \leqq \lambda < -\frac{1}{2}$, $\frac{3}{2} < \mu \leqq \operatorname{Re}\nu + \frac{3}{2}$, the Hankel transform H_ν defined by

$$(H_\nu\phi)(x) = \int_0^\infty (xt)^{1/2} J_\nu(xt)\phi(t)\,dt \tag{4.2}$$

is a homeomorphism of $T(1-\mu, 1-\lambda)$ onto $T(\lambda, \mu)$ and of $T(1-\mu, \infty)$ onto $T_2(-\nu-\frac{1}{2}, \mu)$. Furthermore $H_\nu = H_\nu^{-1}$.

We may weaken these conditions by extending the definition of the Hankel transform as in Remark 1 (see for a modification of this extension § 6). Then we see that the extended Hankel transform is a continuous operator from $T(1-\mu, 1-\lambda)$ into $T(\lambda, \mu)$, if $-\operatorname{Re}\nu - \frac{1}{2} \leqq \lambda < \mu$. However, if

$$-\operatorname{Re}\nu - \tfrac{1}{2} \leqq \lambda < 1, \qquad \lambda < \mu, \tag{4.3}$$

then the extended transform and the transform given by (4.2) coincide, since the differentiations in (2.10) may be performed under the integral sign. This follows from the asymptotic behavior of the Bessel function near the origin and ∞. It is now easy to prove the following result for $T(1-\mu, 1-\lambda)$ and some of its subspaces.

THEOREM 7. *The Hankel transform H_ν defined by* (4.2) *is a continuous operator of*

(i) $T(1-\mu, 1-\lambda)$ *into* $T(\lambda, \mu)$ *if* (4.3) *holds*;
(ii) $T(1-\mu, \infty)$ *into* $T_2(-\nu-\frac{1}{2}, \mu)$ *if* $-\operatorname{Re}\nu - \frac{1}{2} < \mu$;
(iii) $T_2(-\nu-\frac{1}{2}-2h, \mu)$ *into* $T(1-\mu, \infty)$ *if* $-\operatorname{Re}\nu - \frac{1}{2} - 2h < \mu \leqq \operatorname{Re}\nu + \frac{3}{2}, h \in \mathbb{N}$;
(iv) $T_2(-\nu-\frac{1}{2}-2h, \infty)$ *into itself if* $\operatorname{Re}\nu > -h-1, h \in \mathbb{N}$.

In the cases (iii) *and* (iv) *with $h = 0$ it is an involutory isomorphism. It is also an involutory isomorphism of* $T(1-\mu, 1-\lambda)$ *if*

$$-\operatorname{Re}\nu - \tfrac{1}{2} \leqq \lambda < \mu \leqq \operatorname{Re}\nu + \tfrac{3}{2}, \qquad \lambda < 1, \quad \mu > 0.$$

In all these cases,

$$\mathscr{M}H_\nu\phi(s) = \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{4} + \frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu + \frac{3}{4} - \frac{1}{2}s)} 2^{s-(1/2)}(\mathscr{M}\phi)(1-s). \tag{4.4}$$

Remark 3. Let $K(s)$ be as in Example 1 with $K_1(s)$ an entire function, $K_1(s) = K_1^{-1}(1-s)$, $K_1(s) = O(s^\gamma)$, $s \to \infty$ on any set $a \leqq \operatorname{Re}s \leqq b$, where γ depends on a and b. Assume (4.1) holds for some c with $\operatorname{Re}\lambda < c$ and for some c with $c < 1 - \operatorname{Re}\lambda$, ($\operatorname{Re}\lambda < \frac{1}{2}$). Then $A = A^{-1}$ is an automorphism on $T_m(\lambda, \infty)$.

Example 3. Let $m, n, p, q \in \mathbb{N}$, $n \leqq p$, $m \leqq q$. Let $\mathbf{a}, \boldsymbol{\alpha} \in \mathbb{C}^p$, $\mathbf{b}, \boldsymbol{\beta} \in \mathbb{C}^q$; $a_j > 0$, $j = 1, \cdots, p$; $b_j > 0, j = 1, \cdots, q$. Suppose

$$\frac{\operatorname{Re}\alpha_j - 1}{a_j} < c < \frac{\operatorname{Re}\beta_h}{b_h}, \qquad j = 1, \cdots, n; \quad h = 1, \cdots, m. \tag{4.5}$$

Suppose

$$\sum_{j=1}^{n} a_j - \sum_{j=m+1}^{q} b_j > \sum_{j=n+1}^{p} a_j - \sum_{j=1}^{m} b_j \tag{4.6}$$

or the following two conditions are satisfied:

$$\sum_{j=1}^{n} a_j - \sum_{j=m+1}^{q} b_j = \sum_{j=n+1}^{p} a_j - \sum_{j=1}^{m} b_j \tag{4.7}$$

and

$$c\left(\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j\right) < -1 + \frac{q-p}{2} + \sum_1^p \operatorname{Re}\alpha_j - \sum_1^q \operatorname{Re}\beta_j. \tag{4.8}$$

Then we define according to Fox [6]:

$$H_{p,q}^{m,n}\left(x\,\middle|\,\begin{matrix}\mathbf{a},\boldsymbol{\alpha}\\ \mathbf{b},\boldsymbol{\beta}\end{matrix}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\prod_1^n \Gamma(1-\alpha_j+a_js)\prod_1^m \Gamma(\beta_j - b_js)x^{-s}}{\prod_{m+1}^q \Gamma(1-\beta_j+b_js)\prod_{n+1}^p \Gamma(\alpha_j - a_js)}\,ds \tag{4.9}$$

if $x > 0$. This integral is easily seen to be absolutely convergent.

Suppose

$$\frac{\operatorname{Re}\alpha_j - 1}{a_j} \leqq \lambda < \mu \leqq \frac{\operatorname{Re}\beta_h}{b_h}, \qquad j = 1, \cdots, n; \quad h = 1, \cdots, m. \tag{4.10}$$

Then the map A defined by

$$(A\phi)(x) = \int_0^\infty H_{p,q}^{m,n}\left(xt\,\middle|\,\begin{matrix}\mathbf{a},\boldsymbol{\alpha}\\ \mathbf{b},\boldsymbol{\beta}\end{matrix}\right)\phi(t)\,dt \tag{4.11}$$

is a continuous linear map of $T(1-\mu, 1-\lambda)$ into $T(\lambda, \mu)$.

A is an isomorphism of $T(1-\mu, 1-\lambda)$ into $T(\lambda, \mu)$ with

$$(A^{-1}\psi)(x) = \int_0^\infty H_{p,q}^{q-m,p-n}\left(xt\,\middle|\,\begin{matrix}\tilde{\mathbf{a}},\tilde{\boldsymbol{\alpha}}\\ \tilde{\mathbf{b}},\tilde{\boldsymbol{\beta}}\end{matrix}\right)\psi(t)\,dt, \tag{4.12}$$

where

$$\begin{aligned}\tilde{\mathbf{a}} &= (a_{n+1}, \cdots, a_p, a_1, \cdots, a_n),\\ \tilde{\mathbf{b}} &= (b_{m+1}, \cdots, b_q, b_1, \cdots, b_m),\end{aligned} \tag{4.13}$$

$$\begin{aligned}\tilde{\boldsymbol{\alpha}} &= (1 + a_{n+1} - \alpha_{n+1}, \cdots, 1 + a_p - \alpha_p, 1 + a_1 - \alpha_1, \cdots, 1 + a_n - \alpha_n),\\ \tilde{\boldsymbol{\beta}} &= (1 + b_{m+1} - \beta_{m+1}, \cdots, 1 + b_q - \beta_q, 1 + b_1 - \beta_1, \cdots, 1 + b_m - \beta_m)\end{aligned} \tag{4.14}$$

if the following conditions are satisfied:

(i) (4.7), (4.8) and (4.10);

$$\text{(ii)}\quad \begin{cases}\mu \leqq \operatorname{Re}(\alpha_j/a_j), & j = n+1, \cdots, p,\\ \operatorname{Re}((\beta_j - 1)/b_j) \leqq \lambda, & j = m+1, \cdots, q;\end{cases} \tag{4.15}$$

(iii) there exists a real number c_1 such that $1 - \mu < c_1 < 1 - \lambda$ and

$$(c_1 - 1)\left(\sum_1^p a_j - \sum_1^q b_j\right) < -1 + \frac{p-q}{2} + \sum_1^q \operatorname{Re}\beta_j - \sum_1^p \operatorname{Re}\alpha_j. \tag{4.16}$$

Proceeding as in Remark 1, § 2, we may extend the definition of A and A^{-1} in cases where (4.8) and (4.16) are not satisfied (cf. also § 8).

Since the G-function and many other special functions are special cases of the H-function, many integral transforms are contained in this example. Especially the Hankel transform of Example 2 may be considered as a special case of Example 3.

5. Other product convolutions; fractional integrals. The Watson transforms of §§ 2 and 3 have the so-called "product-kernel" $k(xt)$. Another integral transform arises if we replace $k(xt)$ by $k(x/t)$ and $\phi(t)$ by $(1/t)\phi(1/t)$. Both integral transforms are called product convolutions. Since

$$\int_0^\infty k\left(\frac{x}{t}\right)\phi(t)\frac{dt}{t} = \int_0^\infty k(xt)\phi\left(\frac{1}{t}\right)\frac{dt}{t},$$

the new integral transform is a Watson transform applied to $(1/t)\phi(1/t)$. If $\phi \in T(\lambda, \mu)$, then $(1/t)\phi(1/t) \in T(1 - \mu, 1 - \lambda)$ and conversely. Hence we have the following.

THEOREM 8. *If $k(x)$ satisfies the assumptions of Theorem 2, then the map A_1 of $T(\lambda, \mu)$ defined by*

$$A_1\phi(x) = \int_0^\infty k\left(\frac{x}{t}\right)\phi(t)\frac{dt}{t}, \qquad \phi \in T(\lambda, \mu), \tag{5.1}$$

is linear and continuous into $T(\lambda, \mu)$.

Moreover, if $K(s)$ does not have zeros in $\lambda < \operatorname{Re} s < \mu$ and $H(s) = K^{-1}(s)$ satisfies (2.6) uniformly on any strip $\lambda < a_1 \leqq \operatorname{Re} s \leqq b_1 < \mu$ with some constant γ_1 depending on a_1 and b_1 and if $H(c_1 + it) \in L(-\infty, \infty)$ for some c_1 with $\lambda < c_1 < \mu$, then A_1 is an isomorphism of $T(\lambda, \mu)$ onto $T(\lambda, \mu)$ and

$$(A_1^{-1}\phi)(x) = \int_0^\infty h\left(\frac{x}{t}\right)\phi(t)\frac{dt}{t}, \tag{5.2}$$

where h is defined by (2.8).

Remark 4. The maps A_1 and A_1^{-1} are given in Fig. 2;

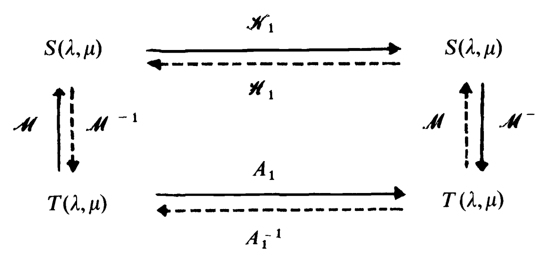

FIG. 2.

where $(\mathscr{K}_1\Phi)(s) = K(s)\Phi(s)$, $(\mathscr{H}_1\Phi)(s) = H(s)\Phi(s)$. The conditions on $K(s)$ and $H(s)$ may be weakened as in Remark 1. If we define k_n and h_m as in Remark 1, then

$$(A_1\phi)(x) = \frac{d^n}{dx^n}\int_0^\infty k_n\left(\frac{x}{t}\right)\phi(t)t^{n-1}\,dt,$$

$$(A_1^{-1}\phi)(x) = \frac{d^m}{dx^m}\int_0^\infty h_m\left(\frac{x}{t}\right)\phi(t)t^{m-1}\,dt,$$

where $\phi \in T(\lambda, \mu)$. It is easy to formulate and to prove the analogues of Theorems 5 and 6 for the transform A_1.

As an application of this type of product convolution we consider the operators of fractional integration, studied among others by Kober [9] and Erdélyi [3].

Let $\alpha, \eta \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$, $\lambda, \mu, m \in \mathbb{R}$, $m > 0$, $\lambda < \mu$ and $m(\operatorname{Re}\eta + 1) > \lambda$. Then

$$
\begin{aligned}
I_m^{\eta,\alpha}\phi(x) &= \frac{m}{\Gamma(\alpha)} x^{-m(\alpha+\eta)} \int_0^x (x^m - t^m)^{\alpha-1} t^{m\eta+m-1}\phi(t)\,dt \\
&= \frac{m}{\Gamma(\alpha)} \int_0^x \left\{ \left(\frac{x}{t}\right)^m - 1 \right\}^{\alpha-1} \left(\frac{x}{t}\right)^{-m(\alpha+\eta)} \phi(t)\frac{dt}{t},
\end{aligned} \tag{5.3}
$$

if we choose $\phi \in T(\lambda, \mu)$, $x > 0$. So we have the special case of Theorem 8 with

$$k(t) = \frac{m}{\Gamma(\alpha)}(t^m - 1)^{\alpha-1} t^{-m(\alpha+\eta)} \quad \text{if } t > 1, \quad k(t) = 0 \quad \text{if } 0 < t < 1,$$

and

$$K(s) = \Gamma\left(1 + \eta - \frac{s}{m}\right)\left\{\Gamma\left(1 + \eta + \alpha - \frac{s}{m}\right)\right\}^{-1}. \tag{5.4}$$

Here $K(s) = O(s^{-\alpha})$ as $s \to \infty$ uniformly on any strip $a \leqq \operatorname{Re} s \leqq b$. Hence, $I_m^{\eta,\alpha}$ is an automorphism of $T(\lambda, \mu)$ if $\operatorname{Re}\alpha > 1$ and

$$\lambda < \mu \leqq m(1 + \operatorname{Re}\eta). \tag{5.5}$$

In order to relax the conditions on η we use the extension of fractional integrals considered by Erdélyi [1]. If $\operatorname{Re}\alpha > 1$, $\phi \in T(\lambda, \mu)$, $h \in \mathbb{N}$ and

$$m(\operatorname{Re}\eta + h) \leqq \lambda < \mu \leqq m(1 + \operatorname{Re}\eta + h), \qquad h \neq 0, \tag{5.5^h}$$

we define

$$
\begin{aligned}
I_{m,h}^{\eta,\alpha}\phi(x) = \frac{m}{\Gamma(\alpha)} x^{-m(\alpha+\eta)} \Bigg[\int_0^x \left\{ (x^m - t^m)^{\alpha-1} - \sum_{j=0}^{h-1} \binom{\alpha-1}{j} (-x^{-m}t^m)^j x^{m(\alpha-1)} \right\} \\
\cdot\, t^{m(1+\eta)-1}\phi(t)\,dt - \int_x^\infty \sum_{j=0}^{h-1} \binom{\alpha-1}{j} (-x^{-m}t^m)^j x^{m(\alpha-1)} t^{m(1+\eta)-1}\phi(t)\,dt \Bigg].
\end{aligned} \tag{5.3^h}
$$

It is easy to show that this so-called cut fractional integral operator is a product convolution and that the Mellin transform of the kernel is given by (5.4). Moreover, the operator $I_{m,h}^{\eta,\alpha}$ is continuous on $T(\lambda, \mu)$, $h \in \mathbb{N}$, $h \neq 0$. For convenience we shall use the notation $I_{m,0}^{\eta,\alpha}$ for $I_m^{\eta,\alpha}$ and (5.5)°, (5.3)° for (5.5), (5.3).

In order to avoid the condition on α we may use Remark 4. However, an adaption of the method in that remark is more useful. The starting point for this extension is the relation

$$x^{-m(\alpha+\eta)} \left(\frac{d}{dx^m}\right)^n x^{m(\alpha+\eta+n)} I_{m,h}^{\eta,\alpha+n} = I_{m,h}^{\eta,\alpha}, \tag{5.6}$$

which is valid on $T(\lambda, \mu)$ if $\operatorname{Re}\alpha > 1$, $n \in \mathbb{N}$ and (5.5)^h is satisfied. For, if we apply the left-hand side of (5.6) to $\phi \in T(\lambda, \mu)$, then we obtain

$$\frac{1}{2\pi i} x^{-m(\alpha+\eta)} \left(\frac{d}{dx^m}\right)^n \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(1 + \eta - (s/m))}{\Gamma(1 + \alpha + \eta + n - (s/m))} \Phi(s)(x^m)^{\alpha+\eta+n-(s/m)}\,ds,$$

where $\lambda < c < \mu$, $\Phi = \mathscr{M}\phi$, and this expression is easily seen to be equal to $I_{m,h}^{\eta,\alpha}\phi$. However, the left-hand side of (5.6) defines a continuous operator on $T(\lambda, \mu)$ if $\operatorname{Re}(\alpha + n) > 1$ and $(5.5)^h$ holds. Therefore we use (5.6) as the definition of $I_{m,h}^{\eta,\alpha}$ on $T(\lambda, \mu)$ if $\operatorname{Re}(\alpha + n) > 1$ and $(5.5)^h$ holds. If $\operatorname{Re}\alpha > 0$ and $(5.5)^h$ holds, then $(5.3)^h$ remains valid.

The operator $I_{m,h}^{\eta,\alpha}$ is continuous on $T(\lambda, \mu)$ and

$$(5.7)\qquad (\mathscr{M}I_{m,h}^{\eta,\alpha}\phi)(s) = \Gamma\left(1 + \eta - \frac{s}{m}\right)\left\{\Gamma\left(1 + \alpha + \eta - \frac{s}{m}\right)\right\}^{-1}(\mathscr{M}\phi)(s),$$

if $(5.5)^h$ is satisfied and $\phi \in T(\lambda, \mu)$. From this it easily follows that

$$(5.8)\qquad I_{m,h_1}^{\eta+\alpha,\beta}I_{m,h}^{\eta,\alpha} = I_{m,h}^{\eta,\alpha}I_{m,h_1}^{\eta+\alpha,\beta} = I_{m,h}^{\eta,\alpha+\beta}$$

on $T(\lambda, \mu)$ if $(5.5)^h$ holds and

$$(5.9)\qquad \begin{aligned} &\lambda < \mu \leqq m(1 + \operatorname{Re}\eta + \operatorname{Re}\alpha) \quad \text{if } h_1 = 0,\\ &m(\operatorname{Re}\eta + \operatorname{Re}\alpha + h_1) \leqq \lambda < \mu \leqq m(1 + \operatorname{Re}\eta + \operatorname{Re}\alpha + h_1) \quad \text{if } h_1 \neq 0. \end{aligned}$$

Then in particular,

$$(5.10)\qquad (I_{m,h}^{\eta,\alpha})^{-1} = I_{m,h_1}^{\eta+\alpha,-\alpha}, \qquad I_{m,h}^{\eta,0} = \text{identity operator},$$

and $I_{m,h}^{\eta,\alpha}$ is a topological automorphism of $T(\lambda, \mu)$.

According to (5.6),

$$(5.11)\qquad I_{m,h}^{\eta,-n} = x^{-m(\eta-n)}\left(\frac{d}{dx^m}\right)^n x^{m\eta}, \qquad n \in \mathbb{N}.$$

We may use this last relation as the definition for arbitrary values of η. Indeed, it is easily verified that the right-hand side of (5.11) represents a continuous operator of $T(\lambda, \mu)$ into itself even if $(5.5)^h$ is not satisfied.

Combining (5.8) and (5.11) we obtain an analogue of (5.6),

$$(5.6)'\qquad I_{m,h}^{\eta,\alpha} = I_{m,h}^{\eta,\alpha+n}x^{-m(\alpha+\eta)}\left(\frac{d}{dx^m}\right)^n x^{m(\alpha+\eta+n)}.$$

From (5.7) we readily deduce that if $\alpha = n \in \mathbb{N}$, then

$$(5.12)\qquad (\mathscr{M}I_{m,h}^{\eta,n}\phi)(s) = \left\{\left(1 + \eta - \frac{s}{m}\right)_n\right\}^{-1}(\mathscr{M}\phi)(s),$$

and consequently, $I_{m,h}^{\eta,n}$ is a continuous operator independent of h on $T(\lambda, \mu)$ if $h \geqq n$,

$$(5.12)'\qquad m(n + \operatorname{Re}\eta) \leqq \lambda < \mu.$$

Finally (5.7) implies

$$(5.13)\qquad I_{m,h}^{\eta,\alpha}x^\beta = x^\beta I_{m,h}^{\eta+(\beta/m),\alpha}$$

on $T(\lambda, \mu)$ if the operators I exist. The above results are collected in the following theorem.

THEOREM 9. *Let $h, n \in \mathbb{N}$, $m > 0$, $\eta, \alpha \in \mathbb{C}$, $\lambda < \mu$. Let the operator $I_{m,h}^{\eta,-n}$ be defined by* (5.11) *on $T(\lambda, \mu)$. It is independent of h. Let the operator $I_{m,h}^{\eta,\alpha}$ be defined by*

$(5.3)^h$ *if* $\operatorname{Re}\alpha > 0$ *and* $(5.5)^h$ *holds. Here* $I_{m,0}^{\eta,\alpha} = I_m^{\eta,\alpha}$ *and* $(5.3)^\circ$, $(5.5)^\circ$ *denote* (5.3), (5.5). *If* $\alpha = n$, $h \geqq n$, *then condition* $(5.5)^h$ *may be replaced by* $(5.12)'$ *and then* $I_{m,h}^{\eta,n}$ *is independent of h. If* $-n < \operatorname{Re}\alpha \leqq 0$, $-\alpha \notin \mathbb{N}$ *and* $(5.5)^h$ *holds, then* $I_{m,h}^{\eta,\alpha}$ *is defined by* (5.6) *on* $T(\lambda, \mu)$. *This definition does not depend on the choice of n.*

In all these cases the operator $I_{m,h}^{\eta,\alpha}$ *is a continuous operator from* $T(\lambda, \mu)$ *into itself. It satisfies* (5.6), $(5.6)'$, (5.7), (5.8) *and* (5.13) *on* $T(\lambda, \mu)$ *provided the operators I involved exist. In particular,* $I_{m,h}^{\eta,\alpha}$ *is an automorphism on* $T(\lambda, \mu)$ *satisfying* (5.10) *if* $(5.5)^h$ *and* (5.9) *hold.*

A second operator of fractional integration studied a.o. by Kober [9] and Erdélyi is given by

$$(5.14)\qquad \begin{aligned} K_m^{\eta,\alpha}\phi(x) &= \frac{m}{\Gamma(\alpha)} x^{m\eta} \int_x^\infty (t^m - x^m)^{\alpha-1} t^{m(1-\alpha-\eta)-1}\phi(t)\,dt \\ &= \frac{m}{\Gamma(\alpha)} \int_x^\infty \left\{1 - \left(\frac{x}{t}\right)^m\right\}^{\alpha-1} \left(\frac{x}{t}\right)^{m\eta} \phi(t)\frac{dt}{t}. \end{aligned}$$

Here we choose $\phi \in T(\lambda, \mu)$, $\operatorname{Re}\alpha > 0$, $m\operatorname{Re}\eta + \mu > 0$. This is the special case of Theorem 8 with

$$k(t) = \frac{m}{\Gamma(\alpha)}(1 - t^m)^{\alpha-1} t^{m\eta} \quad \text{if } 0 < t < 1, \quad k(t) = 0 \quad \text{if } t > 1,$$

and

$$K(s) = \Gamma\left(\eta + \frac{s}{m}\right)\left\{\Gamma\left(\alpha + \eta + \frac{s}{m}\right)\right\}^{-1}.$$

Now $K(s) = O(s^{-\alpha})$ as $s \to \infty$ uniformly on any strip $a \leqq \operatorname{Re}s \leqq b$. Hence, $K_m^{\eta,\alpha}$ is an automorphism of $T(\lambda, \mu)$ if $\operatorname{Re}\alpha > 1$ and

$$(5.15)\qquad -m\operatorname{Re}\eta \leqq \lambda < \mu.$$

The extension to other values of η is given by

$$(5.14)^h\qquad \begin{aligned} K_{m,h}^{\eta,\alpha}\phi(x) = \frac{m}{\Gamma(\alpha)} x^{m\eta} \Bigg[&\int_x^\infty \left\{ (t^m - x^m)^{\alpha-1} - \sum_{j=0}^{h-1} \binom{\alpha-1}{j} (-t^{-m}x^m)^j t^{m(\alpha-1)} \right\} \\ &\cdot t^{m(1-\alpha-\eta)-1}\phi(t)\,dt - \int_0^x \sum_{j=0}^{h-1} \binom{\alpha-1}{j} (-t^{-m}x^m)^j t^{-m\eta-1}\phi(t)\,dt \Bigg], \end{aligned}$$

where $\phi \in T(\lambda, \mu)$, $\operatorname{Re}\alpha > 0$, $h \in \mathbb{N}$ and

$$(5.15)^h\qquad -m(\operatorname{Re}\eta + h) \leqq \lambda < \mu \leqq -m(\operatorname{Re}\eta + h - 1), \qquad h \neq 0.$$

Then $K_{m,h}^{\eta,\alpha}$ is a continuous operator on $T(\lambda, \mu)$. We use for convenience the notation $K_{m,0}^{\eta,\alpha}$ for $K_m^{\eta,\alpha}$ and $(5.14)^\circ$, $(5.15)^\circ$ for (5.14), (5.15). In all cases we have

$$(5.16)\qquad (\mathscr{M} K_{m,h}^{\eta,\alpha}\phi)(s) = \frac{\Gamma(\eta + (s/m))}{\Gamma(\alpha + \eta + (s/m))}(\mathscr{M}\phi)(s).$$

The analogue of (5.6) is

$$(5.17)\qquad x^{m(\alpha+\eta+n)}\left(-\frac{d}{dx^m}\right)^n x^{-m(\alpha+\eta)} K_{m,h}^{\eta,\alpha+n} = K_{m,h}^{\eta,\alpha}, \qquad n \in \mathbb{N}.$$

This relation will be used as the definition of $K_{m,h}^{\eta,\alpha}$ if $-n < \operatorname{Re}\alpha \leqq 0$. Then this operator does not depend on n, it is continuous on $T(\lambda, \mu)$ and (5.16) remains true if $(5.15)^h$ holds.

Using $(5.14)^h$, (5.16) and (5.17) it is easily seen that $K_{m,h}^{\eta,\alpha}$ also defines a continuous operator from $T_m(-m(\alpha + \eta + h_1), \mu)$ into $T_m(-m(\eta + h), \mu)$ if $h_1, h \in \mathbb{N}$ and

$$(5.18)\qquad \begin{aligned} &-m \operatorname{Re}\eta < \mu \quad \text{in case } h = 0,\\ &-m(\operatorname{Re}\eta + h) < \mu \leqq -m(\operatorname{Re}\eta + h - 1) \quad \text{in case } h > 0,\\ &-m \operatorname{Re}(\alpha + \eta + h_1) < \mu. \end{aligned}$$

Analogous to (5.8) we have

$$(5.19)\qquad K_{m,h}^{\eta,\alpha} K_{m,h_1}^{\eta+\alpha,\beta} = K_{m,h}^{\eta,\alpha+\beta}, \ (K_{m,h}^{\eta,\alpha})^{-1} = K_{m,h_1}^{\eta+\alpha,-\alpha},$$

$$(5.19)'\qquad K_{m,h_1}^{\eta+\alpha,\beta} K_{m,h}^{\eta,\alpha} = K_{m,h}^{\eta,\alpha+\beta},$$

on $T(\lambda, \mu)$ if $(5.15)^h$ holds and

$$(5.20)\qquad \begin{aligned} &-m \operatorname{Re}(\alpha + \eta) \leqq \lambda < \mu \quad \text{in case } h_1 = 0,\\ &-m \operatorname{Re}(\alpha + \eta + h_1) \leqq \lambda < \mu \leqq -m \operatorname{Re}(\alpha + \eta + h_1 - 1) \quad \text{in case } h_1 > 0, \end{aligned}$$

whereas (5.19) holds on $T_m(-m(\alpha + \beta + \eta + h_2), \mu)$ if (5.18) holds and

$$(5.21)\qquad \mu \leqq -m \operatorname{Re}(\alpha + \eta + h_1 - 1) \quad \text{if } h_1 > 0 \quad \text{and} \quad -m \operatorname{Re}(\alpha + \beta + \eta + h_2) < \mu.$$

The operator $K_{m,h}^{\eta,\alpha}$ is an automorphism on $T(\lambda, \mu)$ in the first case and it is an isomorphism between $T_m(-m(\alpha + \eta + h_1), \mu)$ and $T_m(-m(\eta + h), \mu)$ in the second case (with $h_2 = h$).

The analogue of (5.11) is

$$(5.22)\qquad K_{m,h}^{\eta,-n} = x^{m\eta}\left(-\frac{d}{dx^m}\right)^n x^{m(n-\eta)}, \qquad n, h \in \mathbb{N}.$$

This relation may be used as the definition of K on $T(\lambda, \mu)$ if $(5.15)^h$ is not satisfied. The analogue of (5.12) shows that $K_{m,h}^{\eta,n}$ is a continuous operator on $T(\lambda, \mu)$ independent of h if $h \geqq n$ and

$$(5.23)\qquad \lambda < \mu \leqq -m(\operatorname{Re}\eta + n - 1).$$

Then

$$(5.24)\qquad K_{m,h}^{\eta,n} = (-1)^n I_m^{-\eta-n,n}.$$

In the same way we obtain

$$(5.25)\qquad I_{m,h}^{\eta,n} = (-1)^n K_m^{-\eta-n,n}$$

on $T(\lambda, \mu)$ if (5.12)′ is satisfied and $h \geqq n$.

We deduce from (5.16),

$$(5.26)\qquad K_{m,h}^{\eta,\alpha} x^\beta = x^\beta K_{m,h}^{\eta-(\beta/m),\alpha},$$

if the operators K exist.

Combining (5.19) and (5.22) we get the analogue of (5.17):

$$K_{m,h}^{\eta,\alpha} = K_{m,h}^{\eta,\alpha+n} x^{m(\alpha+\eta+n)}\left(-\frac{d}{dx^m}\right)^n x^{-m(\alpha+\eta)}. \tag{5.17)'$$

Combining the results above we obtain the following theorem.

THEOREM 10. *Let* $n, h \in \mathbb{N}$, $m > 0$, $\alpha, \eta \in \mathbb{C}$, $\lambda < \mu$. *Let the operator* $K_{m,h}^{\eta,-n}$ *be defined independently of h by* (5.22) *on* $T(\lambda, \mu)$. *Let* $K_{m,h}^{\eta,\alpha}$ *be defined on* $T(\lambda, \mu)$ *by* $(5.14)^h$ *and* (5.17) *respectively if* $(5.15)^h$ *holds and moreover* $\operatorname{Re}\alpha > 0$ *and* $-n < \operatorname{Re}\alpha \leqq 0$ *respectively. Here* $K_{m,0}^{\eta,\alpha} = K_m^{\eta,\alpha}$ *and* $(5.15)^0$ *denotes* (5.15). *If* $\alpha = n$, $h \geqq n$ *the condition* $(5.15)^h$ *may be replaced by* (5.23).

In these cases $K_{m,h}^{\eta,\alpha}$ *is a continuous operator on* $T(\lambda, \mu)$. *It is also a continuous operator from* $T_m(-m(\alpha + \eta + h_1), \mu)$ *into* $T_m(-m(\eta + h), \mu)$ *defined by* $(5.14)^h$ *and* (5.17), *if* $h_1 \in \mathbb{N}$ *and* (5.18) *holds.*

This operator satisfies (5.16), (5.17), (5.17)′, (5.19) *and* (5.26) *in all cases where the expressions involved make sense according to the definitions above. In particular,* (5.19) *holds on* $T(\lambda, \mu)$ *if* $(5.15)^h$ *and* (5.20) *are satisfied, and on* $T_m(-m(\alpha+\beta+\eta+h_2), \mu)$ *if* (5.18) *and* (5.21) *are satisfied. In the first case* $K_{m,h}^{\eta,\alpha}$ *is an automorphism on* $T(\lambda, \mu)$, *in the second case (with* $h_2 = h$*) it is an isomorphism between* $T_m(-m(\alpha + \eta + h_1), \mu)$ *and* $T_m(-m(\eta + h), \mu)$.

We now define subspaces of $T(\lambda, \mu)$ which have useful properties for operators of fractional integration.

DEFINITION. Let a be a positive number. Then $T([0, a], \lambda)$ is the subspace of $T(\lambda, \infty)$ of functions with support contained in $[0, a]$. In the same way $T_m([0, a], \lambda)$ is the subspace of $T_m(\lambda, \infty)$ consisting of functions with support contained in $[0, a]$. Finally, $T([a, \infty), \mu)$ is the subspace of $T(-\infty, \mu)$ consisting of functions with support contained in $[a, \infty)$. It is clear that in this way really closed subspaces are defined.

From the definitions of I and K it follows that

(i) $I_m^{\eta,\alpha}$ is a continuous operator from $T([a, \infty), \mu)$ into itself if

$$\mu \leqq m(1 + \operatorname{Re}\eta) \tag{5.27}$$

and it is an automorphism if moreover

$$\mu \leqq m(1 + \operatorname{Re}\alpha + \operatorname{Re}\eta); \tag{5.28}$$

(ii) $K_m^{\eta,\alpha}$ is a continuous operator from $T([0, a], \lambda)$ into itself if

$$\lambda \geqq -m \operatorname{Re}\eta \tag{5.29}$$

and it is an automorphism if moreover

$$\lambda \geqq -m \operatorname{Re}(\alpha + \eta); \tag{5.30}$$

(iii) $K_m^{\eta,\alpha}$ is an isomorphism from $T_m([0, a], -m\alpha - m\eta)$ onto $T_m([0, a], -m\eta)$.

The translation of the results above to the dual operators is easy. A simplification of the notation may be obtained as follows. Suppose $T(\lambda_0, \mu_0) \subset T'(1 - \mu, 1 - \lambda)$. This is the case iff $\lambda_0 < \mu$, $\lambda < \mu_0$. Suppose $f \in T(\lambda_0, \mu_0)$, $\phi \in T(1 - \mu, 1 - \lambda)$ and $(5.5)^h$ is satisfied, and $(5.5)^h$ also holds with λ and μ replaced by λ_0 and μ_0. Then

$$\int_0^\infty \phi(x) I_{m,h}^{\eta,\alpha} f(x)\, dx = \int_0^\infty f(x) K_{m,h}^{\eta_0,\alpha}\phi(x)\, dx, \qquad \eta_0 = \eta + 1 - \frac{1}{m}. \tag{5.31}$$

Hence$(K_{m,h}^{\eta_0,\alpha})' = I_{m,h}^{\eta,\alpha}$ on any space $T(\lambda_0, \mu_0) \subset T'(1 - \mu, 1 - \lambda)$ and therefore we use this relation as a notation on $T'(1 - \mu, 1 - \lambda)$ if $(5.5)^h$ is satisfied. In the same way (5.31) motivates the notation $(I_{m,h}^{\eta_1,\alpha})' = K_{m,h}^{\eta,\alpha}$ on $T'(1 - \mu, 1 - \lambda)$ if $\eta_1 = \eta - 1 + (1/m)$ and $(5.15)^h$ is satisfied.

THEOREM 11. *Let $n, h, h_1 \in \mathbb{N}, m, a \in \mathbb{R}_+, \lambda < \mu, \alpha, \eta \in \mathbb{C}, \eta_1 = \eta - 1 + (1/m)$. Then the adjoint operator of $I_{m,h}^{\eta_1,\alpha}$, to be denoted by $K_{m,h}^{\eta,\alpha}$, is a continuous operator on $T'(1 - \mu, 1 - \lambda)$ in the following cases:*
(i) $\alpha = -n$; (ii) $\alpha = n, h \geqq n$ and (5.23) holds; (iii) $(5.15)^h$ holds. The operator $K_m^{\eta,\alpha}$ is a continuous operator on $T'([a, \infty), 1 - \lambda)$ if (5.29) holds.

Furthermore, the relations (5.17), (5.17)′, (5.19), (5.19)′, (5.22), (5.24)–(5.26) hold in all cases where the operators involved make sense according to the definitions above. In particular, (5.19) and (5.19)′ hold on $T'(1 - \mu, 1 - \lambda)$ if $(5.15)^h$ and (5.20) are satisfied. In this case $K_{m,h}^{\eta,\alpha}$ is an automorphism. Finally, $K_m^{\eta,\alpha}$ is an automorphism on $T'([a, \infty), 1 - \lambda)$ if (5.29) and (5.30) are satisfied.

THEOREM 12. *Let $n, h, h_1 \in \mathbb{N}, m, a \in \mathbb{R}_+, \lambda < \mu, \alpha, \eta \in \mathbb{C}, \eta_1 = \eta + 1 - (1/m)$. Then the adjoint operator of $K_{m,h}^{\eta_1,\alpha}$, to be denoted by $I_{m,h}^{\eta,\alpha}$, is continuous on $T'(1 - \mu, 1 - \lambda)$ in the following cases:*
(i) $\alpha = -n$; (ii) $\alpha = n, h \geqq n$ and (5.13) holds; (iii) $(5.5)^h$ holds. It is a continuous operator from $T'_m(1 - m(\eta + h + 1), \mu)$ into $T'_m(1 - m(\alpha + \eta + h_1 + 1), \mu)$ if (5.18) with η replaced by η_1 holds.

Furthermore, $I_m^{\eta,\alpha}$ is continuous from $T'([0, a], 1 - \mu)$ into itself if (5.27) holds and an automorphism if moreover (5.28) holds. It is an isomorphism from $T'_m([0, a], 1 - m\eta - m)$ onto $T'_m([0, a], 1 - m\alpha - m\eta - m)$.

The operator $I_{m,h}^{\eta,\alpha}$ satisfies (5.6), (5.6)′, (5.8), (5.10), (5.11) and (5.13) in all cases where the operators involved exist according to the definitions above. In particular, (5.8) holds on $T'(1 - \mu, 1 - \lambda)$ if $(5.5)^h$ and (5.9) are satisfied. In this case, $I_{m,h}^{\eta,\alpha}$ is an automorphism on $T'(1 - \mu, 1 - \lambda)$, whereas it is an isomorphism from $T'_m(1 - m(\eta + h + 1), \mu)$ into $T'_m(1 - m(\alpha + \eta + h_1 + 1), \mu)$ if (5.18) and (5.21) with η replaced by η_1 are satisfied.

6. Extension of the Hankel transform. The extension of the Hankel transform H_ν to arbitrary values of ν has been treated in [10] and [14] by means of auxiliary operators N_ν and M_ν. (For the definitions cf. [14, pp. 135 and 163]). Our approach includes these methods as is easily seen from the behavior of the differential operators N_ν and M_ν with respect to the Mellin transform.

For the extension of the definition of the Hankel transform we use the relation

$$H_\nu = 2^\alpha x^{-\alpha} H_{\nu+\alpha} K_{2,h}^{(1/2)\nu+(1/4)+(1/2)\alpha,-\alpha} x^{-\alpha}. \tag{6.1}$$

This formula is valid on $T(1 - \mu, 1 - \lambda)$ if the following conditions are satisfied: (4.3),

$$-\operatorname{Re}\nu - \tfrac{1}{2} \leqq \lambda < 1 + \operatorname{Re}\alpha, \tag{6.2}$$

$$\begin{aligned} &\lambda < \mu \leqq \tfrac{3}{2} + \operatorname{Re}(\nu + 2\alpha) \quad \text{in case } h = 0,\\ &\operatorname{Re}(\nu + 2\alpha) + 2h - \tfrac{1}{2} \leqq \lambda < \mu \leqq \operatorname{Re}(\nu + 2\alpha) + 2h + \tfrac{3}{2} \quad \text{in case } h \in \mathbb{N}, h \neq 0. \end{aligned} \tag{6.3}$$

The proof is straightforward using Mellin transforms and Theorems 7 and 10.

In particular, if $n \in \mathbb{N}$ we obtain with (5.22),

$$H_\nu = (-2)^n x^{-n} H_{\nu+n} x^{\nu+n+(1/2)} \left(\frac{d}{dx^2}\right)^n x^{-\nu-(1/2)}. \tag{6.4}$$

The right-hand side exists and is a continuous operator on
(i) $T(1-\mu, 1-\lambda)$ if $-\operatorname{Re}\nu - \frac{1}{2} \leqq \lambda < n+1$;
(ii) $T_2(-\nu-\frac{1}{2}, \mu)$ if $-\operatorname{Re}\nu - 2n - \frac{1}{2} < \mu \leqq \operatorname{Re}\nu + \frac{3}{2}$;
(iii) $T_2(-\nu-\frac{1}{2}, \infty)$ if $\operatorname{Re}\nu > -n-1$.
Therefore we define in these cases H_ν by (4.2) and (6.4). By choosing n suitably we thus obtain a continuous operator H_ν:
(i) from $T(1-\mu, 1-\lambda)$ to $T(\lambda, \mu)$ if $\lambda \geqq -\operatorname{Re}\nu - \frac{1}{2}$;
(ii) from $T_2(-\nu-\frac{1}{2}, \mu)$ to $T(1-\mu, \infty)$ if $\mu \leqq \operatorname{Re}\nu + \frac{3}{2}$;
(iii) from $T_2(-\nu-\frac{1}{2}, \infty)$ into itself for arbitrary values of ν.
Then (6.1) holds:
(I) on $T(1-\mu, 1-\lambda)$ if $\lambda \geqq -\operatorname{Re}\nu - \frac{1}{2}$ and (6.3) is satisfied; if $\alpha \in \mathbb{N}$, we may omit (6.3) and then (6.1) reduces to (6.4) with $\alpha = n$; if $-\alpha \in \mathbb{N}$, $0 < -\alpha \leqq h$, we may replace (6.3) by $\lambda \geqq \operatorname{Re}\nu - \frac{1}{2}$ and use (5.24);
(II) on $T_2(-\nu - \frac{1}{2} - 2g, \mu)$ if $g \in \mathbb{N}$,

$$-\operatorname{Re}\nu - 2g - \tfrac{1}{2} < \mu \leqq \operatorname{Re}\nu + \tfrac{3}{2}, \tag{6.5}$$

$$(6.6)^h \quad \begin{aligned} &-\operatorname{Re}(\nu+2\alpha) - \tfrac{1}{2} < \mu \quad \text{if } h = 0,\\ &-\operatorname{Re}(\nu+2\alpha) - 2h - \tfrac{1}{2} < \mu \leqq -\operatorname{Re}(\nu+2\alpha) - 2h + \tfrac{3}{2} \quad \text{if } h \in \mathbb{N}, h \neq 0;\end{aligned}$$

if $\alpha \in \mathbb{N}$ we may omit (6.6), and now (6.1) reduces to (6.4) with $\alpha = n$; if $-\alpha \in \mathbb{N}$, $h \geqq -\alpha > 0$, then we may replace (6.6) by $\mu \leqq \frac{3}{2} - \operatorname{Re}\nu$, and use (5.24).
(III) on $T_2(-\nu-\frac{1}{2}, \infty)$ for arbitrary ν and $h = 0$.

Next we consider the cut Hankel transform (cf. [1]). Suppose $p \in \mathbb{N}$, $p \neq 0$, $\lambda < -\frac{1}{2}$,

$$(6.7)^p \quad -\operatorname{Re}\nu - \tfrac{1}{2} - 2p \leqq \lambda < \mu \leqq -\operatorname{Re}\nu + \tfrac{3}{2} - 2p.$$

If $\phi \in T(1-\mu, 1-\lambda)$, we define

$$H_{\nu,p}\phi(x) = \int_0^\infty (xt)^{1/2}\left\{J_\nu(xt) - \sum_{j=0}^{p-1} \frac{(-1)^j(\frac{1}{2}xt)^{\nu+2j}}{j!\Gamma(\nu+j+1)}\right\}\phi(t)\,dt. \tag{6.8}$$

Now

$$H_{\nu,p}\phi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{4} + \frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu + \frac{3}{4} - \frac{1}{2}s)} 2^{s-(1/2)}\Phi(1-s)x^{-s}\,ds,$$

if $\Phi = \mathscr{M}\phi$, $x > 0$, $\lambda < c < -\frac{1}{2}$, $c < \mu$. So again,

$$(\mathscr{M}H_{\nu,p}\phi)(s) = \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{4} + \frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu + \frac{3}{4} - \frac{1}{2}s)} 2^{s-(1/2)}\Phi(1-s). \tag{6.9}$$

Analogous to (6.1) we have

$$H_{\nu,p} = 2^\alpha x^{-\alpha} H_{\nu+\alpha,p} K_{2,h}^{(1/2)\nu+(1/4)+(1/2)\alpha,-\alpha} x^{-\alpha}, \tag{6.10}$$

if $\lambda < \operatorname{Re}\alpha - \frac{1}{2}$ and $(6.7)^p$ and (6.3) are satisfied. In particular,

$$H_{\nu,p} = (-2)^n x^{-n} H_{\nu+n,p} x^{\nu+n+(1/2)} \left(\frac{d}{dx^2}\right)^n x^{-\nu-(1/2)}, \tag{6.11}$$

if $\lambda < n - \frac{1}{2}$ and $(6.7)^p$ holds. By means of (6.11) with a suitable value of $n \in \mathbb{N}$ we may extend the definition of $H_{\nu,p}$ on $T(1-\mu, 1-\lambda)$ if $(6.7)^p$ holds. Then (6.10) is valid if (6.3) and $(6.7)^p$ are satisfied, and also if $-\alpha \in \mathbb{N}$, $h \geqq -\alpha$, $\lambda \geqq \operatorname{Re}\nu - \frac{1}{2}$.

Since $H_{\nu,0} = H_\nu$ we conclude that $H_{\nu,p}$ is a continuous operator from $T(1-\mu, 1-\lambda)$ into $T(\lambda, \mu)$ if $p \in \mathbb{N}$ and $(6.7)^p$ holds where $(6.7)^0$ is given by

$$-\operatorname{Re}\nu - \tfrac{1}{2} \leqq \lambda < \mu. \tag{6.7^0}$$

It follows that $H_{\nu,q}$, $q \in \mathbb{N}$, is a continuous operator from $T(\lambda, \mu)$ into $T(1-\mu, 1-\lambda)$ if

$$\begin{aligned} &\lambda < \mu \leqq \operatorname{Re}\nu + \tfrac{3}{2} \quad \text{in case } q = 0, \\ &\operatorname{Re}\nu + 2q - \tfrac{1}{2} \leqq \lambda < \mu \leqq \operatorname{Re}\nu + 2q + \tfrac{3}{2} \quad \text{in case } q > 0. \end{aligned} \tag{6.12^q}$$

Using Mellin transforms, (6.9) and the Theorems 9 and 10 we may prove an extension of Theorem 7.

THEOREM 7[a]. *The Hankel transform $H_{\nu,p}$ defined by* (6.8) *and* (6.11) *is a continuous operator from $T(1-\mu, 1-\lambda)$ into $T(\lambda, \mu)$ if $p \in \mathbb{N}$ and $(6.7)^p$ holds. It is an isomorphism between these spaces if moreover $(6.12)^q$ is satisfied for some $q \in \mathbb{N}$. Then*

$$(H_{\nu,p})^{-1} = H_{\nu,q}. \tag{6.13}$$

Furthermore, H_ν is an involutory automorphism on $T_2(-\nu - \frac{1}{2}, \infty)$ for arbitrary ν.

The following relations hold whenever the operators involved make sense: (6.1), (6.10), (6.4), (6.11),

$$H_{\nu,p} = 2^\alpha x^{-\alpha} I_{2,h}^{(1/2)\nu-(1/4)+(1/2)\alpha,-\alpha} H_{\nu+\alpha,p} x^{-\alpha}, \tag{6.14}$$

$$H_{\nu,p} = 2^{-\alpha} x^\alpha K_{2,p}^{(1/2)\nu+(1/4)-(1/2)\alpha,\alpha} H_{\nu+\alpha,h} x^\alpha, \tag{6.15}$$

$$H_{\nu,p} = 2^{-\alpha} x^\alpha H_{\nu+\alpha,h} I_{2,p}^{(1/2)\nu-(1/4)-(1/2)\alpha,\alpha} x^\alpha. \tag{6.16}$$

In particular, (6.1) *holds in the cases* (I), (II), (III) *mentioned above. Moreover,* (6.14) *with $p = 0$ holds in case* (I). *Formulas* (6.10) *and* (6.14) *hold on $T(1-\mu, 1-\lambda)$ if* (6.3) *and $(6.7)^p$ are satisfied; if $\alpha \in \mathbb{N}$, we may omit* (6.3) *(then we may use* (5.22) *and* (5.11)), *and if $-\alpha \in \mathbb{N}$, $0 < -\alpha \leqq h$ we may replace* (6.3) *by $\lambda \geqq \operatorname{Re}\nu - \frac{1}{2}$. In the last case we may transform* (6.10) *and* (6.14) *by means of* (5.24) *and* (5.25).

The relations (6.15) *and* (6.16) *hold on $T(1-\mu, 1-\lambda)$ if $(6.7)^p$ and*

$$\begin{aligned} &-\operatorname{Re}(\nu + 2\alpha) - \tfrac{1}{2} \leqq \lambda < \mu \quad \textit{in case } h = 0, \\ &-\operatorname{Re}(\nu + 2\alpha) - 2h - \tfrac{1}{2} \leqq \lambda < \mu \leqq -\operatorname{Re}(\nu - 2\alpha) - 2h + \tfrac{3}{2} \\ &\qquad\qquad\qquad\qquad \textit{in case } h \in \mathbb{N},\ h \neq 0. \end{aligned} \tag{6.17^h}$$

Furthermore (6.15) *with $p = h = 0$ is also valid on $T_2(-\nu - \frac{1}{2}, \mu)$ if*

$$-\operatorname{Re}\nu - \tfrac{1}{2} < \mu \leqq \operatorname{Re}\nu + \tfrac{3}{2} + \min(0, 2\operatorname{Re}\alpha), \tag{6.18}$$

whereas it holds on $T_2(-\nu - \frac{1}{2}, \infty)$ for arbitrary ν.

Now we consider the adjoint operator $(H_{\nu,p})'$. We may simplify the notation in view of the Parseval relation

$$\int_0^\infty (H_{\nu,p}\phi)(x)\psi(x)\,dx = \int_0^\infty \phi(x)(H_{\nu,p}\psi)(x)\,dx, \tag{6.19}$$

which holds for example if $\phi, \psi \in T(1-\mu, 1-\lambda)$ and $(6.7)^p$ is satisfied (this may be proved using Mellin transforms). Therefore we may denote $(H_{\nu,p})'$ on $T'(\lambda, \mu)$ by $H_{\nu,p}$.

From Theorems 7 and 7^a we now deduce the following.

THEOREM 13. *The Hankel transform $H_{\nu,p}$ is a continuous operator from $T'(\lambda, \mu)$ into $T'(1-\mu, 1-\lambda)$ if $(6.7)^p$ holds. The operator H_ν is continuous from $T'(1-\mu, \infty)$ into $T'_2(-\nu-\frac{1}{2}-2h, \mu)$ if*

$$-\operatorname{Re}\nu - \tfrac{1}{2} - 2h < \mu \leqq \operatorname{Re}\nu + \tfrac{3}{2}, \qquad h \in \mathbb{N}. \tag{6.20}$$

This operator is an involutory isomorphism from $T'(1-\mu, \infty)$ onto $T'_2(-\nu-\frac{1}{2}, \mu)$ if (6.18) *holds, and an involutory automorphism on $T'_2(-\nu-\frac{1}{2}, \infty)$ for arbitrary ν. The operator $H_{\nu,p}$ is an isomorphism from $T'(\lambda, \mu)$ onto $T'(1-\mu, 1-\lambda)$ satisfying* (6.13) *if $(6.7)^p$ and $(6.12)^q$ are satisfied.*

The relation (6.14) *with $p = 0$ holds in the following cases*:

(i) *on $T'(\lambda, \mu)$ if $(6.7)^0$ and* (6.3) *are satisfied. If $\alpha \in \mathbb{N}$ we may omit* (6.3) *and use* (5.11). *If $-\alpha \in \mathbb{N}$, $0 < -\alpha \leqq h$, we may replace* (6.3) *by $\lambda \geqq \operatorname{Re}\nu - \frac{1}{2}$ and use* (5.25).

(ii) *on $T'(1-\mu, \infty)$ if* (6.5) *with some $g \in \mathbb{N}$ and $(6.6)^h$ are satisfied. If $\alpha \in \mathbb{N}$ we may omit* (6.6) *and use* (5.11). *If $-\alpha \in \mathbb{N}$, $0 < -\alpha \leqq h$, we may replace* (6.6) *by $\mu \leqq \frac{3}{2} - \operatorname{Re}\nu$ and use* (5.25).

(iii) *on $T'_2(-\nu-\frac{1}{2}, \infty)$ for arbitrary ν and $h = 0$.*

Furthermore, (6.1) *is valid on $T'(\lambda, \mu)$ if $\lambda \geqq -\operatorname{Re}\nu - \frac{1}{2}$ and* (6.3) *holds. If $\alpha \in \mathbb{N}$ we may omit* (6.3) *and then* (6.1) *reduces to* (6.4). *If $-\alpha \in \mathbb{N}$, $0 < -\alpha \leqq h$, we may replace* (6.3) *by $\lambda \geqq \operatorname{Re}\nu - \frac{1}{2}$ and use* (5.24). *The relations* (6.10) *and* (6.14) *hold on $T'(\lambda, \mu)$, if* (6.3) *and $(6.7)^p$ are satisfied; if $\alpha \in \mathbb{N}$, we may omit* (6.3) (*then we may use* (5.22) *and* (5.11)); *if $-\alpha \in \mathbb{N}$, $0 < -\alpha \leqq h$, we may replace* (6.3) *by $\lambda \geqq \operatorname{Re}\nu - \frac{1}{2}$. In the last case we may use* (5.24) *and* (5.25). *The relations* (6.15) *and* (6.16) *hold on $T'(\lambda, \mu)$ if $(6.7)^p$ and $(6.17)^h$ are satisfied. Finally,* (6.16) *with $p = h = 0$ is valid on $T'(1-\mu, \infty)$ if* (6.20) *holds, whereas it holds on $T'_2(-\nu-\frac{1}{2}, \infty)$ for arbitrary ν.*

7. A dual integral equation involving Hankel functions. Let $c_1, c_2, \nu \in \mathbb{C}$, $a > 0$, $\lambda < \mu$, $p \in \mathbb{N}$ and

$$g_1 \in T'([0, a], 1-\mu-\operatorname{Re}c_1),\ g_2 \in T'([a, \infty), 1-\lambda-\operatorname{Re}c_2). \tag{7.1}$$

Consider the following dual integral equation:

$$H_\nu x^{c_1} f = g_1, \quad H_{\nu,p} x^{c_2} f = g_2, \tag{7.2}$$

where the left-hand sides have to be interpreted as elements of $T'([0, a], 1-\mu-\operatorname{Re}c_1)$ and $T'([a, \infty), 1-\lambda-\operatorname{Re}c_2)$ respectively. This is a distributional analogue of a dual integral equation considered by Titchmarsh [13], Erdélyi and Sneddon [5] and others. Erdélyi and Sneddon use fractional integrals in the solution of their equation. We extend their method to the solution of (7.2). Thus we obtain all solutions $f \in T'(\lambda, \mu)$ of (7.2) if the following conditions are satisfied: $h \in \mathbb{N}$,

$$(7.3) \qquad -\mathrm{Re}(\nu + c_1) - \tfrac{1}{2} \leqq \lambda < \mu \leqq \mathrm{Re}(\nu - c_1) + \tfrac{3}{2},$$

$$(7.4)^h \qquad \begin{array}{l} \mu \leqq \mathrm{Re}(\nu - c_2) + \tfrac{3}{2}, \quad \text{if } h = 0, \\ \mathrm{Re}(\nu - c_2) + 2h - \tfrac{1}{2} \leqq \lambda < \mu \leqq \mathrm{Re}(\nu - c_2) + 2h + \tfrac{3}{2} \quad \text{if } h > 0, \end{array}$$

$$(7.5)^p \qquad \begin{array}{l} -\mathrm{Re}(\nu + c_2) - \tfrac{1}{2} \leqq \lambda \quad \text{if } p = 0, \\ -\mathrm{Re}(\nu + c_2) - 2p - \tfrac{1}{2} \leqq \lambda < \mu \leqq -\mathrm{Re}(\nu + c_2) - 2p + \tfrac{3}{2} \quad \text{if } p > 0. \end{array}$$

First we assume that a solution f of (7.2) exists. Let $c = \frac{1}{2}(c_1 - c_2)$. We apply Theorem 13, formula (6.14) with ν and α replaced by $\nu + c$ and $-c$, $h = p = 0$. Then we get

$$(7.6) \qquad 2^{-c}x^c I_2^{(1/2)\nu-(1/4),c} H_\nu x^{c_1} f = H_{\nu+c} x^{(1/2)(c_1+c_2)} f.$$

The conditions $(6.7)^0$ and (6.3) for formula (6.14) are satisfied because of (7.3).

Next we apply Theorem 13, formula (6.15) with ν, α, h and p replaced by $\nu + c$, $-c$, p and 0. Then we obtain

$$(7.7) \qquad 2^c x^{-c} K_2^{(1/2)\nu+(1/4)+c,-c} H_{\nu,p} x^{c_2} f = H_{\nu+c} x^{(1/2)(c_1+c_2)} f.$$

The conditions $(6.7)^0$ and $(6.17)^p$ for formula (6.15) are satisfied because of (7.3) and $(7.5)^p$.

Now let

$$(7.8) \qquad F = H_{\nu+c} x^{(1/2)(c_1+c_2)} f.$$

Then (7.2), (7.6) and (7.7) imply

$$(7.9) \qquad \begin{array}{ll} F = 2^{-c}x^c I_2^{(1/2)\nu-(1/4),c} g_1 & \text{in } T'([0,a], 1 - \mu - \tfrac{1}{2}\mathrm{Re}(c_1 + c_2)), \\ F = 2^c x^{-c} K_2^{(1/2)\nu+(1/4)+c,-c} g_2 & \text{in } T'([a,\infty), 1 - \lambda - \tfrac{1}{2}\mathrm{Re}(c_1 + c_2)), \end{array}$$

where the right-hand sides exist as elements of these spaces because of Theorems 11 and 12. Hence we know F completely if we can determine F on $\mathscr{D}(\frac{1}{2}a, \frac{3}{2}a)$. However, by (7.9) we know the restriction of F on $\mathscr{D}(\frac{1}{2}a, a)$ and on $\mathscr{D}(a, \frac{3}{2}a)$. Therefore we may write F as the generalized derivative of some order q of regular distributions on these spaces. Consequently F may be extended to a continuous linear functional F_0 on the completions C_1 of $\mathscr{D}(\frac{1}{2}a, a)$ and C_2 of $\mathscr{D}(a, \frac{3}{2}a)$ in $C^q[\frac{1}{2}a, \frac{3}{2}a]$.

Let $\phi \in \mathscr{D}(\frac{1}{2}a, \frac{3}{2}a)$ and $\chi \in \mathscr{D}(\frac{1}{4}a, 2a)$, $\chi(x) = 1$ if $\frac{1}{2}a \leqq x \leqq \frac{3}{2}a$. Then we may write

$$(7.10) \qquad \phi(x) = \sum_{j=0}^{q} \frac{1}{j!} \phi^{(j)}(a)(x-a)^j \chi(x) + \phi_1(x) + \phi_2(x),$$

where $\phi_1 \in C_1$, $\phi_2 \in C_2$. Now (F_0, ϕ_1) and (F_0, ϕ_2) may be calculated using (7.9). If $\phi \in \mathscr{D}(\frac{1}{2}a, \frac{3}{2}a)$, then we define

$$(7.11) \qquad (F_0, \phi) = (F_0, \phi_1) + (F_0, \phi_2).$$

Now $(F, \phi) = (F_0, \phi)$ if $\phi \in \mathscr{D}(\frac{1}{2}a, \frac{3}{2}a)$ and ϕ vanishes in a neighborhood of a. So $F - F_0 \in \mathscr{D}'(\frac{1}{2}a, \frac{3}{2}a)$ is concentrated in a. Therefore $F - F_0$ is a linear combination of the delta-functional and a finite number of its derivatives concentrated in a. Apart from these terms now F is uniquely determined on $\mathscr{D}(\frac{1}{2}a, \frac{3}{2}a)$, and consequently as an element of $T'(1 - \mu - \frac{1}{2}\operatorname{Re}(c_1 + c_2),\ 1 - \lambda - \frac{1}{2}\operatorname{Re}(c_1 + c_2))$ by means of g_1 and g_2.

From Theorem 13 and (7.8) we now deduce

$$f = x^{-(1/2)(c_1+c_2)} H_{v+c,h} F \in T'(\lambda, \mu). \tag{7.12}$$

So if a solution of (7.2) exists in $T'(\lambda, \mu)$ it is given by (7.12). Conversely, it is easy to check that the distributions f constructed above from g_1 and g_2 by means of (7.9) and (7.12) are solutions of (7.2). Extensions to other dual integral equations as in [2] may be given in an analogous way.

8. Dual integral equations involving *H*-functions. Before considering such integral equations we first extend the definition of the operator A of § 4, Example 3. In what follows we use the notation of that example and

$$I(\eta, \alpha, m) = I_m^{\eta,\alpha}, \quad K(\eta, \alpha, m) = K_m^{\eta,\alpha}. \tag{8.1}$$

Suppose (4.7) and (4.10) are satisfied. If $n < j \leqq p$,

$$\mu \leqq \operatorname{Re} \tilde{\alpha}_j / a_j \tag{8.2}$$

and (4.8), and (4.8) with α_j replaced by $\tilde{\alpha}_j$ are satisfied, then

$$A = \tilde{A} K(\tilde{\alpha}_j - a_j, \alpha_j - \tilde{\alpha}_j, a_j^{-1}) \quad \text{on } T(1 - \mu, 1 - \lambda), \tag{8.3}$$

where $\tilde{A}$ is defined by (4.11) with α_j replaced by $\tilde{\alpha}_j$. This may be shown using Mellin transforms, (5.16) and (4.9).

If $\tilde{\alpha}_j - \alpha_j \in \mathbb{N}$, we may omit (8.2) and use (5.22). Choosing $\tilde{\alpha}_j$ sufficiently large, the right-hand side of (8.3) exists on $T(1 - \mu, 1 - \lambda)$ even if (4.8) does not hold. Hence we may use (8.3) to define A in case only (4.7) and (4.10) are satisfied. It is a continuous operator of $T(1 - \mu, 1 - \lambda)$ into $T(\lambda, \mu)$ satisfying (8.3) if (4.7) and (4.10) are fulfilled.

In the same way we have

$$A = A^* I(b_h - \beta_h^*, \beta_h^* - \beta_h, b_h^{-1}) \quad \text{on } T(1 - \mu, 1 - \lambda), \tag{8.4}$$

if $m < h \leqq q$, A^* denotes the operator A with β_h replaced by β_h^*, (4.7), (4.10), (4.8) and (4.8) with β_h replaced by β_h^* are satisfied and

$$(\operatorname{Re} \beta_h^* - 1)/b_h \leqq \lambda. \tag{8.5}$$

If $\beta_h - \beta_h^* \in \mathbb{N}$ we may omit (8.5) and use (5.11). If $n = p$ and (4.7) holds, then $m < q$. If in this case (4.10) is fulfilled but (4.8) is not satisfied, we may use (8.4) with a suitably chosen β_h^* as definition of A. Hence A is defined as a continuous operator of $T(1 - \mu, 1 - \lambda)$ into $T(\lambda, \mu)$ if (4.10) and either (4.6) or (4.7) hold. The relations (8.3) and (8.4) are valid on $T(1 - \mu, 1 - \lambda)$ if (4.10) and either (4.6) or (4.7) are satisfied and in case of (8.3) also (8.2), in case of (8.4) also (8.5).

In case (4.7), (4.10) and (4.15) are fulfilled, the inverse of A exists as a continuous operator from $T(\lambda, \mu)$ into $T(1 - \mu, 1 - \lambda)$ and it is given by (4.12) with (4.13) and (4.14).

Now we consider the adjoint A' of A. It is a continuous operator from $T'(\lambda, \mu)$ into $T'(1-\mu, 1-\lambda)$ if (4.10) and either (4.6) or (4.7) hold. Using Parseval's formula we may show that

$$(8.6)\qquad (A\phi, \psi) = (\phi, A\psi) \quad \text{if } \psi \in T(1-\mu, 1-\lambda), \quad \phi \in T(\lambda_1, \mu_1), \quad \lambda + \lambda_1 < 1 < \mu + \mu_1 .$$

Therefore we denote A' by A on $T'(\lambda, \mu)$. The dual relations of (8.3) and (8.4) are

$$(8.7)\qquad A = I(\tilde{\alpha}_j - 1, \alpha_j - \tilde{\alpha}_j, a_j^{-1})\tilde{A}$$

and

$$(8.8)\qquad A = K(1 - \beta_h^*, \beta_h^* - \beta_h, b_h^{-1})A^*$$

which hold on $T'(\lambda, \mu)$ and on $T(1-\mu, 1-\lambda)$ if either (4.6) or (4.7) holds, (4.10) is satisfied, whereas in case of (8.7) we assume $n < j \leqq p$ and (8.2) and in case of (8.8) we assume $m < h \leqq q$ and (8.5). Also (8.3) and (8.4) are valid on $T'(\lambda, \mu)$ with corresponding conditions.

Let B be the operator which arises from A by replacing α_j and β_h by γ_j and δ_h for $j = 1, \cdots, p$ and $h = 1, \cdots, q$, where

$$(8.9)\qquad \operatorname{Re}\frac{\gamma_j - 1}{a_j} \leqq \lambda < \mu \leqq \operatorname{Re}\frac{\delta_h}{b_h}, \qquad j = 1, \cdots, n; \quad h = 1, \cdots, m.$$

Now we consider the dual integral equation,

$$(8.10)\qquad Af = g_1 \quad \text{in } T'([0, a], 1-\mu), \qquad Bf = g_2 \quad \text{in } T'([a, \infty), 1-\lambda),$$

where $a > 0$ and g_1 and g_2 are given elements in these spaces and (4.7) holds. Integral equations of this type for ordinary functions have been treated by Fox [6] and Saxena [12]. We use here a construction of solutions which is analogous to their formal solution.

Let C be the operator A with α_j replaced by $\gamma_j (j = n+1, \cdots, p)$ and β_h replaced by δ_h $(h = 1, \cdots, m)$. Define

$$(8.11)\qquad \begin{aligned} P_1 &= \left\{\prod_{j=n+1}^{p} I(\alpha_j - 1, \gamma_j - \alpha_j, a_j^{-1})\right\}\left\{\prod_{h=1}^{m} I(\delta_h - 1, \beta_h - \delta_h, b_h^{-1})\right\}, \\ P_2 &= \left\{\prod_{j=1}^{n} K(1 - \alpha_j, \alpha_j - \gamma_j, a_j^{-1})\right\}\left\{\prod_{h=m+1}^{q} K(1 - \delta_h, \delta_h - \beta_h, b_h^{-1})\right\}. \end{aligned}$$

For the existence of these operators on $T'(1-\mu, 1-\lambda)$ we assume (cf. Theorems 11 and 12) besides (4.10) and (8.9) also

$$(8.12)\qquad \operatorname{Re}\frac{\delta_h - 1}{b_h} \leqq \lambda < \mu \leqq \operatorname{Re}\frac{\alpha_j}{a_j}, \qquad h = m+1, \cdots, q; \quad j = n+1, \cdots, p.$$

Then

$$(8.13)\qquad P_1 Af = Cf = P_2 Bf.$$

From this, (8.10) and Theorems 11 and 12 it follows that

$$(8.14)\qquad Cf = P_1 g_1 \quad \text{in } T'([0, a], 1-\mu), \qquad Cf = P_2 g_2 \quad \text{in } T'([a, \infty), 1-\lambda).$$

As in § 7 we may now determine $F = Cf$ in $T'(1 - \mu, 1 - \lambda)$ from (8.14) apart from a linear combination of the delta-functional with center a and a finite number of its derivatives. From F we now obtain the solution f of (8.10) by means of

$$f = C_0 F,$$

where C_0 is the adjoint of the operator defined by

$$\psi \to \int_0^\infty H_{p,q}^{q-m,p-n}\left(xt \middle| \begin{matrix} \tilde{\mathbf{a}}, \boldsymbol{\alpha}^* \\ \tilde{\mathbf{b}}, \boldsymbol{\beta}^* \end{matrix}\right) \psi(t)\, dt$$

with $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ given by (4.13) and

$$\boldsymbol{\alpha}^* = (1 + a_{n+1} - \gamma_{n+1}, \cdots, 1 + a_p - \gamma_p, 1 + a_1 - \alpha_1, \cdots, 1 + a_n - \alpha_n),$$
$$\boldsymbol{\beta}^* = (1 + b_{m+1} - \beta_{m+1}, \cdots, 1 + b_q - \beta_q, 1 + b_1 - \delta_1, \cdots, 1 + b_m - \delta_m).$$

This solution exists if (4.7), (4.10), (8.9), (8.12) and

$$\operatorname{Re} \frac{\beta_h - 1}{b_h} \leqq \lambda < \mu \leqq \operatorname{Re} \frac{\gamma_j}{a_j}, \qquad h = m + 1, \cdots, q; j = n + 1, \cdots, p$$

are satisfied.

9. A convolution map involving a hypergeometric function. Finally we consider another special case of the product convolution (5.1), viz. a hypergeometric integral transform considered among others by Love [11a] and [11b]. Let $\operatorname{Re} c > 1$,

$$-\operatorname{Re} a \leqq \lambda, \quad -\operatorname{Re} b \leqq \lambda < \mu. \tag{9.1}$$

Then if $\phi \in T(\lambda, \mu)$, we define

$$(A\phi)(x) = \frac{1}{\Gamma(c)} \int_x^\infty \left(1 - \frac{x}{t}\right)^{c-1} F\left(a, b; c; 1 - \frac{t}{x}\right) \phi(t) \frac{dt}{t}. \tag{9.2}$$

Now we have the special case of Theorem 8 where

$$\begin{aligned} k(x) &= \frac{1}{\Gamma(c)}(1 - x)^{c-1} F\left(a, b; c; 1 - \frac{1}{x}\right) \quad \text{if } 0 < x \leqq 1, \\ k(x) &= 0 \quad \text{if } x > 1. \end{aligned} \tag{9.3}$$

The Mellin transform $K(s)$ of $k(x)$ is given by

$$K(s) = \frac{\Gamma(a + s)\Gamma(b + s)}{\Gamma(c + s)\Gamma(a + b + s)}. \tag{9.4}$$

This may be shown using Euler's integral for the hypergeometric function or Barnes' integral representation for this function and Barnes' lemma.

The condition $\operatorname{Re} c > 1$ may be removed as in Remark 4. However, we may also use a modification of the method in Remark 4. If $\operatorname{Re} c + n > 0$, we define

$$(A\phi)(x) = \frac{(-1)^n x^{c+n}}{\Gamma(c + n)} \frac{d^n}{dx^n} x^{-c} \int_x^\infty \left(1 - \frac{x}{t}\right)^{c+n-1} F\left(a, b; c + n; 1 - \frac{t}{x}\right) \phi(t) \frac{dt}{t}. \tag{9.5}$$

This is consistent with the first definition in (9.2) since (9.5) implies $\mathscr{M}(A\phi)(s) = K(s)(\mathscr{M}\phi)(s)$. Hence A defines a continuous mapping of $T(\lambda, \mu)$ into itself if (9.1) holds.

From Theorem 10, formula (5.16), and (9.4) we deduce

$$A = K_1^{a,c-a} K_1^{b,a} \quad \text{on } T(\lambda, \mu). \tag{9.6}$$

This relation may also be proved directly using the definition (9.2) and Euler's integral for the hypergeometric function.

Now we consider the inverse of A, if it exists. First we assume

$$-\operatorname{Re} c \leqq \lambda, \quad -\operatorname{Re}(a+b) \leqq \lambda. \tag{9.7}$$

Then it follows from (9.6) and Theorem 10 that A is an automorphism on $T(\lambda, \mu)$ with

$$A^{-1} = K_1^{a+b,-a} K_1^{c,a-c}. \tag{9.8}$$

From (9.4) and Theorem 8 we may also deduce that A is an automorphism, and if moreover $\operatorname{Re} c < -1$,

$$A^{-1}\phi(x) = \int_0^\infty h\left(\frac{x}{t}\right)\phi(t)\frac{dt}{t}, \tag{9.9}$$

where

$$h(x) = \left\{\mathscr{M}^{-1}\frac{\Gamma(c+s)\Gamma(a+b+s)}{\Gamma(a+s)\Gamma(b+s)}\right\}(x). \tag{9.10}$$

Using residue calculus we obtain

$$\begin{aligned} &h(x) = 0 \quad \text{if } x > 1, \\ &h(x) = \frac{1}{\Gamma(-c)} x^c (1-x)^{-c-1} F(-a, -b; -c; 1-x) \quad \text{if } 0 < x < 1. \end{aligned} \tag{9.11}$$

Hence if (9.7) holds and $\operatorname{Re} c < 0$, then the inverse B of A on $T(\lambda, \mu)$ is given by

$$B\phi(x) = \frac{1}{\Gamma(-c)} x^c \int_x^\infty (t-x)^{-c-1} F\left(-a, -b; -c; 1 - \frac{x}{t}\right)\phi(t)\, dt. \tag{9.12}$$

If $\operatorname{Re} c < m$, $m \in \mathbb{N}$, we easily see using (9.10) that

$$A^{-1} = (-1)^m x^c \frac{d^m}{dx^m} x^{m-c} B_m \quad \text{on } T(\lambda, \mu), \tag{9.13}$$

where B_m is defined by (9.12) with c replaced by $c - m$ and B by B_m.

Now we consider cases where (9.7) need not be fulfilled. Then we suppose that λ and μ satisfy the following condition with p and $q \in \mathbb{N}$:

$$\begin{aligned} &-p - \operatorname{Re} c \leqq \lambda < \mu \leqq 1 - p - \operatorname{Re} c, \\ &-q - \operatorname{Re}(a+b) \leqq \lambda < \mu \leqq 1 - q - \operatorname{Re}(a+b). \end{aligned} \tag{9.14}$$

If $p = 0$ or $q = 0$ we may omit the expression "$\leqq 1 - p - \operatorname{Re} c$" or "$\leqq 1 - q - \operatorname{Re}(a+b)$" respectively in this condition. Now (9.6) and Theorem 10 imply that A is an automorphism on $T(\lambda, \mu)$ with

$$A^{-1} = K_{1,q}^{a+b,-a} K_{1,p}^{c,a-c}. \tag{9.15}$$

If $\operatorname{Re} c < -1$ and (9.14) holds, we deduce (9.9) with (9.10) from Theorem 8 and (9.4). Using residue calculus we get

$$\begin{aligned} h(x) &= -P(x) \quad \text{if } x > 1, \\ h(x) &= \frac{1}{\Gamma(-c)} x^c (1-x)^{-c-1} F(-a, -b; -c; 1-x) - P(x) \end{aligned} \tag{9.16}$$

if $0 < x < 1$, where

$$\begin{aligned} P(x) = {} & \frac{\Gamma(a+b-c)}{\Gamma(a-c)\Gamma(b-c)} \sum_{j=0}^{p-1} \frac{(1+c-a)_j(1+c-b)_j}{j!(1+c-a-b)_j} x^{c+j} \\ & + \frac{\Gamma(c-a-b)}{\Gamma(-b)\Gamma(-a)} \sum_{j=0}^{q-1} \frac{(1+a)_j(1+b)_j}{j!(1+a+b-c)_j} x^{a+b+j}. \end{aligned} \tag{9.17}$$

Hence if (9.14) is fulfilled and $\operatorname{Re} c < 0$, the inverse B of A on $T(\lambda, \mu)$ is given by

$$\begin{aligned} B\phi(x) = {} & \int_x^\infty \left\{ \frac{1}{\Gamma(-c)} \frac{t}{x} \left(\frac{t}{x} - 1 \right)^{-c-1} F\left(-a, -b; -c; 1 - \frac{x}{t} \right) - P\left(\frac{x}{t} \right) \right\} \phi(t) \frac{dt}{t} \\ & - \int_0^x P\left(\frac{x}{t} \right) \phi(t) \frac{dt}{t}. \end{aligned} \tag{9.18}$$

If $\operatorname{Re} c < m$, $m \in \mathbb{N}$, we have (9.13) where B_m is defined by (9.18) and (9.17) with B, c and p replaced by B_m, $c - m$ and $p + m$.

Finally, we consider the adjoint A' of A on $T'(\lambda, \mu)$. Assuming (9.1),

$$\phi \in T(\lambda, \mu), \quad f \in T(\lambda', \mu') \subset T'(\lambda, \mu) \quad (\text{hence } \lambda + \lambda' < 1 < \mu + \mu'),$$

we have according to Parseval's formula,

$$\begin{aligned} \int_0^\infty f(x) A\phi(x)\, dx &= \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} K(s)\Phi(s)F(1-s)\, ds \\ &= \frac{1}{2\pi i} \int_{1-\nu-i\infty}^{1-\nu+i\infty} K(1-s)F(s)\Phi(1-s)\, ds = \int_0^\infty (\tilde{A} f)(x)\phi(x)\, dx, \end{aligned}$$

where $\lambda < \nu < \mu$, $1 - \mu' < \nu < 1 - \lambda'$, $F = \mathscr{M} f$, $\Phi = \mathscr{M}\phi$,

$$(\mathscr{M}\tilde{A} f) = \frac{\Gamma(a+1-s)\Gamma(b+1-s)}{\Gamma(c+1-s)\Gamma(a+b+1-s)} F(s).$$

Hence $A' = \tilde{A}$ on $T(\lambda', \mu')$ where (cf. (9.3) and (9.4))

$$\tilde{A} f(x) = \frac{1}{x\Gamma(c)} \int_0^x \left(1 - \frac{t}{x} \right)^{c-1} F\left(a, b; c; 1 - \frac{x}{t} \right) f(t)\, dt, \tag{9.19}$$

if $f \in T(\lambda', \mu')$, $\lambda' < 1 + \min(\operatorname{Re} a, \operatorname{Re} b)$, $\operatorname{Re} c > 0$,

$$\tilde{A} f(x) = \frac{x^{-c}}{\Gamma(c+n)} \frac{d^n}{dx^n} x^{c+n-1} \int_0^x \left(1 - \frac{t}{x} \right)^{c+n-1} F\left(a, b; c+n; 1 - \frac{x}{t} \right) f(t)\, dt \tag{9.20}$$

if $f \in T(\lambda', \mu')$, $\lambda' < 1 + \min(\operatorname{Re} a, \operatorname{Re} b)$, $\operatorname{Re} c + n > 0$, $n \in \mathbb{N}$. If (9.1) holds, $\tilde{A}$ is a continuous operator of $T(1 - \mu, 1 - \lambda)$ into itself.

Now A' is a continuous mapping of $T'(\lambda, \mu)$ into itself if (9.1) is satisfied, and according to Theorem 12 and (9.8):

$$A' = I_1^{b,a} I_1^{a,c-a}. \tag{9.21}$$

If moreover (9.14) holds, then A' is an automorphism on $T'(\lambda, \mu)$ with

$$(A')^{-1} = I_{1,p}^{c,a-c} I_{1,q}^{a+b,-a}. \tag{9.22}$$

Analogous to $\tilde{A}$ we define an operator $\tilde{B}$ which plays the same role with respect to $(A')^{-1}$ as $\tilde{A}$ plays with respect to A'.

Suppose $\lambda' < \mu'$, $p, q \in \mathbb{N}$,

$$\lambda' < 1 + \min\{p + \operatorname{Re} c, q + \operatorname{Re}(a + b)\},$$

$$p + \operatorname{Re} c < \mu' \quad \text{if } p \neq 0, \quad q + \operatorname{Re}(a + b) < \mu' \quad \text{if } q \neq 0.$$

Let P be defined by (9.17) and $g \in T(\lambda', \mu')$. If $\operatorname{Re} c < 0$, then

$$\begin{aligned} \tilde{B}g(x) = \int_0^x \left\{ \frac{1}{\Gamma(-c)} \left(\frac{x}{t} - 1 \right)^{-c-1} F\left(-a, -b; -c; 1 - \frac{t}{x} \right) - \frac{t}{x} P\left(\frac{t}{x} \right) \right\} g(t) \frac{dt}{t} \\ - \frac{1}{x} \int_x^\infty P\left(\frac{t}{x} \right) g(t)\, dt. \end{aligned} \tag{9.23}$$

If $\operatorname{Re} c < m$, $m \in \mathbb{N}$, we define

$$\tilde{B} = x^{m-c} \frac{d^m}{dx^m} x^c \tilde{B}_m, \tag{9.24}$$

where $\tilde{B}_m$ is defined by (9.23) and (9.17) with $\tilde{B}$, c and p replaced by $\tilde{B}_m$, $c - m$ and $p + m$. Then the operator $(A')^{-1}$ on $T'(\lambda, \mu)$ coincides with $\tilde{B}$ on $T(\lambda', \mu')$ if $\lambda + \lambda' < 1 < \mu + \mu'$, (9.1) and (9.14) are satisfied. Furthermore, $\tilde{B}$ is a continuous operator on $T(1 - \mu, 1 - \lambda)$ and it is the inverse of $\tilde{A}$ if (9.1) and (9.14) are satisfied.

It is obvious that instead of starting with the transformation A we could also start with B, $\tilde{A}$ or $\tilde{B}$ and apply an analogous reasoning as above. We obtain similar results by extending the definition of A and $\tilde{A}$ in the same way as the definition of B is extended from (9.12) and (9.18).

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ERROR BOUNDS FOR ASYMPTOTIC EXPANSIONS OF HANKEL TRANSFORMS*

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Abstract. An explicit expression is derived for the error term associated with the asymptotic expansions of the Hankel transform

$$I(x) = \int_0^\infty J_\nu(xt)q(t)\,dt,$$

where $J_\nu(t)$ is the Bessel function of the first kind and x is a large positive parameter. From the explicit expression, realistic error bounds are also obtained.

1. Introduction. Consider the Hankel transform of order ν defined by

$$I(x) = \int_0^\infty J_\nu(xt)q(t)\,dt, \tag{1.1}$$

where $J_\nu(t)$ is the Bessel function of the first kind and ν is a fixed complex number. The function $q(t)$ may be real or complex, and x is a positive parameter. Asymptotic expansions of $I(x)$ as $x \to \infty$ have been obtained recently by several authors, using different methods and assumptions; see Slonovskii [5], Handelsman and Lew [2], and Mackinnon [3].

The main purpose of the present paper is to supply an explicit expression for the error term associated with the expansion of $I(x)$ from which an error bound can readily be obtained. The conditions which we shall impose on $q(t)$ are weaker than those given in [2], [3] and [5], and the method employed here also differs considerably from the methods used in the papers mentioned above. Our approach is motivated by a recent article of Olver [4] on stationary phase approximations.

2. Preliminaries. Throughout the paper, we assume that the integral $I(x)$ in (1.1) exists uniformly for all large values of x and that $q(t)$ has the following properties:

(Q_1) $q^{(m)}(t)$ is continuous on $(0, \infty)$, where m is a nonnegative integer.

(Q_2) As $t \to 0+$,

$$q(t) \sim \sum_{s=0}^{\infty} q_s t^{s+\lambda-1}, \tag{2.1}$$

where $q_0 \neq 0$, $\operatorname{Re}(\nu + \lambda) > 0$ and $m \geqq \operatorname{Re}\lambda$. Moreover, the expansion in (2.1) is m-times differentiable.

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(Q_3) $t^{-1/2}q^{(j)}(t) \to 0$ as $t \to \infty$, $j = 0, 1, \cdots, m-1$; and

$$\int J_{\nu+m}(xt)q^{(m)}(t)\,dt \tag{2.2}$$

converges at $t = \infty$ uniformly for all sufficiently large values of x.

These conditions are similar to those adopted by Olver for stationary phase approximations in [4], and the following lemmas are analogues of Lemmas 1 and 2 in Olver's paper.

LEMMA 1. *For $x > 0$, η real and* $\operatorname{Re}(\mu + \alpha) > 0$,

$$\lim_{\eta\to 0}\int_0^\infty e^{-\eta^2t^2}J_\alpha(xt)t^{\mu-1}\,dt = \frac{\Gamma(\frac{1}{2}\mu + \frac{1}{2}\alpha)2^{\mu-1}}{\Gamma(\frac{1}{2}\alpha - \frac{1}{2}\mu + 1)x^\mu}. \tag{2.3}$$

Proof. The integral in (2.3) can be evaluated by means of the confluent hypergeometric function [1, p. 50] to be

$$\int_0^\infty e^{-\eta^2t^2}J_\alpha(xt)t^{\mu-1}\,dt = \frac{x^\alpha\Gamma(\frac{1}{2}\mu + \frac{1}{2}\alpha)\,e^{-x^2/(4\eta^2)}}{2^{\alpha+1}\eta^{\alpha+\mu}\Gamma(\alpha+1)}\,{}_1F_1(\tfrac{1}{2}\alpha - \tfrac{1}{2}\mu + 1; \alpha + 1; x^2/(4\eta^2)).$$

In view of the asymptotic formula

$${}_1F_1(a; c; z) \sim \frac{\Gamma(c)}{\Gamma(a)}e^z z^{a-c} \quad \text{as } z \to +\infty,$$

we immediately obtain (2.3).

LEMMA 2. *If $\varphi(t)$ is piecewise continuous on $(0, \infty)$ and $\int_0^\infty \varphi(t)\,dt$ converges, then $\int_0^\infty e^{-\eta^2t^2}\varphi(t)\,dt$ converges for every real number η and tends to $\int_0^\infty \varphi(t)\,dt$ as $\eta \to 0$.*

Proof. This follows from Lemma 2 in [4] by a simple substitution $t^2 = \tau$.

3. Main theorem. For each $n > 0$, set

$$q(t) = \sum_{s=0}^{n-1} q_s t^{s+\lambda-1} + \varphi_n(t), \qquad 0 < t < \infty. \tag{3.1}$$

Then for $j = 0, 1, \cdots, m$, we have

$$\varphi_n^{(j)}(t) = q^{(j)}(t) - \sum_{s=0}^{n-1}\frac{\Gamma(s+\lambda)}{\Gamma(s+\lambda-j)}q_s t^{s+\lambda-1-j}, \tag{3.2}$$

and by condition (Q_2),

$$\varphi_n^{(j)}(t) \sim \frac{\Gamma(n+\lambda)}{\Gamma(n+\lambda-j)}q_n t^{n+\lambda-1-j} + \cdots \quad \text{as } t \to 0+. \tag{3.3}$$

Now define inductively $\varphi_{0,n}(t) = \varphi_n(t)$, and

$$\varphi_{j+1,n}(t) = \varphi_{j,n}'(t) - (\nu + j + 1)\varphi_{j,n}(t)t^{-1}, \qquad j = 0, 1, \cdots, m-1. \tag{3.4}$$

THEOREM 1. *Assume that conditions* (Q_1), (Q_2) *and* (Q_3) *hold, and let n be a positive integer satisfying*

$$m - \operatorname{Re}\lambda < n < m + \tfrac{3}{2} - \operatorname{Re}\lambda. \tag{3.5}$$

Then

$$I(x) = \sum_{s=0}^{n-1} q_s \frac{\Gamma(\frac{1}{2}s + \frac{1}{2}\lambda + \frac{1}{2}\nu)2^{s+\lambda-1}}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s - \frac{1}{2}\lambda + 1)x^{s+\lambda}} + \delta_{m,n}(x), \tag{3.6}$$

where

$$\delta_{m,n}(x) = \left(\frac{-1}{x}\right)^m \int_0^\infty J_{\nu+m}(xt)\varphi_{m,n}(t)\,dt. \tag{3.7}$$

Proof. Let $\eta \neq 0$ be an arbitrary real number. Then we have, from (3.1),

$$\int_0^\infty e^{-\eta^2 t^2} J_\nu(xt)q(t)\,dt = \sum_{s=0}^{n-1} q_s \int_0^\infty e^{-\eta^2 t^2} J_\nu(xt)t^{s+\lambda-1}\,dt + E_n(\eta, x), \tag{3.8}$$

where

$$E_n(\eta, x) = \int_0^\infty e^{-\eta^2 t^2} J_\nu(xt)\varphi_n(t)\,dt. \tag{3.9}$$

Applying Lemmas 1 and 2, we obtain, by passing to the limit as $\eta \to 0$,

$$I(x) = \sum_{s=0}^{n-1} q_s \frac{\Gamma(\frac{1}{2}s + \frac{1}{2}\lambda + \frac{1}{2}\nu)2^{s+\lambda-1}}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s - \frac{1}{2}\lambda + 1)x^{s+\lambda}} + E_n(x), \tag{3.10}$$

where

$$E_n(x) = \lim_{\eta \to 0} E_n(\eta, x). \tag{3.11}$$

The convergence of the above integrals is assured by Lemma 2 and the convergence of $\int_0^\infty J_\nu(xt)q(t)\,dt$.

From the well-known identity

$$\frac{d}{dt}[t^{\nu+1}J_{\nu+1}(t)] = t^{\nu+1}J_\nu(t), \tag{3.12}$$

it follows by integration by parts that

$$\begin{aligned}\int e^{-\eta^2 t^2} J_\nu(xt)\varphi_n(t)\,dt &= \frac{1}{x}J_{\nu+1}(xt)\,\varphi_n(t)\,e^{-\eta^2 t^2} \\ &\quad - \frac{1}{x}\int e^{-\eta^2 t^2} J_{\nu+1}(xt)\varphi_{1,n}(t)\,dt \\ &\quad + \frac{2\eta^2}{x}\int e^{-\eta^2 t^2} J_{\nu+1}(xt)\varphi_n(t)t\,dt.\end{aligned} \tag{3.13}$$

By (3.3) and the asymptotic formula

$$J_\alpha(t) \sim \frac{t^\alpha}{2^\alpha \Gamma(\alpha + 1)} \quad \text{as } t \to 0+, \tag{3.14}$$

we have $J_{\nu+1}(xt)\varphi_n(t)\,e^{-\eta^2t^2} \to 0$ as $t \to 0$ for any real η and any nonnegative x. Furthermore, since $t^{-1/2}q(t) \to 0$ as $t \to \infty$ and

$$J_\alpha(t) = \sqrt{\frac{2}{\pi t}}\cos(t - \tfrac{1}{2}\alpha\pi - \tfrac{1}{4}\pi) + O(t^{-3/2}) \quad \text{as } t \to \infty, \tag{3.15}$$

we also have $J_{\nu+1}(xt)\varphi_n(t)\,e^{-\eta^2t^2} \to 0$ as $t \to \infty$ for any nonnegative x and real $\eta \neq 0$. Thus it follows from (3.13) that

$$\begin{aligned} E_n(\eta, x) = \left(\frac{-1}{x}\right)\int_0^\infty e^{-\eta^2t^2}J_{\nu+1}(xt)\varphi_{1,n}(t)\,dt \\ + \frac{2\eta^2}{x}\int_0^\infty e^{-\eta^2t^2}J_{\nu+1}(xt)\varphi_n(t)t\,dt. \end{aligned} \tag{3.16}$$

Note that $J_{\nu+1}(xt)q(t) \to 0$ as $t \to \infty$, and hence

$$\lim_{\eta\to 0}\eta^2\int_0^\infty e^{-\eta^2t^2}J_{\nu+1}(xt)q(t)t\,dt = 0.$$

By Lemma 1, the second term on the right of (3.16) also tends to zero as $\eta \to 0$. We thus obtain

$$\lim_{\eta\to 0}E_n(\eta, x) = \left(\frac{-1}{x}\right)\lim_{\eta\to 0}\int_0^\infty e^{-\eta^2t^2}J_{\nu+1}(xt)\varphi_{1,n}(t)\,dt. \tag{3.17}$$

This procedure can be repeated m times and finally leads to

$$\lim_{\eta\to 0}E_n(\eta, x) = \left(\frac{-1}{x}\right)^m\lim_{\eta\to 0}\int_0^\infty e^{-\eta^2t^2}J_{\nu+m}(xt)\varphi_{m,n}(t)\,dt. \tag{3.18}$$

Returning to (3.4), it is easy to see that there are constants $c_1, \cdots, c_m$ such that

$$\varphi_{m,n}(t) = \varphi_n^{(m)}(t) + \sum_{j=1}^m c_j\varphi_n^{(m-j)}(t)t^{-j}. \tag{3.19}$$

Hence, from (3.3),

$$\varphi_{m,n}(t) = O(t^{n+\lambda-1-m}) \quad \text{as } t \to 0, \tag{3.20}$$

and the integral $\int J_{\nu+m}(xt)\varphi_{m,n}(t)\,dt$ converges at $t = 0$. From (3.2), we also have constants $d_0, \cdots, d_{n-1}$ such that

$$\varphi_{m,n}(t) = q^{(m)}(t) + \sum_{j=1}^m c_jq^{(m-j)}(t)t^{-j} + \sum_{s=0}^{n-1} d_st^{s+\lambda-1-m}. \tag{3.21}$$

Since $q^{(m-j)}(t)t^{-j} = o(t^{-j+1/2})$ as $t \to \infty$, the integrals $\int J_{\nu+m}(xt)q^{(m-j)}(t)t^{-j}\,dt$, $j = 1, \cdots, m$, all converge at $t = \infty$ uniformly for all sufficiently large values of x. Furthermore, since the powers of t in the last sum in (3.21) are all less than $-\frac{1}{2}$ on account of (3.5), condition (Q_3) implies that $\int J_{\nu+m}(xt)\varphi_{m,n}(t)\,dt$ converges also at $t = \infty$ uniformly for all sufficiently large values of x. By Lemma 2, we have

$$E_n(x) = \lim_{\eta \to 0} E_n(\eta, x) = \left(\frac{-1}{x}\right)^m \int_0^\infty J_{\nu+m}(xt)\varphi_{m,n}(t)\,dt. \tag{3.22}$$

This completes the proof.

4. Asymptotic nature of the expansion. For $t \geqq 0$ and $\operatorname{Re}\alpha \geqq 0$, the function $J_\alpha(t)$ is bounded. Hence there is a finite number A_α such that

$$A_\alpha = \sup_{0 \leqq t < \infty} |J_\alpha(t)|. \tag{4.1}$$

If α is real and greater than or equal to zero, then $A_\alpha \leqq 1$ and $A_{\alpha+1} \leqq 1/\sqrt{2}$ (see [6, p. 406]).

In §3 we have shown that the integral

$$\int_0^\infty J_{\nu+m}(xt)\varphi_{m,n}(t)\,dt$$

converges uniformly for all sufficiently large values of x. Hence for any $\varepsilon > 0$, there exists a constant c independent of x such that

$$\left| \int_c^\infty J_{\nu+m}(xt)\varphi_{m,n}(t)\,dt \right| < \frac{\varepsilon}{2}.$$

Furthermore, since $J_{\nu+m}(xt) \to 0$ as $x \to \infty$ for every fixed $t \geqq 0$, we also have, by (4.1) and the Lebesgue dominated convergence theorem,

$$\left| \int_0^c J_{\nu+m}(xt)\varphi_{m,n}(t)\,dt \right| < \frac{\varepsilon}{2}$$

for sufficiently large x. Therefore

$$\lim_{x \to \infty} \int_0^\infty J_{\nu+m}(xt)\varphi_{m,n}(t)\,dt = 0$$

and

$$\delta_{m,n}(x) = o(x^{-m}) \quad \text{as } x \to \infty. \tag{4.2}$$

The following result is an analogue of Watson's lemma for Laplace transforms.

THEOREM 2. *Assume that* (i) $q(t)$ *is infinitely differentiable on* $(0, \infty)$; (ii) *as* $t \to 0$,

$$q(t) \sim \sum_{s=0}^{\infty} q_s t^{s+\lambda-1}, \tag{4.3}$$

where $\operatorname{Re}(\nu + \lambda) > 0$ *and the expansion can be differentiated any number of times*; (iii) $t^{-1/2}q^{(j)}(t) \to 0$ *as* $t \to \infty$ *for* $j = 0, 1, 2, \cdots$. *Then as* $x \to \infty$, *the asymptotic expansion of* $I(x)$ *is obtained by substituting* (4.3) *in* (1.1) *and integrating formally term by term in the generalized sense of* (2.3).

Proof. The conditions $t^{-1/2}q^{(j)}(t) \to 0$ as $t \to \infty$ for $j = 0, 1, 2, \cdots$ imply that the integrals

$$\int J_{\nu+m}(xt)q^{(m)}(t)\,dt, \qquad m = 0, 1, 2, \cdots,$$

all converge uniformly for sufficiently large values of x. Thus the result follows immediately from Theorem 1 and (4.2).

Remarks. (i) From our analysis, it is easily seen that the asymptotic expansion (2.1) can be replaced by its more general form

$$q(t) \sim \sum_{s=0}^{\infty} q_s t^{\lambda_s - 1} \quad \text{as } t \to 0+, \tag{4.4}$$

where $\operatorname{Re}(\lambda_0 + \nu) > 0$, $\operatorname{Re}\lambda_{s+1} > \operatorname{Re}\lambda_s$ for $s = 0, 1, 2, \cdots$, and $\operatorname{Re}\lambda_s \to \infty$ as $s \to \infty$.

(ii) It is well known that the Hankel transform (1.1) is a generalization of the Fourier transform

$$I(x) = \int_0^\infty e^{ixt} q(t)\,dt.$$

For this particular case, Condition (iv) in [4] is equivalent to our condition (Q_3) of § 2.

Example 1. As an illustration of Theorem 2, we consider the Hankel transform of $\sin\sqrt{t}$:

$$I(x) = \int_0^\infty J_\nu(xt)\sin\sqrt{t}\,dt, \qquad \nu > -\tfrac{3}{2}. \tag{4.5}$$

The function $q(t) = \sin\sqrt{t}$ has the convergent expansion

$$\sin\sqrt{t} = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} t^{s+1/2}$$

for all values of t, and the conditions of Theorem 2 are clearly satisfied. Therefore

$$I(x) \sim \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} \frac{\Gamma(\frac{1}{2}s + \frac{3}{4} + \frac{1}{2}\nu)}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s + \frac{1}{4})} \frac{2^{s+1/2}}{x^{s+3/2}} \tag{4.6}$$

as $x \to \infty$. This result does not seem to follow from the theorems given in references [2], [3] and [5].

5. Bounds for $\delta_{m,n}(x)$. In order to bound the error term $\delta_{m,n}(x)$, we recall the quantity A_α as given in (4.1), and define

$$B_\alpha = \sup_{0 \leq t < \infty} |t^{1/2} J_\alpha(t)|. \tag{5.1}$$

Numerical computations yield

$$
\begin{aligned}
&A_0 = 1.00000, && B_0 = 0.79788,\\
&A_1 = 0.58187, && B_1 = 0.82503,\\
&A_2 = 0.48650, && B_2 = 0.86842,\\
&A_3 = 0.43439, && B_3 = 0.90238.
\end{aligned} \tag{5.2}
$$

Since $J_\alpha(t)$ is continuous on $[0, \infty)$ for $\operatorname{Re}\alpha \geqq 0$, relations (3.14) and (3.15) show that A_α and B_α are finite. For the sake of simplicity, we shall restrict ourselves to real parameters.

THEOREM 3. *Assume that conditions* (Q_1) *and* (Q_2) *hold, and replace condition* (Q_3) *by*

(Q_3') *for each* $j = 0, 1, \cdots, m$, $q^{(j)}(t) = O(t^{-j-1})$ *as* $t \to \infty$.

Let n be a positive integer satisfying $m - \lambda < n \leqq m - \lambda + 1$. *Then*

$$|\delta_{m,n}(x)| \leqq \frac{B_{\nu+m}}{x^{m+1/2}} \int_0^\infty t^{-1/2} |\varphi_{m,n}(t)|\, dt \tag{5.3}$$

if $n = m - \lambda + 1$, *or*

$$|\delta_{m,n}(x)| \leqq \frac{A_{\nu+m}}{x^m} \int_0^\infty |\varphi_{m,n}(t)|\, dt \tag{5.4}$$

if $m - \lambda < n < m - \lambda + 1$.

Proof. If $n = m - \lambda + 1$, then (3.20) gives $\varphi_{m,n}(t) = O(1)$ as $t \to 0+$. Thus the integral $\int t^{-1/2}|\varphi_{m,n}(t)|\, dt$ is convergent at $t = 0$. Furthermore, from (3.21), we have

$$|\varphi_{m,n}(t)| \leqq |q^{(m)}(t)| + \sum_{j=1}^{m} |c_j|\, |q^{(m-j)}(t)| t^{-j} + \sum_{s=0}^{n-1} |d_s| t^{s+\lambda-1-m}, \tag{5.5}$$

where the exponents in the last sum are all less than or equal to -1. Therefore, by condition (Q_3'), the integral $\int t^{-1/2}|\varphi_{m,n}(t)|\, dt$ also converges at $t = \infty$. That is,

$$\int_0^\infty t^{-1/2}|\varphi_{m,n}(t)|\, dt < \infty.$$

The error bound (5.3) now follows from (3.7) and (5.1).

The proof of (5.4) is similar. From (3.20), $\varphi_{m,n}(t) = O(^{n+\lambda-1-m})$ as $t \to 0+$, and hence the integral $\int |\varphi_{m,n}(t)|\, dt$ is convergent at $t = 0$. Since the exponents in the last sum of (5.5) are all less than -1 in this case, the integral $\int |\varphi_{m,n}(t)|\, dt$ also converges at $t = \infty$. Therefore (5.4) holds in view of (3.7) and (4.1).

Example 2. Consider the integral

$$I(x) = \int_0^\infty \frac{J_0(xt)}{1+t} dt. \tag{5.6}$$

In the notation of §§ 2 and 3, we have

$$q(t) = \frac{1}{1+t} \sim 1 - t + t^2 - \cdots .$$

Thus $\lambda = 1$,

$$\varphi_{0,1}(t) = -\frac{t}{1+t} \quad \text{and} \quad \varphi_{1,1}(t) = \frac{t}{(1+t)^2}.$$

Therefore

$$\int_0^\infty |\varphi_{1,1}(t)|\,dt = \infty, \qquad \int_0^\infty t^{-1/2}|\varphi_{1,1}(t)|\,dt = \frac{\pi}{2}. \tag{5.7}$$

From (3.6) it follows that

$$I(x) = \frac{1}{x} + \delta_{1,1}(x) \tag{5.8}$$

where

$$\delta_{1,1}(x) = -\frac{1}{x}\int_0^\infty J_1(xt)\varphi_{1,1}(t)\,dt.$$

The bound (5.3) yields

$$|\delta_{1,1}(x)| \leqq \frac{\pi}{2} B_1 x^{-3/2}. \tag{5.9}$$

Note that the complete asymptotic expansion in this example is

$$I(x) \sim \sum_{s=0}^{\infty} \frac{\Gamma(\frac{1}{2}+s)2^{2s}}{\Gamma(\frac{1}{2}-s)x^{2s+1}}. \tag{5.10}$$

Thus the actual error in the expansion $I(x) \sim x^{-1}$ is $O(x^{-3})$, and the estimate (5.9) seems crude. One way to arrive at a bound involving the correct power of the asymptotic variable is to continue the expansion to one more term, and then use this term plus the bound for the new remainder term (which is of a lower asymptotic order of magnitude).

Simple calculations give

$$\varphi_{3,3}(t) = -\frac{3t^3 + 12t^2 + 3t}{(1+t)^4} \tag{5.11}$$

and

$$\int_0^\infty t^{-1/2}|\varphi_{3,3}(t)|\,dt = \tfrac{15}{8}\pi. \tag{5.12}$$

From (3.6) it follows that

$$I(x) = \frac{1}{x} - \frac{1}{x^3} + \delta_{3,3}(x), \tag{5.13}$$

where

$$\delta_{3,3}(x) = \left(-\frac{1}{x}\right)^3 \int_0^\infty J_3(xt)\varphi_{3,3}(t)\,dt$$

Thus we derive, from (5.3),

$$|\delta_{3,3}(x)| \leqq \tfrac{15}{8}\pi B_3 x^{-7/2}. \tag{5.14}$$

This estimate allows us to rewrite (5.13) as

$$I(x) = \frac{1}{x} + \delta^*_{1,1}(x), \tag{5.15}$$

where

$$|\delta^*_{1,1}(x)| \leqq x^{-3}[1 + \tfrac{15}{8}\pi B_3 x^{-1/2}]. \tag{5.16}$$

Example 3. Finally, we consider the function

$$I(x) = \int_0^\infty \frac{J_0(xt)}{\sqrt{t}(1+t)}\,dt. \tag{5.17}$$

Here we have

$$q(t) = \frac{1}{\sqrt{t}(1+t)} \sim t^{-1/2} - t^{1/2} + t^{3/2} - \cdots.$$

Thus $\lambda = \frac{1}{2}$,

$$\varphi_{0,1}(t) = -\frac{\sqrt{t}}{1+t} \quad \text{and} \quad \varphi_{1,1}(t) = \frac{3t+1}{2\sqrt{t}(1+t)^2}.$$

Therefore, in contrast with (5.7),

$$\int_0^\infty |\varphi_{1,1}(t)|\,dt = \pi, \qquad \int_0^\infty t^{-1/2}|\varphi_{1,1}(t)|\,dt = \infty. \tag{5.18}$$

From (3.6) and (5.4) it follows that

$$I(x) = \frac{\Gamma^2(\frac{1}{4})}{\pi\sqrt{x}} + \delta_{1,1}(x), \tag{5.19}$$

where

$$|\delta_{1,1}(x)| \leqq \frac{A_1}{x}\int_0^\infty |\varphi_{1,1}(t)|\,dt = \frac{\pi A_1}{x}. \tag{5.20}$$

The complete asymptotic expansion in this example is

$$I(x) \sim \sum_{s=0}^{\infty} (-1)^s \frac{\Gamma(\frac{1}{4} + s/2)}{\Gamma(\frac{3}{4} - s/2)} 2^{s-1/2} x^{-1/2-s}. \tag{5.21}$$

Thus the actual error in the expansion (5.19) is $O(x^{-3/2})$, and the estimate (5.20) again falls short of the actual result. To improve the estimate in (5.20), we calculate $\varphi_{2,2}(t)$ and evaluate $\int_0^\infty |\varphi_{2,2}(t)|\,dt$. The results are

$$\varphi_{2,2}(t) = \frac{5t^2 - 6t - 3}{4\sqrt{t}(1+t)^3}$$

and

$$\int_0^\infty |\varphi_{2,2}(t)|\,dt = 2.05833.$$

The last numerical value is accurate to five decimal places. From (3.6) and (5.4), we have

$$I(x) = \frac{\Gamma^2(\frac{1}{4})}{\pi\sqrt{x}} - \frac{2\pi}{\Gamma^2(\frac{1}{4})x^{3/2}} + \delta_{2,2}(x), \tag{5.22}$$

where

$$|\delta_{2,2}(x)| \leqq \frac{A_2}{x^2}\int_0^\infty |\varphi_{2,2}(t)|\,dt = 1.00138x^{-2}. \tag{5.23}$$

Thus (5.22) may be rewritten as

$$I(x) = \frac{\Gamma^2(\frac{1}{4})}{\pi\sqrt{x}} + \delta^*_{1,1}(x), \tag{5.24}$$

where

$$|\delta^*_{1,1}(x)| \leqq x^{-3/2}\left[\frac{2\pi}{\Gamma^2(\frac{1}{4})} + 1.00138x^{-1/2}\right]. \tag{5.25}$$

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A CLASS OF CARDINAL TRIGONOMETRIC SPLINES*

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Abstract. About a decade ago Schoenberg introduced trigonometric splines which are related to the differential operator $\Delta_m = D(D^2 + 1^2) \cdots (D^2 + m^2)$. Here we introduce the cardinal trigonometric splines and show that cardinal trigonometric interpolation at the nodes to data of power growth is not unique. We also study trigonometric Euler splines and an extremal property for its restriction to $[0, \eta]$. We prove a similar result for cardinal L-splines of Micchelli.

1. Introduction. The subject of trigonometric spline interpolation was first discussed in a very elegant paper by Schoenberg [4] with respect to the operators Δ_m and Δ_m^2. Since then, many generalizations of the same have appeared in the literature. Now that the fever and excitment of these generalizations has abated, it is time to return to the original memoir of Schoenberg. This has been done to some extent by Micchelli [1] when he introduced the cardinal $\mathscr{L}$-splines, and also by Schoenberg [3] who develops a new approach to the results of Micchelli. However, Schoenberg refers to the trigonometric splines of degree 1 related to the operator $D(D^2 + 1)$ only as an example, with the remark that "this example suggests that Micchelli's theory will extend to operators $\mathscr{L}$ with pairs of imaginary conjugates γ_ν". The object of this study is to begin such an extension for the operator Δ_m defined in § 2. However we shall be more interested in finding some extremal properties of the trigonometric polynomials associated with cardinal trigonometric splines. We also extend our study to Micchelli's cardinal $\mathscr{L}$-splines.

In § 2 we give the preliminaries and the definition of the trigonometric B-splines due to Schoenberg [4]. Section 3 deals with the trigonometric Euler splines which are analogous to the exponential Euler splines of Schoenberg [5]. We prove incidentally that cardinal trigonometric interpolation at the nodes to data of power growth is not unique. This has been shown by Schoenberg [3] for trigonometric splines of degree 1. In § 4, we study the restriction $A_m(x; t)$ of the trigonometric Euler spline to the interval $[0, \eta]$ and study its extremal property. Lastly, § 5 is devoted to the corresponding result for cardinal $\mathscr{L}$-splines of Micchelli.

2. The class $\mathscr{S}(\Delta_m, \eta)$. Let η be a positive constant $< 2\pi/(2m + 1)$, and let

$$\Delta_m = D(D^2 + 1^2) \cdots (D^2 + m^2), \qquad D = \frac{d}{dx},$$

for some integer m. We define the class $\mathscr{S}(\Delta_m, \eta)$ of functions $S(x)$ satisfying the following conditions:

(i) $\Delta_m S(x) = 0$ in $(\nu\eta, \nu\eta + \eta)$ for every integer ν,

(ii) $S(x) \in C^{2m-1}(\mathbb{R})$.

The functions $S(x)$ will be called cardinal trigonometric splines.

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Set

$$\phi(x) = \frac{2^m}{(2m)!}(1 - \cos x)^m = \frac{1}{(2m)!}x^{2m} + \cdots$$

and let $\phi_+(x) = \phi(x)$ for $x > 0$ and $= 0$ for $x \leqq 0$. Then every $\sigma(x) \in \mathscr{S}(\Delta_m, \eta)$ can be written uniquely in the following form:

$$\sigma(x) = \tau(x) + \sum_{\nu=1}^{\infty} c_\nu \phi_+(x - \nu\eta) + \sum_{\nu=1}^{\infty} d_\nu \phi_+(-x - \nu\eta),$$

where $\tau(x)$ is a trigonometric polynomial of degree m. We formulate the problem of cardinal trigonometric spline interpolation as follows.

Problem. Given a sequence $\{y_\nu\}$, find $S(x) \in \mathscr{S}(\Delta_m, \eta)$ such that

$$S(\nu\eta) = y_\nu, \qquad \nu = 0, \pm 1, \pm 2, \cdots.$$

It is easy to see that this problem has infinitely many solutions forming a linear manifold in $\mathscr{S}(\Delta_m, \eta)$ of dimension $2m - 1$.

We follow Schoenberg [4] and define the generalized differences $D\{f(x_j); j = 1, \cdots, 2m + 2\}$ by

$$D\{f(x_j); j = 1, \cdots, 2m + 2\} = \begin{vmatrix} 1 & \cos x_1 & \sin x_1 & \cdots \cos mx_1 & \sin mx_1 & f(x_1) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cos x_{2m+2} & \sin x_{2m+2} & \cdots \cos mx_{2m+2} & \sin mx_{2m+2} & f(x_{2m+2}) \end{vmatrix}.$$

Set $x_j = j\eta$ $(j = 0, \pm 1, \pm 2, \cdots)$ and

$$M_\nu(x) = -\pi(m!)^2 \frac{D\{\phi_+(x - x_j); j = \nu, \cdots, \nu + 2m + 1\}}{D\{x_j; j = \nu, \cdots, \nu + 2m + 1\}}, \qquad M(x) = M_0(x).$$

Since $(2m + 1)\eta < 2\pi$, it follows from Lemma 6 of Schoenberg [4] that the denominator on the right side is not equal to zero. Schoenberg has shown that $M_\nu(x) = 0$ for $x \notin (x_\nu, x_{\nu+2m+1})$ and that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} M_\nu(x)\, dx = 1,$$

$$\frac{D\{f(x_j); j = \nu, \cdots, \nu + 2m + 1\}}{D\{x_j; j = \nu, \cdots, \nu + 2m + 1\}} = \frac{1}{\pi(m!)^2}\int_{-\infty}^{\infty} M_\nu(x)\, \Delta_m f(x)\, dx.$$

It is clear that $M_\nu(x) = M(x - \nu\eta)$.

The following theorem can be proved on the same lines as that of Schoenberg [5].

THEOREM 1. *If $S(x) \in \mathscr{S}(\Delta_m, \eta)$, then $S(x)$ admits a unique representation*

$$S(x) = \sum_{-\infty}^{\infty} c_\nu M(x - \nu\eta),$$

where c_ν's are appropriate constants. Conversely every such expansion represents an element of $\mathscr{S}(\Delta_m, \eta)$.

The proof depends upon a lemma of Schoenberg [4, p. 804, Lemma 7] which says that the functions $M(x - \eta), \cdots, M(x - \overline{2m+1}\eta)$ are linearly independent for $(2m + 1)\eta \leqq x \leqq (2m + 2)\eta$. One has only to repeat then the reasoning of Schoenberg [5, p. 13–15].

3. Trigonometric Euler splines. We see from Theorem 1 that the spline $M(x)$ plays the role of a B-spline. We shall now construct some special splines which will be useful later.

Let t be a constant, $t \neq 0$, $t \neq 1$. We want to find a spline $S(x) \in \mathscr{S}(\Delta_m, \eta)$ such that

$$S(x + \eta) = tS(x). \tag{3.1}$$

It is clear that the most general nontrivial element $S(x)$ satisfying (3.1) is given by

$$S(x) = c_0 \sum_{-\infty}^{\infty} t^\nu M(x - \nu\eta), \qquad c_0 \neq 0. \tag{3.2}$$

Let $A_m(x; t)$ denote the restriction of $S(x)$ to $[0, \eta]$, and let

$$A_m(x; t) = a_0 + 2 \sum_{\nu=1}^{m} \frac{\binom{2m}{m+\nu}}{\binom{2m}{m}} (a_\nu \cos \nu x + b_\nu \sin \nu x). \tag{3.3}$$

By (3.1), $A_m(x; t)$ must satisfy the conditions

$$A_m^{(\nu)}(\eta; t) = tA_m^{(\nu)}(0; t), \qquad \nu = 0, 1, \cdots, 2m - 1, \tag{3.4}$$

where the differentiations are with respect to the variable x. Applying the operators $\Delta_m/D(D^2 + j^2)$ and $\Delta_m/(D^2 + j^2)$, $j = 1, \cdots, m$, in succession to $A_m(x; t)$, we see easily that a_j, b_j are given by

$$a_j(\cos j\eta - t) + b_j \sin j\eta = (-1)^{j-1}(t - 1)a_0,$$
$$a_j \sin j\eta - b_j(\cos j\eta - t) = 0.$$

Hence,

$$a_j = \frac{(-1)^{j-1}(t - 1)(\cos j\eta - t)a_0}{1 - 2t \cos j\eta + t^2},$$
$$b_j = \frac{(-1)^{j-1}(t - 1)\sin j\eta \cdot a_0}{1 - 2t \cos j\eta + t^2}. \tag{3.5}$$

For the sake of convenience, we shall choose $a_0 = 1$, so that

$$A_m(x; t) = 1 + 2 \sum_{\nu=1}^{m} \frac{\binom{2m}{m+\nu}}{\binom{2m}{m}} \frac{(-1)^{\nu-1}(t - 1)\{\cos \nu(x - \eta) - t \cos \nu x\}}{1 - 2t \cos \nu\eta + t^2}. \tag{3.6}$$

We have thus shown that there is a unique trigonometric polynomial $A_m(x; t)$ of degree m with constant term 1 which satisfies (3.4).

The expression (3.6) recalls the de la Vallee Poussin means of a function $f(x)$. It is known [2] that if the Fourier series of $f(x)$ is given by

$$f(x) \sim \tfrac{1}{2}\alpha_0 + \sum_{\nu=1}^{\infty} (\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x),$$

then the de la Vallee Poussin means are given by

$$V_m(f; x) = \tfrac{1}{2}\alpha_0 + \sum_{\nu=1}^{m} \frac{\binom{2m}{m+\nu}}{\binom{2m}{m}} (\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x)$$

(3.7a)
$$= \frac{1}{\binom{2m}{m}} \cdot \frac{1}{2\pi} \int_0^{2\pi} \left(2 \cos \frac{x - \tau}{2}\right)^{2m} f(\tau)\, d\tau.$$

Thus if

$$f(x) \sim 1 + 2 \operatorname{Re} \sum_{\nu=1}^{\infty} \frac{(1 - t)\, e^{i\nu(x+\eta)}}{e^{i\nu\eta} - t},$$

then

(3.7) $$A_m(x, t) = V_m(f; x).$$

We shall prove a simple lemma.

LEMMA 1. *If $t > 0$, $A_m(x; t) > 0$ for all real x.*

Proof. For $t = 1$, $A_m(x; 1) = 1$. From (3.6), it follows that

(3.8) $$A_m(x; t) = A_m(\eta - x; t^{-1}).$$

It is therefore enough to prove the result for $0 < t < 1$. It follows easily from (3.6) that

$$A_m(x; t) = 1 + \operatorname{Re}\left[2 \sum_{\nu=1}^{m} \frac{\binom{2m}{m+\nu}}{\binom{2m}{m}} (-1)^\nu (1 - t) \frac{e^{i\nu(x-\eta)}}{1 - te^{-\nu\eta i}}\right]$$

$$= (1 - t) \sum_{k=0}^{\infty} t^k \operatorname{Re}\left[1 + 2 \sum_{\nu=1}^{m} \frac{\binom{2m}{m+\nu}}{\binom{2m}{m}} e^{i\nu(x+\pi-\eta(k+1))}\right]$$

$$= (1 - t) \sum_{k=0}^{\infty} t^k \left[1 + 2 \sum_{\nu=1}^{m} \frac{\binom{2m}{m+\nu}}{\binom{2m}{m}} \cos \nu(x + \pi - \eta(k + 1))\right].$$

From a well-known identity [2], it follows that

$$A_m(x;t) = \frac{1-t}{\binom{2m}{m}} \sum_{k=0}^{\infty} t^k \left(2 \sin \frac{(k+1)\eta - x}{2}\right)^{2m}, \tag{3.9}$$

which proves the lemma. This leads us to formulate Theorem 2.

THEOREM 2. *If $t > 0$, $t \neq 1$, then the unique trigonometric Euler spline $S_m(x;t)$ which interpolates the data $\{t^{\nu\eta}\}$ at the nodes $\{\nu\eta\}$ is uniquely determined by*

$$S_m(x;t) = \frac{A_m(x;t^\eta)}{A_m(0;t^\eta)}, \qquad 0 < x < \eta,$$

$$S_m(x+\eta) = t^\eta S_m(x;t).$$

Remark. It follows from (3.6) that

$$(D^2 + m^2)A_m(x;t) = m^2 A_{m-1}(x;t). \tag{3.10}$$

In Theorem 2, the data is of exponential growth and we require the interpolatory spline to satisfy a certain difference equation. It is interesting to inquire whether interpolation by trigonometric splines is unique when the data is of power growth and the interpolatory spline is also of power growth. The analogous problem for polynomials and $\mathscr{L}$-splines has been solved by Schoenberg [3], [5]. For trigonometric splines of order 1, Schoenberg has shown the nonuniqueness of the trigonometric spline which interpolates the data at the nodes. More generally we can prove the next theorem.

THEOREM 3. *If the data $\{y_\nu\}$ is of power growth, then the interpolation problem of finding $S(x) \in \mathscr{S}(\Delta_m, \eta)$ such that*

(i) *$S(\nu\eta) = y_\nu$ for all ν, and*

(ii) *$S(x)$ is of power growth,*

does not admit a unique solution.

Proof. There exists a nonzero trigonometric spline $\tilde{S}(x)$ which is bounded with $\tilde{S}(\nu\eta) = 0$ for all η. The restriction of this spline to $[0, \eta]$ is given by the trigonometric polynomial $A_m(x; -1)$, where

$$A_m(x;-1) = 1 + 2 \sum_{\nu=1}^{m} \frac{\binom{2m}{m+\nu}}{\binom{2m}{m}} (-1)^\nu \cdot \frac{\cos \nu x + \cos \nu(x-\eta)}{1 + \cos \nu\eta}.$$

For $x \notin [0, \eta)$, $\tilde{S}(x+\eta) = -\tilde{S}(x)$. It is easy to verify that $\tilde{S}(x) \in \mathscr{S}(\Delta_m, \eta)$ and $\tilde{S}(\nu\eta) = 0$ for all ν since $A_m(0, -1) = 0$.

4. An extremum property of $A_m(x;t)$. The trigonometric polynomials $A_m(x;t)$ are analogues of the algebraic exponential polynomials which were introduced by Schoenberg [5]. We have retained the same notation as his, and a natural question arises whether there is an analogue of the extremum property in the present trigonometric case. In order to formulate our answer to this question, we introduce the class $\mathscr{F}(A_m)$ of functions $f(x)$ which satisfy the following conditions:

$$\text{(4.1)}\qquad \begin{aligned} &\text{(i)}\quad f(x) \in C^{2m-1}[0,\eta], f^{(2m)}(x) \text{ integrable in } [0,\eta],\\ &\text{(ii)}\quad \textstyle\int_0^\eta f(x)\,dx \geqq \int_0^\eta A_m(x;t)\,dx,\\ &\text{(iii)}\quad f^{(2\nu)}(\eta) = tf^{(2\nu)}(0),\ \nu = 0,1,\cdots,m-1,\\ &\text{(iv)}\quad f^{(2\nu+1)}(\eta) \geqq tf^{(2\nu+1)}(0),\ \nu = 0,1,\cdots,m-1, \end{aligned}$$

where t is a positive constant. We can now prove Theorem 4.

THEOREM 4. *The trigonometric polynomial $A_m(x;t)$ is the unique element of the class $\mathscr{F}(A_m)$ that minimizes the norm*

$$\|\Delta_m^* f\|_\infty = \sup_{0 \leqq x \leqq \eta} |\Delta_m^* f|, \qquad f \in \mathscr{F}(A_m), \tag{4.2}$$

where $\Delta_m^* = (D^2 + 1^2)\cdots(D^2 + m^2)$. The least value of $\|\Delta_m^* f\|_\infty$ is

$$\|\Delta_m^* A_m(x;t)\| = (m!)^2. \tag{4.3}$$

Remark 1. In the definition of the class $\mathscr{F}(A_m)$ in (4.1), it would be interesting to replace the sign of equality in (iii) in (4.1) by the sign of inequality. However it would be necessary, for this to be possible, to prove that $DA_m(x;t) \leqq 0$ for $0 \leqq x \leqq \eta$ and $t > 0$. This is true for $m = 0$ and $m = 1$.

In order to prove Theorem 4, we shall need two simple lemmas.

LEMMA 2. *Suppose $F(x)$, $G(x) \in C^1[0,\eta]$ and $F''(x)$, $G''(x)$ are integrable on $[0,\eta]$. Then*

$$\int_0^\eta G(x)(D^2 + k^2)F(x)\,dx = [G(x)F'(x) - G'(x)F(x)]_0^\eta + \int_0^\eta F(x)(D^2 + k^2)G(x)\,dx. \tag{4.4}$$

We omit the quite elementary proof. Set $\Delta_{k,m} = D(D^2 + k^2)\cdots(D^2 + m^2)$, $\Delta_{k,m}^* = (D^2 + k^2)\cdots(D^2 + m^2)$. In particular, we have $\Delta_{1,m} = \Delta_m$ and $\Delta_{1,m}^* = \Delta_m^*$. We shall use the convention that $\Delta_{m+1,m} = \Delta_{m+1,m}^* = 1$. By repeated use of Lemma 2, we get Lemma 3.

LEMMA 3. *If $f(x)$, $K_m(x) \in C^{2m-1}[0,\eta]$, and if $f^{(2m)}(x)$ and $K_m^{(2m)}(x)$ are integrable on $[0,\eta]$, then*

$$\int_0^\eta K_m(x)(\Delta_m^* f)\,dx = \sum_{\nu=0}^{m-1} [(\Delta_{m+1-\nu,m}^* K_m)(\Delta_{m-1-\nu} f) - (\Delta_{m+1-\nu,m} K_m)(\Delta_{m-1-\nu}^* f)]_0^\eta$$
$$+ \int_0^\eta (\Delta_m^* K_m) f\,dx. \tag{4.5}$$

Proof of Theorem 4. Choose $K_m = A_m(x;t^{-1})$ in (4.5) and observe that by (3.4), we have

$$\text{(4.6)}\qquad \begin{aligned} &\Delta_{m+1-\nu,m}^* K_m(\eta) = t^{-1}\,\Delta_{m+1-\nu,m}^* K_m(0), \qquad \nu = 0,1,\cdots,m-1,\\ &\Delta_{m+1-\nu,m} K_m(\eta) = t^{-1}\,\Delta_{m+1-\nu,m} K_m(0), \qquad \nu = 0,1,\cdots,m-1,\\ &\Delta_m^* K_m(x) = (m!)^2. \end{aligned}$$

Hence by Lemma 3, we have

$$\int_0^\eta K_m(x)(\Delta_m^* f)\,dx = t^{-1}\sum_{v=0}^{m-1} \Delta_{m+1-v,m}^* K_m(0)\{\Delta_{m-1-v}f(\eta) - t\,\Delta_{m-1-v}f(0)\}$$

$$(4.7)\qquad - t^{-1}\sum_{v=0}^{m-1} \Delta_{m+1-v,m}K_m(0)\{\Delta_{m-1-v}^* f(\eta) - t\,\Delta_{m-1-v}^* f(0)\}$$

$$+ (m!)^2\int_0^\eta f(x)\,dx.$$

If we choose $f(x) = A_m(x;t)$ in (4.5), then from (3.4) we have

$$(4.8)\qquad \begin{aligned} \Delta_{m-1-v}f(\eta) &= t\,\Delta_{m-1-v}f(0) \\ \Delta_{m-1-v}^* f(\eta) &= t\,\Delta_{m-1-v}^* f(0) \end{aligned} \qquad v = 0, 1, \cdots, m-1.$$

Hence,

$$(4.9)\qquad \int_0^\eta K_m(x)\,\Delta_m^* A_m(x;t)\,dx = (m!)^2\int_0^\eta A_m(x;t)\,dx.$$

Suppose there exists $g(x) \in \mathscr{F}(A_m)$ different from $A_m(x;t)$ with

$$(4.10)\qquad \|\Delta_m^* g\|_\infty \leqq (m!)^2.$$

Then from (ii) in (4.1), we have

$$(4.11)\qquad (m!)^2\int_0^\eta A_m(x;t)\,dx \leqq (m!)^2\int_0^\eta g(x)\,dx.$$

Now using (3.10), Lemma 1 and properties (iii) and (iv) in (4.1), we have

$$(m!)^2\int_0^\eta A_m(x;t)\,dx$$

$$\leqq (m!)^2\int_0^\eta g(x)\,dx + t^{-1}\sum_{v=0}^{m-1}\Delta_{m+1-v,m}^* K_m(0)\{\Delta_{m-1-v}g(\eta) - t\Delta_{m-1-v}g(0)\}$$

$$- t^{-1}\sum_{v=0}^{m-1}\Delta_{m+1-v,m}K_m(0)\{\Delta_{m-1-v}^* g(\eta) - t\,\Delta_{m-1-v}^* g(0)\}$$

$$= \int_0^\eta K_m(x)\,\Delta_m^* g(x)\,dx$$

$$\leqq \int_0^\eta K_m(x)\,\Delta_m^* A_m(x;t)\,dx = (m!)^2\int_0^\eta A_m(x;t)\,dx.$$

We have used (4.3), (4.9) and (4.10) also in obtaining the above inequalities.

Hence the inequalities are in fact equalities, so that

$$(4.12)\qquad \Delta_{m-1-v}g(\eta) = t\Delta_{m-1-v}g(0), \qquad v = 0, 1, \cdots, m-1,$$

and

$$\int_0^\eta K_m(x)[\Delta_m^* A_m(x;t) - \Delta_m^* g(x)]\,dx = 0. \tag{4.13}$$

Since $K_m(x) = A_m(x; t^{-1}) > 0$ in $[0, \eta]$, and by our assumption, the second expression in the above integral is also nonnegative, it follows that

$$\Delta_m^* g(x) = \Delta_m^* A_m(x;t) \tag{4.14}$$

almost everywhere. Hence $g(x)$ is a trigonometric polynomial of degree $\leqq m$ with constant term 1. From (4.12) and from (iii) in (4.1), we have

$$g^{(\nu)}(\eta) = tg^{(\nu)}(0), \qquad \nu = 0, 1, \cdots, 2m-1. \tag{4.15}$$

Since $A_m(x;t)$ is the unique trigonometric polynomial with constant term 1 satisfying (4.15), it follows that $g(x) = A_m(x;t)$. This completes the proof of Theorem 4.

5. Cardinal $\mathscr{L}$-splines. We shall devote this section to obtaining an extremum property related to the exponential $\mathscr{L}$-splines. We use the notation of Micchelli [1] and Schoenberg [3] and set

$$\mathscr{L}_k = D\prod_1^k (D - \gamma_\nu), \qquad \mathscr{L}_k^* = \prod_1^k (D - \gamma_\nu), \qquad k = 0, 1, \cdots, n, \tag{5.1}$$

where $\gamma_1, \cdots, \gamma_n$ are real nonzero distinct constants. Let $\mathscr{S}(\mathscr{L}_n, \eta)$ denote the class of cardinal $\mathscr{L}$-splines $S(x)$ such that

(i) $\mathscr{L}_n S(x) = 0$ in $(\nu\eta, \nu\eta + \eta)$ for every integer,

(ii) $S(x) \in C^{n-1}(\mathbb{R})$.

For a given real number t, denote by $\Phi(x;t)$ the $\mathscr{L}$-spline satisfying the functional equation

$$\Phi(x + \eta; t) = t\Phi(x;t).$$

Using the method of § 3 with minor modifications, we can show that the restriction of $\Phi(x;t)$ to $[0, \eta]$ is given explicitly by

$$\tilde{A}_n(x;t) = 1 + (1 - t)\sum_{\nu=1}^{n} \frac{e^{\gamma_\nu x}}{t - e^{\gamma_\nu \eta}} \cdot \frac{1}{\omega_{\nu,n}}, \tag{5.2}$$

where we set

$$\omega_{\nu,n} = \prod_{\substack{k=1\\k\neq\nu}}^{n} \left(1 - \frac{\gamma_\nu}{\gamma_k}\right). \tag{5.3}$$

$\tilde{A}_n(x;t)$ is the unique generalized exponential polynomial having constant term 1 and satisfying the conditions

$$\tilde{A}_n^{(\nu)}(\eta;t) = t\tilde{A}_n^{(\nu)}(0;t), \qquad \nu = 0, 1, \cdots, n-1, \tag{5.4}$$

where the derivatives are with respect to x. We can rewrite (5.2) easily and show that for $0 < t < 1$,

$$\tilde{A}_n(x;t) = (1 - t) \sum_{k=0}^{\infty} t^k \left[1 - \sum_{\nu=1}^{n} \frac{e^{\gamma_\nu(x - k\eta - \eta)}}{\omega_{\nu,n}} \right].$$

Since

$$\frac{(-1)^n}{\prod_1^n \gamma_\nu} \left[1 - \sum_{\nu=1}^{n} \frac{e^{\gamma_\nu(x - k\eta - \eta)}}{\omega_{\nu,n}} \right]$$

is the divided difference of the function $e^{\gamma(x - k\eta - \eta)}$ at the $n + 1$ values of γ, namely, $0, \gamma_1, \cdots, \gamma_n$, it follows that

$$\tilde{A}_n(x;t) = (1 - t) \sum_{k=0}^{\infty} t^k e^{\xi_k(x - k\eta - \eta)} \frac{(x - k\eta - \eta)^n}{n!} (-1)^n \left(\prod_1^n \gamma_\nu \right),$$

where ξ_k lies between the maximum and minimum of the numbers $0, \gamma_1, \cdots, \gamma_n$. Hence for $0 < t < 1$ and $0 \leqq x \leqq \eta$,

$$\operatorname{sgn} \tilde{A}_n(x;t) = \operatorname{sgn} \left(\prod_1^n \gamma_\nu \right). \tag{5.5}$$

Similarly we can also verify that for $t > 1$, we have

$$\operatorname{sgn} \tilde{A}_n(x;t) = (-1)^n \operatorname{sgn} \left(\prod_1^n \gamma_\nu \right). \tag{5.5a}$$

We can now introduce the class $\mathscr{F}(\tilde{A}_n)$ of functions $f(x)$ satisfying the conditions:

(a) $f \in C^{n-1}[0, \eta]$ and $f^{(n)}$ is integrable in $[0, \eta]$,

(b) $\mathscr{L}_k^* f(\eta) \geqq t \mathscr{L}_k^* f(0)$, $k = 0, 1, \cdots, n - 1$,

(c) $\int_0^\eta f(x)\, dx \geqq \int_0^\eta A_n(x;t)\, dx$.

We shall now formulate Theorem 5,

THEOREM 5. *The generalized exponential polynomial $\tilde{A}_n(x;t)$ is the unique element of the class $\mathscr{F}(\tilde{A}_n)$ that minimizes the norm*

$$\|\mathscr{L}_n^* f\|_\infty = \sup_{0 \leqq x \leqq \eta} |\mathscr{L}_n^* f|, \qquad f \in \mathscr{F}(A_n). \tag{5.6}$$

The least value of $\|\mathscr{L}_n^* f\|_\infty$ is

$$\|\mathscr{L}_n^* \tilde{A}_n(x;t)\| = \prod_1^n |\gamma_\nu|. \tag{5.7}$$

The proof follows the same idea as that of Theorem 4 and requires the use of the following identities:

$$\int_0^\eta G(x) \cdot (D - \gamma_k) F(x)\, dx = G(\eta)F(\eta) - G(0)F(0) - \int_0^\eta F(x)(D + \gamma_k) G(x)\, dx, \tag{5.8}$$

$$(D - \gamma_n)\tilde{A}_n(x;t) = -\gamma_n \tilde{A}_{n-1}(x;t). \tag{5.9}$$

Using (5.8) we can easily get an analogue of (4.5), and the proof can be completed on using (5.5) and (5.9). We omit the details.

Remark 2. An interesting special case arises when $\mathscr{L}_n = D(D^2 - 1^2) \cdots (D^2 - n^2)$. In this case, the results of § 3 and § 4 are easily modified by changing trigonometric functions into hyperbolic functions.

Remark 3. It follows from (3.6) that $A_m(x; t)$, for a fixed x, has at most $2m$ zeros in the variable t if $0 < x < \eta$ and has at most $2m - 1$ zeros if $x = 0$. We have shown that these zeros are not positive. An analogue of the result of Shoenberg [5] on the nature of the zeros of the Euler–Frobenius polynomials would suggest that for a fixed x, $0 \leqq x < \eta$, the zeros of $A_m(x; t)$ will be real, simple and negative.

6. Conclusion. The results of Theorems 4 and 5 can be generalized in the light of our paper [6]. More precisely, we consider a given trigonometric polynomial $A_m(x)$ of degree m with constant term 1 and introduce the class $\mathscr{F}(A_m)$ of functions $f(x)$ which satisfy the following conditions:

(i) $f \in C^{2m-1}[0, \eta], f^{(2m)}(x)$ is integrable on $[0, \eta]$,
(ii) $\int_0^\eta f(x)\,dx \geqq \int_0^\eta A_m(x)\,dx$,
(iii) $f^{(2\nu)}(\eta) - tf^{(2\nu)}(0) = A_m^{(2\nu)}(\eta) - tA_m^{(2\nu)}(0)$, $\nu = 0, 1, \cdots, m-1$,
(iv) $f^{(2\nu+1)}(\eta) - tf^{(2\nu+1)}(0) \geqq A_m^{(2\nu+1)}(\eta) - tA_m^{(2\nu-1)}(0)$, $\nu = 0, 1, \cdots, m-1$.

Then we have Theorem 6.

THEOREM 6. *The trigonometric polynomial $A_m(x)$ of degree m and constant term one is the unique element in $\mathscr{F}(A_m)$ that minimizes the norm*

$$\|\Delta_m^* f\|_\infty = \sup_{0 \leqq x \leqq \eta} |\Delta_m^* f|, \qquad f \in \mathscr{F}(A_m).$$

The least values of $\|\Delta_m^* f\|_\infty$ is $(m!)^2$.

Similarly using the notation of § 5, let $\tilde{A}_n(x)$ be a generalized exponential polynomial with constant term 1, and let $\mathscr{F}(\tilde{A}_n)$ be the class of functions satisfying the following three conditions:

(i) $f \in C^{n-1}[0, \eta]$ and $f^{(n)}$ integrable on $[0, \eta]$,
(ii) $\mathscr{L}_k^* f(h) - t\mathscr{L}_k^* f(0) \geqq \mathscr{L}_k^* \tilde{A}_n(\eta) - t\mathscr{L}_k^* \tilde{A}_n(0)$, $k = 0, 1, \cdots, n-1$,
(iii) $\int_0^\eta f(x)\,dx \geqq \int_0^\eta \tilde{A}_n(x)\,dx$.

Then we can prove Theorem 7.

THEOREM 7. *The generalized trigonometric polynomial $\tilde{A}_n(x)$ with constant term 1 is the unique element in the class $\mathscr{F}(\tilde{A}_n)$ which minimizes the norm*

$$\|\mathscr{L}_n^* f\|_\infty = \sup_{0 \leqq x \leqq \eta} |\mathscr{L}_n^* f|, \qquad f \in \mathscr{F}(\tilde{A}_n),$$

and with minimum value equal to $\prod_1^n |\gamma_\nu|$.

The details of the proofs are omitted for obvious reasons. We may remark that if all γ_i's are negative, then the conditions (ii) can be replaced by

(ii′) $f^{(k)}(\eta) - tf^{(k)}(0) \geqq \tilde{A}_n^{(k)}(\eta) - t\tilde{A}_n^{(k)}(0)$, $k = 0, 1, \cdots, n-1$.

We may remark that the case when some of γ_ν's are coincident or become zero requires minor suitable modification of the above results.

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Note added in proof. We would like to thank Professor B. Kuttner for kindly sending us a simple proof of the inequality $DA_m(x; t) \leqq 0$ for $0 \leqq x \leqq \eta < \pi/m$. This implies that in (4.1) (iii) and in condition (iii) § 6 preceding Theorem 6, the sign of equality can be replaced by $\geqq$.

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A NOTE ON CARLITZ'S FORMULA*

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Abstract. In this paper we show the importance of Carlitz's formula [5] in obtaining new transformation formulas for hypergeometric series of two variables. The results (3), (6), (9) and (11) are believed to be new.

1. Introduction. Professor Carlitz [5] has proved the formula

$$F\begin{bmatrix} \alpha; -m, \beta_1; -n, \beta_2; 1, 1 \\ \beta_1+\beta_2; 1+\alpha-\beta_2-m; 1+\alpha-\beta_1-n; \end{bmatrix} = \frac{(\beta_1+\beta_2-\alpha)_{m+n}(\beta_2)_m(\beta_1)_n}{(\beta_1+\beta_2)_{m+n}(\beta_2-\alpha)_m(\beta_1-\alpha)_n}. \tag{1}$$

In (1), we have used the following notation due to Chaundy [4] to represent the hypergeometric series of higher order and of two variables:

$$F\begin{bmatrix} (a_p); (b_q); (c_r); x, y \\ (d_s); (e_h); (f_k); \end{bmatrix} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{[(a_p)]_{m+n}[(b_q)]_m[(c_r)]_n x^m y^n}{[(d_s)]_{m+n}[(e_h)]_m[(f_k)]_n m!n!}, \tag{2}$$

where (a_p) and $[(a_p)]_{m+n}$ will mean $a_1, \cdots, a_p$ and $(a_1)_{m+n}, \cdots, (a_p)_{m+n}$.

The object of this paper is to obtain transformation formulas for hypergeometric series of two variables with the help of (1). On specializing the parameters, we obtain two new summation formulas for hypergeometric series. It should be remarked here that whenever the hypergeometric functions reduce to gamma products, the results are important from the point of view of applications. Summation formulas have been used to solve optimization problems in management science [6], in the expansion of hypergeometric functions [3] and in the summation of finite and infinite series involving special functions [8]. The summation formulas (6) and (9) of this paper are likely to prove quite useful in solving problems of optimization in management science, in the expansion of functions and in summation of series.

2. In this section we discuss the transformation formulas for hypergeometric series of two variables. First of all we prove the formula

$$\begin{aligned} &F\begin{bmatrix} \beta_1+\beta_2-\alpha; \beta_2, \lambda_2; \beta_1, \lambda_1; 1, 1 \\ \beta_1+\beta_2; \mu_2; \mu_1; \end{bmatrix} \\ &\quad = \frac{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_1-\lambda_1-\beta_1+\alpha)\Gamma(\mu_2-\lambda_2-\beta_2+\alpha)}{\Gamma(\mu_1-\lambda_1)\Gamma(\mu_2-\lambda_2)\Gamma(\mu_1-\beta_1+\alpha)\Gamma(\mu_2-\beta_2+\alpha)} \\ &\quad\quad \cdot F\begin{bmatrix} \alpha; \beta_2, \lambda_1; \beta_1, \lambda_2; 1, 1 \\ \beta_1+\beta_2; \mu_1-\beta_1+\alpha; \mu_2-\beta_2+\alpha; \end{bmatrix}, \end{aligned} \tag{3}$$

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valid for $R(\mu_1 - \lambda_1 - \beta_1 + \alpha) > 0$, $R(\mu_2 - \lambda_2 - \beta_2 + \alpha) > 0$.

Proof. To prove (3), we start with the left side of (3):

$$F\begin{bmatrix}\beta_1 + \beta_2 - \alpha; \beta_2, \lambda_2; \beta_1, \lambda_1; 1, 1\\ \beta_1 + \beta_2; \mu_2; \mu_1;\end{bmatrix}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta_1 + \beta_2 - \alpha)_{m+n}(\beta_2)_m(\lambda_2)_m(\beta_1)_n(\lambda_1)_n}{(\beta_1 + \beta_2)_{m+n}(\mu_2)_m(\mu_1)_n m!n!}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\lambda_2)_m(\lambda_1)_n}{(\mu_2)_m(\mu_1)_n} \sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(\alpha)_{r+s}(\beta_1)_r(\beta_2)_s}{(\beta_1 + \beta_2)_{r+s}\,\overline{m - r}!r!}$$

$$\cdot \frac{(\beta_2 - \alpha)_{m-r}(\beta_1 - \alpha)_{n-s}}{\overline{n - s}!s!} \quad \text{(by (1))}$$

$$\text{(a)} \qquad = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\alpha)_{r+s}(\beta_1)_r(\lambda_2)_r(\beta_2)_s(\lambda_1)_s}{(\beta_1 + \beta_2)_{r+s}(\mu_2)_r(\mu_1)_s r!s!}$$

$$\cdot {}_2F_1(\lambda_2 + r, \beta_2 - \alpha; \mu_2 + r; 1)$$

$$\cdot {}_2F_1(\lambda_1 + s, \beta_1 - \alpha; \mu_1 + s; 1).$$

We use Gauss's theorem [10, p. 243. (III.3)]:

$$\text{(4)} \qquad {}_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)},$$

valid for $R(\gamma - \alpha - \beta) > 0$ and we have

$$\text{(a)} = \frac{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_2 - \lambda_2 - \beta_2 + \alpha)\Gamma(\mu_1 - \lambda_1 - \beta_1 + \alpha)}{\Gamma(\mu_1 - \lambda_1)\Gamma(\mu_2 - \lambda_2)\Gamma(\mu_1 - \beta_1 + \alpha)\Gamma(\mu_2 - \beta_2 + \alpha)}$$

$$\cdot F\begin{bmatrix}\alpha; \beta_2, \lambda_1; \beta_1, \lambda_2; 1, 1\\ \beta_1 + \beta_2; \mu_1 - \beta_1 + \alpha; \mu_2 - \beta_2 + \alpha;\end{bmatrix},$$

by interpreting the double series with the help of (2). This completes the proof of (3).

We shall mention some interesting particular cases of (3). If $\beta_1 = 0$ in (3), it reduces to a well-known formula due to Hardy [2, p. 98, Q.7]. If $\beta_1 = \mu_1, \beta_2 = \mu_2$ and we use the formula due to Appell and Kampé de Fériet [1, p. 22. (4)],

$$\text{(5)} \qquad F_1[\alpha; \beta, \gamma; \delta; 1, 1] = \frac{\Gamma(\delta)\Gamma(\delta - \alpha - \beta - \gamma)}{\Gamma(\delta - \alpha)\Gamma(\delta - \beta - \gamma)},$$

valid for $R(\delta - \alpha - \beta - \gamma) > 0$. In (3), it gives a new summation formula

$$\text{(6)} \qquad F\begin{bmatrix}\alpha; \mu_2, \lambda_1; \mu_1, \lambda_2; 1, 1\\ \mu_1 + \mu_2; \alpha; \alpha;\end{bmatrix}$$

$$= \frac{\Gamma(\alpha)\Gamma(\mu_1 - \lambda_1)\Gamma(\mu_2 - \lambda_2)\Gamma(\mu_1 + \mu_2)\Gamma(\alpha - \lambda_1 - \lambda_2)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\alpha - \lambda_1)\Gamma(\alpha - \lambda_2)\Gamma(\mu_1 + \mu_2 - \lambda_1 - \lambda_2)}$$

valid for $R(\alpha - \lambda_1) > 0$, $R(\alpha - \lambda_2) > 0$ and $R(\alpha - \lambda_1 - \lambda_2) > 0$.

In case $\lambda_1 = -n$ and $\lambda_2 = -m$ in (6), it reduces to the following form:

$$F\begin{bmatrix} \alpha; -n, \mu_2; -m, \mu_1; 1, 1 \\ \mu_1 + \mu_2; \alpha; \alpha; \end{bmatrix} = \frac{(\mu_1)_n(\mu_2)_m(\alpha)_{m+n}}{(\alpha)_n(\alpha)_m(\mu_1 + \mu_2)_{m+n}}. \tag{7}$$

Further, we put $\alpha = -n$, $\mu_1 = 1 + \lambda_1$, $\mu_2 = 1 + \lambda_2$ in (3) and using the formula due to author [9]

$$\begin{aligned} &F\begin{bmatrix} -n; \alpha_1, \beta_1; \alpha_2, \beta_2; 1, 1 \\ \alpha_1 + \alpha_2; 1 + \beta_1 - \alpha_2 - n; 1 + \beta_2 - \alpha_1 - n; \end{bmatrix} \\ &\qquad = \frac{(\alpha_1 + \alpha_2 - \beta_1 - \beta_2)_n(\alpha_1)_n(\alpha_2)_n}{(\alpha_1 + \alpha_2)_n(\alpha_1 - \beta_2)_n(\alpha_2 - \beta_1)_n} \end{aligned} \tag{8}$$

in (3), it yields another new summation formula

$$\begin{aligned} &F\begin{bmatrix} \beta_1 + \beta_2 + n; \beta_2, \lambda_2; \beta_1, \lambda_1, 1, 1 \\ \beta_1 + \beta_2; 1 + \lambda_2; 1 + \lambda_1; \end{bmatrix} \\ &\qquad = \frac{\Gamma(1 + \lambda_1)\Gamma(1 + \lambda_2)\Gamma(1 - \beta_1)\Gamma(1 - \beta_2)(\beta_1 + \beta_2 - \lambda_1 - \lambda_2)_n}{\Gamma(1 + \lambda_2 - \beta_2)\Gamma(1 + \lambda_1 - \beta_1)(\beta_1 + \beta_2)_n} \end{aligned} \tag{9}$$

valid for $R(1 - \beta_1 - n) > 0$, $R(1 - \beta_2 - n) > 0$ and $n = 0, 1, \cdots$. In case $\lambda_1 = 0$ or $\lambda_2 = 0$ in (9), it reduces to a known result due to Kala and Saxena [7, p. 234, (10)].

In place of Gauss's theorem (4), if we use Saalschutz's theorem [10, p. 49, (2.3.1.4)]

$${}_3F_2\begin{bmatrix} -n, \alpha, \beta; 1 \\ \gamma, 1 + \alpha + \beta - \gamma - n; \end{bmatrix} = \frac{(\gamma - \alpha)_n(\gamma - \beta)_n}{(\gamma)_n(\gamma - \alpha - \beta)_n}, \tag{10}$$

and proceed on the same lines, we get the following new transformation for a double series:

$$\begin{aligned} &F\begin{bmatrix} \beta_1 + \beta_2 - \alpha; -n, a_1, \beta_1; -m, a_2, \beta_2; 1, 1 \\ \beta_1 + \beta_2; c_1, 1 + a_1 + \beta_1 - \alpha - c_1 - n; c_2, 1 + a_2 + \beta_2 - \alpha - c_2 - m; \end{bmatrix} \\ &\quad = \frac{(c_1 - a_1)_n(c_2 - a_2)_m(c_1 - \beta_1 + \alpha)_n(c_2 - \beta_2 + \alpha)_m}{(c_1)_n(c_2)_m(c_1 - a_1 - \beta_1 + \alpha)_n(c_2 - a_2 - \beta_2 + \alpha)_m} \\ &\qquad \cdot F\begin{bmatrix} \alpha; -n, a_1, \beta_2; -m, a_2, \beta_1; 1, 1 \\ \beta_1 + \beta_2; c_1 - \beta_1 + \alpha, 1 - c_1 + a_1 - n; \\ c_2 - \beta_2 + \alpha, 1 - c_2 + a_2 - m; \end{bmatrix}. \end{aligned} \tag{11}$$

If we put $m = 0$ in (11), we get

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix}\beta_1, a, \beta_1+\beta_2-\alpha, -n;\\ \beta_1+\beta_2, c, 1+a+\beta_1-\alpha-c-n;\end{matrix}\right]\\
&\qquad = \frac{(c-a)_n(c-\beta_1+\alpha)_n}{(c)_n(c-a-\beta_1+\alpha)_n}\\
&\qquad\quad \cdot {}_4F_3\left[\begin{matrix}-n, a, \alpha, \beta_2; 1\\ \beta_1+\beta_2, c-\beta_1+\alpha, 1-c+a-n;\end{matrix}\right].
\end{aligned} \tag{12}
$$

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ORTHOGONAL POLYNOMIAL EXPANSIONS WITH NONNEGATIVE COEFFICIENTS*

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Abstract. This paper presents general theorems implying that the coefficients in the expansions of one set of orthogonal polynomials in terms of another are positive or nonnegative. These theorems imply several results, previously obtained by special arguments, for the classical orthogonal polynomials. A special case of one of the theorems settles affirmatively a conjecture of Askey.

1. Introduction. The question of when the coefficients in the expansions

$$q_n(x) = \sum_{r=0}^{n} a_{rn} p_r(x) \tag{1}$$

of one set of orthogonal polynomials in terms of another are nonnegative has been studied in several recent papers (for examples and applications, see the references); in addition, there are several older results on this question for the classical orthogonal polynomials. Most of these results have been obtained by special arguments, often involving explicit computation of the coefficients. Askey [3] and Askey and Gasper [5] observed as recently as 1971 that there were only two general theorems [4], [11] implying nonnegativity of the coefficients in (1), and that many of the classical results had not been shown to follow from them. Since then, the author [8] has considered the case where $\{p_n(x)\}$ and $\{q_n(x)\}$ are orthogonal with respect to distributions $du(x)$ and $dv(x) = w(x)\,du(x)$, and has given conditions on $w(x)$ which imply that the coefficients in (1) are nonnegative for all n, while those in the "inverse" expansions

$$p_n(x) = \sum_{r=0}^{n} b_{rn} q_r(x)$$

alternate in sign; i.e., $(-1)^{n-r} b_{rn} \geqq 0$.

2. Main results. Here we present general theorems which imply several known results on the classical polynomials. Our starting point is the following lemma.

LEMMA 1. *For $s = 0, 1, \cdots, n$, let $p_s(x)$ be a polynomial of degree s with s roots in an interval (a, b). Suppose $x_0 \notin (a, b)$, $p_s(x_0) > 0$ $(0 \leqq s \leqq n)$, and $Q(x)$ is a polynomial of degree n. Then*

$$Q(x) = \sum_{s=0}^{n} c_s p_s(x), \qquad c_s \geqq 0, \quad 0 \leqq s \leqq n, \tag{2}$$

if there is a distribution function $F(x)$ with at least $n + 1$ points of increase in (a, b) such that $\int_a^b x^k\, dF(x)$ exists for $0 \leqq k \leqq 2n$,

$$(-1)^j \int_a^b |x - x_0|^j p_s(x)\, dF(x) \leqq 0, \qquad 0 \leqq j < s \leqq n, \tag{3}$$

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and

$$(4) \qquad (-1)^j \int_a^b |x - x_0|^j Q(x)\, dF(x) \geqq 0, \qquad 0 \leqq j \leqq n.$$

Moreover, $c_s > 0$ $(0 \leqq s \leqq n)$ *if at least one of the inequalities in* (3) *is strict for each s.*

Proof. Descartes' rule of signs implies that

$$p_r(x) = \sum_{j=0}^{r} p_{jr}|x - x_0|^j, \qquad a < x < b,$$

with

$$(5) \qquad (-1)^j p_{jr} > 0;$$

therefore (3) and symmetry imply that

$$(6) \qquad \int_a^b p_r(x)p_s(x)\, dF(x) \leqq 0, \qquad 0 \leqq r, s \leqq n, \quad r \neq s,$$

and (4) implies that

$$(7) \qquad \int_a^b p_r(x)Q(x)\, dF(x) \geqq 0, \qquad 0 \leqq r \leqq n.$$

From (2), $c_0, \cdots, c_n$ satisfy the system

$$\int_a^b p_r(x)Q(x)\, dF(x) = \sum_{s=0}^{n} c_s \int_a^b p_r(x)p_s(x)\, dF(x), \qquad 0 \leqq r \leqq n,$$

which, since $F(x)$ has at least $n + 1$ points of increase, has a positive definite Gram matrix G with nonpositive off-diagonal elements (cf. (6)). Stieltjes [7] (see also [10, § 3.5]) showed that the inverse of such a matrix is nonnegative. This and (7) imply that $c_s \geqq 0$, $0 \leqq s \leqq n$. If at least one of the inequalities in (3) is strict for each s, then all of those in (6) are strict because of (5), and so the off-diagonal elements of G are negative; in this case, Stieltjes' result implies that $G^{-1} > 0$. Since at least one of the inequalities in (7) must be strict, it then follows that $c_s > 0$, $0 \leqq s \leqq n$.

The idea of applying Stieltjes' theorem here came from a paper by M. W. Wilson [11].

Except where stated otherwise, we assume throughout the rest of the paper that $\{p_r(x)\}$ and $\{q_r(x)\}$ are orthogonal over a finite or semi-infinite interval (a, b) with respect to distributions $du(x)$ and $dv(x)$, respectively, and normalized so as to be positive at some point $x_0 \notin (a, b)$. It is to be understood that the distributions have enough moments and points of increase so that the polynomials are defined and unique up to normalization.

For convenience below, we state the following obvious "principle of composition": *If* $p_k(x)$, $q_k(x)$ *and* $r_k(x)$ *are polynomials of exact degree* $k = 0, 1, 2, \cdots$ *such that*

$$q_k(x) = \sum_{i=0}^{k} a_{ik} p_i(x)$$

and

$$r_k(x) = \sum_{i=0}^{k} b_{ik} q_i(x),$$

with

(8) (a) $a_{ik} \geqq 0$, (b) $b_{ik} \geqq 0$, $0 \leqq i \leqq k \leqq n$,

then

$$r_k(x) = \sum_{i=0}^{k} c_{ik} p_i(x),$$

with

(9) $c_{ik} \geqq 0, \qquad 0 \leqq i \leqq k \leqq n;$

moreover, if the inequalities in either (8a) *or* (8b) *are all strict, then so are those in* (9).

THEOREM 1. *If*

(10) $$(-1)^j \int_a^b |x - x_0|^j p_s(x)\, dv(x) \leqq 0, \qquad 0 \leqq j < s \leqq n,$$

then $a_{rn} \geqq 0$ *in* (1); *moreover,* $a_{rn} > 0$ *if at least one of the inequalities in* (10) *is strict for each s.*

Proof. The polynomials $\{p_s(x)\}$ satisfy the conditions of Lemma 1, with $F(x) = v(x)$. Since

$$\int_a^b (x - x_0)^j q_n(x)\, dv(x) = \frac{\delta_{jn} n!}{q_n^{(n)}(x_0)} \int_a^b (q_n(x))^2\, dv(x), \qquad 0 \leqq j \leqq n,$$

inequality (4) also holds with $F(x) = v(x)$ and $Q(x) = q_n(x)$; to see this, observe that since $q_n(x)$ has n roots in (a, b), Descartes' rule of signs implies that $q_n^{(n)}(x_0) > 0$ if $x_0 \geqq b$ and $(-1)^n q_n^{(n)}(x_0) > 0$ if $x_0 \leqq a$. Now the conclusion follows from Lemma 1.

Because of the difficulty of verifying (10), Theorem 1 may be too general to yield specific results; however, the following special case is applicable, as we will see below from examples.

THEOREM 2. *Suppose*

(11) $$dv(x) = \sigma(x)\, du(x),$$

where $\sigma(x)$ *is nonnegative* ($\not\equiv 0$) *and n times differentiable on* (a, b). *If* $x_0 \leqq a$, *then* $a_{rn} \geqq 0$ *in* (1) *if*

(12) $$(-1)^{s-j}[(x - x_0)^j \sigma(x)]^{(s)} \leqq 0, \qquad a < x < b, \quad 0 \leqq j < s \leqq n;$$

moreover, $a_{rn} > 0$ *if at least one of the inequalities in* (12) *is strict for each s. The same conclusions hold if* $x_0 \geqq b$, *and* (12) *is replaced by*

(13) $$[(x - x_0)^j \sigma(x)]^{(s)} \leqq 0, \qquad a < x < b, \quad 0 \leqq j < s \leqq n.$$

Proof. Let $h(x)$ be the projection of $|x - x_0|^j\sigma(x)$ on the space of polynomials of degree $\leqq s$ ($\leqq n$) with respect to the inner product

$$(f, g) = \int_a^b f(x)g(x)\,du(x);$$

thus,

$$h(x) = b_0p_0(x) + \cdots + b_sp_s(x),$$

where

$$b_s = \frac{1}{\|p_s\|^2}\int_a^b |x - x_0|^j\sigma(x)p_s(x)\,du(x). \tag{14}$$

Since $x_0 \notin (a, b)$, $|x - x_0|^j\sigma(x)$ has n derivatives on (a, b). Moreover, the function

$$|x - x_0|^j\sigma(x) - h(x)$$

is orthogonal to every polynomial of degree $\leqq s$, and therefore has at least $s + 1$ zeros in (a, b); hence, its sth derivative has at least one, and so

$$([|x - x_0|^j\sigma(x)]^{(s)} - b_sp_s^{(s)}(x_0))|_{x=x_1} = 0$$

for some x_1 in (a, b), which implies that the sign of b_s is the same as that of

$$p_s^{(s)}(x_0)([|x - x_0|^j\sigma(x)]^{(s)})|_{x=x_1}.$$

If $x_0 \leqq a$, then $(-1)^sp_s^{(s)}(x_0) > 0$, and (10) follows from (11), (12) and (14); if $x_0 \geqq b$, then $p_s^{(s)}(x_0) > 0$, and (10) follows from (11), (13) and (14). Theorem 1, therefore, implies that $a_{rn} \geqq 0$ in (1) in either case. It is straightforward to verify that the statements concerning strict positivity of a_{rn} also follow from Theorem 1.

The following corollary settles affirmatively a conjecture of Askey [1], [3]; its proof has also been given separately elsewhere [9].

COROLLARY 1. *If $c > 0$ and*

$$dv(x) = |x - x_0|^c\,du(x),$$

then $a_{rn} > 0$ in (1) for all n.

Proof. With $\sigma(x) = |x - x_0|^c$ and $0 < c < 1$, the inequalities in (12) hold strictly for all n if $x_0 \leqq a$, and those in (13) hold strictly for all n if $x_0 \geqq b$. This gives the result for $0 < c < 1$, and it follows from this for all positive c, by the principle of composition.

Example 1. Corollary 1 implies known results for the Laguerre polynomials $\{L_r^\alpha(x)\}$ and the Jacobi polynomials $\{P_r^{(\alpha,\beta)}(x)\}$ (for definitions, see [6]); namely, that

$$L_n^{\alpha+\mu}(x) = \sum_{r=0}^{n} a_{rn}L_r^\alpha(x),$$

with $a_{rn} > 0$ if $\mu > 0$ and $\alpha > -1$, and that

$$P_n^{(\alpha+\mu,\beta)}(x) = \sum_{r=0}^{n} b_{rn} P_r^{(\alpha,\beta)}(x)$$

if $\mu > 0$ and $\alpha, \beta > -1$.

Askey [1] cited these results as evidence supporting his conjecture of Corollary 1.

COROLLARY 2. *Suppose $x_1, x_2, \cdots, x_m$ are in an interval I which does not intersect (a, b), $\{p_r(x)\}$ and $\{q_r(x)\}$ are normalized so as to be positive on I, and*

$$dv(x) = \sigma(x)\, du(x),$$

where

$$\sigma(x) = \sum_{k=1}^{m} b_k |x - x_k|^{c_k} \tag{15}$$

with

$$\text{(a)}\quad b_k > 0, \qquad \text{(b)}\quad 0 < c_k < 1, \qquad k = 1, \cdots, m. \tag{16}$$

Then $a_{rn} > 0$ in (1) for all n.

Proof. If $a > -\infty$ and $x_k \leqq a$, then

$$\begin{aligned} &(-1)^{s-j}[(x-a)^j (x - x_k)^{c_k}]^{(s)} \\ &= (-1)^{s-j} \sum_{i=0}^{j} \binom{j}{i} (x_k - a)^{j-i} [(x - x_k)^{c_k + i}]^{(s)} \\ &= (-1)^{s-j} s! \sum_{i=0}^{j} \binom{j}{i} (x_k - a)^{j-i} \binom{c_k + i}{s} (x - x_k)^{c_k + i - s}. \end{aligned} \tag{17}$$

Because of (16b),

$$(-1)^{s-i} \binom{c_k + i}{s} < 0, \qquad i = 0, \cdots, s-1,$$

and, therefore, the last member of (17) is negative if $0 \leqq j < s$ and $x \geqq a$. Now (16a) implies that $\sigma(x)$ satisfies (12) (with strict inequality) for all n if $I = (-\infty, a]$, and so the conclusion follows from Theorem 2. The proof for the case where $I = [b, \infty)$ is similar.

Corollary 1 and the following lemma enable us to improve on a result obtained in [8].

LEMMA 2. *Suppose $y_0 \notin (a, b)$ and*

$$(-1)^r p_r(y_0) > 0, \qquad (-1)^r q_r(y_0) > 0, \qquad r = 0, 1, \cdots. \tag{18}$$

Let m be a positive integer, and suppose the distribution

$$dv(x) = \frac{du(x)}{|x - y_0|^m}$$

has moments of all orders on (a, b). *Then* $a_{rn} \geqq 0$ *in* (1) *for all* n.

Proof. For $m = 1$, it is known [6, Thm. 3.1.4, § 3.1] that

$$q_n(x) = A_n p_n(x) - B_n \frac{q_{n-1}(y_0)}{q_n(y_0)} p_{n-1}(x), \qquad (A_n, B_n > 0),$$

and the conclusion follows from (18); it follows for all positive integrals m from this and the principle of composition.

Lemma 2 is not valid for arbitrary positive m. For a counter-example, see [8].

The following theorem improves on Theorem 1 of [8].

THEOREM 3. *Suppose* (a, b) *is finite. Let* I *be one of the intervals* $(-\infty, a]$ *or* $[b, \infty)$, *and let* J *be the other. Suppose* $p_r(x) > 0$ *and* $q_r(x) > 0$ $(r = 0, 1, \cdots)$ *for* x *in* I. *Let* $x_1, \cdots, x_r$ *be in* I *and* $z_1, \cdots, z_s$ *be in* J, *and define*

$$dv(x) = \frac{\prod_{i=1}^{r} |x - x_i|^{c_i}}{\prod_{j=1}^{s} |x - z_j|^{m_j}} du(x), \qquad a < x < b,$$

where $m_1, \cdots, m_s$ *are nonnegative integers and* $c_1, \cdots, c_s$ *are arbitrary nonnegative numbers. Suppose* $du(x)$ *and* $dv(x)$ *have moments of all orders on* (a, b). *Then* $a_{rn} \geqq 0$ *in* (1) *for all* n; *moreover,* $a_{rn} > 0$ *if at least one of* $c_1, \cdots, c_s$ *is positive.*

The proof consists of a straightforward application of Corollary 1, Lemma 2, and the principle of composition. (Notice that the assumptions imply that $(-1)^r \cdot p_r(x) > 0$ and $(-1)^r q_r(x) > 0$ on J, so that Lemma 2 is applicable.)

Example 2. With

$$du(x) = (1 - x)^{\alpha}(1 + x)^{\beta}\, dx$$

and

$$dv(x) = (1 - x)^{\mu}(1 + x)^{-k}\, du(x),$$

Theorem 3 implies a known result for the Jacobi polynomials; namely, that

$$P_n^{(\alpha+\mu,\beta-k)}(x) = \sum_{r=0}^{n} a_{rn} P_r^{(\alpha,\beta)}(x),$$

with $a_{rn} > 0$ if $\mu > 0$, k is a nonnegative integer, $\alpha > -1$ and $\beta > k - 1$.

Example 3. We introduce a class of orthogonal polynomials which includes Jacobi's and Heine's polynomials [6]. Suppose $k \geqq 2$ and $a_1 < a_2 < \cdots < a_k$. Let v be a fixed integer in $\{1, \cdots, k - 1\}$ and let $A = (\alpha_1, \cdots, \alpha_k)$ be a k-tuple of real numbers restricted only by the requirement that $\alpha_v > -1$ and $\alpha_{v+1} > -1$. Let $\{P_r^{(A)}(x)\}$ be a sequence of polynomials orthogonal over $[a_v, a_{v+1}]$ with respect to

$$du(x) = \prod_{j=1}^{k} |x - a_j|^{\alpha_j}\, dx,$$

and normalized so that $P_r^{(A)}(\infty) = \infty$. Then Theorem 1 implies that

$$P_n^{(B)}(x) = \sum_{r=0}^{n} a_{rn} P_r^{(A)}(x),$$

with $a_{rn} \geqq 0$ for all n if

$$B = (\alpha_1 - k_1, \cdots, \alpha_\nu - k_\nu, \alpha_{\nu+1} + \mu_{\nu+1}, \cdots, \alpha_k + \mu_k),$$

provided $k_1, \cdots, k_\nu$ are nonnegative integers, $k_\nu < 1 + \alpha_\nu$, and $\mu_{\nu+1}, \cdots, \mu_k$ are arbitrary nonnegative numbers; moreover, $a_{rn} > 0$ if at least one of the latter is positive.

3. Special results concerning even distributions. The case where $du(x)$ and $dv(x)$ are even distributions deserves special attention. If

$$(a, b) = (-R, R), \quad u(-x) = -u(x), \quad v(-x) = -v(x), \tag{19}$$

then

$$p_n(-x) = (-1)^n p_n(x), \qquad q_n(-x) = (-1)^n q_n(x),$$

and it is appropriate to consider separately the expansions

$$q_{2n}(x) = \sum_{r=0}^{n} b_{rn} p_{2r}(x) \tag{20}$$

and

$$q_{2n+1}(x) = \sum_{r=0}^{n} c_{rn} p_{2r+1}(x). \tag{21}$$

In this case, the sequences $\{P_n(y)\}$ and $\{Q_n(y)\}$, defined by

$$P_n(y) = p_{2n}(y^{1/2}), \qquad Q_n(y) = q_{2n}(y^{1/2}),$$

are orthogonal over $(0, R^2)$ with respect to $du(y^{1/2})$ and $dv(y^{1/2})$, and the sequences $\{\tilde{P}_n(y)\}$ and $\{\tilde{Q}_n(y)\}$, defined by

$$\tilde{P}_n(y) = y^{-1/2} p_{2n+1}(y^{1/2}), \qquad \tilde{Q}_n(y) = y^{-1/2} q_{2n+1}(y^{1/2}),$$

are orthogonal over $(0, R^2)$ with respect to $y\,du(y^{1/2})$ and $y\,dv(y^{1/2})$. Our earlier results, applied separately to these two pairs of sequences, yield conclusions not directly obtainable by considering $\{p_n(x)\}$ and $\{q_n(x)\}$.

The next two theorems follow from Theorem 1.

THEOREM 4. *Suppose* (19) *holds and*

$$p_r(R) > 0, \qquad q_r(R) > 0, \qquad r = 0, 1, \cdots. \tag{22}$$

Then: (i) $b_{rn} \geqq 0$ *in* (20) *if*

$$\int_0^R (x^2 - x_1^2)^j p_{2r}(x)\, dv(x) \leqq 0, \qquad 0 \leqq j < r \leqq n,$$

for some $x_1 \geqq R$, *and* $b_{rn} > 0$ *if at least one of these inequalities is strict for each* r; (ii) $c_{rn} \geqq 0$ *in* (21) *if*

$$\int_0^R (x^2 - x_2^2)^j x p_{2j+1}(x)\, dv(x) \leqq 0, \qquad 0 \leqq j < r \leqq n,$$

for some $x_2 \geqq R$, *and* $c_{rn} > 0$ *if at least one of these inequalities is strict for each r.*

THEOREM 5. *Suppose* (19) *holds and*

$$p_{2r}(0) > 0, \qquad q_{2r}(0) > 0, \qquad r = 0, 1, \cdots. \tag{23}$$
$$p'_{2r+1}(0) > 0, \qquad q'_{2r+1}(0) > 0,$$

Then: (i) $b_{rn} \geqq 0$ *in* (20) *if*

$$(-1)^j \int_0^R (x^2 + x_1^2)^j p_{2r}(x)\, dv(x) \leqq 0, \qquad 0 \leqq j < r \leqq n,$$

for some number x_1, *and* $b_{rn} > 0$ *if at least one of these inequalities is strict for each* r; (ii) $c_{rn} \geqq 0$ *in* (21) *if*

$$(-1)^j \int_0^R (x^2 + x_2^2)^j x p_{2r+1}(x)\, dv(x) \leqq 0, \qquad 0 \leqq j < r \leqq n,$$

for some x_2, *and* $c_{rn} > 0$ *if at least one of these inequalities is strict for each r.*

The next theorem follows from Theorem 2.

THEOREM 6. *Suppose* (19) *holds and*

$$dv(x) = \rho(x^2)\, du(x),$$

where $\rho(y)$ *has n derivatives on* $(0, R^2)$. *Then*: (i) $b_{rn} \geqq 0$ *and* $c_{rn} \geqq 0$ *in* (20) *and* (21) *if* (23) *holds and*

$$(-1)^{s-j}[(y + \gamma^2)^j \rho(y)]^{(s)} \leqq 0, \qquad 0 < y < R^2, \quad 0 \leqq j < s \leqq n, \tag{24}$$

for some number γ; *moreover,* $b_{rn} > 0$ *and* $c_{rn} > 0$ *if at least one of the inequalities in* (24) *is strict for each s.* (ii) *The same conclusions hold if* (22) *holds and* (24) *is replaced by*

$$[(y - \gamma^2)^j \rho(y)]^{(s)} \leqq 0, \qquad 0 < y < R^2, \quad 0 \leqq j < s \leqq n,$$

for some $\gamma \geqq R$.

Corollaries 1 and 2 and Theorem 3 can also be adapted to the special case (19). We present only the following adaptation of Corollary 1.

COROLLARY 3. *Suppose* (19) *holds. Then* $a_{rn} > 0$ *and* $b_{rn} > 0$ *for all n in* (20) *and* (21) *if*: (i) (22) *holds and*

$$dv(x) = (x_0^2 - x^2)^c\, du(x)$$

with $c > 0$ *and* $x_0 \geqq R$; *or,* (ii) (23) *holds and*

$$dv(x) = (x_0^2 + x^2)^c\, du(x)$$

where $c > 0$ *and* x_0 *is any real number.*

Example 4. By taking

$$du(x) = (1 - x^2)^\alpha\, dx$$

and

$$dv(x) = (1 - x^2)^\mu\, du(x),$$

we obtain from (i) of Corollary 3 the following known result for Jacobi polynomials, due to Gegenbauer:

$$P_{2n}^{(\alpha+\mu,\alpha+\mu)}(x) = \sum_{r=0}^{n} b_{rn} P_{2r}^{(\alpha,\alpha)}(x)$$

and

$$P_{2n+1}^{(\alpha+\mu,\alpha+\mu)}(x) = \sum_{r=0}^{n} c_{rn} P_{2r+1}^{(\alpha,\alpha)}(x),$$

where $b_{rn} > 0$ and $c_{rn} > 0$ for all n if $\alpha > -1$ and $\mu > 0$.

Example 5. As applications of Corollary 3 (ii), we show that

$$P_n^{(\alpha,\beta)}(2x^2 - 1) = \sum_{r=0}^{n} (-1)^{n-r} \phi_{rn} P_{2r}^{(\alpha,\alpha)}(x) \tag{25}$$

with $\phi_{rn} > 0$ for all n if $\alpha > -1$ and $\beta > -1/2$, and

$$xP_n^{(\alpha,\beta)}(2x^2 - 1) = \sum_{r=0}^{n} (-1)^{n-r} \psi_{rn} P_{2r+1}^{(\alpha,\alpha)}(x) \tag{26}$$

with $\psi_{rn} > 0$ for all n if $\alpha < -1$ and $\beta > 1/2$.

By substituting $y = 2x^2 - 1$ in the orthogonality relation

$$\int_{-1}^{1} P_r^{(\alpha,\beta)}(y) P_s^{(\alpha,\beta)}(y)(1 - y)^\alpha (1 + y)^\beta \, dy = 0, \qquad r \neq s,$$

and using the evenness of the resulting integrand, we find that

$$\int_{-1}^{1} P_r^{(\alpha,\beta)}(2x^2 - 1) P_s^{(\alpha,\beta)}(2x^2 - 1)(1 - x^2)^\alpha (x^2)^{\beta+1/2} \, dx = 0, \qquad r \neq s. \tag{27}$$

This can be interpreted to mean that $\{P_r^{(\alpha,\beta)}(2x^2 - 1)\}$ is the "even-degree" subsequence of a sequence of polynomials orthogonal over $(-1, 1)$ with respect to

$$dv(x) = (1 - x^2)^\alpha (x^2)^{\beta+1/2} \, dx.$$

Since $\{P_r^{(\alpha,\alpha)}(x)\}$ is orthogonal over $(-1, 1)$ with respect to

$$du(x) = (1 - x^2)^\alpha \, dx,$$

we infer the stated conclusion concerning (25) by applying (ii) of Corollary 3, with $x_0 = 0$ and $c = \beta + 1/2$, to the sequences $\{p_{2n}(x)\}$ and $\{q_{2n}(x)\}$ defined by

$$p_{2n}(x) = (-1)^n P_{2n}^{(\alpha,\alpha)}(x)$$

and

$$q_{2n}(x) = (-1)^n P_n^{(\alpha,\beta)}(2x^2 - 1).$$

(The factor $(-1)^n$ adjusts the normalizations of $\{p_{2n}(x)\}$ and $\{q_{2n}(x)\}$ so that they satisfy (23), as required in (ii).)

To prove the assertion concerning (26), we interpret (27) to mean that $\{xP_n^{(\alpha,\beta)}(2x^2 - 1)\}$ is the "odd-degree" subsequence of a sequence of polynomials orthogonal over $(-1, 1)$ with respect to

$$dv(x) = (1 - x^2)^\alpha (x^2)^{\beta - 1/2}\, dx,$$

and apply (ii) of Corollary 3, with $x_0 = 0$ and $c = \beta - 1/2$, to the sequences $\{\tilde{P}_{2n+1}(x)\}$ and $\{\tilde{q}_{2n+1}(x)\}$ defined by

$$\tilde{p}_{2n+1}(x) = (-1)^n P_{2n+1}^{(\alpha,\alpha)}(x)$$

and

$$\tilde{q}_{2n+1}(x) = (-1)^n x P_n^{(\alpha,\beta)}(2x^2 - 1).$$

The results in this example can also be deduced from earlier known properties of the Jacobi polynomials.

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NOTE ON RECIPROCITY RELATIONS ASSOCIATED WITH A LINEAR HYPERBOLIC, PARTIAL DIFFERENTIAL EQUATION*

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Many problems of mathematical physics are special cases of the problem defined as follows. ϕ is a function of position $\underline{x}$ and time t which satisfies the differential equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(p\frac{\partial \phi}{\partial x_i}\right) - \alpha\phi - \beta\frac{\partial \phi}{\partial t} - \gamma\frac{\partial^2 \phi}{\partial t^2} = f, \qquad x \in D, \quad t > 0, \tag{1}$$

subject to the initial conditions

$$[\phi]_{t=0} = f_1, \qquad x \in \bar{D}, \tag{2}$$

$$\left[\frac{\partial \phi}{\partial t}\right]_{t=0} = f_2, \qquad x \in \bar{D}, \tag{3}$$

and the boundary conditions

$$\phi = g_1, \qquad x \in \Gamma_1, \quad t > 0, \tag{4}$$

$$\frac{\partial \phi}{\partial v} + k\phi = g_2, \qquad x \in \Gamma_2, \quad t > 0, \tag{5}$$

$p, \alpha, \beta, \gamma, f_1, f_2$ are all functions of position only and f, g_1, g_2 are functions of position and time. $\bar{D}$ is the closure of D and its boundary ∂D which comprises the disjoint union of two portions Γ_1 and Γ_2—either of which may comprise the whole of the boundary.

The conditions for the uniqueness of the solution to this problem have been discussed by Chambers (1970), Murray (1972) and Murray (1974) for the case p unity. (The same conditions are clearly valid if p is real and positive everywhere except over a set of measure zero and differentiable everywhere.) α, γ, k must be real and nonnegative. β may be complex, but its real part must be nonnegative. The uniqueness of the solution is equivalent to saying that if all of f, f_1, f_2, g_1, g_2 are zero then ϕ is zero. Thus each of f, f_1, f_2, g_1, g_2 may be regarded in a generalized sense as a source. (f, f_1, f_2, g_1, g_2) may be referred to as a source system. The question of reciprocity relationships in such time-dependent problems have been considered in a few special cases by Aĭnola (1967) and Murray (1972) but no systematic listing of reciprocity relationships appears to have been given.

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If the convolution of two functions is defined in the usual way by

$$\phi^*\psi = \int_0^t \phi(t-\tau)\psi(\tau)\,d\tau = \psi^*\phi \tag{6}$$

and, if superfixes A and B refer to two different source systems and the associated excitations, by applications of Green's Theorem it can be shown that

$$\begin{aligned}\int_{\Gamma_1}\left(g_1^A * \frac{\partial\phi^B}{\partial v} - g_1^B * \frac{\partial\phi^A}{\partial v}\right)p\,d\sigma + \int_{\Gamma_2}(g_2^B * \phi^A - g_2^A * \phi^B)p\,d\sigma \\ = \int_D (f^A * \phi^B - f^B * \phi^A)\,d\tau + \int_D (f_2^A\phi^B - f_2^B\phi^A)\gamma\,d\tau \\ + \int_D\left[f_1^A\left(\beta\phi^B + \gamma\frac{\partial\phi^B}{\partial t}\right) - f_1^B\left(\beta\phi^A + \gamma\frac{\partial\phi^A}{\partial t}\right)\right]d\tau.\end{aligned} \tag{7}$$

By putting all except one of the different source functions f, f_1, f_2, g_1, g_2 zero in turn, five different reciprocity relations follow. It will be noted that if γ is zero everywhere, there is not any reciprocity relationship involving source functions of type f_2. This would be expected for if γ vanishes, equation (1) would only need one initial condition for its solution. The relations are

$$\int_D f^A * \phi^B\,d\tau = \int_D f^B * \phi^A\,d\tau, \tag{8a}$$

$$\int_D f_1^A\left(\beta\phi^B + \gamma\frac{\partial\phi^B}{\partial t}\right)d\tau = \int_D f_1^B\left(\beta\phi^A + \gamma\frac{\partial\phi^A}{\partial t}\right)d\tau, \tag{8b}$$

$$\int_D f_2^A\phi^B\,d\tau = \int_D f_2^B\phi^A\,d\tau, \tag{8c}$$

$$\int_{\Gamma_1} g_1^A * \frac{\partial\phi^B}{\partial v}p\,d\sigma = \int_{\Gamma_1} g_1^B * \frac{\partial\phi^A}{\partial v}p\,d\sigma, \tag{8d}$$

$$\int_{\Gamma_2} g_2^A * \phi^B p\,d\sigma = \int_{\Gamma_2} g_2^B * \phi^A p\,d\sigma. \tag{8e}$$

By applications of Green's theorem it may also be shown that, if $G(\underset{\sim}{x}, \underset{\sim}{x}', t)$ is the Greens function defined by

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(p\frac{\partial G}{\partial x_i}\right) - \alpha G - \beta\frac{\partial G}{\partial t} - \gamma\frac{\partial^2 G}{\partial t^2} = \delta(\underset{\sim}{x} - \underset{\sim}{x}')\delta(t), \qquad x, x' \in D, \quad t > 0, \tag{9}$$

the initial conditions

$$G = 0, \qquad x \in \bar{D}, \tag{10}$$

$$\frac{\partial G}{\partial t} = 0, \qquad x \in \bar{D}, \tag{11}$$

and the boundary conditions

(12)
$$G = 0, \qquad x \in \Gamma_1, \quad t > 0,$$

(13)
$$\frac{\partial G}{\partial v} + kG = 0, \qquad x \in \Gamma_2, \quad t > 0,$$

then

$$\phi = f_1(x) + tf_2(x) + \int_D G * [f(x',t) - f_1(x') - tf_2(x')]\, d\tau'$$

(14)
$$- \int_{\Gamma_1} \frac{\partial G}{\partial v} * [g_1(x',t) - f_1(x') - tf_2(x')]p\, d\sigma'$$

$$+ \int_{\Gamma_2} G * \left[g_2(x',t) - \frac{\partial f_1(x')}{\partial v'} + kf_1(x') - t\left\{ \frac{\partial f_2(x')}{\partial v'} + kf_2(x') \right\} \right] p\, d\sigma'.$$

Note. The full analysis can be obtained from the author.

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DISTRIBUTION THEORY AS THE BASIS OF A POSTULATIONAL FOUNDATION OF LINEAR OPTICAL SYSTEMS*

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Abstract. Linear operator theory in the framework of Schwartz's distribution theory is applied in order to model the physical behavior of the optical linear system. A distinction is made between coherently and incoherently illuminated systems. The incoherent system admits a unique set of postulates which leads to new concepts of positivity and an L^1-type of passivity. On the other hand, the coherent system exhibits a characterization essentially similar to the scattering formalism of the electrical network.

1. Introduction. When linear operators are applied to model the behavior of a physical system, the common example treated is the electrical network. Youla, Castriota and Carlin [7], Zemanian [8], Wohlers and Beltrami [6] propose sets of postulates motivated by the physical nature of the electrical network. These postulates are imposed on the operator, and their effect on the characterization of the operator is then analyzed.

In this work, the optical system is investigated. It may serve as an additional physical example for the application of linear operator theory. The optical system constributes two types of linear systems depending on the type of illumination, whether it is coherent or incoherent. As will be shown, the coherently illuminated optical system provides a set of postulates which resembles the scatter-passive formalism of the electrical network. However, the physical behavior of the incoherently illuminated system is described in terms of a unique set of postulates leading to new concepts of positivity and passivity.

The physics of the optical system under consideration is outlined in § 2. In § 3 the coherent system is briefly treated comparing it to the familiar electrical network. Section 4 constitutes the main part of the work. It is devoted to the unique postulates connected with the incoherent system and to the characterizations of the operator which follow. The framework of the analysis is Schwartz's distribution theory. This is compatible with the abovementioned works on electrical systems.

2. The physics of the optical system. The optical system under consideration consists of optical materials which perform a linear operation on the electromagnetic field. These are composed of lenses, glass surfaces, pupils, partially absorbing materials, prisms, etc. Nonlinear devices like photomultipliers are excluded. Two physical planes are used to display the input and output functions, respectively. The input function is introduced in the object plane whose two spatial coordinates (x, y) constitute the two-dimensional argument of the input function. The output function is obtained on the image plane and is again a function of the two spatial coordinates (ξ, η) of the plane. In the medium between these two planes, various optical materials are introduced. The system is illuminated by an external light

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source which illuminates the object plane and generates an electromagnetic wave in the visible range. A transparency which is introduced in the object plane serves to locally modulate the amplitude and phase of the electromagnetic illumination. The result of this configuration is an electromagnetic field $V_0(x, y, t)$ at each point (x, y) of the object plane. This field is transformed by the optical system to an electromagnetic field $V_i(\xi, \eta, t)$ in the image plane.

One distinguishes between a coherent illumination, like the light of the laser, and an incoherent illumination, like the light of the sun. The interested reader is referred to Beran and Parrent [1] for these considerations. The facts which are relevant to our work here are as follows:

(I) The coherently illuminated optical system determines a linear operation between the complex amplitudes of the electromagnetic fields in the object and image planes, respectively. Hence the system defines a linear operator which maps a complex-valued function on R^2 into another such function.

(II) In contrast, the incoherently illuminated system defines a linear transformation between the intensity distributions of the two planes. Since the intensity at a spatial point is the square of the modulus of the complex amplitude, the input and output functions have real nonnegative values.

The last physical consideration is the concept of passivity. Let $I_0(x, y)$ denote the light intensity at the object plane. Its integral over the object plane determines the total power entering the system. Similarly, $I_i(x, y)$ denotes the intensity at the image plane, and its integral over the image plane expresses the total power leaving the system. Since there are no internal power sources, we obtain the following passivity condition:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_0(x, y)\, dx\, dy \geqq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_i(\xi, \eta)\, d\xi\, d\eta. \tag{1}$$

3. The coherent system. Zemanian [8] pursues the "postulational foundations of linear systems", based on the electrical network. It is instructive to compare the properties of the optical operator to the postulates of the electrical one. First, the optical system operates on functions of two-dimensional variables. Second, since the argument consists of the spatial coordinates of a physical plane, the postulate of causality is not applicable here. The concept of passivity holds in a weak rather than in a strong sense. By weak passivity we mean that the upper limits of the integration in the passivity constraint of (1) are at infinity; this is in contrast to the use of finite upper limits to express the so-called strong sense of passivity used in reference to the electrical system.

The above considerations are common to both types of illumination. Specific to the coherent system are the following:

(i) The coherent operator exhibits a linear transformation between complex amplitudes, which are complex-valued functions of the spatial coordinates of the planes. Hence unlike the electrical operator, it need not be real; namely, it does not necessarily map real functions into real functions.

(ii) The passivity constraint of (1) involves intensities. Hence the coherent passivity is the usual L^2-type of passivity familiar from the scattering formalism of the electrical system.

4. The incoherent system. In the preceding section, we saw that the coherent system is controlled by postulates which are quite similar to the scattering formalism of the linear passive electrical network. The incoherent system, however, provides a unique set of postulates which leads to several new concepts. We start by introducing the concept of positivity.

Following Zemanian [8], we will start by considering a restrictive domain set and a broad range set for the operator. This initially allows a large class of operators for the analysis. Let $\mathscr{D}$ denote the space of infinitely differentiable testing functions on R^2 with compact support and $\mathscr{D}'$ its dual. $\mathscr{D}$ is equipped with the usual testing function topology, and $\mathscr{D}'$ is equipped with its weak dual topology. Let $\mathscr{P}$ denote the positive cone in $\mathscr{D}$, namely, the set of nonnegative testing functions. Let $\mathscr{Q}$ denote the positive cone in $\mathscr{D}'$ of positive distributions, namely, elements of $\mathscr{D}'$ which assume nonnegative values on the cone $\mathscr{P}$ of positive testing functions. It is well known (Gel'fand and Vilenkin [2]) that $\mathscr{Q}$ consists of the positive Radon measures on R^2. Let L be an operator which maps $\mathscr{P}$ into $\mathscr{Q}$. L is said to be *sublinear* if it is linear on the cone $\mathscr{P}$; namely, let $f_1, f_2 \in \mathscr{P}$ and λ_1, λ_2 be nonnegative scalars. L is sublinear if $L(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 L f_1 + \lambda_2 L f_2$. The cone $\mathscr{P}$ is generating in $\mathscr{D}$; i.e., every testing function in $\mathscr{D}$ can be expressed as the difference of two positive functions of $\mathscr{P}$. It follows that every sublinear operator L from $\mathscr{P}$ into $\mathscr{Q}$ can be uniquely extended as a linear and *positive* operator from $\mathscr{D}$ into $\mathscr{D}'$. We say that an operator from $\mathscr{D}$ into $\mathscr{D}'$ is positive if it maps $\mathscr{P}$ into $\mathscr{Q}$.

We show now that the positivity of a linear operator implies its continuity. Actually, this is a special case of a more general theorem regarding positive operators in ordered topological vector spaces (e.g., Peressini [5]). However the positive operators here exhibit an extendibility property in the sense that they can be uniquely extended as continuous operators from $\mathscr{C}_c$ into $\mathscr{C}_c'$. $\mathscr{C}_c$ denotes the space of continuous functions with compact support and $\mathscr{C}_c'$ its dual. $\mathscr{C}_c$ is equipped by the usual inductive limit topology and $\mathscr{C}_c'$ by the weak topology generated by $\mathscr{C}_c$.

THEOREM 1. *Let L be a positive linear operator from $\mathscr{D}$ into $\mathscr{D}'$. Then it is continuous. Moreover, it is uniquely and continuously extendible from $\mathscr{C}_c$ into $\mathscr{C}_c'$.*

Proof. Let $\{f_n\}$ be a sequence converging to zero in $\mathscr{D}$. We prove that $\{Lf_n\}$ converges to zero in $\mathscr{D}'$. Indeed,

$$-M_n \leqq f_n(x, y) \leqq M_n,$$

where $M_n = \max_{x,y} f_n(x, y)$.

Let $\lambda(x, y)$ be a positive testing function which is equal to unity over the compact set which contains all supports of $\{f_n\}$. Then

$$-M_n\lambda(x, y) \leqq f_n(x, y) \leqq M_n\lambda(x, y).$$

We apply the operator L on the terms of the inequality. Since L is positive it is order preserving,

$$-M_n L\lambda \leqq Lf_n \leqq M_n L\lambda,$$

where the inequality is interpreted in the sense of the order induced in $\mathscr{D}'$ by the cone $\mathscr{Q}$ of positive distributions. Hence if g is a positive testing function, then

$$-M_n\langle L\lambda, g\rangle \leqq \langle Lf_n, g\rangle \leqq M_n\langle L\lambda, g\rangle.$$

Since $\{f_n\}$ converges to zero by hypothesis, M_n converges to zero. $\langle L\lambda, g\rangle$ is a positive constant. It follows that $\{\langle Lf_n, g\rangle\}$ converges to zero with respect to positive testing functions g. But since every testing function in $\mathscr{D}$ can be represented as the difference between two positive testing functions, we have that $\{Lf_n\}$ converges weakly to zero in $\mathscr{D}'$. This establishes the continuity of L as an operator from $\mathscr{D}$ into $\mathscr{D}'$.

From the above, it follows that $\{Lf_n\}$ converges to zero if $\{\max f_n\}$ converges to zero and if there exists a compact set which contains all supports of $\{f_n\}$. But this is equivalent to saying that L is continuous on $\mathscr{D}$ for the relative topology induced by $\mathscr{C}_c$. Since $\mathscr{D}$ is dense in $\mathscr{C}_c$, it follows that it is uniquely extendible onto $\mathscr{C}_c$, where the extended operator is continuous from $\mathscr{C}_c$ into $\mathscr{D}'$. It can easily be verified that the extended operator is positive as well. Hence we have that $L(\mathscr{C}_c) \subset \mathscr{C}_c'$. By an application of the closed graph theorem, we have that L is in fact a continuous operator from $\mathscr{C}_c$ into $\mathscr{C}_c'$.

At this point the assumption of regularity is introduced. Let L be an operator from $\mathscr{D}$ into $\mathscr{D}'$. It is said to be *regular* if its range is contained in $\mathscr{C}$, the space of continuous functions, considered here as a subspace of $\mathscr{D}'$. This assumption is justified from a physical viewpoint in view of the consideration that the output of a physical system should at least be a continuous function when the input is an infinitely differentiable testing function (e.g., Newcomb [4]). For regular operators, we have a representation in terms of a scalar product (Meidan [3]). Combined with the positivity we can state Theorem 2.

THEOREM 2. *The following statements are equivalent*:

(i) *L is a linear positive and regular operator from $\mathscr{D}$ into $\mathscr{D}'$.*

(ii) *L is extendible as a linear continuous and positive operator from $\mathscr{C}_c$ into $\mathscr{C}$, where the latter is equipped by the countable norm topology generated by the uniform compact seminorms.*

(iii) *There exists a mapping $(\xi, \eta) \to K_{\xi,\eta}(x, y)$ from R^2 into $\mathscr{Q}$ which is weakly continuous and such that the operator L is representable by*

$$(Lf)(\xi, \eta) = \langle K_{\xi\eta}(x, y), f(x, y)\rangle, \qquad f \in \mathscr{C}_c. \tag{2}$$

(iv) *L^t is the transpose of L. It is a linear, positive and weakly continuous operator from $\mathscr{C}'$ into $\mathscr{C}_c'$ such that*

$$K_{\xi\eta} = L^t \delta_{\xi\eta}, \tag{3}$$

where $\delta_{\xi\eta}$ are the shifted impulse functionals in $\mathscr{C}'$,

$$\langle \delta_{\xi\eta}(x, y), f(x, y)\rangle = f(\xi, \eta), \qquad f \in \mathscr{C}. \tag{4}$$

Proof. The main part of the theorem is proved in [3]. It remains only to consider the connection with the assumption of positivity. It can easily be verified that the operator L is positive if and only if L^t is positive and if and only if the family $K_{\xi\eta}(x, y)$ is a family of positive distributions in $\mathscr{D}'$. By the positivity of L^t it is meant, of course, that it maps the positive cone of $\mathscr{C}'$ into the positive cone of $\mathscr{C}_c'$.

We consider now the concept of passivity connected with the incoherent operator. Let $f(x, y)$ be an input function. Then by the passivity constraint of (1) and the fact that $f(x, y)$ represents an intensity function, we obtain the following inequality:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Lf|\, d\xi\, d\eta \leqq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f|\, dx\, dy. \tag{5}$$

DEFINITION. Let L be a linear operator from $\mathscr{D}$ into $\mathscr{D}'$. L is said to be *incoherently passive* if Lf is an integrable function and inequality (5) holds for all $f \in \mathscr{D}$.

Youla et al. [7], in their work on linear, time-invariant, passive electrical networks, were mainly concerned with the usual L^2 scattering type of passivity. However they also mentioned an L^1-type of passivity, leaving the question of physical significance of such passivity unsettled. Hence the incoherently illuminated optical system provides the requested physical example for it.

Next, the investigation of the operator under the assumption of incoherent passivity is pursued. First, the passivity enables one to extend the domain of definition onto L^1, as is expressed in the next theorem for which the proof is obvious.

THEOREM 3. *There is a linear isomorphism between the class of linear incoherently passive operators and contractions in L^1. The isomorphism is obtained by continuously extending the domain of definition from $\mathscr{D}$ onto L^1.*

Incoherent passivity combined with regularity allows the definition of an integral of the operator in the following sense.

THEOREM 4. *Let L be a linear regular and incoherently passive operator from $\mathscr{D}$ into $\mathscr{D}'$. Let $K_{\xi,\eta}(x, y)$ denote the family of distributions connected to L by Theorem 2. Then the integral*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle K_{\xi,\eta}(x, y), f(x, y)\rangle\, d\xi\, d\eta \tag{6}$$

exists for each $f \in \mathscr{D}$ and defines a distribution I in $\mathscr{D}'$. Symbolically we write

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{\xi,\eta}(x, y)\, d\xi\, d\eta, \tag{7}$$

where $\langle I, f\rangle$ is expressed by (6) for each $f \in \mathscr{D}$.

Proof. Since L is incoherently passive,

$$\langle K_{\xi,\eta}(x, y), f(x, y)\rangle$$

is an integrable function for each $f \in \mathscr{D}$. It follows that the integral in (6) exists. The linearity of L is obvious. It remains to verify its continuity. Indeed, I can be considered as a composite operation $I = J \circ L$, where J denotes the functional on L^1 obtained by taking the integral of the absolute value of the elements of L^1. Since L is continuous from $\mathscr{D}$ into L^1 and J is a continuous functional on L^1, it follows that I is a continuous functional on $\mathscr{D}$.

THEOREM 5. *Let L be a linear, regular and continuous operator from $\mathscr{D}$ into $\mathscr{D}'$. L is positive and incoherently passive if and only if* $0 \leqq I \leqq 1$.

Proof. The direct statements are fairly obvious; hence only the converse needs proof. Following the hypotheses with respect to L, I has a meaning. For $I \geqq 0$, it is necessary that $K_{\xi,\eta}(x, y)$ be positive distributions for each (ξ, η). This follows from the continuity of the mapping $(\xi, \eta) \to K_{\xi,\eta}(x, y)$ from R^2 into $\mathscr{D}'$, which is a consequence of the regularity of L (Theorem 2). But if $K_{\xi,\eta}(x, y)$ are positive, L is positive, again by Theorem 2.

Since $I \leqq 1$,

$$\int_{\infty}^{\infty} \int_{\infty}^{\infty} (Lf)\, d\xi\, d\eta \leqq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\, dx\, dy$$

for each $f \in \mathscr{P}$ in $\mathscr{D}$. Assume any $f \in \mathscr{D}$, and decompose it into its positive and negative parts,

$$f = f_p - f_n.$$

Then, due to the linearity and positivity of L, f_p and f_n as continuous functions are in the domain of the extended operator and

$$|Lf| \leqq Lf_p + Lf_n.$$

But

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Lf|\, d\xi\, d\eta \leqq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Lf_p + Lf_n)\, d\xi\, d\eta \leqq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f_p + f_n)\, dx\, dy$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f|\, dx\, dy,$$

which establishes the requested passivity of L.

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AN ASYMPTOTIC FLOQUET THEOREM FOR LINEAR ALMOST PERIODIC SYSTEMS*

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Abstract. We consider the differential system $x' = [A + B(t)]x$ with $B(t)$ continuous and almost periodic. Conditions are given under which there exist n independent solutions of the form $q(t) \exp \int_0^t u(s)\, ds$ with q and u almost periodic. The analogy with Floquet's theorem for periodic systems is indicated.

1. Introduction. We consider the differential equation

$$x'(t) = [A + B(t)]x(t), \qquad t \in (-\infty, \infty), \tag{1}$$

where $x(t) = \operatorname{col}(x_1(t), \cdots, x_n(t))$ is a complex n-vector and A and $B(t)$ are complex $n \times n$ matrices with A constant and the entries of $B(t)$ continuous. Floquet's theorem of 1883 [7, Chap. III] states that if $B(t)$ is periodic of period T then every fundamental matrix solution of (1) has the form

$$X(t) = P(t)e^{Ct}, \tag{2}$$

where $P(t)$ is periodic of period T and C is constant. However Floquet's theorem does not in general hold if $B(t)$ is assumed only to be almost periodic (a.p.) (see, for example, [2] or [5]). Partial analogues have been given, prominent among which is Coppel's [3] which states, in part, that the a.p. system (1) has a fundamental matrix of the form (2) with $P(t)$ a.p. if the distance of the set of differences $\{\alpha_j - \alpha_k\}$ of the eigenvalues of A from the extended spectrum of $B(t)$ is positive.

The problem described here is to be distinguished from the almost constant coefficient case although similar results have been obtained in this area. Bellman [1] has shown that if A has simple eigenvalues and $\|B(t)\| \to 0$ as $t \to +\infty$, then corresponding to each eigenvalue λ, there is a solution x_λ of (1) satisfying

$$\lim_{t \to \infty} t^{-1} \log \|x_\lambda(t)\| = \operatorname{Re} \lambda.$$

Hartman and Wintner [8] and later Sibuya [9] have established similar results for systems containing a parameter u and have relaxed the condition $\|B(t, u)\| \to 0$ to $\|B(t, u)\| \leqq f(t)$, where

$$\sup_{p \geqq t} (1 + p - t)^{-1} \int_t^p f(s)\, ds \to 0 \quad \text{as } t \to +\infty.$$

In this article, we show how the implicit function theorem can be used to characterize solutions of system (1) when $B(t)$ is (Bohr) almost periodic. Our approach is elementary in that only the most immediate properties of almost periodic functions are used. Our result is to be distinguished from those for the almost constant coefficient case described above since we require only that $\|B\|$ be small.

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2. The theorem. Specifically we prove the following theorem.

THEOREM. *Let $B(t)$ in* (1) *be a.p. Let* $A = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$ *with* $\operatorname{Re}(\lambda_i - \lambda_j) \neq 0$ *for $i \neq j$. Then for $\|B\|$ sufficiently small, the equation* (1) *possesses n independent solutions x_k of the form*

$$x_k(t) = (p_k(t) + e_k) \exp\left(\lambda_k t + \int_0^t v_k(s)\, ds\right), \tag{3}$$

where $e_k = \operatorname{col}(\delta_{1k}, \delta_{2k}, \cdots, \delta_{nk})$, $v_k(s)$ *is an a.p. scalar function, and*

$$p_k(t) = \operatorname{col}(p_{k1}(t), \cdots, p_{k,k-1}(t), 0, p_{k,k+1}(t), \cdots, p_{kn}(t)), \tag{4}$$

where $p_{kj}(t)$ and $p'_{kj}(t)$ are a.p., $1 \leqq j \leqq n$.

Before turning to the proof, we point out the analogy with Floquet's theorem. We first recall that if $v(t)$ is an almost periodic scalar function, then the mean value M of $v(t)$ defined by

$$M = \lim_{T \to \infty} \frac{1}{T} \int_a^T v(t)\, dt$$

exists independent of a [4]. Denote by M_k the mean value of $v_k(t)$ and let $q_k(t) = p_k(t) + e_k$. From our theorem, the solutions x_k approach the form

$$x_k(t) \to q_k(t)\, e^{(\lambda_k + M_k)t} \tag{5}$$

as $t \to \infty$ with q_k a.p. and $\lambda_k + M_k$ constant. For the case $B(t)$ periodic, we would have, from Floquet's theorem, equality in (5) with $q_k(t)$ periodic and $\lambda_k + M_k$ a characteristic exponent of the system (1).

Note also the distinction between our result and those above for the almost constant case. We have that

$$\lim_{t \to \infty} t^{-1} \log \|x_k(t)\| = \operatorname{Re} \lambda_k + M_k.$$

Obviously, imposing $\|B(t)\| \to 0$ would force $M_k = 0, k = 1, \cdots, n$.

Proof of the theorem. Let $k \in \{1, \cdots, n\}$. Denote by $p_k(t)$ any complex n-vector function of the form (4), where p_{kj} and p'_{kj} are a.p. functions of t. Denote by $v_k(t)$ any a.p. scalar function of t. Since an almost periodic function is necessarily bounded, we may define

$$\|f\| = \sup_t \|f(t)\|$$

for $f(t)$ an a.p. scalar or vector function of t. We can now define the Banach spaces B_1^k for $k = 1, \cdots, n$ by

$$B_1^k = \{(p_k, v_k)\}$$

with norm defined by

$$\|(p_k, v_k)\| = \|p_k\| + \|p'_k\| + \|v_k\|.$$

The completeness of B_1^k follows from the properties that (a) the uniform limit of a sequence of a.p. functions is again a.p., and (b) if p_k is a.p., then p'_k is a.p. if and only if p'_k is uniformly continuous (see [5, Chap. 1]).

Let B_2 denote the Banach space of all almost periodic $n \times n$ matrix functions with norm consistent, with the norm chosen on complex n-space, and let C denote the Banach space defined by

$$C = \{\text{col}\,(y_1(t), \cdots, y_n(t)) \mid y_j(t) \text{ is a.p., } 1 \leqq j \leqq n\}.$$

We now consider the mapping $F^k : B_1^k \times B_2 \to C$ for each $1 \leqq k \leqq n$ defined by

$$F^k((p_k, v_k), B) = p_k' - [\Lambda_k + B](p_k + e_k) - (p_k + e_k)v_k,$$

where $e_k = (\delta_{1k}, \delta_{2k}, \cdots, \delta_{nk})$ and

$$\Lambda_k = A - \lambda_k I, \qquad I = \text{identity}.$$

Observe that $F^k(\bar{0}) = \bar{0}$. Since F^k is linear in its second component, it has a continuous partial derivative with respect to its second component in a neighborhood of zero. We next observe that

$$\begin{aligned} F^k((p_k, v_k), \bar{0}) - F^k((\bar{0}, \bar{0}), \bar{0}) &= F^k((p_k, v_k), \bar{0}) \\ &= p_k' - [\Lambda_k + 0](p_k + e_k) - v_k e_k - v_k p_k. \end{aligned}$$

We therefore define the linear map $L^k : B_1^k \to C$ by

$$L^k(p_k, v_k) = p_k' - \Lambda_k(p_k + e_k) - v_k e_k$$

and note that

$$\frac{\|v_k p_k\|_C}{\|(p_k, v_k)\|_{B_1^k}} \leqq \frac{\|v_k\| \cdot \|p_k\|}{\|p_k\| + \|p_k'\| + \|v_k\|} \leqq \|p_k\| \to 0$$

as $\|(p_k, v_k)\|_{B_1^k} \to 0$. Thus $v_k p_k = F^k((p_k, v_k), 0) - L^k(p_k, v_k)$ is $o(\|(p_k, v_k)\|)$ from which it follows that F^k has a partial derivative in a neighborhood of zero with respect to its first component. We next show that L^k is a bijection. Let $f = \text{col}\,(f_1, \cdots, f_n)$ be arbitrary in C. The equation

$$L^k(p_k, v_k) = f$$

is equivalent to the system

$$\begin{aligned} &-v_k = f_k, \\ &p_{k1}' + \gamma_1 p_{k1} = f_1, \\ &\quad \vdots \\ &p_{k,k-1}' + \gamma_{k-1} p_{k,k-1} = f_{k-1}, \\ &p_{k,k+1} + \gamma_{k+1} p_{k,k+1} = f_{k+1}, \\ &\quad \vdots \\ &p_{k,n}' + \gamma_n p_{kn} = f_n, \end{aligned}$$

where $\gamma_j = \lambda_j - \lambda_k$, $j = 1, \cdots, n$.

We thus let $v_k = -f_k$ and recall (see [5, Thm. 5.9]) that the differential equation

$$p'_{kj}(t) + \gamma_j p_{kj}(t) = f_j(t), \qquad 1 \leqq j \leqq n, \quad j \neq k, \tag{6}$$

has the unique solution

$$p_{kj}(t) = e^{-\gamma_j t}\left[-\int_0^\infty e^{\gamma_j s} \cdot f_j(s)\, ds + \int_0^t e^{\gamma_j s} \cdot f_j(s)\, ds\right]. \tag{7}$$

Letting $p_k = \operatorname{col}(p_{k1}, \cdots, p_{k,k-1}, 0, p_{k,k+1}, \cdots, p_{k,n})$, we now have that (p_k, v_k) is the unique element in B_1^k so that $L^k(p_k, v_k) = f$. Since f was arbitrary in C, L^k is a bijection. Since L^k is clearly continuous by our preceding observations and the implicit function theorem for Banach spaces (see [6]), there exists a bounded open neighborhood S of the origin in B_1^k such that to each $B \in B_2$ with $\|B\|$ sufficiently small and positive corresponds a unique nonzero element (p_k, v_k) in S, so that $F^k((p_k, v_k), B) = \bar{0}$, i.e., so that

$$p'_k(t) = [\Lambda_k + B(t)](p_k(t) + e_k) - (p_k(t) + e_k)v_k(t). \tag{8}$$

Now via the substitution $x(t) = y(t)\, e^{\lambda_k t}$, the equation (1) transforms into the equation

$$y'(t) = [\Lambda_k + B(t)]y(t), \tag{9}$$

and via the second substitution, $y(t) = (p_k(t) + e_k) \exp \int_0^t v_k(s)\, ds$, the equation (9) transforms into (8). Since (p_k, v_k) is the unique solution of (8) in B_1^k for $k = 1, \cdots, n$, we have that

$$x_k(t) = (p_k(t) + e_k) \exp\left(\lambda_k t + \int_0^t v_k(s)\, ds\right)$$

satisfies (1) for $k = 1, \cdots, n$.

Finally we indicate the independence of $x_1, \cdots, x_n$. Let $c_1, \cdots, c_n$ be constants for which the equation

$$c_1 x_1(t) + \cdots + c_n x_n(t) = 0 \tag{10}$$

holds for all t. If $c_j \neq 0$ for some $j \in \{1, \cdots, n\}$, we have from (3) and (10) that

$$p_j(t) + e_j = \sum_{k \neq j} c_j^{-1} c_k (p_k(t) + e_k) \exp\left[(\lambda_k - \lambda_j)t + \int_0^t (v_k(s) - v_j(s))\, ds\right]. \tag{11}$$

Now let $B(t) = (b_{ij}(t))$, and note from (8) that

$$\sum_{\substack{r=1 \\ r \neq k}}^{n} b_{kr}(t) p_{kr}(t) + b_{kk}(t) - v_k(t) = 0 \tag{12}$$

holds for all t and each $k = 1, \cdots, n$. Since p_k remains in a bounded neighborhood of the origin as $\|B\| \to 0$, it follows from (12) that $\|v_k\| \to 0$ as $\|B\| \to 0, k = 1, \cdots, n$. This implies the existence of an $\varepsilon > 0$ so that

$$M_{kj} \underset{\text{Df}}{=} \lim_{T \to \infty} \frac{1}{T} \int_0^T (v_k(s) - v_j(s))\, ds < \operatorname{Re}(\lambda_k - \lambda_j) \tag{13}$$

holds for all $k = 1, \cdots, n$, $k \neq j$, whenever $\|B\| < \varepsilon$. Using the equality in (13), we have from (11) that

$$p_j(t) + e_j \to \sum_{k \neq j} c_j^{-1} c_k (p_k(t) + e_k)\, e^{(\lambda_k - \lambda_j + M_{kj})t} \tag{14}$$

as $t \to \infty$. Since $p_k(t)$ is bounded and $\mathrm{Re}\,(\lambda_k - \lambda_j) \neq 0$, it follows from inequality (13) that for $\|B\| < \varepsilon$, the right side of (14) must either approach zero or become infinite as $t \to \infty$ contradicting the almost periodicity of p_j. The contradiction implies the stated independence. Q.E.D.

An immediate extension of the preceding theorem is the following corollary.

COROLLARY. *Let $\lambda_1, \cdots, \lambda_n$ denote the eigenvalues of A. If* $\mathrm{Re}\,(\lambda_i - \lambda_j) \neq 0$ *for all $i \neq j$, then for $\|B\|$ sufficiently small, the equation* (1) *possesses n independent solutions of the form $x_k(t) = q_k(t) \exp\,(\lambda_k t + \int_0^t v_k(s)\,ds)$ with q_k and v_k almost periodic.*

Indication of proof. Since the eigenvalues of A are distinct, we can find a matrix S so that $S^{-1}AS = D$, where D is diagonal with eigenvalues $\lambda_1, \cdots, \lambda_n$. Now $C(t) = S^{-1}B(t)S$ is again almost periodic, so by our theorem the equation

$$y'(t) = [D + C(t)]y(t) \tag{15}$$

has n independent solutions of the form (3) for $\|C\|$ sufficiently small. But $\|B\| \to 0$ as $\|C\| \to 0$ and y is a solution of (15) iff $x = S^{-1}y$ is a solution of (1). Since y has the form (3), x has the stated form. Q.E.D.

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OSCILLATION OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $y''+p(x)y=f(x)$*

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Abstract. In this paper we study the behavior of solutions of the differential equation

(NH) $$y''+p(x)y=f(x),$$

where $p(x)$ and $f(x)$ are continuous and $p(x)$ is positive. In particular, the role of the *concavity quotient* $A(x)=f(x)/p(x)$ is delineated.

The first part of the paper relates boundedness of $A(x)$ to boundedness of solutions of (NH). One result is that all solutions of (NH) are unbounded when $A(x)$ is unbounded, $|A'(x)|<B$, and $p(x)>\varepsilon>0$.

In the second part of the paper, general sufficient conditions for oscillation are examined, for the two cases $f(x)$ positive and $f(x)$ oscillatory. Some results extend oscillation theorems in a recent paper of Leighton and Skidmore, while others are unrelated to that paper. The following is proved: If derivatives of solutions of

(H) $$z''+p(x)z=0$$

are bounded and $A(x)\to 0^+$, or if solutions of (H) are bounded and $\int_a^\infty |f(x)|\,dx<\infty$, then oscillation of (H) is a necessary and sufficient condition for oscillation of all solutions of (NH) except possibly one.

1. Introduction. In this paper we study the behavior of solutions of the differential equation

(NH) $$y''+p(x)y=f(x),$$

where $p(x)$ and $f(x)$ are continuous on the real line, and $p(x)$ is a positive function. With this equation we associate the corresponding homogeneous equation

(H) $$z''+p(x)z=0.$$

Several papers have appeared recently dealing with equation (NH), notably those by Keener [5], and by Leighton and Skidmore [8].

In the present paper we discuss boundedness of solutions of (NH) and extend some earlier oscillation theorems. In this connection, the role played by the *concavity quotient* $f(x)/p(x)$ in determining the behavior of solutions of (NH) is delineated.

In our analysis we shall frequently employ the following basic formula, which may be verified by differentiation:

(1.1) $$\int_a^x f(x)z(x)\,dx=[y'(x)z(x)-y(x)z'(x)]_a^x.$$

Recall that if $z(x)$ is a nonnull solution of (H) such that $z(a)=0$ and the first zero of $z(x)$ following a occurs at $x=c_a$, c_a is the (first) *conjugate point of* a.

If $u(x)$ is a nonnull solution of (NH) or of (H), we say $u(x)$ is *oscillatory* if $u(x)$ has arbitrarily large zeros on (a,∞); and we say $u(x)$ is *nonoscillatory* if there is a

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number b such that $u(x) \neq 0$ on (b, ∞). We say (H) or (NH) is oscillatory (or nonoscillatory) if all solutions are oscillatory (nonoscillatory). The differential equation $y'' + y = 1$ that has, for example, solutions 1 and $1 + 2 \sin x$ shows that (NH) may be neither oscillatory nor nonoscillatory.

The following basic result is known, and follows easily from (1.1).

LEMMA 1.1. *If* $f(x) > 0$ *on* (a, ∞) *and* (H) *is oscillatory, then*

(i) *any solution of* (NH) *which is positive at a point* b *must be positive over some interval of the form* $[x, c_x]$ *containing* b; *and,*

(ii) *no solution of* (NH) *can be negative over any interval* (x, c_x).

We note that any statement corresponding to $f(x) \geqq 0$ has a dual corresponding to $f(x) \leqq 0$, since $w(x) = -y(x)$ satisfies $w'' + p(x)w = -f(x)$ whenever $y(x)$ satisfies (NH).

2. Boundedness of solutions and the function $f(x)/p(x)$. If we define $A(x) = f(x)/p(x)$, we can rewrite (NH) as

$$y''(x) = p(x)[A(x) - y(x)]. \tag{2.1}$$

It follows that any solution of (NH) is concave down or up at x according as it is above or below $A(x)$; all solutions "bend toward" $A(x)$. Consequently, boundedness of solutions of (NH) is closely related to the behavior of the concavity quotient $A(x)$.

The following result is fundamental and is an immediate consequence of (2.1).

LEMMA 2.1. *If* $y(x)$ *is any solution of* (NH), *then at least one of the following occurs*:

(i) $y(x) - A(x)$ *oscillates*;

(ii) $y(x) \to L$ *as* $x \to \infty$, L *finite*;

(iii) $y(x) \to -\infty$ *as* $x \to \infty$;

(iv) $y(x) \to \infty$ *as* $x \to \infty$.

Keener [5] noted that $\int_a^\infty f(x)\,dx = \infty$ implies that any nonoscillatory solution of (NH) is eventually positive; hence, when $f(x)$ is a positive function, condition (ii) with $L < 0$, or condition (iii) can occur only if $\int_a^\infty f(x)\,dx < \infty$ and (H) is nonoscillatory.

We come to the following group of theorems related to Lemma 2.1.

THEOREM 2.2. *If* $A(x) > bx + c$ *on* (a, ∞) *for some constants* b *and* c, *and if* $\int_a^\infty p(x)\,dx = \infty$, *no solution of* (NH) *stays above* $A(x) + \varepsilon$, *for any* $\varepsilon > 0$.

Assume the contrary, that there is a solution $y(x)$ of (NH) and a number $\varepsilon > 0$ such that $y(x) > A(x) + \varepsilon$ on (b, ∞) for some $b > a$. We have

$$y''(x) = f(x) - p(x)y(x) < f(x) - p(x)[A(x) + \varepsilon] = -p(x)\varepsilon.$$

Then $y'(x) < y'(a) - \varepsilon \int_a^x p(x)\,dx \to -\infty$ as $x \to \infty$. Eventually, $y(x)$ must fall below any straight line—hence, below $A(x)$. From this contradiction, we infer the truth of the theorem.

Example. No solution of $y'' + e^x y = xe^x$ stays below $(x - \varepsilon)$ on (b, ∞) for any $\varepsilon > 0$ and any b. Note that all solutions of $z'' + e^x z = 0$ are bounded, and indeed, approach zero as $x \to \infty$ (Cesari [3, p. 85].)

If $A(x)$ is bounded and $\int_a^\infty p(x)\,dx = \infty$, it follows from Lemma 2.1 and

Theorem 2.2 that for any solution $y(x)$ of (NH), $[y(x) - A(x)]$ oscillates or $y(x) \to L$ as $x \to \infty$, where $\liminf_{x\to\infty} A(x) \leqq L \leqq \limsup_{x\to\infty} A(x)$. That both types of behavior can occur may be seen from the equation $y'' + y = L + 2e^{-x}$, which has the general solution $y(x) = L + e^{-x} + c_1 \cos x + c_2 \sin x$.

A well-known result of Leighton [6, p. 230] states that if $\int_a^\infty p(x)\,dx = \infty$, then (H) is oscillatory. Theorems 2.2 and 2.3 can be considered extensions of this theorem to (NH).

THEOREM 2.3. *If* $\int_a^\infty p(x)\,dx = \infty$, *and* $\int_a^\infty |f(x)|\,dx < \infty$, *then no solution of* (NH) *is bounded away from zero.*

To prove the theorem, assume there is a solution $y(x)$ such that $|y(x)| > \varepsilon > 0$ on (a, ∞). Since $y(x) \neq 0$ on (a, ∞), it satisfies the homogeneous equation

$$y'' + P(x)y = 0, \tag{2.2}$$

where $P(x) = p(x) - (f(x)/y(x))$. Now,

$$\int_a^\infty P(x)\,dx > \int_a^\infty \left[p(x) - \frac{|f(x)|}{\varepsilon}\right] dx = \infty.$$

It follows that (2.2) must be oscillatory, and hence that $y(x)$ must oscillate. This contradicts the original assumption.

We note that if the finiteness conditions on $f(x)$ and $p(x)$ are interchanged in Theorem 2.3 (i.e., if $\int_a^\infty p(x)\,dx < \infty$ and $\int_a^\infty f(x)\,dx = \infty$) an analogous proof shows that no solution of (NH) is bounded above.

In the next group of theorems we prove several sufficient conditions for boundedness of solutions of (NH).

THEOREM 2.4. *If* $A(x)$ *is of bounded variation on* $[a, \infty)$ *and all solutions of* (H) *and their derivatives are bounded, then all solutions of* (NH) *are bounded.*

The proof follows from applying an integration by parts to the variation of constants formula.

A sufficient condition for all solutions of (H) and their derivatives to be bounded is $p(x) = q(x) + r(x)$, where $q(x) \to L > 0$ as $x \to \infty$, $q(x)$ is of bounded variation, and $\int_a^\infty |r(x)|\,dx < \infty$ (Cesari [3, p. 81]).

We apply the theorem to the equation

$$y'' + \frac{x+1}{x}y = \frac{\sin x}{x^2}.$$

Since $A(x) = \sin x/(x(x+1))$, all solutions are bounded for $x > \varepsilon > 0$.

A sufficient condition that all solutions of (H) be bounded is that $p(x) = q(x) + r(x)$, where $q(x)$ is a monotone function, $q(x) > \varepsilon > 0$, and $\int_a^\infty |r(x)|\,dx < \infty$ (Bellman [1, pp. 112, 113]).

Hammett [4] proved that if solutions of (H) are bounded and $\int_a^\infty |f(x)|\,dx < \infty$, there is a solution $y(x)$ of (NH) such that $y(x) \to 0$ as $x \to \infty$; hence, that all solutions of (NH) are bounded. We note that if $y(x)$ solves (NH), then $w(x) = [y(x) - A(x)]$ solves $w'' + p(x)w = -A''(x)$. If we apply Hammett's result to this equation, we can conclude that if $A(x)$ is C'' on $[a, \infty)$, $\int_a^\infty |A''(x)|\,dx < \infty$, and solutions of (H) are bounded, then $[y(x) - A(x)]$ is bounded for all solutions $y(x)$

of (NH). In particular, if $A(x)$ is increasing and concave down (or decreasing and concave up), boundedness of (NH) is equivalent to boundedness of $A(x)$.

It follows, for example, that all solutions of the equation $y'' + x^{\varepsilon}y = \ln x$ are bounded for all $\varepsilon > 0$, while all solutions of the limiting equation $y'' + y = \ln x$ are unbounded.

We note that if there is an unbounded solution $u(x)$ of (H), then there is an unbounded solution of (NH). For, if $y(x)$ is any solution of (NH), it follows that $y(x)$ or $[y(x) + u(x)]$ must be unbounded. The following lemma gives sufficient conditions that there be an unbounded solution of (H).

LEMMA 2.5. *If* $\int_a^\infty p(x)\,dx < \infty$, *or if* $p(x) \to 0$ *as* $x \to \infty$, *there is an unbounded solution of* (H).

The proof is quite direct, using the well-known result that if $u(x)$ and $v(x)$ are any linearly independent solutions of (H), $W \equiv u'(x)v(x) - v'(x)u(x)$, and $y(x) = \sqrt{u^2(x) + v^2(x)}$, then $y(x)$ satisfies the differential equation $y'' + p(x)y = W^2/y^3$.

Lemma 2.5 can be used to obtain sufficient conditions that there be an unbounded solution of (NH).

THEOREM 2.6. *If* $A(x) \to \infty$ *as* $x \to \infty$, *there is an unbounded solution of* (NH). *If, in addition,* $\int_a^\infty f(x)\,dx = \infty$, *all solutions of* (NH) *are unbounded above.*

We prove the second statement first. Let us assume there exists a solution $y(x)$ of (NH) that is $< B$ on (a, ∞). Since $A(x) \to \infty$, there is a number $b > a$ such that $A(x) > 2B$ on (b, ∞). On this interval, we have $f(x)/2 > Bp(x)$ and $y''(x) > f(x)/2$. Then $y'(x) \to \infty$ as $x \to \infty$, a contradiction. Hence, all solutions of (NH) are unbounded above.

Now, suppose $\int_a^\infty f(x)\,dx < \infty$. Since $A(x) > 1$ on some interval $[c, \infty)$, it follows that $f(x) > p(x)$ on this interval. Consequently, $\int_a^\infty p(x)\,dx < \infty$. By Lemma 2.5, there is an unbounded solution of (H). Hence, there is an unbounded solution of (NH). The theorem is proved.

We note that, if $A(x) \to \infty$ and (NH) is oscillatory, then all solutions of (NH) are unbounded (since oscillation of $y(x)$ implies oscillation of $y''(x)$.)

Along these lines, it can be proved in a straightforward manner that all solutions of (NH) are unbounded whenever $A(x)$ is unbounded, $|A'(x)| < B$, and $p(x) > \varepsilon > 0$.

3. Oscillation of solutions and the concavity quotient. In the remainder of this paper we examine the effects of the function $A(x)$ on oscillatory behavior of solutions of (NH).

If an integration by parts is performed on the left-hand member of (1.1), we obtain

$$\int_a^x A'(t)z'(t)\,dt = [y'(t)z(t) - \{y(t) - A(t)\}z'(t)]_a^x, \tag{3.1}$$

where $z(x)$ is any solution of (H) and $y(x)$ is any solution of (NH).

One application of this identity is the following.

THEOREM 3.1. *Suppose* (H) *is oscillatory and* $A(x)$ *is* C'.

(i) *If* $A(x)$ *is decreasing and bounded below, then no solution stays above* $A(x)$ *on* $[a, \infty)$.

(ii) *If $A(x)$ is increasing and bounded above, then no solution stays above $A(x)$ on $[a, \infty)$.*

We prove only (i); the proof of (ii) is strictly analogous. Let $z(x)$ be a solution of (H) with $z(a) > 0$, $z'(a) = 0$. Then there is a point $d > a$ such that $z'(d) = 0$ and $z'(x) < 0$ on (a, d). If $y(x)$ is a solution of (NH) with $y(a) > A(a)$, then either $y'(a) \leqq 0$ or, by (3.1) with $x = d$, $y'(d) < 0$. In either case, since $y''(x) < 0$ for $y(x) > A(x)$, it follows that $y(x)$ must cut $A(x)$. The proof is complete.

Next, we extend an idea used by Burton [2], Hammett [4], and Tefteller [9]. It is well-known that if $u(x)$ and $v(x)$ are any functions of class C' and

$$W[u, v](x) \equiv u'(x)v(x) - u(x)v'(x) \neq 0$$

on (a, ∞), then the zeros of $u(x)$ and $v(x)$ separate each other on (a, ∞). If $y_0(x)$ is the solution of (NH) with a double zero at a, we have from (1.1) that

$$W[y_0, z](x) = \int_a^x f(t)z(t)\, dt \quad \text{for any solution } z(x) \text{ of (H).}$$

Consequently, $y_0(x)$ is oscillatory if and only if (H) is oscillatory, whenever there is a solution $z(x)$ of (H) for which $\int_a^x fz\, dt$ is eventually of one sign.

Suppose there is such a solution $z(x)$, and let $u(x)$ be any solution of (H) such that $W[z, u] \equiv 1$. If $y(x)$ is any solution of (NH), $y(x) = y_0(x) + Cz(x) + Du(x)$ for some constants C and D, and

$$W[y, z] = W[y_0, z] - D = \int_a^x fz\, dt - D. \tag{3.2}$$

It follows that oscillation of (H) is equivalent to oscillation of $y(x)$ for all solutions $y(x)$ of (NH) for which the right-hand member of (3.2) is eventually of one sign, that is, whenever $D < L_*$ or $D > L^*$, where

$$L_* = \liminf_{x \to \infty} \int_a^x fz\, dt \quad \text{and} \quad L^* = \limsup_{x \to \infty} \int_a^x fz\, dt.$$

Hence, unless $L_* = -\infty$ and $L^* = +\infty$ for all solutions $z(x)$ of (H), oscillation of (H) is a necessary and sufficient condition for oscillation of infinitely many solutions of (NH).

A consequence of these remarks is that whenever $\lim_{x \to \infty} \int_a^x fz\, dt = L$ (L finite), oscillation of (H) is equivalent to oscillation of all solutions of (NH) except possibly those for which $D = L$. And if $\int_a^x fz\, dt$ converges to L from one side (say, from above) we can let $D = L$ as well, and oscillation of (H) is equivalent to oscillation of (NH).

In the same way, we observe that oscillation of (H) and (NH) are equivalent when $\int_a^\infty fz\, dt = \pm\infty$. (Tefteller [9].)

That there may exist a unique nonoscillatory solution of (NH) can be seen from the example $y'' + y = 2e^{-x}$, for which the only nonoscillatory solution is

$y = e^{-x}$. We note that if $a = 0$ and $z(x) = \sin x$,

$$\int_a^\infty fz\,dx = \int_0^\infty 2\,e^{-x}\sin x\,dx = 1.$$

This computation illustrates the following result.

THEOREM 3.2. *If solutions of* (H) *are bounded and* $\int_a^\infty |f(x)|\,dx < \infty$, *then all solutions of* (NH) *are oscillatory except possibly one.*

The proof is an application of the preceding observations. If all solutions of (H) are bounded, then $\int_a^\infty p(x)\,dx = \infty$, by Lemma 2.5; consequently, (H) is oscillatory. Now,

$$0 \leqq \int_a^x |fz|\,dt < B\int_a^\infty |f|\,dt < \infty.$$

Hence, $\lim_{x\to\infty}\int_a^x fz\,dt$ exists and is finite, for all solutions $z(x)$ of (H) with $z(a) = 0$.

If we now define the solutions $z(x)$ and $u(x)$ of (H) by $z(a) = 0$, $z'(a) = 1$, $u(a) = 1$, $u'(a) = 0$, it follows that all solutions of (NH) are oscillatory except possibly those solutions $y(x)$ for which $y(a) = D = \int_a^\infty fz\,dx$.

If c is the first conjugate point of a, and if $\tilde{a}$ is any value in (a, c), we define the solution $\tilde{z}(x)$ of (H) by $\tilde{z}(\tilde{a}) = 0$, $\tilde{z}'(\tilde{a}) = 1$. By the same argument as before, $\lim_{x\to\infty}\int_{\tilde{a}}^x f(t)\tilde{z}(t)\,dt$ exists and is finite; and, for any nonoscillatory solution $y(x)$ of (NH), we have $y(\tilde{a}) = \int_{\tilde{a}}^\infty f(t)\tilde{z}(t)\,dt$. But there is only one solution of (NH) through the points $(a, \int_a^\infty fz\,dt)$ and $(\tilde{a}, \int_{\tilde{a}}^\infty f\tilde{z}\,dt)$. We conclude that there is at most one nonoscillatory solution of (NH).

For the remainder of the paper, let us assume that $f(x)$ is a positive function. Under this restriction, some of the foregoing observations can be sharpened.

If (H) is oscillatory and $z(x)$ is the solution of (H) with $z(a) = 0$, $z'(a) = 1$, let $c_1, c_2, c_3, \cdots$ denote the successive zeros of $z(x)$ following a; that is, let c_n be the nth conjugate point of a.

For $n = 1, 2, 3, \cdots$, define the solution $y_n(x)$ of (NH) by the conditions $y_n(c_n) = y_n'(c_n) = 0$. We note that $y_n(x) > 0$ for x near $c_n (x \neq c_n)$, since $y_n''(x)$ has the same sign as $f(x)$ near c_n. It follows from Lemma 1.1 that $y_n(x) > 0$ on $[c_{n-1}, c_n)$ and $(c_n, c_{n+1}]$.

LEMMA 3.3. *Suppose* (H) *is oscillatory*, $y(x)$ *is a solution of* (NH), *and* $Y_n = y_n(a)$. *Then*

(i) *if* $y(a) \geqq Y_{2m+1}$, $y(x)$ *has one zero on* $(c_{2m}, c_{2m+1}]$;

(ii) *if* $y(a) \leqq Y_{2m}$, $y(x)$ *has one zero on* $(c_{2m-1}, c_{2m}]$; *and*,

(iii) $Y_{2m} < Y_{2m+1}$ *and* $Y_{2m} < Y_{2m-1}$.

To prove (i), we define $u(x) = y(x) - y_{2m+1}(x)$. We note that $u(x)$ is a solution of (H) with $u(a) \geqq 0$. Then, by the Sturm separation theorem, $u(c_{2m}) \geqq 0$ and $u(c_{2m+1}) \leqq 0$. But $y_{2m+1}(c_{2m}) > 0$; hence, $y(c_{2m}) > 0$ and $y(c_{2m+1}) \leqq 0$. Consequently, $y(x)$ has a zero on $(c_{2m}, c_{2m+1}]$. That $y(x)$ has only one zero on the interval follows from Lemma 1.1.

The proofs of (ii) and (iii) are similar. Part (iii) also follows if we note that $Y_n = y_n(a) = \int_a^{c_n} fz\,dt$, from (1.1); hence, $Y_{n+1} - Y_n = \int_{c_n}^{c_{n+1}} fz\,dt$.

A consequence of the lemma is that if (H) is oscillatory and the sequence $\{Y_n\}$ has a (finite) cluster point P, then there are infinitely many oscillatory solutions of (NH). For at least one of the sequences $\{Y_{2m}\}$, $\{Y_{2m+1}\}$ must cluster at P, say, $\{Y_{2m+1}\}$. It follows that for every solution $y(x)$ of (NH) with $y(a) > P$, we have $y(a) > Y_{2m+1}$ for infinitely many integers m. Hence, (*i*) implies that $y(x)$ is oscillatory.

It also follows from the lemma that if $Y_n \to \infty$ or $Y_n \to -\infty$, then (NH) is oscillatory; and, if $Y_n \to L$ (L finite), then all solutions of (NH) are oscillatory except possibly those solutions $y(x)$ for which $y(a) = L$. More generally, we observe that (NH) has infinitely many oscillatory solutions unless both $Y_{2m+1} \to +\infty$ and $Y_{2m} \to -\infty$; in other words, unless $\lim_{n\to\infty} |\int_{c_n}^{c_{n+1}} fz\, dt| = \infty$.

For example, $y'' + y = f(x)$ must have oscillatory solutions whenever $f(x)$ is bounded. Another application of these ideas is the following.

THEOREM 3.4. *Suppose* (H) *is oscillatory and derivatives of solutions of* (H) *are bounded.*

(i) *If $A(x)$ is bounded, then there are oscillatory solutions of* (NH).

(ii) *If $A(x) \to 0$ as $x \to \infty$, then all solutions of* (NH) *are oscillatory except possibly one.*

To prove the theorem, let $A_n = \max\{A(x);\ x \text{ in } [c_n, c_{n+1}]\}$. We have

$$\left| \int_{c_n}^{c_{n+1}} fz\, dt \right| = \left| \int_{c_n}^{c_{n+1}} Az''\, dt \right| < A_n |z'(c_{n+1}) - z'(c_n)|.$$

Conclusion (*i*) follows immediately from this expression. To prove (ii), we need only observe that the left-hand member tends to zero, and apply an argument like the one used to establish Theorem 3.2.

The same argument can be used to sharpen Theorem 3.2 when $f(x) > 0$: it can be shown that if solutions of (H) are bounded and $\int_{c_n}^{c_{n+1}} f(x)\, dx \to 0$, then all solutions of (NH) are oscillatory except possibly one. Consequently, all solutions except possibly one are oscillatory for the equation $y'' + y = f(x)$ whenever $f(x) \to 0$ as $x \to \infty$.

Sufficient conditions that $|z'| + |z|$ be bounded for all solutions $z(x)$ of (H) have been mentioned (Cesari [3, p. 81]). Another condition that $|z'|$ be bounded for all solutions $z(x)$ of (H) is that $p(x)$ be positive and decreasing. This is known, and follows from the identity $\int_a^x p'z^2\, dt = [z'^2 + pz^2]_a^x$.

To derive further oscillation properties, we require two identities. Suppose $g(x)$ is C' and $z(x)$ is any solution of the self-adjoint homogeneous differential equation $[r(x)z']' + p(x)z = 0$. It is readily verified by differentiation that

$$\text{(3.3)} \qquad \int_a^x \left[\left[\frac{g}{r} \right]' (rz')^2 + (gp)'z^2 \right] dx = [g\{rz'^2 + pz^2\}]_a^x.$$

This identity is due to Leighton. If we extend this result to equation (NH), we have

$$\text{(3.4)} \qquad \int_a^x [g'y'^2 + (gp)'y^2 + 2gfy']\, dx = [g\{y'^2 + py^2\}]_a^x,$$

and

$$\text{(3.5)} \qquad \int_a^x [g'y'^2 + (gp)'y^2 - 2(gf)'y]\, dx = [g\{y'^2 + py^2 - 2fy\}]_a^x.$$

For the remainder of the paper, let us assume that $p(x)$ and $f(x)$ are continuously differentiable (and positive).

Setting $g(x) = 1/f(x)$ in (3.5), we get

$$\int_a^x \left[\frac{1}{f}\right]' y'^2 + \left[\frac{1}{A}\right]' y^2\, dx = \left[\frac{y'^2}{f} + \frac{y}{A}(y - 2A)\right]_a^x. \tag{3.6}$$

It follows immediately that if $f(x)$ and $A(x)$ are both increasing or both decreasing, then no solution of (NH) has two double zeros.

The general solution of the differential equation $y'' + y = 1$ is $y(x) = 1 + c_1 \sin x + c_2 \cos x$. The nonoscillatory solutions are 1 and those which have relative extrema in the region $0 < y < 2$. This example is a limiting case of Theorems 3.5 and 3.6, which follow. We note that Theorem 3.5 extends a result of Keener [5, p. 63].

THEOREM 3.5. *If $f(x)$ and $A(x)$ are increasing and if $y(x)$ is any solution of* (NH) *such that $y'(a) = 0$ and $0 \leqq y(a) \leqq 2A(a)$, then $0 < y(x) < 2A(x)$ on (a, ∞).*

To prove the theorem, we suppose that at some point $m > a$, we have either $y(m) = 0$ or $y(m) = 2A(m)$. We apply (3.6) to the interval $[a, m]$ and have

$$0 > \int_a^m \left[\frac{1}{f}\right]' y'^2 + \left[\frac{1}{A}\right]' y^2\, dx = \frac{[y'(m)]^2}{f(m)} - \frac{y(a)}{A(a)}[y(a) - 2A(a)] > 0,$$

since $0 \leqq y(a) \leqq 2A(a)$ and $y'(a) = 0$. From this contradiction we infer the truth of the theorem.

To illustrate the theorem, we note that if $y(x)$ is any solution of $y'' + xy = (x - 1/x)$ with $y'(a) = 0$ and $0 \leqq y(a) \leqq 2[1 - (1/a^2)]$, then $0 < y(x) < 2$ on (a, ∞). On the other hand, we note that any solution of the equation $y'' + y = 2e^{-x}$ that has a horizontal tangent at any point must be oscillatory.

We observe that $w(x) = y(-x)$ is a solution of the equation $w''(x) + p(-x)w(x) = f(-x)$ whenever $y(x)$ is a solution of (NH). Thus, for any statement involving $f'(x)$ or $p'(x)$ having one sign on (a, ∞), there is a dual statement involving $f'(x)$ or $p'(x)$ having the opposite sign on $(-\infty, a)$. For example, if we make $A(x)$ decreasing instead of increasing in the hypothesis of Theorem 3.5, the conclusion is that $0 < y(x) < 2A(x)$ on $(-\infty, a)$. We use this observation in the next theorem.

THEOREM 3.6. *Suppose that $A(x)$ and $f(x)$ are decreasing and that* (H) *is oscillatory. Let $y(x)$ be a solution of* (NH) *such that $y(a)$ is outside the y-interval* $(0, 2A(a))$. *Then $y(x)$ oscillates, and each relative extremum of $y(x)$ is outside the region $0 \leqq y \leqq 2A(x)$.*

The proof is an application of Lemma 3.3. By the dual to Theorem 3.5, we have $0 < Y_n < 2A(a)$ for all n. From the lemma, it follows that for any solution $y(x)$ of (NH) with $y(a)$ outside $(a, 2A(a))$, $y(x)$ has one zero in each of the intervals $[a, c_1)$, (c_1, c_2), (c_2, c_3), $\cdots$. To complete the proof, we suppose that $y(x)$ has a relative extremum at T, and apply (3.6) to the interval $[a, T]$. Since $y'(T) = 0$, we have

$$0 < \int_a^T \left[\frac{1}{f}\right]' y'^2 + \left[\frac{1}{A}\right]' y^2 \, dx$$
$$= \frac{y(T)}{A(T)}[y(T) - 2A(T)] - \frac{y'^2(a)}{f(a)} - \frac{y(a)}{f(a)}[y(a) - 2A(a)].$$

The last two terms on the right are nonpositive, since $y(a)$ is outside $(0, 2A(a))$. Consequently,

$$y(T)[y(T) - 2A(T)] > 0,$$

and each relative extremum of $y(x)$ is outside the y-interval $[0, 2A(x)]$. The proof is complete.

In [8], Leighton and Skidmore developed an analogue of the Bôcher–Osgood theorem [6, p. 226]. Theorem 3.7, that follows, extends their result.

THEOREM 3.7. *Suppose* $p'(x) \geqq 0$ *and* $y(x)$ *is a solution of* (NH) *with successive relative maximum, minimum, maximum values at* a, m, *and* M. *Then*

(i) $|y(a)| > |y(m)|$; *and*

(ii) *if, in addition,* $A'(x) \geqq 0$, *then* $y(a) > y(M)$.

If we set $g(x) = 1/p(x)$ in (3.4), we have

$$0 > \int_a^m \left[\frac{1}{p}\right]' y'^2 + 2Ay' \, dx = y^2(m) - y^2(a),$$

and (i) follows at once.

To prove (ii), we assume that $y(M) \geqq y(a)$. Then $y(M) > y(x)$ on (a, M). If we set $g(x) = 1/p(x)$ in (3.5), we have

$$[y^2(x) - 2y(x)A(x)]_a^M = \int_a^M \left[\frac{1}{p}\right]' y'^2 - 2A'y \, dx$$
$$< 2y(M)[A(a) - A(M)].$$

Simplifying this inequality, we obtain

$$y^2(M) - y^2(a) < 2A(a)[y(M) - y(a)].$$

Hence $2A(a) > [y(M) + y(a)] \geqq 2y(a)$. It follows that $y(a) < A(a)$. But this is impossible since $y(a)$ is a relative maximum. We conclude that $y(M) < y(a)$, and the proof of the theorem is complete.

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MAXIMIZATION OF GREEN'S FUNCTION OVER CLASSES OF MULTIPOINT BOUNDARY VALUE PROBLEMS*

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Abstract. An inequality is established for the Green's function of a multipoint boundary value problem, which estimates the multipoint Green's function by two-point Green's functions. This basic inequality leads to a maximization principle for Green's function over classes of boundary conditions.

1. Introduction. The purpose of this paper is to establish a maximization identity for the Green's function $G(t, s; \alpha, T)$ of the multipoint boundary value problem

$$y^{(k)} + \sum_{i=0}^{k-1} p_i(t)y^{(i)} = f(t), \tag{1.1}$$

$$y^{(j)}(s_i) = 0, \qquad 0 \leqq j \leqq n_i - 1, \quad 0 \leqq i \leqq v, \tag{1.2}$$

where $\alpha = (n_0, \cdots, n_v)$, $\sum_{i=0}^{v} n_i = k$, $T = \{s_0 < \cdots < s_v\}$, $f \in C[a, b]$.

The maximization identity, in the simplest case, takes the form

$$\sup_{(\alpha,T)\in\mathscr{F}_k} |G(t, s; \alpha, T)| = \max_{(\beta,S)\in\mathscr{F}_2} |G(t, s; \beta, S)|, \tag{1.3}$$

pointwise in (t, s). In the space L^1, the identity is

$$\sup_{(\alpha,T)\in\mathscr{F}_k} \int_a^b |G(t, s; \alpha, T)|\, ds = \max_{(\beta,S)\in\mathscr{F}_2} \int_a^b |G(t, s; \beta, S)|\, ds. \tag{1.4}$$

In both (1.3) and (1.4), $\mathscr{F}_k$ is the class of all boundary conditions (1.2) with $a = s_0 < \cdots < s_v = b$, $v + 1 \leqq k$, and $\mathscr{F}_2$ is the class of two-point boundary conditions of the form (1.2). The operator on the left in (1.1) is assumed to be disconjugate on $[a, b]$.

The function $F(t) = \sup_{(\alpha,T)\in\mathscr{F}_k} \int_a^b |G(t, s; \alpha, T)|\, ds$ appears in the study of boundary value problems in the following way. Suppose $y(t)$ is a solution of (1.1) which arises by a purely existential argument. For example, suppose (1.1) is obtained from a nondisconjugate equation $y^{(k)} + q_{k-1}(t)y^{(k-1)} + \cdots + q_0(t)y = 0$ by selecting $f(t) = \sum_{i=0}^{k-1} [p_i(t) - q_i(t)]y^{(i)}(t)$.

Then problem (1.1), (1.2) is inverted to

$$y(t) = \int_a^b G(t, s; \alpha, T) \cdot f(s)\, ds \tag{1.5}$$

and we obtain the operator inequality

$$|[\mathscr{G}y](t)| \leqq \int_a^b |G(t, s; \alpha, T)|\, ds \|f\| \tag{1.6}$$

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for the linear integral operator $\mathcal{G}$ on the right side of (1.5). To obtain an estimate for $\|\mathcal{G}\|$, we would have to maximize the integral in the right side of (1.6). The problem that emerges is that $[a, b]$ is fixed, but we really did not know the location of T, or the multiplicities assigned by α. So the best that can be said is

$$|[\mathcal{G}y](t)| \leqq F(t)\|f\|. \tag{1.7}$$

On the other hand, writing out $\int_a^b |G(t, s; \alpha, T)|\, ds$ is to no avail, because the best known formula has terms which change sign, and it does not respond to simple algebraic manipulation. Therefore, relation (1.4) makes a precise and essential contribution to the estimation of $\|\mathcal{G}\|$.

Ramifications of (1.3) and (1.4) and their generalizations are to appear in a forthcoming paper by Bogar and Gustafson [4], which treats the question of estimating the length of the disconjugacy interval, and the length of intervals of uniqueness for multipoint boundary value problems (1.1), (1.2).

An intermediate step in obtaining (1.3) and (1.4) is to prove an inequality, which for the special case $v = k - 1$ is

$$|G(t, s; \alpha, T)| \leqq \sum_{i=1}^{k-1} \chi_{(s_{i-1}, s_i)}(t)|G(t, s; \alpha_i, S)|, \tag{1.8}$$

where $S = \{a < b\}$, $\alpha_i = (i, k - i)$. Inequality (1.8) leads to estimates in the spaces $C[a, b]$ and $L^1[a, b]$ which imply (1.3) and (1.4). Accordingly, (1.8) may prove to be of independent interest, aside from the maximization problem discussed in this paper.

Attempts to compare (1.8) with the Beesack inequality [2] for D^n,

$$|G(t, s; \alpha, T)| \leqq \frac{|\prod_{i=0}^{v} (t - s_i)^{n_i}|}{k!(b - a)}, \tag{1.9}$$

have led us to believe that each inequality has its virtues, with (1.9) being quite good near each interior point s_i, and (1.8) being quite good near the ends a, b. Of course, Beesack's inequality (1.9) does not apply to maximization problems of the type considered in this paper.

Potential use of (1.8) has been greatly increased by recent research of the authors [1] on two-point problems. We have illustrated the contribution of [1] to inequality (1.8) in the examples of § 6.

The program to establish (1.3) and (1.4) is as follows. First, in § 2 we introduce notation and terminology. In §§ 3 and 4 we show that $G(t, s; \alpha, T)$ is differentiable in the boundary data variable s_i, provided $n_i = 1$, and we determine the sign of this derivative. The differentiation result is used in § 5 in a monotonicity argument to establish (1.8) and certain generalizations. Then we proceed to prove the maximization formulas (1.3) and (1.4) in § 6 with the aid of the Green's function convergence theorem in Gustafson [7].

Illustrations for nonspecialists appear in § 6.

2. Preliminaries. Consider the linear ordinary differential equation

$$Ku = 0; \quad Ku \equiv u^{(k)} + \sum_{i=0}^{k-1} p_i(t)u^{(i)}, \tag{2.1}$$

with $p_i \in C[a, b]$, $0 \leqq i \leqq k - 1$.

Throughout the paper, it is assumed that K is *disconjugate on* $[a, b]$, that is, K admits a *Libri–Frobenius–Mammana factorization*

$$Ku = b_k^{-1}(\cdots(b_1^{-1}(b_0^{-1}u)')'\cdots)' \tag{2.2}$$

with $b_i(t) > 0$ on $[a, b]$, $0 \leqq i \leqq k$. This condition can be tested in terms of the coefficients of K (see Willett [19], Hartman [8], Levin [10]). A direct attack is to show that no solution has k zeros on $[a, b]$, counting multiplicities; this has been shown equivalent to (2.2) by Polya [16].

The terminology "*u has a zero of order α at T*" will be used to denote the *Nicoletti boundary conditions*

$$u^{(i)}(s_j) = 0, \qquad 0 \leqq i \leqq n_j - 1, \quad 0 \leqq j \leqq \nu, \tag{2.3}$$

where we write $\alpha = (n_0, \cdots, n_\nu)$, $T = \{s_0 < \cdots < s_\nu\}$, and demand always that $|\alpha| \equiv \sum_{i=0}^{\nu} n_i = k$. Unless otherwise stated, we shall assume that $\nu \geqq 1$, $a = s_0$, $b = s_\nu$.

Green's function. The *Green's function* $G(t, s; \alpha, T)$ for problem (2.1), (2.3) always exists under the disconjugacy assumption. This function is the kernel of the linear integral operator which inverts the problem

$$Ku = f, \quad u \text{ has a zero of order } \alpha \text{ at } T, \tag{2.4}$$

hence the unique solution of (2.4) is

$$u(t) = \int_a^b G(t, s; \alpha, T) f(s)\, ds, \tag{2.5}$$

for each $f \in C[a, b]$.

Define for each $u \in C^{k-2}[a, b]$ the operator $\mathscr{L} \equiv \mathscr{L}[\alpha, T] : C^{k-2}[a, b] \to R^k$ by the relation

$$\mathscr{L}u = (u(s_0), \cdots, u^{(n_0-1)}(s_0), \cdots, u(s_\nu), \cdots, u^{(n_\nu-1)}(s_\nu))^T. \tag{2.6}$$

Define for each fixed basis $U = (u_1, \cdots, u_k)$ of $Ku = 0$ a $k \times k$ matrix $Z \equiv Z[U; \alpha, T]$ as follows: the columns of Z are the vectors $\mathscr{L}(u_1), \cdots, \mathscr{L}(u_k)$.

Let $W(s)$ denote the Wronskian matrix of $U(s)$, define $\varepsilon(u) = 1$ for $u \geqq 0$, $\varepsilon(u) = 0$ for $u < 0$, and put $h(t, s) = U(t)W^{-1}(s)e$, $e \equiv (0, \cdots, 0, 1)^T \in R^k$, $\mathscr{H}(t, s) \equiv \varepsilon(t - s)h(t, s)$. Define $V \equiv V[\alpha, T]$ by

$$V(s) \equiv \operatorname{diag}(\varepsilon(s_0 - s)I_{n_0}, \cdots, \varepsilon(s_\nu - s)I_{n_\nu}),$$

where I_j is the $j \times j$ identity matrix.

The following representations of Green's function are valid (see Gustafson [7, §§ 2 and 5]):

$$G(t, s; \alpha, T) = \varepsilon(t - s)h(t, s) - U(t)Z^{-1}V(s)ZW^{-1}(s)e, \tag{2.7}$$

$$G(t, s; \alpha, T) = \mathscr{H}(t, s) - U(t)Z^{-1}\mathscr{L}[\mathscr{H}(\cdot, s)]. \tag{2.8}$$

The function $h(t, s)$ is the *Cauchy function* for the operator K; it satisfies $h^{(j)}(s, s) = \delta_{j,k-1}$ [Kronecker's delta] where $h^{(j)} \equiv [\partial/\partial t]^j h$. The connection between (2.7) and (2.8) is that $\mathscr{L}[\mathscr{H}(\cdot, s)] = V(s)\mathscr{L}[h(\cdot, s)]$ and $\mathscr{L}[h(\cdot, s)] = ZW^{-1}(s)e$. Relation (2.7) is efficient for explicit computation, provided one selects U such that $Z[U; \alpha, T] = I$. In practice, the use of (2.8) in computation reduces to the use of

(2.7). Relation (2.8) can be considered as a reformulation of a scalar identity for two-point problems due to Westfall [18] (see also Birkhoff [3, p. 378]); it has definite advantages in proofs. For example, the following lemma follows easily from (2.8), but is not evident from (2.7).

LEMMA 2.1. *The Green's function* $G(t, s; \alpha, T)$ *is continuous on* $[a, b] \times [a, b]$.

LEMMA 2.2. *Let* K *be disconjugate on* $[a, b]$. *Then*

$$\prod_{i=0}^{v} (t - s_i)^{-n_i} G(t, s; \alpha, T) > 0, \qquad a < s < b, \quad a \leqq t \leqq b, \tag{2.9}$$

where (2.9) *is interpreted as a limit for* $t = s_i$, $0 \leqq i \leqq v$.

Proof. See Coppel [5] or Pokornyi [15].

3. The boundary data derivative of G. The purpose of this section is to establish some special differentiability results for $G(t, s; \alpha, T)$ in the boundary data variable T.

Let us now state this more precisely. Put

$$\alpha = (n_0, \cdots, n_{i_0-1}, 1, n_{i_0+1}, \cdots, n_v),$$
$$T(s^*) = \{s_0 < \cdots < s_{i_0-1} < s^* < s_{i_0+1} < \cdots < s_v\}.$$

The project is to define for each fixed t, s the function $f: s^* \to G(t, s; \alpha, T(s^*))$ on the open interval (s_{i_0-1}, s_{i_0+1}), and establish some differentiability properties of the function f.

Throughout,

$$k_0 \equiv 1 + \sum_{j=0}^{i_0-1} n_j,$$

which denotes the row location of $u(s^*)$ in the vector $\mathscr{L}(u)$. The standard unit vectors in R^k are denoted by e_i, $i = 1, \cdots, k$.

The set difference $T(s^*)\setminus\{s^*\}$ is defined to be the ordered set $\{s_0 < \cdots < s_{i_0-1} < s_{i_0+1} < \cdots < s_v\}$. Terms not defined here are located in § 2, in Gustafson [7], or else are standard terms of analysis.

LEMMA 3.1. *Using the notation as given above, let* $\mathscr{M} : C^{k-2}[a, b] \to R^k$ *be defined by* $\mathscr{M}u = (0, \cdots, 0, u'(s^*), 0, \cdots, 0)$ $(u'(s^*)$ *in position* $k_0)$. *Then the following relation is valid for* $(t, s) \in [a, b] \times [a, b]$:

$$\frac{\partial}{\partial s^*} G(t, s; \alpha, T) = U^*(t)\left\{\frac{\partial Z}{\partial s^*} Z^{-1} \mathscr{L}[\mathscr{H}(\cdot, s)] - \mathscr{M}[\mathscr{H}(\cdot, s)]\right\}, \tag{3.1}$$

where $U^*(t) = U(t)Z^{-1}$ *and* $Z = Z[U; \alpha, T]$.

Proof. Compute, using relation (2.8).

LEMMA 3.2. *Using the notation given above, let* $y(t; s^*) = \int_a^b G(t, s; \alpha, T(s^*)) \cdot f(s)\, ds$, *and denote by* $u(t)$ *the unique solution of* $Ku = 0$ *with a zero of order* $(n_0, \cdots, n_{i_0-1}, n_{i_0+1}, \cdots, n_v)$ *at* $T(s^*)\setminus\{s^*\}$, $u(s^*) = 1$. *Then for each* $f \in C[a, b]$,

$$\frac{\partial y}{\partial s^*}(t; s^*) = -u(t) y'(s^*; s^*), \qquad a \leqq t \leqq b. \tag{3.2}$$

Proof. To prove (3.2), employ (3.1), then after a justification of differentiation under the integral sign one has

$$\frac{\partial y}{\partial s^*}(t; s^*) = U^*(t) \int_a^b \left\{\frac{\partial Z}{\partial s^*} Z^{-1} \mathscr{L}[\mathscr{H}(\cdot, s)] - \mathscr{M}[\mathscr{H}(\cdot, s)]\right\} f(s)\, ds.$$

To evaluate the integral, observe that $\partial Z/\partial s^*$ has k_0th row $U'(s^*)$ and all other rows zero; therefore the portion $\{\cdots\}$ of the integrand is a multiple of the standard unit vector $e_{k_0} \in R^k$. Since $u(t) = U^*(t)e_{k_0}$, the desired identity is a consequence of the equality

$$-y'(s^*; s^*) = \int_a^b \{U'(s^*)Z^{-1}\mathscr{L}[\mathscr{H}(\cdot, s)] - \mathscr{H}'(s^*, s)\}f(s)\,ds,$$

which follows immediately from (2.8).

4. The sign of the boundary data derivative. The purpose of this section is to establish a theorem on the sign of the boundary data derivative $(\partial/\partial s^*) \times G(t, s; \alpha, T(s^*))$ of Green's function. Throughout, let

$$P(t; \alpha, T) \equiv \prod_{i=0}^{v} (t - s_i)^{n_i}.$$

THEOREM 4.1. *Let K be disconjugate on $[a, b]$. Then*

$$\frac{\partial P}{\partial s^*}(t; \alpha, T(s^*))\left[\frac{\partial}{\partial s^*}\right]G(t, s; \alpha, T(s^*)) \geqq 0. \tag{4.1}$$

Remark. The question of strict inequality in relation (4.1) will not concern us, since inequality is sufficient for the purposes of this paper.

The proof of relation (4.1) will use the following result.

LEMMA 4.2. *Let K be disconjugate on $[a, b]$ and put*

$$y(t; s^*) = \int_a^b G(t, s; \alpha, T(s^*))f(s)\,ds,$$

with $f \in C[a, b]$, $f > 0$. Then

$$0 < \left[\frac{\partial P}{\partial s^*}(t; \alpha, T(s^*))\right]^{-1}\frac{\partial y}{\partial s^*}(t; s^*) = \int_a^b \left[\frac{\partial P}{\partial s^*}\right]^{-1}\left[\frac{\partial G}{\partial s^*}\right]f, \tag{4.2}$$

for $a \leqq t \leqq b$ (interpret as a limit when necessary).

Proof. The hypothesis of disconjugacy implies that $y(t; s^*)$ cannot have additional zeros, counting multiplicities. Indeed, otherwise $f(t) = Ky$ would have a zero in $[a, b]$, by Rolle's theorem, and the Libri factorization of K; see Coppel [5].

In particular, $y'(s^*; s^*) \neq 0$, so $\partial y/\partial s^*$ is a nontrivial solution of $Ku = 0$, by virtue of (3.2). But, according to Lemma 3.2,

$$\frac{\partial y}{\partial s^*} = -u(t)y'(s^*; s^*),$$

and $[P(t; \alpha, T(s^*))/(t - s^*)]^{-1}u(t)$ is never zero, so (4.2) will be proved if it can be shown to hold for a single value of t; we intend to verify (4.2) for $t = s^*$.

Since K is disconjugate, one can apply Lemma 2.2 to obtain $P(t; \alpha, T(s^* + h))G(t, s; \alpha, T(s^* + h)) \geqq 0$ for $h > 0$ and small.

On the other hand,

$$\frac{\partial y}{\partial s^*}(s^*; s^*) = \lim_{h \to 0+} \frac{y(s^*; s^* + h)}{h},$$

because $y(s^*; s^*) = 0$. But

$$\left[-\frac{\partial P}{\partial s^*}(t; \alpha, T(s^*))\right][t - s^* - h] = P(t; \alpha, T(s^* + h)),$$

and taking $t = s^*$ in this relation, we have

$$y(s^*; s^* + h)\frac{\partial P}{\partial s^*}(s^*; \alpha, T(s^*))$$

$$= \frac{1}{h}\int_a^b P(s^*; \alpha, T(s^* + h))G(s^*, s; \alpha, T(s^* + h))f(s)\,ds \geqq 0.$$

Now divide by h, limit, and obtain (4.2) for $t = s^*$. This completes the proof.

Proof of Theorem 4.1. Let

$$\mathscr{K}(t, s) = \frac{\partial P}{\partial s^*}(t; \alpha, T(s^*))\left[\frac{\partial}{\partial s^*}\right]G(t, s; \alpha, T(s^*)).$$

Since $\mathscr{K}(t, s) = 0$ for $t \in T(s^*)\diagdown\{s^*\}$, it suffices to prove (4.1) in case $t \notin T(s^*)\diagdown\{s^*\}$. For such fixed values of t, $\mathscr{K}(t, s)$ is sectionally continuous as a function of s, and by Lemma 4.2, $\int_a^b \mathscr{K}(t, s)f(s)\,ds > 0$ for every $f \in C[a, b]$, $f > 0$. Therefore, $\mathscr{K}(t, s)$ cannot take on a negative value; this completes the proof.

5. General inequality theory for Green's function.

LEMMA 5.1. *Let* $S = \{s_0 < s_1 < \cdots < s_{r-1} < s_{r+1} < \cdots < s_v\}$, $S(s^*) = S \cup \{s^*\}$ *for* $s_{r-1} < s^* < s_{r+1}$, *and put*

$$\alpha = (n_0, \cdots, n_{r-1}, 1, n_{r+1}, \cdots, n_v),$$

$$\alpha+ = (n_0, \cdots, n_{r-1}, n_{r+1} + 1, \cdots, n_v),$$

$$\alpha- = (n_0, \cdots, n_{r-1} + 1, n_{r+1}, \cdots, n_v).$$

$$Q(t) \equiv Q(t; \alpha, S) = \prod_{\substack{i=0\\ i\neq r}}^{v} (t - s_i)^{n_i}.$$

If K is disconjugate on $[a, b]$, $s_r \in (s_{r-1}, s_{r+1})$, *then*

(5.1) $$Q(t)[G(t, s; \alpha+, S) - G(t, s; \alpha, S(s_r))] \leqq 0,$$

(5.2) $$Q(t)[G(t, s; \alpha, S(s_r)) - G(t, s; \alpha-, S)] \leqq 0.$$

Proof. It is easy to verify that $Q(t) = -(\partial P/\partial s^*)(t; \alpha, S(s^*))$, hence $Q(t) \times [\partial/\partial s^*]\ G(t, s; \alpha, S(s^*)) \leqq 0$ by Theorem 4.1. By virtue of the *Green's function convergence theorem* in Gustafson [7], the mapping $s^* \to G(t, s; \alpha, S(s^*))$ is continuous on $s_{r-1} \leqq s^* \leqq s_{r+1}$; therefore (5.1) and (5.2) follow from (4.1) and the mean value theorem.

Remark. The proof of the Green's function convergence theorem in [7] for the special case considered in this paper can be obtained from (2.8) using L'Hospital's rule.

LEMMA 5.2. *Let* $T = \{s_0 < \cdots < s_v\}$, $\alpha = (n_0, \cdots, n_v)$, $S = T\diagdown\{s_r\}$, $\alpha- = (n_0, \cdots, n_{r-1} + n_r, n_{r+1}, \cdots, n_v)$, $\alpha+ = (n_0, \cdots, n_{r-1}, n_r + n_{r+1}, \cdots, n_v)$.

If K is disconjugate on $[a, b]$, *then*

$$|G(t, s; \alpha, T)| \leqq \chi_{[a,s_r)}(t)|G(t, s; \alpha+, S)| + \chi_{(s_r,b]}(t)|G(t, s; \alpha-, S)|. \tag{5.3}$$

Proof. Define $S(s^*) = \{s_0 < \cdots < s_{r-1} < s^* < s_{r+1} < \cdots < s_v\}$. The result is proved by finite induction on the integer n_r.

Consider first the case when $n_r = 1$. If $Q(t)$ is the polynomial of Lemma 5.1, then by Lemma 2.2,

$$(t - s^*)Q(t)G(t, s; \alpha, S(s^*)) \geqq 0.$$

If $t - s_r > 0$ and $s_{r-1} < s^* < s_r$, then $t - s^* > 0$, so $Q(t)G(t, s; \alpha, S(s^*)) \geqq 0$. An application of Lemma 5.1 gives

$$0 \leqq Q(t)G(t, s; \alpha, T) \leqq Q(t)G(t, s; \alpha-, S), \tag{5.4}$$

valid for $s_r < t \leqq s_v$. We may take absolute values in (5.4) to obtain

$$\chi_{(s_r,s_v]}(t)|G(t, s; \alpha, T)| \leqq \chi_{(s_r,s_v]}(t)|G(t, s; \alpha-, S)|. \tag{5.5}$$

In a similar way, consider $t < s_r$ and $s_r \leqq s^* < s_{r+1}$. Then

$$-Q(t)G(t, s; \alpha, S(s^*)) \geqq 0,$$

and Lemma 5.1 gives

$$-Q(t)G(t, s; \alpha+, S) \geqq -Q(t)G(t, s; \alpha, T) \geqq 0. \tag{5.6}$$

Taking absolute values in (5.6) gives

$$\chi_{[s_0,s_r)}(t)|G(t, s; \alpha, T)| \leqq \chi_{[s_0,s_r)}(t)|G(t, s; \alpha+, S)|. \tag{5.7}$$

Now let's use the fact that $G(s_r, s; \alpha, T) = 0$ and add (5.5) and (5.7) to obtain relation (5.3) in the special case $n_r = 1$.

Suppose now that relation (5.3) has been proved for $n_r \leqq l$. We prove it for $n_r = l + 1$.

Let $T(s^*) = T \cup \{s^*\}$ for $s_{r-1} < s^* < s_r$. Define

$$\beta = (n_0, \cdots, n_{r-1}, 1, n_r - 1, n_{r+1}, \cdots, n_v),$$

$$\beta+ = (n_0, \cdots, n_v) = \alpha, \qquad \beta- = (n_0, \cdots, n_{r-1} + 1, n_r - 1, n_{r+1}, \cdots, n_v).$$

By relations (5.1), (5.2), for $s_{r-1} < s^* < s_r < t$,

$$Q_1(t)[G(t, s; \beta+, T) - G(t, s; \beta, T(s^*))] \leqq 0,$$

$$Q_1(t)[G(t, s; \beta, T(s^*)) - G(t, s; \beta-, T)] \leqq 0,$$

where $Q_1(t) = (t - s^*)Q(t)$. Adding these two inequalities and employing Lemma 2.1, we have for $t > s_r > s^*$ the inequality

$$|G(t, s; \beta+, T)| \leqq |G(t, s; \beta-, T)|. \tag{5.8}$$

Let us now invoke the induction hypothesis that inequality (5.3) holds whenever l zeros are assigned at s_r. Then (5.3) applied to $(\beta-, T)$ gives

$$\chi_{(s_r,s_v]}(t)|G(t, s; \beta-, T)| \leqq \chi_{(s_r,s_v]}(t)|G(t, s; \alpha-, T)|.$$

However, $\beta+ = \alpha$, so the preceding inequality and (5.8) give

$$\chi_{(s_r,s_v]}(t)|G(t,s;\alpha,T)| \leqq \chi_{(s_r,s_v]}(t)|G(t,s;\alpha-,T)|. \tag{5.9}$$

In a similar way, one can show that

$$\chi_{[s_0,s_r)}(t)|G(t,s;\alpha,T)| \leqq \chi_{[s_0,s_r)}(t)|G(t,s;\alpha+,S)|. \tag{5.10}$$

Adding relations (5.9) and (5.10) proves the lemma.

THEOREM 5.3. *Let $q \geqq 2$ and $0 \leqq p \leqq v$ be integers, $p + q \leqq v$, and put $\alpha = (n_0, \cdots, n_v)$, $T = \{s_0 < \cdots < s_v\}$,*

$$S = \{s_0 < \cdots < s_p < s_{p+q} < \cdots < s_v\},$$

$$\alpha_j = \left(n_0, \cdots, n_{p-1}, \sum_{i=p}^{p+j-1} n_i, \sum_{i=p+j}^{p+q} n_i, n_{p+q+1}, \cdots, n_v\right), \qquad 1 \leqq j \leqq q.$$

If K is disconjugate on $[a, b]$, and $r_0 \equiv s_0$, $r_1 \equiv s_{p+1}, \cdots, r_{q-1} \equiv s_{p+q-1}$, $r_q \equiv s_v$, then

$$|G(t,s;\alpha,T)| \leqq \sum_{j=1}^{q} \chi_{(r_{j-1},r_j)}(t)|G(t,s;\alpha_j,S)|. \tag{5.11$_q$}$$

Proof. This is easily proved by induction on q, using the preceding lemma. Indeed, Lemma 5.2 establishes the result for $q = 2$. If the inequality is valid for all integers $\leqq q - 1$, then to prove it for q, proceed as follows. By the induction hypothesis, the zeros $r_0, \cdots, r_{q-2}$ will satisfy an inequality (5.11), with α_j and S replaced by $\bar{\alpha}_j$ and $\bar{S}$, to help keep the notation appropriate. The Green's function $G(t, s; \bar{\alpha}_j, \bar{S})$ on the right side of $(5.11)_{q-1}$ will have a multiple zero at s_{p+q-1}. By Lemma 5.2, applied to the point $r_{q-1} = s_{p+q-1}$, the following inequality is valid:

$$|G(t,s;\bar{\alpha}_j,\bar{S})| \leqq \chi_{(r_0,r_{q-1})}(t)|G(t,s;\bar{\alpha}_j+,S)| + \chi_{(r_{q-1},r_q)}(t)|G(t,s;\bar{\alpha}_j-,S)|, \tag{5.12}$$

where S has the meaning given in the hypothesis of the theorem. Therefore, using (5.12) for $1 \leqq j \leqq q - 1$, relation $(5.11)_{q-1}$ gives

$$\begin{aligned} |G(t,s;\alpha,T)| &\leqq \sum_{j=1}^{q-2} \chi_{(r_{j-1},r_j)}(t)|G(t,s;\bar{\alpha}_j,\bar{S})| + \chi_{(r_{q-2},r_q)}(t)|G(t,s;\bar{\alpha}_{q-1},\bar{S})| \\ &\leqq \sum_{j=1}^{q-2} \chi_{(r_{j-1},r_j)}(t)|G(t,s;\bar{\alpha}_j+,S)| + \chi_{(r_{q-2},r_{q-1})}(t)|G(t,s;\bar{\alpha}_{q-1}+,S)| \\ &\quad + \chi_{(r_{q-1},r_q)}(t)|G(t,s;\bar{\alpha}_{q-1}-,S)|. \end{aligned}$$

It is easily verified that $\alpha_j = \bar{\alpha}_j+$ for $1 \leqq j \leqq q - 1$, and $\alpha_q = \bar{\alpha}_{q-1}-$; therefore the induction is completed, and the theorem is proved.

Remark. At first appearance, it would seem that relation (5.11) is not the most general relation possible. However, because of the Green's function convergence theorem in Gustafson [7], every inequality of this same kind can be obtained from (5.11) by a limiting procedure.

Remark. Simple induction using relation (5.11) is sometimes convenient. For example, let us start with $\alpha = (2, 1, 3, 1, 5)$, $T = \{s_0 < s_1 < s_2 < s_3 < s_4\}$. It is clear from relation (5.11) that we can estimate $G(t, s; \alpha, T)$ by 3-point problems at

$S = \{s_0 < s_2 < s_4\}$. In fact, using $\bar{\alpha}_1 = (2, 1, 3, 6)$, $\bar{\alpha}_2 = (2, 1, 4, 5)$ and $\bar{S} = \{s_0 < s_1 < s_2 < s_4\}$, one has

$$\begin{aligned}|G(t, s; \alpha, T)| &\leqq \chi_{(s_0,s_3)}(t)|G(t, s; \bar{\alpha}_1, \bar{S})| + \chi_{(s_3,s_4)}(t)|G(t, s; \bar{\alpha}_2, \bar{S})| \\ &\leqq \chi_{(s_0,s_1)}(t)|G(t, s; \bar{\alpha}_1 +, S)| + \chi_{(s_1,s_3)}(t)|G(t, s; \bar{\alpha}_1 -, S)| \\ &\quad + \chi_{(s_3,s_4)}(t)|G(t, s; \bar{\alpha}_2 -, S)|,\end{aligned}$$

where $\bar{\alpha}_1 - = (3, 3, 6)$, $\bar{\alpha}_1 + = (2, 4, 6)$, $\bar{\alpha}_2 - = (3, 4, 5)$.

6. Maximization of $G(t, s; \alpha, T)$. Let us now reap the benefits of the results of § 5, and establish the following maximization theorems.

DEFINITION 6.1. Define $S = \{s_0 < \cdots < s_p < t_0 < \cdots < t_r\}$, $p = 0$ or $r = 0$ allowed. Set

$$\alpha_j = \left(m_0, \cdots, m_p + \sum_{i=0}^{j-1} n_i, l_0 + \sum_{i=j}^{q} n_i, l_1, \cdots, l_r\right),$$

where $n_0, \ldots, n_q$ are fixed integers with $|\alpha_j| = k$, and $m_0, \cdots, m_p, l_0, \cdots, l_r$ are fixed, $0 \leqq j \leqq q + 1 (\sum_{i=0}^{-1} = \sum_{i=q+1}^{q} = 0)$.

Let $\mathscr{F}(\alpha)$ be the class of all T, where

$$\alpha = (m_0, \cdots, m_p, n_0, \cdots, n_q, l_0, \cdots, l_r)$$

and

$$T = \{s_0 < \cdots < s_p < x_0 < \cdots < x_q < t_0 < \cdots < t_r\},$$

and $x_0, \cdots, x_q$ are arbitrary within the interval (s_p, t_0).

THEOREM 6.2. *Let K be disconjugate on $[a, b]$. Then*

$$\sup \{|G(t, s; \alpha, T)| : T \in \mathscr{F}(\alpha)\} = \max_{0 \leqq j \leqq q+1} |G(t, s; \alpha_j, S)|. \tag{6.1}$$

Proof. By inequality (5.11) of Theorem 5.3 we have

$$|G(t, s; \alpha, T)| \leqq \max_{0 \leqq j \leqq q+1} |G(t, s; \alpha_j, S)|.$$

On the other hand, by the Green's function convergence theorem in Gustafson [7],

$$|G(t, s; \alpha_j, S)| \leqq \sup \{|G(t, s; \alpha, T)| : T \in \mathscr{F}(\alpha)\},$$

for $0 \leqq j \leqq q + 1$. This completes the proof of (6.1).

THEOREM 6.3. *Let K be disconjugate on $[a, b]$. Then*

$$\sup \left\{\int_a^b |G(t, s; \alpha, T)|\, ds : T \in \mathscr{F}(\alpha)\right\} = \max_{0 \leqq j \leqq q+1} \int_a^b |G(t, s; \alpha_j, S)|\, ds. \tag{6.2}$$

Proof. The proof of (6.2) parallels that of (6.1). To reverse the inequality supplied by integration of (5.11), apply the convergence theorem in Gustafson [7] for the space L^1.

COROLLARY 6.4. *Relations* (1.3) *and* (1.4) *are valid.*

Illustrations for nonspecialists. Consider the special case $p = 0$, $q = \nu$ in Theorem 5.3. Write $m_j = \sum_{i=0}^{j-1} n_i$. Then $S = \{a, b\}$, $\alpha_j = (m_j, k - m_j)$ and the following inequality is obtained:

$$(6.3) \qquad |G(t,s;\alpha,T)| \leqq \sum_{j=1}^{\nu} \chi_{E_j}(t)|G(t,s;\alpha_j,S)|.$$

An application of Theorem 3.1 in Bates and Gustafson [1] gives

$$(6.4) \qquad |G(t,s;\alpha,T)| \leqq \sum_{j=1}^{\nu} \chi_{E_j}(t) \min\left\{\frac{v_{m_j}(t)v^*_{m_j}(s)}{v^{(m_j)}_{m_j}(a)}, \frac{w_{m_j}(t)w^*_{m_j}(s)}{|w^{(k-m_j)}_{m_j}(b)|}\right\},$$

where $E_j = (s_{j-1}, s_j)$. The functions v_{m_j}, $v^*_{m_j}$, w_{m_j}, $w^*_{m_j}$ satisfy $Kv_{m_j} = K^*v^*_{m_j} = Kw_{m_j} = K^*w^*_{m_j} = 0$ and have zeros of order $(m_j, k - m_j - 1)$, $(k - m_j - 1, m_j)$, $(m_j - 1, k - m_j)$, $(k - m_j, m_j - 1)$, at S, respectively. Furthermore,

$$(-1)^{k-m_j-1}v^{(k-m_j-1)}_{m_j}(b) = v^{*(k-m_j-1)}_{m_j}(a) = w^{(m_j-1)}_{m_j}(a) = (-1)^{m_j-1}w^{*(m_j-1)}_{m_j}(b) = 1.$$

The solutions v_{m_j}, $v^*_{m_j}$, w_{m_j}, $w^*_{m_j}$ are easily found, so (6.4) gives a simple piecewise separable estimate for $|G(t,s;\alpha,T)|$, whenever K is disconjugate on $[a,b]$ and the adjoint K^* of K is defined.

Example 6.5. The operator $Kv = v''' + v'$ on $[0,X]$, $\alpha = (1,1,1)$, $T = \{0 < c < X\}$.

A short calculation gives

$$v_1(t) = \frac{2\sin(t/2)\sin[(X-t)/2]}{\sin(X/2)} = v^*_1(t) = w_2(t) = w^*_2(t),$$

$$w^*_1(t) = v_2(t) = \frac{\sin^2(t/2)}{\sin^2(X/2)}, \qquad w_1(t) = v^*_2(t) = v_2(X-t).$$

Therefore, as long as $0 < X < 2\pi$,

$$|G(t,s;\alpha,T)| \leqq \frac{2\sin[(X-t)/2]\sin(s/2)}{\sin^2(X/2)} \min\left\{2\sin\left(\frac{t}{2}\right)\sin\left(\frac{X-s}{2}\right),\right.$$

$$\left.\sin\left(\frac{X-t}{2}\right)\sin\left(\frac{s}{2}\right)\right\}\chi_{(0,c)}(t) + \frac{2\sin(t/2)\sin[(X-s)/2]}{\sin^2(X/2)}$$

$$\cdot\min\left\{2\sin\left(\frac{s}{2}\right)\sin\left(\frac{X-t}{2}\right), \sin\left(\frac{X-s}{2}\right)\sin\left(\frac{t}{2}\right)\right\}\chi_{(c,X)}(t).$$

Example 6.6. The operator $Kv = v^{(k)}$ on $[a,b]$,

$$\alpha = (n_0, n_1, \cdots, n_\nu), \qquad T = \{a = s_0 < s_1 < \cdots < s_\nu = b\}.$$

Using the boundary conditions we compute the following:

$$v_{m_j}(t) = \frac{(t-a)^{m_j}(b-t)^{k-m_j-1}}{(b-a)^{m_j}(k-m_j-1)!}, \qquad v^*_{m_j}(s) = \frac{(s-a)^{k-m_j-1}(b-s)^{m_j}}{(b-a)^{m_j}(k-m_j-1)!},$$

$$w_{m_j}(t) = \frac{(t-a)^{m_j-1}(b-t)^{k-m_j}}{(b-a)^{k-m_j}(m_j-1)!}, \qquad w^*_{m_j}(s) = \frac{(s-a)^{k-m_j}(b-s)^{m_j-1}}{(b-a)^{k-m_j}(m_j-1)!}.$$

Thus,

$$v_{m_j}^{(m_j)}(a) = \frac{m_j!(b-a)^{k-m_j-1}}{(k-m_j-1)!(b-a)^{m_j}}$$

and

$$w_{m_j}^{(k-m_j)}(b) = \frac{(-1)^{k-m_j}(b-a)^{m_j-1}(k-m_j)!}{(b-a)^{k-m_j}(m_j-1)!}$$

for $1 \leqq j \leqq \nu$. This gives

$$|G(t,s;\alpha,T)| \leqq \sum_{j=1}^{\nu} \chi_{E_j}(t) \frac{[(t-a)(b-s)]^{m_j-1}[(b-t)(s-a)]^{k-m_j-1}}{(m_j-1)!(k-m_j-1)!(b-a)^{k-1}} \cdot \min\left\{\frac{(t-a)(b-s)}{m_j}, \frac{(b-t)(s-a)}{k-m_j}\right\}.$$

For the special case $k = 4$, $\alpha_1 = (1,1,1,1)$, $T_1 = \{a < c < d < b\}$,

$$\alpha_2 = (1,1,2), \quad \alpha_3 = (1,2,1), \quad \alpha_4 = (2,1,1), \quad T_2 = T_3 = T_4 = \{a < c < b\},$$

we obtain the estimates:

$$|G(t,s;\alpha_1,T_1)| \leqq \chi_{(a,c)}(t)f_1(t,s) + \chi_{(c,d)}(t)f_2(t,s) + \chi_{(d,b)}(t)f_3(t,s),$$
$$|G(t,s;\alpha_2,T_2)| \leqq \chi_{(a,c)}(t)f_1(t,s) + \chi_{(c,b)}(t)f_2(t,s),$$
$$|G(t,s;\alpha_3,T_3)| \leqq \chi_{(a,c)}(t)f_1(t,s) + X_{(c,b)}(t)f_3(t,s),$$
$$|G(t,s;\alpha_4,T_4)| \leqq \chi_{(a,c)}(t)f_2(t,s) + \chi_{(c,b)}(t)f_3(t,s),$$

where

$$f_1(t,s) = \frac{(b-t)^2(s-a)^2}{2(b-a)^3}\min\left\{(t-a)(b-s), \frac{(b-t)(s-a)}{3}\right\},$$

$$f_2(t,s) = \frac{(t-a)(b-s)(b-t)(s-a)}{2(b-a)^3}\min\left\{(t-a)(b-s), (b-t)(s-a)\right\},$$

$$f_3(t,s) = \frac{(t-a)^2(b-s)^2}{2(b-a)^3}\min\left\{\frac{(t-a)(b-s)}{3}, (b-t)(s-a)\right\}.$$

Example 6.7. The operator $Ky = y^{\text{iv}} - y$ on $[0, X]$, $X \leqq 4.73$,

$$\alpha = (1,1,1,1), \qquad T = \{0 < c < d < X\}.$$

The boundary conditions give the following identities:

$$v_1(t) = \frac{\cosh X - \cos X}{2} \cdot \left[\frac{\cosh(X-t) - \cos(X-t)}{\cosh X - \cos X} - \frac{\sinh(X-t) - \sin(X-t)}{\sinh X - \sin X}\right],$$

$$v_3(t) = \frac{\sinh t - \sin t}{\sinh X - \sin X},$$

$$v_1^*(t) = w_3(t) = v_1(X - t), \qquad w_3^*(t) = v_1(t),$$
$$v_3^*(t) = w_1(t) = v_3(X - t), \qquad w_1^*(t) = v_3(t),$$
$$v_2(t) = w_2^*(t) = \frac{\sinh X - \sin X}{1 - \cosh X \cos X} \cdot v_1(X - t),$$
$$v_2^*(t) = w_2(t) = v_2(X - t).$$

Thus

$$|G(t, s; \alpha, T)| \leqq \chi_{(0,c)}(t) \min\left\{\frac{v_1(t)v_1(X - s)}{1 - \cosh X \cos X}, \frac{v_3(X - t)v_3(s)}{2}\right\} \cdot [\sinh X - \sin X]$$
$$+ \chi_{(c,d)}(t) \min\{v_1(X - t)v_1(s), v_1(t)v_1(X - s)\} \cdot \left[\frac{\sinh X - \sin X}{1 - \cosh X \cos X}\right]$$
$$+ \chi_{(d,X)}(t) \min\left\{\frac{v_3(t)v_3(X - s)}{2}, \frac{v_1(X - t)v_1(s)}{1 - \cosh X \cos X}\right\} \cdot [\sinh X - \sin X].$$

For the cases $\alpha = (1, 1, 2), (1, 2, 1)$ and $(2, 1, 1)$ with $T = \{0 < c < X\}$, the bounds are obtained from the preceding estimate by letting $d \to X$, $d \to c$ and $c \to 0$ resp., and putting $d = c$ in the latter.

Example 6.8. The operator $Kv = v^{\text{iv}} + v$ on $[0, X]$, $X \leqq 5.553$,

$$\alpha = (1, 1, 1, 1), \qquad T = \{0 < c < d < X\}.$$

This operator will be disconjugate on $[0, X]$ where X is the first positive root of $\tan[(X/\sqrt{2}) + (\pi/4)] = e^{\sqrt{2}X}$. Solving numerically, $5.5530 < X < 5.5531$.

Let $\gamma = \sqrt{2}$, $\beta = 1/\sqrt{2}$, and put

$$z(t) = [\sin \beta t](1 - e^{\gamma t})\left[\sin\left(\beta X + \frac{\pi}{4}\right) - e^{\gamma X} \cos\left(\beta X + \frac{\pi}{4}\right)\right]$$
$$- [\sin \beta X](1 - e^{\gamma X})\left[\sin\left(\beta t + \frac{\pi}{4}\right) - e^{\gamma t} \cos\left(\beta t + \frac{\pi}{4}\right)\right],$$
$$y(t) = \sin\left(\beta t + \frac{\pi}{4}\right) - e^{\gamma t} \cos\left(\beta t + \frac{\pi}{4}\right).$$

Then we get

$$v_1(t) = -\frac{e^{-\beta t}}{2} z(t), \qquad v_2(t) = e^{-\beta(t - X)} \frac{z(t)}{z'(X)},$$

and

$$v_3(t) = e^{-\beta(t - X)} \frac{y(t)}{y(X)}.$$

Further,

$$w_i(t) = v_{4-i}(X - t), \quad v_i^*(t) = v_i(X - t), \quad w_i^*(t) = w_i(X - t),$$

for $i = 1, 2, 3$. Thus

$$\begin{aligned} |G(t, s; \alpha, T)| \leqq\ & \chi_{(0,c)}(t) \min \tfrac{1}{2}\left\{ e^{-\beta(X+t-s)} \frac{z(t)z(X - s)}{z'(X)},\ e^{\beta(X+t-s)} \frac{y(X - t)y(s)}{y(X)} \right\} \\ & + \chi_{(c,d)}(t) \min \left\{ e^{\beta(s-t)} z(t)z(X - s),\ e^{\beta(t-s)} z(X - t)z(s) \right\} \frac{1}{2|z'(X)|} \\ & + \chi_{(d,X)}(t) \min \tfrac{1}{2}\left\{ e^{-\beta(X+s-t)} \frac{z(X - t)z(s)}{z'(X)},\ e^{\beta(X+s-t)} \frac{y(t)y(X - s)}{y(X)} \right\}, \end{aligned}$$

where

$$z'(X) = [e^{\gamma X} - 1 - 2\sin^2 \beta X]^2 + \sin^2 (2\beta X).$$

The bounds for different (α, T) are found as in Example 6.7.

Remark 6.9. For the operators $(D - \gamma)^k$, $[D^2 - 2\gamma D + \gamma^2 + \beta^2]^2$, and $(D - \gamma)^2(D - \beta)^2$, the reader is referred to Bates and Gustafson [1, § 5].

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SOME A PRIORI BOUNDS FOR NONLINEAR VOLTERRA EQUATIONS*

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Abstract. A priori bounds are obtained for the solutions, x, of the real vector Volterra equation $x(t) + \int_0^t a(t - s)g(x(s))\, ds = f(t)$ under several different sets of hypotheses on the prescribed functions a, g and f. In each case, the key conditions are on a and g. Thus there is a positivity assumption on the matrix $a(0)$ and an assumption on g which, in the scalar case, is typified by (but less restrictive than) the hypothesis $xg(x) \geqq 0\ (-\infty < x < \infty)$. The known procedure of reducing the problem of global existence to that of finding an a priori bound is briefly reviewed together with a relevant local existence theorem.

1. Introduction. In this paper, some a priori bounds are obtained for continuous solutions of the real Volterra equation

$$x(t) + \int_0^t a(t - s)g(x(s))\, ds = f(t), \qquad 0 \leqq t < t^*, \tag{1.1}$$

where $t^* \in (0, \infty]$, g and f are prescribed vectors with values in $\mathbb{R}^n$, $a = (a_{ij})$ is a prescribed matrix and x is the unknown vector. Although $t^* = \infty$ is often the situation in applications, the case $t^* < \infty$ occurs when a and/or f have appropriate singularities for finite t. In the present context, it is convenient to consider both cases simultaneously.

An a priori bound for continuous solutions of (1.1), or simply, an a priori bound for (1.1), is defined by

DEFINITION 1. A nondecreasing function

$$\Gamma : [0, t^*) \to [0, \infty) \tag{1.2}$$

is called an a priori bound for (1.1) if every pair $(\hat{t}, \varphi)$ that satisfies

$$\hat{t} \in (0, t^*], \qquad \varphi \in C([0, \hat{t}), \mathbb{R}^n) \text{ is a solution of (1.1) on } [0, \hat{t}), \tag{1.3}$$

must also satisfy

$$|\varphi(t)| = \left(\sum_{i=1}^n \varphi_i^2(t)\right)^{1/2} \leqq \Gamma(t) \quad \text{on } [0, \hat{t}). \tag{1.4}$$

An important application of an a priori bound is to establish the existence of a solution of (1.1) on $[0, t^*)$, i.e., to establish global rather than local existence. A well-known procedure for proving global existence, which involves a local existence theorem, a continuation argument and an a priori bound, will be given in a preliminary lemma and some corollaries below. It will then be unnecessary to state the global existence theorems which correspond to most of the a priori bounds obtained here.

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For the scalar case of (1.1), the key hypotheses will be that $a(0) > 0$ and that, loosely speaking, $g(x)$ is not too negative when $x > 0$ or too positive when $x < 0$. Analogous hypotheses are obtained in the vector case. Examples are given which show that global existence need not hold when these conditions are violated. If in addition to $a(0) > 0$, it is further assumed that a is nonincreasing, then a much stronger bound than the present ones is given in Levin [2]. Monotonicity assumptions on a will not be made here.

The following notation is employed: For $x = \mathrm{col}\,(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, let

$$|x| = (x^T x)^{1/2} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}, \tag{1.5}$$

where $x^T = (x_1 x_2 \cdots x_n)$. For real $n \times n$ matrices $a = (a_{ij})$, let

$$|a| = \left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{1/2}. \tag{1.6}$$

$L^1_{\mathrm{loc}}([0, t^*), \mathbb{R}^n)$ denotes the set of real Lebesgue measurable functions (or matrices, depending on the context) z on $[0, t^*)$, or on $(0, t^*)$, with values in $\mathbb{R}^n$, such that

$$\int_0^{t_0} |z(t)|\, dt < \infty \quad \text{for all } t_0 \in [0, t^*). \tag{1.7}$$

Similarly, $BV_{\mathrm{loc}}([0, t^*), \mathbb{R}^n)$ denotes the set of real functions (or matrices) on $[0, t^*)$ with values in $\mathbb{R}^n$ which are of bounded variation on $[0, t_0]$ for every $t_0 \in [0, t^*)$. The sets $AC_{\mathrm{loc}}([0, t^*), \mathbb{R}^n)$, denoting local absolute continuity, and $L^2_{\mathrm{loc}}([0, t^*), \mathbb{R}^n)$ are similarly defined.

Local existence for (1.1) is established in the following result of Nohel [5], whose proof is an extension to Volterra equations of a well-known one for ordinary differential equations. For convenience, a proof is given in § 2; see also Miller [4, p. 36] and Hartman [1, p. 10].

LEMMA 1. *Let g, a and f satisfy the conditions*

$$g \in C(\mathbb{R}^n, \mathbb{R}^n), \tag{1.8}$$

$$a \in L^1_{\mathrm{loc}}([0, t^*), \mathbb{R}^n), \qquad a(t) = (a_{ij}(t)), \tag{1.9}$$

$$f \in C([0, t^*), \mathbb{R}^n). \tag{1.10}$$

Then there exists $\bar{t} \in (0, t^*)$ such that (1.1) *has a continuous solution on* $[0, \bar{t}]$.

Hypothesis (1.10) can be weakened to $f \in L^\infty_{\mathrm{loc}}([0, t^*), \mathbb{R}^n)$ at the expense of dropping the continuity assertion on the solution of (1.1). This is not done here as the a priori bounds below require even stronger conditions than (1.10).

That uniqueness is not a consequence of the hypothesis of Lemma 1 is shown by setting

$$n = 1, \quad t^* = \infty, \quad g(x) = x^{1/3}, \quad a(t) \equiv -1, \quad f(t) \equiv 0 \tag{1.11}$$

in (1.1). Example (1.11) is obviously equivalent to the ordinary differential equation problem

$$x' = x^{1/3}, \quad x(0) = 0, \quad \left(' = \frac{d}{dt}\right),$$

among whose solutions are $x(t) \equiv 0$ and $x(t) = (\frac{2}{3}t)^{3/2}$.

That global existence is not implied by the hypothesis of Lemma 1 is seen from the example

$$n = 1, \quad t^* = \infty, \quad g(x) = x|x|, \quad a(t) \equiv -1, \quad f(t) \equiv 1, \tag{1.12}$$

which has $x(t) = (1 - t)^{-1}$ as its unique solution. Note that (1.12) is equivalent to

$$x' = x|x|, \qquad x(0) = 1.$$

The example

$$n = 1, \quad t^* = \infty, \quad g(x) = x|x|, \quad a(t) = -t, \quad f(t) \equiv 1 \tag{1.13}$$

also illustrates the nonglobal nature of Lemma 1. Let $t_0 \in (0, \infty)$ and $\varphi \in C^2([0, t_0), \mathbb{R}^1)$ be defined by

$$t_0 = \left(\frac{3}{2}\right)^{1/2} \int_1^\infty (\xi^3 - 1)^{-1/2}\, d\xi, \qquad \left(\frac{3}{2}\right)^{1/2} \int_1^{\varphi(t)} (\xi^3 - 1)^{-1/2}\, d\xi = t.$$

Then $\varphi(t) \to \infty$ as $t \to t_0-$. A little calculation shows that φ is the unique solution of (1.13) on $[0, t_0)$ and that (1.13) is equivalent to

$$x'' = x|x|, \quad x(0) = 1, \quad x'(0) = 0.$$

Lemma 1 has several well-known (see, e.g., Miller [4, p. 93]) consequences concerning the continuation of continuous solutions of (1.1).

COROLLARY 1. *Let* (1.8), (1.9) *and* (1.10) *hold and let* $\varphi \in C([0, \hat{t}], \mathbb{R}^n)$ *be a solution of* (1.1) *for some* $\hat{t} \in (0, t^*)$. *Then there exists* $\bar{t} \in (\hat{t}, t^*)$ *such that* φ *has a continuous extension which is a solution of* (1.1) *on* $[0, \bar{t}]$.

The proof given in § 2 is an easy adaptation of that of Lemma 1 to the equation

$$x(t) + \int_{\hat{t}}^t a(t - s) g(x(s))\, ds = \hat{f}(t), \qquad \hat{t} \leqq t < t^*, \tag{1.14}$$

where

$$\hat{f}(t) = -\int_0^{\hat{t}} a(t - s) g(\varphi(s))\, ds + f(t), \qquad \hat{t} \leqq t < t^*. \tag{1.15}$$

COROLLARY 2. *Let* (1.8), (1.9) *and* (1.10) *hold and let* $\varphi \in C([0, \hat{t}), \mathbb{R}^n)$ *be a solution of* (1.1) *for some* $\hat{t} \in (0, t^*)$. *In addition suppose that* $\limsup_{t \to \hat{t}-} |\varphi(t)| < \infty$. *Then*

(i) φ *is uniformly continuous on* $[0, \hat{t})$,

(ii) *there exists* $\bar{t} \in (\hat{t}, t^*)$ *such that* φ *has a continuous extension which is a solution of* (1.1) *on* $[0, \bar{t}]$.

The next result completes the abovementioned procedure for obtaining global existence from local existence, a continuation argument and an a priori bound. Its proof is an immediate consequence of Definition 1, Lemma 1 and Corollaries 1 and 2.

COROLLARY 3. *Let* (1.8), (1.9) *and* (1.10) *hold and let* Γ *be an a priori bound for* (1.1). *Then*

(i) (1.1) *has a continuous solution on* $[0, t^*)$,

(ii) *any continuous solution of* (1.1) *on* $[0, \hat{t})$, *for some* $\hat{t} \in (0, t^*)$, *may be extended continuously as a solution of* (1.1) *onto* $[0, t^*)$.

As a final preliminary, it is convenient to state without proof the following well-known global existence (and uniqueness) result for the linear equation

$$x(t) + \int_0^t a(t-s)x(s)\,ds = f(t), \qquad 0 \leqq t < t^*. \tag{1.16}$$

LEMMA 2. *Let* (1.9) *and* (1.10) *hold. Then* (1.16) *has a unique continuous solution on* $[0, t^*)$.

Suppose $n = 1$. Then the g of (1.12), (1.13) and (1.16) satisfy

$$xg(x) \geqq 0, \qquad -\infty < x < \infty. \tag{1.17}$$

Observe that while the sign of $a(0)$ is arbitrary in Lemma 2, $a(0) \leqq 0$ in (1.12) and (1.13).

The following result establishes an a priori bound for (1.1) when $n = 1$. The crucial hypotheses are that $a(0) > 0$ and that the graph of g is bounded from above in the second quadrant and from below in the fourth quadrant. Since g is assumed to be continuous in this theorem, the latter condition is less restrictive than the hypothesis

$$xg(x) \geqq 0 \quad \text{on } (-\infty, -X) \cup (X, \infty) \text{ for some } X \in [0, \infty),$$

which reduces to (1.17) when $X = 0$.

THEOREM 1. *Let g, a and f satisfy the conditions*

$$g \in C(\mathbb{R}^1, \mathbb{R}^1), \qquad \sup_{\{x \mid xg(x) \leqq 0\}} |g(x)| = K_1 < \infty, \tag{1.18}$$

$$a \in C^1([0, t^*), \mathbb{R}^1), \qquad a(0) > 0, \tag{1.19}$$

$$f \in C^1([0, t^*), \mathbb{R}^1). \tag{1.20}$$

Then (1.1) *has an a priori bound*, Γ_1, *given by* (3.2) *below.*

The next theorem is closely related to the preceding one.

THEOREM 1′. *Let g, a and f satisfy the conditions*

$$g : \mathbb{R}^1 \to \mathbb{R}^1 \textit{ is nondecreasing}, \tag{1.21}$$

$$a \in AC_{\text{loc}}([0, t^*), \mathbb{R}^1), \quad a(0) > 0, \quad a' \in L^\infty_{\text{loc}}([0, t^*), \mathbb{R}^1), \tag{1.22}$$

$$f \in AC_{\text{loc}}([0, t^*), \mathbb{R}^1), \qquad f' \in L^\infty_{\text{loc}}([0, t^*), \mathbb{R}^1). \tag{1.23}$$

Then (1.1) *has an a priori bound*, $\tilde{\Gamma}_1$, *given by* (4.1) *below.*

Observe that the assumptions on a and f have been relaxed in passing from Theorem 1 to Theorem 1′. Further note that hypothesis (1.21) on g implies that the second, but not the first, part of (1.18) is satisfied. However, the assumption $g \in C(\mathbb{R}^1, \mathbb{R}^1)$ must be added to (1.21) in order to invoke (the existence) Lemma 1 and its corollaries. Existence theorems for (1.1), with $\mathbb{R}^1$ replaced by a Hilbert space and (1.21) replaced by the assumption that g is a maximal monotone operator, have recently been studied by London [3] using the theory of such operators.

Theorem 2 is concerned with $\mathbb{R}^n$ for $n \geqq 1$.

THEOREM 2. *Let g, a and f satisfy the conditions*

$$g = \operatorname{grad} G \quad (\textit{i.e.}, g_i(x) = \frac{\partial}{\partial x_i} G(x)), \quad \textit{where} \tag{1.24}$$
$$G \in C^1(\mathbb{R}^n, \mathbb{R}^1), \quad \lim_{|x| \to \infty} G(x) = \infty, \quad \min_{x \in \mathbb{R}^n} G(x) > 0,$$

$$\sup_{x \in \mathbb{R}^n} \frac{|g(x)|}{G(x)} = K_2 < \infty, \tag{1.25}$$

$$a \in BV_{\mathrm{loc}}([0, t^*), \mathbb{R}^{n^2}), \qquad a(0) = a(0+), \tag{1.26}$$

$$\inf_{x \neq 0} \frac{x^T a(0) x}{|x|^2} = \mu > 0, \tag{1.27}$$

$$f \in C \cap BV_{\mathrm{loc}}([0, t^*), \mathbb{R}^n). \tag{1.28}$$

Then (1.1) *has a priori bound*, Γ_2, *given by* (5.11) *and* (5.12) *below.*

When $n = 1$, the assumptions on a and f are considerably weaker in Theorem 2 than in Theorems 1 or 1′. Although $a(0+) = \infty$ is excluded by (1.26), it will be allowed in Theorem 3, which is a modification of Theorem 2 when $n = 1$.

The hypotheses of Theorem 1 and Theorem 2 for $n = 1$ concerning g overlap; however, neither implies the other. The assumption $\min_{x \in \mathbb{R}^n} G(x) > 0$ is made for convenience and, in view of the other conditions of (1.24), is not a serious requirement as it can always be achieved by adding a sufficiently large constant to G. In this connection, observe that the usual formula

$$G(x) = \int_0^x g(\xi)\, d\xi + G(0), \qquad x \in \mathbb{R}^1, \tag{1.29}$$

now requires that $G(0)$ be added to the right-hand side of (1.29). In view of (1.24), hypothesis (1.25) is equivalent to

$$\limsup_{|x| \to \infty} \frac{|g(x)|}{G(x)} < \infty.$$

The constant K_2 is required in the definition of Γ_2. Hypothesis (1.25) will be dropped in Theorem 4 at the expense of a more stringent hypothesis on f than (1.28).

Since $a(0)$ is not assumed to be symmetric, (1.27) is not equivalent to the assumption that the real part of each of the characteristic roots of $a(0)$ is positive. The following example, together with Corollary 3, shows that the latter condition cannot be substituted for (1.27) in Theorem 2.

$$n = 2, \quad t^* = \infty, \quad G(x) = G(x_1, x_2) = \tfrac{1}{4}x_1^4 + \tfrac{1}{4}(x_1 + x_2)^4 + 1, \tag{1.30}$$
$$a(t) \equiv \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}, \qquad f(t) \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Except for (1.27), the hypotheses of Theorem 2 are satisfied by (1.30). The real part of each of the characteristic roots of $a(0)$ is $\frac{1}{2}$. A calculation shows that (1.30) is equivalent to

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1^3 \\ 3x_1^3 + (x_1 + x_2)^3 \end{pmatrix}, \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

from which it follows that $x_1(t) = (1 - 2t)^{-1/2}$. Thus the unique continuous solution of (1.30) only exists on $[0, \frac{1}{2})$ and cannot be continued onto $[0, t^*) = [0, \infty)$.

The next result, unlike the preceding or succeeding ones, treats the situation in which a is singular at $t = 0$. It covers, for example, the important kernel $a(t) = t^{-1/2}$. For convenience set $a = b + c$ in (1.1), where b will be the "singular part" of a; thus we obtain the equation

$$x(t) + \int_0^t [b(t-s) + c(t-s)]g(x(s))\,ds = f(t), \qquad 0 \leqq t < t^*. \tag{1.31}$$

The assumptions on g and f are the same as in Theorem 2, except that here the setting is $\mathbb{R}^1$.

THEOREM 3. *Let g satisfy* (1.24) *and* (1.25) *and f satisfy* (1.28), *all with $n = 1$. Further, let b and c satisfy the conditions*

$$\begin{gathered} b \in L^1_{\mathrm{loc}}([0, t^*), \mathbb{R}^1), \qquad b \text{ is nonincreasing}, \\ b(0) = \lim_{t \to 0+} b(t) = \infty, \end{gathered} \tag{1.32}$$

$$c \in BV_{\mathrm{loc}}([0, t^*), \mathbb{R}^1). \tag{1.33}$$

Then (1.31) *has a continuous solution on* $[0, t^*)$.

The existence of an a priori bound, in the sense of Definition 1, for (1.31) is left open in Theorem 3. Only the global existence of a continuous solution is asserted. If uniqueness is also assumed, then the bound Γ_3, (6.16) below, becomes an a priori one.

Returning to $\mathbb{R}^n$ with $n \geqq 1$, the following result shows that the hypothesis (1.25) on g in Theorem 2 can be dropped if more is assumed about f.

THEOREM 4. *Let g and a satisfy* (1.24), (1.26) *and* (1.27) *and let f satisfy the condition*

$$f \in AC_{\mathrm{loc}}([0, t^*), \mathbb{R}^n), \qquad f' \in L^2_{\mathrm{loc}}([0, t^*), \mathbb{R}^n). \tag{1.34}$$

Then (1.1) *has an a priori bound*, Γ_4, *given by* (6.2) *and* (6.3) *below*.

2. Proofs of Lemma 1 and Corollaries 1 and 2. A combination of the proofs from [1] and [4] which were cited in § 1 is used to establish Lemma 1. For $m = 1, 2, \cdots$, set

$$x_m(t) = \begin{cases} f(0), & -1 \leqq t \leqq 0, \\ -\displaystyle\int_0^t a(t-s)g\left(x_m\left(s - \frac{1}{m}\right)\right) ds + f(t), & 0 \leqq t < t^*. \end{cases} \tag{2.1}$$

Since for any $t_0 \in [0, t^*)$ the value of $x_m(t_0)$ only depends on $x_m(s)$ for $s \in [-1/m,$

$t_0 - 1/m]$, (2.1) is a valid definition of $x_m(t)$ on $[-1, t^*)$. The hypothesis readily implies that each $x_m(t) \in C([-1, t^*), \mathbb{R}^n)$.

Let

$$M(t) = 1 + \max_{0 \leqq s \leqq t} |f(s)|, \qquad M_1(t) = \max_{|x| \leqq 2M(t)} |g(x)|, \qquad 0 \leqq t < t^*. \tag{2.2}$$

Choose $\bar{t} \in (0, t^*)$ so that $\bar{M} = M(\bar{t})$, $\bar{M}_1 = M_1(\bar{t})$ satisfy

$$\bar{M}_1 \int_0^{\bar{t}} |a(s)|\, ds \leqq \bar{M}. \tag{2.3}$$

Let $t_0 \in [0, \bar{t}]$ and suppose that

$$|x_m(s)| \leqq 2\bar{M} \quad \text{for } -1 \leqq s \leqq t_0 - \frac{1}{m}. \tag{2.4}$$

Clearly (2.1) and (2.2) imply that (2.4) holds for $t_0 = 0$. From (2.1)–(2.4) it follows that

$$|x_m(t_0)| \leqq \max_{0 \leqq s \leqq t_0} \left| g\left(x_m\left(s - \frac{1}{m}\right)\right)\right| \int_0^{t_0} |a(t_0 - s)|\, ds + |f(t_0)| \leqq 2\bar{M}.$$

Thus (2.4) implies $|x_m(t_0)| \leqq 2\bar{M}$. Therefore an induction over subintervals of $[0, \bar{t}]$ of length not greater than $1/m$ yields

$$|x_m(t)| \leqq 2\bar{M}, \qquad -1 \leqq t \leqq \bar{t}, \ m = 1, 2, \cdots. \tag{2.5}$$

Hence the sequence $\{x_m(t)\}$ is uniformly bounded on $[-1, \bar{t}]$.

In order to establish equicontinuity of $\{x_m(t)\}$ on $[-1, \bar{t}]$, observe first that for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$0 \leqq t_1 \leqq t_2 \leqq \bar{t}, \qquad t_2 \leqq t_1 + \delta(\varepsilon) \tag{2.6}$$

implies

$$|f(t_2) - f(t_1)| + \bar{M}_1 \int_0^{t_2 - t_1} |a(s)|\, ds + \bar{M}_1 \int_0^{t_1} |a(t_2 - t_1 + s) - a(s)|\, ds < \varepsilon. \tag{2.7}$$

Then (2.1) and (2.6) imply

$$x_m(t_2) - x_m(t_1) = f(t_2) - f(t_1) - \int_{t_1}^{t_2} a(t_2 - s) g\left(x_m\left(s - \frac{1}{m}\right)\right) ds$$
$$- \int_0^{t_1} [a(t_2 - s) - a(t_1 - s)] g\left(x_m\left(s - \frac{1}{m}\right)\right) ds,$$

which together with (2.2) and (2.5)–(2.7) yields

$$|x_m(t_2) - x_m(t_1)| < \varepsilon. \tag{2.8}$$

The asserted equicontinuity follows immediately from (2.6), (2.8) and the first line of (2.1).

The Ascoli–Arzela lemma now implies the existence of a subsequence $\{x_{m_k}(t)\}$ of $\{x_m(t)\}$ and an $x(t) \in C[-1, \bar{t}]$ such that

$$\lim_{k \to \infty} \left\{ \max_{-1 \leqq t \leqq \bar{t}} |x_{m_k}(t) - x(t)| \right\} = 0. \tag{2.9}$$

From (2.1), with m replaced by m_k, and (2.9) it follows on letting $k \to \infty$, that $x(t)$ satisfies (1.1) on $[0, \bar{t}]$, which completes the proof of Lemma 1.

For the proof of Corollary 1, replace (2.1) by

$$x_m(t) = \begin{cases} \varphi(\hat{t}), & -1 + \hat{t} \leqq t \leqq \hat{t}, \\ -\displaystyle\int_{\hat{t}}^{t} a(t - s) g\left(x_m\left(s - \frac{1}{m}\right)\right) ds + \hat{f}(t), & \hat{t} \leqq t < t^*, \end{cases} \tag{2.10}$$

where $\hat{f}$ is given in (1.15), and (2.2) by

$$M(t) = 1 + \max_{\hat{t} \leqq s \leqq t} |\hat{f}(s)|, \qquad M_1(t) = \max_{|x| \leqq 2M(t)} |g(x)|, \qquad \hat{t} \leqq t < t^*.$$

Instead of (2.3), now choose $\bar{t} \in (\hat{t}, t^*)$ so that $\bar{M} = M(\bar{t})$, $\bar{M}_1 = M_1(\bar{t})$ satisfy

$$\bar{M}_1 \int_0^{\bar{t} - \hat{t}} |a(s)|\, ds \leqq \bar{M}.$$

As in the proof of Lemma 1, the sequence $\{x_m(t)\}$ of (2.10) possesses a subsequence $\{x_{m_k}(t)\}$ which converges uniformly on $[-1 + \hat{t}, \bar{t}]$ to a continuous function $\hat{x}(t)$, which satisfies (1.14) on $[\hat{t}, \bar{t}]$. Let

$$\psi(t) = \varphi(t) \quad \text{on } [0, \hat{t}], \qquad \psi(t) = \hat{x}(t) \quad \text{on } [\hat{t}, \bar{t}]. \tag{2.11}$$

It follows easily from (1.1), (1.14), (1.15), (2.10) and (2.11) that $\psi(t)$ satisfies the conclusion of Corollary 1.

Part (i) of Corollary 2 is not really a consequence of Lemma 1 or Corollary 1. The hypothesis implies that

$$\sup_{0 \leqq t < \hat{t}} |g(\varphi(t))| = M_2 < \infty. \tag{2.12}$$

Let $\varepsilon > 0$. As in (2.6) and (2.7), there exists a $\delta(\varepsilon) > 0$ such that

$$0 \leqq t_1 \leqq t_2 < \hat{t}, \qquad t_2 \leqq t_1 + \delta(\varepsilon) \tag{2.13}$$

implies

$$\begin{aligned} &|f(t_2) - f(t_1)| + M_2 \int_0^{t_2 - t_1} |a(s)|\, ds \\ &\quad + M_2 \int_0^{t_1} |a(t_2 - t_1 + s) - a(s)|\, ds < \varepsilon. \end{aligned} \tag{2.14}$$

From (1.1) and (2.13), one has

$$\begin{aligned} \varphi(t_2) - \varphi(t_1) = f(t_2) - f(t_1) &- \int_{t_1}^{t_2} a(t_2 - s) g(\varphi(s))\, ds \\ &- \int_0^{t_1} [a(t_2 - s) - a(t_1 - s)] g(\varphi(s))\, ds, \end{aligned}$$

which together with (2.12)–(2.14) implies $|\varphi(t_2) - \varphi(t_1)| < \varepsilon$ and thus establishes (i).

From (i) it follows that

$$\varphi(\hat{t}-) = \lim_{t \to \hat{t}-} \varphi(t)$$

exists, which together with the hypothesis implies

$$\varphi(\hat{t}-) + \int_0^{\hat{t}} a(\hat{t} - s) g(\varphi(s))\, ds = f(\hat{t}). \tag{2.15}$$

Extend $\varphi(t)$ to $[0, \hat{t}]$ by setting $\varphi(\hat{t}) = \varphi(\hat{t}-)$. Then $\varphi(t) \in C([0, \hat{t}], \mathbb{R}^n)$ and, in view of (1.1) and (2.15), is a solution of (1.1) on $[0, \hat{t}]$. Assertion (ii) is now an immediate consequence of Corollary 1.

3. Proof of Theorem 1. On $[0, t^*)$ set

$$\begin{aligned} \alpha_1(t) &= \max_{0 \leqq s \leqq t} \frac{|a'(s)|}{a(0)}, \\ \alpha_2(t) &= \max\left(K_1, \max_{|x| \leqq |f(0)|} |g(x)|, \max_{0 \leqq s \leqq t} \frac{|f'(s)|}{a(0)}\right), \\ \alpha_3(t) &= \alpha_1(t) \int_0^t \alpha_2(\tau) \exp\left\{\int_\tau^t \alpha_1(s)\, ds\right\} d\tau + \alpha_2(t), \end{aligned} \tag{3.1}$$

and

$$\Gamma_1(t) = \int_0^t |a(t - s)| \alpha_3(s)\, ds + \max_{0 \leqq s \leqq t} |f(s)|. \tag{3.2}$$

Then Γ_1 is nondecreasing and satisfies (1.2).

Let $(\hat{t}, \varphi)$ satisfy (1.3) with $n = 1$. Then

$$\varphi(t) + \int_0^t a(t - s) g(\varphi(s))\, ds = f(t), \qquad 0 \leqq t < \hat{t}. \tag{3.3}$$

It will be shown that

$$|g(\varphi(t))| \leqq \alpha_1(t) \int_0^t |g(\varphi(s))|\, ds + \alpha_2(t), \qquad 0 \leqq t < \hat{t}. \tag{3.4}$$

From (3.1), (3.4) and the Gronwall inequality, it follows that

$$|g(\varphi(t))| \leqq \alpha_3(t), \qquad 0 \leqq t < \hat{t}, \tag{3.5}$$

which together with (3.2) and (3.3) implies (1.4), for $\Gamma = \Gamma_1$ and hence completes the proof. It may be noted that while the maxima in (3.1) are required in the following argument, the maximum in (3.2) only serves to guarantee that Γ_1 is nondecreasing. Thus (1.4) holds with the second term of (3.2) replaced by $|f(t)|$.

It is evident from (3.3) and the hypothesis that $\varphi \in C^1([0, \hat{t}), \mathbb{R}^1)$ and that

$$\varphi'(t) + a(0) g(\varphi(t)) = -\int_0^t a'(t - s) g(\varphi(s))\, ds + f'(t), \qquad 0 \leqq t < \hat{t}. \tag{3.6}$$

Define the following eight disjoint subsets of $[0, \hat{t})$:

$$
\begin{aligned}
S_1 &= \{t \mid \varphi(t) \geqq 0,\, g(\varphi(t)) < 0\}, \\
S_2 &= \{t \mid \varphi(t) \geqq 0,\, g(\varphi(t)) \geqq 0,\, \varphi'(t) \geqq 0\}, \\
S_3 &= \{t \mid \varphi(t) \geqq 0,\, g(\varphi(t)) \geqq 0,\, \varphi'(t) < 0,\, f(0) \geqq \varphi(t)\}, \\
S_4 &= \{t \mid \varphi(t) \geqq 0,\, g(\varphi(t)) \geqq 0,\, \varphi'(t) < 0,\, f(0) < \varphi(t)\}, \\
S_5 &= \{t \mid \varphi(t) < 0,\, g(\varphi(t)) > 0\}, \\
S_6 &= \{t \mid \varphi(t) < 0,\, g(\varphi(t)) \leqq 0,\, \varphi'(t) \leqq 0\}, \\
S_7 &= \{t \mid \varphi(t) < 0,\, g(\varphi(t)) \leqq 0,\, \varphi'(t) > 0,\, f(0) \leqq \varphi(t)\}, \\
S_8 &= \{t \mid \varphi(t) < 0,\, g(\varphi(t)) \leqq 0,\, \varphi'(t) > 0,\, f(0) > \varphi(t)\}.
\end{aligned} \tag{3.7}
$$

Clearly,

$$
[0, \hat{t}) = \bigcup_{k=1}^{8} S_k. \tag{3.8}
$$

Inequality (3.4) will be established for an arbitrary $t_0 \in [0, \hat{t})$. Each case, $t_0 \in S_k$, is treated separately. Since the reasoning required for $k = 5, 6, 7, 8$ parallels that employed for $k = 1, 2, 3, 4$, respectively, only the latter cases are considered here.

Let $t_0 \in S_1$. Then (1.18), (3.1) and (3.7) imply

$$
|g(\varphi(t_0))| \leqq K_1 \leqq \alpha_2(t_0) \leqq \alpha_1(t_0) \int_0^{t_0} |g(\varphi(s))|\, ds + \alpha_2(t_0). \tag{3.9}
$$

Let $t_0 \in S_2$. Then (1.19), (3.1), (3.6) and (3.7) imply

$$
\begin{aligned}
|g(\varphi(t_0))| = g(\varphi(t_0)) &\leqq \frac{1}{a(0)}\{\varphi'(t_0) + a(0)g(\varphi(t_0))\} \\
&= -\frac{1}{a(0)} \int_0^{t_0} a'(t_0 - s) g(\varphi(s))\, ds + \frac{1}{a(0)} f'(t_0) \\
&\leqq \alpha_1(t_0) \int_0^{t_0} |g(\varphi(s))|\, ds + \alpha_2(t_0).
\end{aligned} \tag{3.10}
$$

Let $t_0 \in S_3$. Even without invoking the condition $\varphi'(t_0) < 0$ of (3.7), it follows from (3.1) that

$$
\begin{aligned}
|g(\varphi(t_0))| &\leqq \max_{|x| \leqq |f(0)|} |g(x)| \leqq \alpha_2(t_0) \\
&\leqq \alpha_1(t_0) \int_0^{t_0} |g(\varphi(s))|\, ds + \alpha_2(t_0).
\end{aligned} \tag{3.11}
$$

Let $t_0 \in S_4$. Since $\varphi(0) = f(0)$, it is evident from (3.7) that there exists a unique $t_1 \in (0, t_0)$ such that

$$
\varphi(t_1) = \varphi(t_0) \quad \text{and} \quad \varphi(t) > \varphi(t_0) \quad \text{on } (t_1, t_0). \tag{3.12}
$$

Since $\varphi \in C^1([0, \hat{t}), \mathbb{R}^1)$, (3.12) implies

$$\varphi'(t_1) \geqq 0. \tag{3.13}$$

Reasoning similar to that employed for $t_0 \in S_2$, but now also using (3.12), (3.13), and the monotonicity of $\alpha_1(t)$ and $\alpha_2(t)$ yields

$$\begin{aligned} |g(\varphi(t_0))| &= g(\varphi(t_0)) = g(\varphi(t_1)) \\ &\leqq \frac{1}{a(0)}\{\varphi'(t_1) + a(0)g(\varphi(t_1))\} \\ &= -\frac{1}{a(0)}\int_0^{t_1} a'(t_1 - s)g(\varphi(s))\,ds + \frac{1}{a(0)}f'(t_1) \\ &\leqq \alpha_1(t_1)\int_0^{t_1} |g(\varphi(s))|\,ds + \alpha_2(t_1) \\ &\leqq \alpha_1(t_0)\int_0^{t_0} |g(\varphi(s))|\,ds + \alpha_2(t_0). \end{aligned} \tag{3.14}$$

Thus (3.9), (3.10), (3.11) and (3.14) establish (3.4) for $t_0 \in S_k$ ($k = 1, 2, 3, 4$), respectively. This, together with similar reasoning for $k = 5, 6, 7, 8$, completes the proof as noted above.

4. Proof of Theorem 1′. Analogous to (3.1) and (3.2), on $[0, t^*)$ set

$$\begin{aligned} \tilde{\alpha}_1(t) &= \operatorname*{ess\,sup}_{0 \leqq s \leqq t} \frac{|a'(s)|}{a(0)}, \\ \tilde{\alpha}_2(t) &= \max\left(|g(0)|, |g(f(0))|, \operatorname*{ess\,sup}_{0 \leqq s \leqq t} \frac{|f'(s)|}{a(0)}\right) \\ \tilde{\alpha}_3(t) &= \tilde{\alpha}_1(t)\int_0^t \tilde{\alpha}_2(\tau)\exp\left\{\int_\tau^t \tilde{\alpha}_1(s)\,ds\right\}d\tau + \tilde{\alpha}_2(t), \\ \tilde{\Gamma}_1(t) &= \int_0^t |a(t-s)|\tilde{\alpha}_3(s)\,ds + \max_{0 \leqq s \leqq t} |f(s)|. \end{aligned} \tag{4.1}$$

Then $\tilde{\Gamma}_1$ is nondecreasing and satisfies (1.2).

Let $(\hat{t}, \varphi)$ satisfy (1.3) with $n = 1$. Then (3.3) holds. Instead of (3.4), it will now be shown that

$$|g(\varphi(t))| \leqq \tilde{\alpha}_1(t)\int_0^t |g(\varphi(s))|\,ds + \tilde{\alpha}_2(t) \quad \text{a.e. on } [0, \hat{t}). \tag{4.2}$$

As in § 3, however, (3.3), (4.1), (4.2) and the Gronwall inequality yield (1.4), for $\Gamma = \tilde{\Gamma}_1$, which completes the proof.

From (3.3) and the hypothesis it follows that

$$\varphi \in AC_{\text{loc}}([0, \hat{t}), \mathbb{R}^1) \tag{4.3}$$

and that

$$\text{(4.4)} \qquad \varphi'(t) + a(0)g(\varphi(t)) = -\int_0^t a'(t-s)g(\varphi(s))\,ds + f'(t) \quad \text{a.e. on } [0, \hat{t}).$$

The a.e. nature of (4.4) necessitates making some modifications in the argument of § 3. Let

$$\text{(4.5)} \qquad \begin{aligned} A &= \{t \in [0, \hat{t}) \mid \varphi'(t) \text{ exists and (4.4) holds}\}, \\ \tilde{S}_k &= A \cap S_k, \qquad k = 1, \cdots, 8, \end{aligned}$$

where the S_k are defined by (3.7). Then the S_k are disjoint and, instead of (3.8),

$$\text{(4.6)} \qquad \operatorname{Meas}\left\{[0, \hat{t}) \cap \bigcap_{k=1}^{8} \tilde{S}_k^c\right\} = 0.$$

Inequality (4.2) will be established a.e. on $\bigcup_{k=1}^{8} \tilde{S}_k$, which in view of (4.6), establishes (4.2) a.e. on $[0, \hat{t})$.

Let $t_0 \in \tilde{S}_1$. Then (1.21), (4.1) and (4.5) imply

$$|g(\varphi(t_0))| \leqq |g(0)| \leqq \tilde{\alpha}_2(t_0) \leqq \tilde{\alpha}_1(t_0) \int_0^{t_0} |g(\varphi(s))|\,ds + \tilde{\alpha}_2(t_0).$$

Let $t_0 \in \tilde{S}_2 \cap B$ where

$$\text{(4.7)} \qquad B = \{t \in [0, t^*) \mid |f'(t)| \leqq \operatorname*{ess\,sup}_{0 \leqq s \leqq t} |f'(s)|\}.$$

It is an elementary consequence of (1.23) that Meas $(B) = t^*$. The same reasoning employed in (3.10) shows that (4.2) holds for this t_0.

Let $t_0 \in \tilde{S}_3$. Then (1.21), (4.1) and (4.5) imply

$$\begin{aligned} |g(\varphi(t_0))| = g(\varphi(t_0)) &\leqq g(f(0)) \\ &\leqq \tilde{\alpha}_1(t_0) \int_0^{t_0} |g(\varphi(s))|\,ds + \tilde{\alpha}_2(t_0). \end{aligned}$$

Let $t_0 \in \tilde{S}_4$. Thus as before, (4.5) implies the existence of a unique $t_1 \in (0, t_0)$ such that (3.12) holds. From (3.12) and (4.3) it follows that

$$\text{(4.8)} \qquad 0 = \int_{t_1}^{t_0} \varphi'(s)\,ds.$$

From (3.12), (4.3) and (4.8), there exists a set of positive measure $C \subset [t_1, t_0)$ such that

$$\text{(4.9)} \qquad \varphi'(t_2) \text{ exists}, \quad \varphi'(t_2) \geqq 0, \quad \varphi(t_2) \geqq \varphi(t_0) \quad \text{for } t_2 \in C.$$

Since Meas $(B) = t^*$ (recall (4.7)), there exists a $t_2 \in B \cap C$. Reasoning similar to (3.14), but now invoking (4.9) and (1.21), implies

$$\begin{aligned} |g(\varphi(t_0))| &= g(\varphi(t_0)) \leqq g(\varphi(t_2)) \\ &\leqq \frac{1}{a(0)}\{\varphi'(t_2) + a(0)g(\varphi(t_2))\} \\ &= -\frac{1}{a(0)}\int_0^{t_2} a'(t_2 - s)g(\varphi(s))\,ds + \frac{1}{a(0)}|f'(t_2)| \\ &\leqq \tilde{\alpha}_1(t_2)\int_0^{t_2} |g(\varphi(s))|\,ds + \tilde{\alpha}_2(t_2) \\ &\leqq \tilde{\alpha}_1(t_0)\int_0^{t_0} |g(\varphi(s))|\,ds + \tilde{\alpha}_2(t_0). \end{aligned}$$

Analogous arguments hold for $\tilde{S}_k$ ($k = 5, 6, 7, 8$). Thus (4.2) is established, which completes the proof.

5. Proof of Theorem 2. The hypotheses imply the existence of several auxiliary functions that are employed in the definition (5.11), (5.12) of Γ_2. These auxiliary functions are also employed in the succeeding sections.

From (1.24) there exists a function Ω with the properties

$$\begin{aligned} &\Omega \in C([0, \infty), [0, \infty)), \quad \Omega(0) = 0, \quad \Omega(r) \leqq \min_{|x| \geqq r} G(x), \\ &\Omega \text{ is strictly increasing}, \qquad \lim_{r \to \infty} \Omega(r) = \infty. \end{aligned} \tag{5.1}$$

Let ω denote the inverse of Ω. Then

$$\begin{aligned} &\omega \in C([0, \infty), [0, \infty)), \quad \omega(0) = 0, \quad \omega \text{ is strictly increasing}, \\ &\lim_{r \to \infty} \omega(r) = \infty, \qquad \omega(\Omega(r)) = r, \quad \Omega(\omega(r)) = r. \end{aligned} \tag{5.2}$$

From (5.1) and (5.2), it follows that

$$G(x) \leqq y \quad \text{implies } |x| \leqq \omega(y). \tag{5.3}$$

Let

$$\gamma(r) = \max_{|x| \leqq r} |g(x)|. \tag{5.4}$$

Then

$$\gamma \in C([0, \infty), [0, \infty)), \quad \gamma \text{ is nondecreasing}, \quad |g(x)| \leqq \gamma(|x|) \tag{5.5}$$

Note that (1.25) has not been invoked in (5.1)–(5.5).

Let

$$\begin{aligned} v_{ij}(t) &= \text{total variation of } a_{ij} \text{ on } [0, t], \\ v(t) &= \sum_{i,j=1}^{n} v_{ij}(t). \end{aligned} \tag{5.6}$$

Then (1.26) implies $v:[0,t^*)\to[0,\infty)$ and $v(t)\to v(0)=0$ as $t\to 0+$. Hence a $\lambda\in(0,t^*)$ may be chosen which satisfies

$$v(\lambda)\leqq \tfrac{1}{2}\mu n^{-1/2}, \tag{5.7}$$

where μ is as in (1.27).

Let

$$\begin{aligned} \beta_i(t) &= \text{total variation of } f_i \text{ on } [0,t],\\ \beta(t) &= \sum_{i=1}^{n}\beta_i(t). \end{aligned} \tag{5.8}$$

Then by (1.28), $\beta\in C([0,t^*),[0,\infty))$ is nondecreasing and there exists an infinite sequence $\{t_m\}$ such that

$$\begin{aligned} &0=t_0<t_1<t_2<\cdots<t^*, \qquad \lim_{m\to\infty} t_m = t^*,\\ &t_{m+1}-t_m\leqq\lambda, \qquad \beta(t_{m+1})-\beta(t_m)\leqq[2K_2n^{1/2}]^{-1}, \qquad m=0,1,\cdots, \end{aligned} \tag{5.9}$$

where K_2 is as in (1.25). Let

$$I_m=[t_m,t_{m+1}), \qquad \bar I_m=[t_m,t_{m+1}], \qquad m=0,1,\cdots. \tag{5.10}$$

It is convenient for later purposes, see (5.15), to always take the sequence $\{t_m\}$ to be an infinite one, even though a finite one would satisfy the requirements of (5.9) if $f\in BV([0,t^*],\mathbb{R}^n)$.

Γ_2 may now be defined by proceeding inductively from I_0 to I_1, to I_2, etc. Observe that $\Gamma_2\in C([0,t^*),[0,\infty))$ is a consequence of the definition. It is convenient to simultaneously define nondecreasing functions $\alpha_m\in C(\bar I_m,[0,\infty))$ and $\zeta\in C([0,t^*),[0,\infty))$. On $\bar I_0$ define

$$\begin{aligned} \alpha_0(t) &= G(f(0)),\\ \zeta(t) &= \alpha_0(t)+2K_2n^{1/2}\int_0^t\alpha_0(\tau)\,d\beta(\tau),\\ \Gamma_2(t) &= \omega(\zeta(t)), \end{aligned} \tag{5.11}$$

and on $\bar I_m$ ($m\geqq 1$) define

$$\begin{aligned} \alpha_m(t) &= \zeta(t_m)+\frac{1}{2\mu}n\gamma^2(\Gamma_2(t_m))\int_{t_m}^t v^2(\tau)\,d\tau,\\ \zeta(t) &= \alpha_m(t)+2K_2n^{1/2}\int_{t_m}^t\alpha_m(\tau)\,d\beta(\tau),\\ \Gamma_2(t) &= \omega(\zeta(t)). \end{aligned} \tag{5.12}$$

Let $(\hat t,\varphi)$ satisfy (1.3). Then

$$\varphi(t)-f(t)=-\int_0^t a(t-s)g(\varphi(s))\,ds, \qquad 0\leqq t<\hat t, \tag{5.13}$$

and

$$\frac{d}{dt}[\varphi(t) - f(t)] = -a(0)g(\varphi(t)) - \int_0^t da(s)g(\varphi(t-s)) \quad \text{a.e. on } [0, \hat{t}). \tag{5.14}$$

That $\varphi - f \in AC_{\text{loc}}([0, \hat{t}), \mathbb{R}^n)$ and that (5.14) holds may be established by integrating the right-hand side of (5.14) and invoking Fubini's theorem and (5.13). It now follows from (1.28) that $\varphi \in C \cap BV_{\text{loc}}([0, \hat{t}), \mathbb{R}^n)$. In (5.14) and the Stieltjes integrals below, the convention $\int_\alpha^\beta = \int_{[\alpha,\beta]}$ is used when $\alpha \leqq \beta$.

Define $\hat{m}$ by

$$t_{\hat{m}} < \hat{t} \leqq t_{\hat{m}+1} \quad \text{if } \hat{t} \in (0, t^*), \quad \hat{m} = \infty \quad \text{if } \hat{t} = t^*. \tag{5.15}$$

Let

$$\begin{aligned} h_0(t) &\equiv 0, && 0 \leqq t < \hat{t}, \\ h_m(t) &= -\int_{(t-t_m,t]} da(s)g(\varphi(t-s)), && t_m \leqq t < \hat{t}, \quad m = 1, 2, \cdots, \hat{m}, \end{aligned} \tag{5.16}$$

where the second line of (5.16) is vacuous if $\hat{m} = 0$ and $m = 1, 2, \cdots$ if $\hat{m} = \infty$. From (1.5), (5.6) and $\sum_{i=1}^n |c_i| \leqq n^{1/2}|c|$ ($c \in \mathbb{R}^n$) it follows that

$$|h_m(t)| \leqq n^{1/2}\int_{t-t_m}^{t} dv(s)|g(\varphi(t-s))| \leqq n^{1/2}v(t) \max_{0 \leqq s \leqq t_m} |g(\varphi(s))| \tag{5.17}$$

on $[t_m, \hat{t})$ for $m = 1, 2, \cdots, \hat{m}$. From (5.14) and (5.16) one has

$$\frac{d}{dt}[\varphi(t) - f(t)] = -a(0)g(\varphi(t)) - \int_0^{t-t_m} da(s)g(\varphi(t-s)) + h_m(t) \tag{5.18}$$

a.e. on $[t_m, \hat{t})$ for $m = 0, 1, \cdots, \hat{m}$. Multiplying (5.18) by $g^T(\varphi(t))$ and integrating yields

$$\int_{t_m}^t g^T(\varphi(\tau))\, d\varphi(\tau) = -E_1(m, t) - E_2(m, t) + E_3(m, t) + E_4(m, t) \tag{5.19}$$

on $[t_m, \hat{t})$ for $m = 0, 1, \cdots, \hat{m}$, where

$$\begin{aligned} E_1(m, t) &= \int_{t_m}^t g^T(\varphi(\tau))a(0)g(\varphi(\tau))\, d\tau, \\ E_2(m, t) &= \int_{t_m}^t g^T(\varphi(\tau))\left\{\int_0^{\tau-t_m} da(s)g(\varphi(\tau-s))\right\} d\tau, \\ E_3(m, t) &= \int_{t_m}^t g^T(\varphi(\tau))h_m(\tau)\, d\tau, \\ E_4(m, t) &= \int_{t_m}^t g^T(\varphi(\tau))\, df(\tau). \end{aligned} \tag{5.20}$$

For any $\theta \in C^1([0, \hat{t}), \mathbb{R}^n)$, it follows from (1.24) that

$$\frac{d}{dt}G(\theta(t)) = \sum_{i=1}^{n} \frac{\partial G}{\partial x_i}(\theta(t))\theta_i'(t) = g^T(\theta(t))\theta'(t), \qquad 0 \leqq t < \hat{t},$$

and hence that

$$\int_{t_m}^{t} g^T(\theta(\tau))\, d\theta(\tau) = G(\theta(t)) - G(\theta(t_m)), \qquad t_m \leqq t < \hat{t}. \tag{5.21}$$

Since $\varphi \in C \cap BV_{\text{loc}}([0, \hat{t}), \mathbb{R}^n)$, there exists for each $\varepsilon \in (0, \hat{t})$ (see Hartman [1, p. 7] for more details concerning this argument) a sequence $\theta^{(j)} \in C^1([0, \hat{t} - \varepsilon], \mathbb{R}^n)$ $(j = 1, 2, \cdots)$ such that

$$\theta^{(j)}(t) \to \varphi(t)\ (j \to \infty) \quad \text{uniformly on } [0, \hat{t} - \varepsilon],$$

$$\sup_j \int_0^{\hat{t}-\varepsilon} \left| \frac{d}{dt}\theta^{(j)}(t) \right| dt < \infty.$$

Hence replacing $\theta(t)$ by $\theta^{(j)}(t)$ in (5.21), letting $j \to \infty$, and then letting $\varepsilon \to 0$ we have

$$\int_{t_m}^{t} g^T(\varphi(\tau))\, d\varphi(\tau) = G(\varphi(t)) - G(\varphi(t_m)), \qquad t_m \leqq t < \hat{t}. \tag{5.22}$$

Combining (5.19) and (5.22) yields

$$G(\varphi(t)) = G(\varphi(t_m)) - E_1(m, t) - E_2(m, t) + E_3(m, t) + E_4(m, t) \tag{5.23}$$

on $[t_m, \hat{t})$ for $m = 0, 1, \cdots, \hat{m}$.

Hypothesis (1.27) and (5.20) imply

$$-E_1(m, t) \leqq -\mu \int_{t_m}^{t} |g(\varphi(\tau))|^2\, d\tau, \qquad t_m \leqq t < \hat{t}. \tag{5.24}$$

Schwarz's inequality, the argument employed in establishing (5.17), and Fubini's theorem yield

$$\begin{aligned} |E_2(m, t)| &\leqq \int_{t_m}^{t} |g(\varphi(\tau))| \left| \int_0^{\tau - t_m} da(s) g(\varphi(\tau - s)) \right| d\tau \\ &\leqq n^{1/2} \int_{t_m}^{t} |g(\varphi(\tau))| \left\{ \int_0^{\tau - t_m} dv(s) |g(\varphi(\tau - s))| \right\} d\tau \\ &= n^{1/2} \int_0^{t - t_m} \left\{ \int_{s + t_m}^{t} |g(\varphi(\tau))|\, |g(\varphi(\tau - s))|\, d\tau \right\} dv(s). \end{aligned}$$

Hence

$$\begin{aligned} |E_2(m, t)| \leqq \frac{1}{2} n^{1/2} \int_0^{t - t_m} &\left\{ \int_{s + t_m}^{t} |g(\varphi(\tau))|^2\, d\tau \right\} dv(s) \\ &+ \frac{1}{2} n^{1/2} \int_0^{t - t_m} \left\{ \int_{s + t_m}^{t} |g(\varphi(\tau - s))|^2\, d\tau \right\} dv(s), \end{aligned}$$

which readily implies

$$(5.25)\qquad |E_2(m, t)| \leqq n^{1/2}v(t - t_m)\int_{t_m}^{t} |g(\varphi(\tau))|^2\, d\tau, \qquad t_m \leqq t < \hat{t}.$$

From (5.7), (5.9), (5.10), (5.15) and (5.25) one has

$$(5.26)\qquad |E_2(m, t)| \leqq \frac{1}{2}\mu\int_{t_m}^{t} |g(\varphi(\tau))|^2\, d\tau, \qquad t \in \bar{I}_m \cap [0, \hat{t}), \quad m = 0, 1, \cdots, \hat{m}.$$

Schwarz's inequality implies

$$(5.27)\qquad \begin{aligned} |E_3(m, t)| &\leqq \int_{t_m}^{t} |g(\varphi(\tau))|\, |h_m(\tau)|\, d\tau \\ &\leqq \frac{1}{2}\mu\int_{t_m}^{t} |g(\varphi(\tau))|^2\, d\tau + \frac{1}{2\mu}\int_{t_m}^{t} |h_m(\tau)|^2\, d\tau \end{aligned}$$

on $[t_m, \hat{t})$.

Hypotheses (1.25) and (1.28) together with (5.8) and the argument employed in establishing (5.17) yield

$$(5.28)\qquad |E_4(m, t)| \leqq n^{1/2}\int_{t_m}^{t} |g(\varphi(\tau))|\, d\beta(\tau) \leqq K_2 n^{1/2}\int_{t_m}^{t} G(\varphi(\tau))\, d\beta(\tau)$$

on $[t_m, \hat{t})$.

Combining (5.23), (5.24), (5.26), (5.27) and (5.28) yields

$$(5.29)\qquad G(\varphi(t)) \leqq \rho_m(t) + K_2 n^{1/2}\int_{t_m}^{t} G(\varphi(\tau))\, d\beta(\tau)$$

on $\bar{I}_m \cap [0, \hat{t})$ for $m = 0, 1, \cdots, \hat{m}$, where

$$(5.30)\qquad \rho_m(t) = G(\varphi(t_m)) + \frac{1}{2\mu}\int_{t_m}^{t} |h_m(\tau)|^2\, d\tau, \qquad t_m \leqq t < \hat{t}.$$

Upon integrating (5.29) against $d\beta(\tau)$, interchanging the order of integration in the last integral, and invoking (5.9) one finds

$$(5.31)\qquad \int_{t_m}^{t} G(\varphi(\tau))\, d\beta(\tau) \leqq 2\int_{t_m}^{t} \rho_m(\tau)\, d\beta(\tau).$$

From (5.29) and (5.31), it follows that

$$(5.32)\qquad G(\varphi(t)) \leqq \rho_m(t) + 2K_2 n^{1/2}\int_{t_m}^{t} \rho_m(\tau)\, d\beta(\tau)$$

on $\bar{I}_m \cap [0, \hat{t})$ for $m = 0, 1, \cdots, \hat{m}$.

It will be shown below that

$$(5.33)\qquad G(\varphi(t)) \leqq \zeta(t) \quad \text{on } \bar{I}_m \cap [0, \hat{t}) \text{ for } m = 0, 1, \cdots, \hat{m}.$$

From (5.3), (5.11), (5.12) and (5.33), it then follows that

$$(5.34)\qquad |\varphi(t)| \leqq \Gamma_2(t) \quad \text{on } \bar{I}_m \cap [0, \hat{t}) \text{ for } m = 0, 1, \cdots, \hat{m}.$$

Clearly (5.9), (5.15) and (5.34) imply

$$|\varphi(t)| \leqq \Gamma_2(t), \qquad 0 \leqq t < \hat{t},$$

which is (1.4) with Γ replaced by Γ_2. Thus establishing (5.33) will complete the proof.

From (5.16), (5.30) and (5.32) with $m = 0$, $\varphi(0) = f(0)$ and (5.11), one has

$$G(\varphi(t)) \leqq G(f(0)) + 2K_2 n^{1/2} \int_0^t G(f(0))\, d\beta(\tau)$$

on $\bar{I}_0 \cap [0, \hat{t})$, i.e.,

$$G(\varphi(t)) \leqq \alpha_0(t) + 2K_2 n^{1/2} \int_0^t \alpha_0(\tau)\, d\beta(\tau) = \zeta(t) \quad \text{on } \bar{I}_0 \cap [0, \hat{t}). \tag{5.35}$$

Thus (5.33) holds for $m = 0$.

In view of (5.35), there is no loss of generality in now assuming that $\hat{m} \geqq 1$. Suppose, therefore, that (5.33) holds for $m = 0, 1, \cdots, m_0 - 1$ where $0 \leqq m_0 - 1 < \hat{m}$. Then $\hat{t} > t_{m_0}$ (by (5.15)) and

$$G(\varphi(t)) \leqq \zeta(t), \qquad 0 \leqq t \leqq t_{m_0}. \tag{5.36}$$

From (5.3), (5.11), (5.12) and (5.36), one has

$$|\varphi(t)| \leqq \Gamma_2(t), \qquad 0 \leqq t \leqq t_{m_0}. \tag{5.37}$$

From (5.5) and (5.37), it follows that

$$|g(\varphi(t))| \leqq \gamma(\Gamma_2(t)) \leqq \gamma(\Gamma_2(t_{m_0})), \qquad 0 \leqq t \leqq t_{m_0}. \tag{5.38}$$

Setting $m = m_0$ in (5.17) and then invoking (5.38) yields

$$|h_{m_0}(t)| \leqq n^{1/2} v(t) \gamma(\Gamma_2(t_{m_0})), \qquad t_{m_0} \leqq t < \hat{t}. \tag{5.39}$$

Setting $m = m_0$ in (5.30) and (5.32), and then invoking (5.36) with $t = t_{m_0}$, (5.39) and (5.12) readily yields

$$G(\varphi(t)) \leqq \alpha_{m_0}(t) + 2K_2 n^{1/2} \int_{t_{m_0}}^t \alpha_{m_0}(\tau)\, d\beta(\tau) = \zeta(t)$$

on $\bar{I}_{m_0} \cap [0, \hat{t})$, which is (5.33) for $m = m_0$. This completes the proof of (5.33) and the theorem.

6. Proof of Theorem 3. Without loss of generality, it is assumed that $c(0) = c(0+)$ and that b is continuous from the right.

It is convenient to first rewrite (1.31) in an equivalent form. Choose $\bar{t} \in (0, t^*)$ so that $b(\bar{t}) + c(0) \geqq 1$. Let

$$\begin{aligned} \bar{b}(t) &= b(t) + c(0) - 1, && 0 \leqq t \leqq \bar{t}, \\ \bar{b}(t) &= 0, && \bar{t} < t < 2t^*, \\ \bar{c}(t) &= c(t) - c(0) + 1, && 0 \leqq t \leqq \bar{t}, \\ \bar{c}(t) &= b(t) + c(t), && \bar{t} < t < t^*. \end{aligned}$$

Then

$$\bar{b}(t) + \bar{c}(t) = b(t) + c(t), \qquad 0 \leqq t < t^*, \tag{6.1}$$

$$\begin{gathered} \bar{b} \in L^1([0, 2t^*), \mathbb{R}^1), \quad \bar{b} \text{ is nonincreasing}, \quad \bar{b}(t) \geqq 0, \\ \bar{b}(0) = \lim_{t \to 0+} \bar{b}(t) = \infty, \qquad \bar{b}(t+) = \bar{b}(t) \quad \text{on } (0, 2t^*) \end{gathered} \tag{6.2}$$

$$\bar{c} \in BV_{\text{loc}}([0, t^*), \mathbb{R}^1), \qquad \bar{c}(0) = \bar{c}(0+) = 1. \tag{6.3}$$

Thus $\bar{b}$ satisfies (1.32) on a larger interval (if $t^* < \infty$) than b does, as well as some additional properties, and $\bar{c}$ satisfies (1.33) as well as hypotheses (1.26) and (1.27) of Theorem 2 on a with $\mu = 1$ for $n = 1$. In view of (6.1), (1.31) is equivalent to

$$x(t) + \int_0^t [\bar{b}(t-s) + \bar{c}(t-s)]g(x(s))\,ds = f(t), \qquad 0 \leqq t < t^*. \tag{6.4}$$

For each $\varepsilon \in (0, t^*)$, (6.2) and (6.3) imply that $\bar{b}(t+\varepsilon) + \bar{c}(t)$ satisfies (1.26) and (1.27) with $\mu = 1 + \bar{b}(\varepsilon)$. Therefore Theorem 2 implies that the equation

$$x(t) + \int_0^t [\bar{b}(t-s+\varepsilon) + \bar{c}(t-s)]g(x(s))\,ds = f(t), \qquad 0 \leqq t < t^*, \tag{6.5}$$

has an a priori bound, $\Gamma_2(\cdot, \varepsilon)$, on $[0, t^*)$. Hence by Corollary 3, there exists a solution $\varphi_\varepsilon \in C([0, t^*), \mathbb{R}^1)$ of (6.5). Thus

$$\varphi_\varepsilon(t) + \int_0^t [\bar{b}(t-s+\varepsilon) + \bar{c}(t-s)]g(\varphi_\varepsilon(s))\,ds = f(t), \qquad 0 \leqq t < t^*, \tag{6.6}$$

$$|\varphi_\varepsilon(t)| \leqq \Gamma_2(t, \varepsilon), \qquad 0 \leqq t < t^*, \tag{6.7}$$

for each $\varepsilon \in (0, t^*)$.

The main task of the present proof is to obtain a function Γ_3 which satisfies

$$\Gamma_3 \in C([0, t^*), \mathbb{R}^1), \qquad \Gamma_3 \text{ is nondecreasing and independent of } \varepsilon, \tag{6.8}$$

such that

$$|\varphi_\varepsilon(t)| \leqq \Gamma_3(t), \qquad 0 \leqq t < t^*, \quad 0 < \varepsilon < t^*. \tag{6.9}$$

Using (5.11) and (5.12), we readily can show that, in general, $\lim_{\varepsilon \to 0+} \Gamma_2(t, \varepsilon) = \infty$ on $(0, t^*)$. Hence (6.7) does not imply (6.8), (6.9).

Assuming the validity of (6.8), (6.9) for the moment, the existence of a solution $\varphi \in C([0, t^*), \mathbb{R}^1)$ of (1.31) which satisfies

$$|\varphi(t)| \leqq \Gamma_3(t), \qquad 0 \leqq t < t^*, \tag{6.10}$$

is readily demonstrated. Let $t_0 \in (0, t^*)$. Then (6.8), (6.9) implies the uniform boundedness of $\{\varphi_\varepsilon(t)\}$ on $[0, t_0]$. Let $0 \leqq t_1 \leqq t_2 \leqq t_0$. Then (6.6) implies

$$\varphi_\varepsilon(t_2) - \varphi_\varepsilon(t_1) = f(t_2) - f(t_1) - F_1(t_1, t_2, \varepsilon) + F_2(t_1, t_2, \varepsilon),$$

where

$$F_1(t_1, t_2, \varepsilon) = \int_{t_1}^{t_2} [\bar{b}(t_2 - s + \varepsilon) + \bar{c}(t_2 - s)]g(\varphi_\varepsilon(s))\,ds,$$

$$F_2(t_1, t_2, \varepsilon) = \int_0^{t_1} [\bar{b}(t_1 - s + \varepsilon) - \bar{b}(t_2 - s + \varepsilon) + \bar{c}(t_1 - s) - \bar{c}(t_2 - s)]g(\varphi_\varepsilon(s))\,ds.$$

From (6.8), (6.9), (1.24) and (6.3), one has

$$M_1 = \sup_{\substack{0 \leqq s \leqq t_0 \\ 0 < \varepsilon < t^*}} |g(\varphi_\varepsilon(s))| < \infty, \qquad M_2 = \sup_{0 \leqq s \leqq t_0} |\bar{c}(s)| < \infty,$$

which together with (6.2) implies

$$|F_1(t_1, t_2, \varepsilon)| \leqq M_1 \int_0^{t_2 - t_1} \bar{b}(s)\,ds + M_1 M_2 (t_2 - t_1),$$

$$|F_2(t_1, t_2, \varepsilon)| \leqq M_1 \int_0^{t_2 - t_1} \bar{b}(s)\,ds + M_1 \int_0^{t_1} |\bar{c}(s) - \bar{c}(t_2 - t_1 + s)|\,ds.$$

It now follows from (6.2) and (6.3) that

$$F_1(t_1, t_2, \varepsilon) \to 0 \quad \text{and} \quad F_2(t_1, t_2, \varepsilon) \to 0 \quad \text{as } t_2 - t_1 \to 0$$

uniformly on $0 \leqq t_1 \leqq t_2 \leqq t_0$, $0 < \varepsilon < t^*$. Thus the family $\{\varphi_\varepsilon(t)\}$ is equicontinuous on $[0, t_0]$. The Ascoli–Arzela theorem and an obvious diagonalization argument imply the existence of a sequence $\varepsilon_n \downarrow 0$ $(n \to \infty)$ and a function $\varphi \in C([0, t^*), \mathbb{R}^1)$ such that $\varphi_{\varepsilon_n}(t) \to \varphi(t)$ as $n \to \infty$ uniformly on every compact subinterval of $[0, t^*)$. Setting $\varepsilon = \varepsilon_n$ in (6.6) and letting $n \to \infty$ readily yields the result. Thus only (6.8), (6.9) remains to be proved.

The following definitions, which will be employed in the definition of Γ_3, are analogous to (5.6)–(5.10); however, they do not involve $\bar{b}$ or ε. Let

$$\bar{v}(t) = \text{total variation of } \bar{c} \text{ on } [0, t], \qquad 0 \leqq t < t^*.$$

Then (6.3) implies the existence of a $\lambda \in (0, t^*)$ such that

$$\bar{v}(\lambda) \leqq \tfrac{1}{2}. \tag{6.11}$$

Let

$$\beta(t) = \text{total variation of } f \text{ on } [0, t], \qquad 0 \leqq t < t^*.$$

Then (1.28) implies the existence of an infinite sequence $\{t_m\}$ such that

$$\begin{gathered} 0 \leqq t_0 < t_1 < t_2 < \cdots < t^*, \qquad \lim_{m \to \infty} t_m = t^*, \\ t_{m+1} - t_m \leqq \lambda, \qquad \beta(t_{m+1}) - \beta(t_m) \leqq \frac{1}{2K_2}, \qquad m = 0, 1, \cdots, \end{gathered} \tag{6.12}$$

where K_2 is as in (1.25). As in (5.10), let $I_m = [t_m, t_{m+1})$.

Let Ω, ω and γ be defined by (5.1), (5.2) and (5.4), respectively. Define the nondecreasing functions α_m, ζ_m and $\tilde{\Gamma}_m$, which belong to $C(\bar{I}_m, [0, \infty))$, for $m = 0$ by

$$\text{(6.13)}\qquad \begin{aligned} &\alpha_0(t) = G(f(0)),\\ &\zeta_0(t) = \alpha_0(t) + 2K_2 \int_0^t \alpha_0(\tau)\, d\beta(\tau),\\ &\tilde{\Gamma}_0(t) = \omega(\zeta_0(t)), \end{aligned}$$

and for $m \geqq 1$ by

$$\text{(6.14)}\qquad \begin{aligned} \alpha_m(t) &= \zeta_{m-1}(t_m) + G(f(0)) + \gamma(\tilde{\Gamma}_{m_0-1}(t_m))\beta(t_m)\\ &\quad + \gamma^2(\tilde{\Gamma}_{m_0-1}(t_m)) \int_0^{t_m} \bar{v}(\tau)(1 + \tfrac{1}{2}\bar{v}(\tau))\, d\tau,\\ \zeta_m(t) &= \alpha_m(t) + 2K_2 \int_{t_m}^t \alpha_m(\tau)\, d\beta(\tau),\\ \tilde{\Gamma}_m(t) &= \omega(\zeta_m(t)). \end{aligned}$$

Observe that (6.13) and (6.14) imply

$$\text{(6.15)}\qquad \alpha_{m-1}(t_m) \leqq \alpha_m(t_m), \quad \zeta_{m-1}(t_m) \leqq \zeta_m(t_m), \quad \tilde{\Gamma}_{m-1}(t_m) \leqq \tilde{\Gamma}_m(t_m)$$

for $m = 1, 2, \cdots$.

Let Γ_3 satisfy

$$\text{(6.16)}\qquad \begin{aligned} &\Gamma_3 \in C([0, t^*), \mathbb{R}^1), \qquad \Gamma_3 \text{ is nondecreasing},\\ &\tilde{\Gamma}_m(t) \leqq \Gamma_3(t) \quad \text{on } \bar{I}_m \text{ for } m = 0, 1, \cdots. \end{aligned}$$

Clearly, there exist many such functions. The assertion is that any Γ_3 which satisfies (6.16) also satisfies (6.8), (6.9). This will be proven by showing that

$$\text{(6.17)}\qquad |\varphi_\varepsilon(t)| \leqq \tilde{\Gamma}_m(t) \quad \text{for } t \in \bar{I}_m, \qquad m = 0, 1, \cdots, \quad 0 < \varepsilon < t^*.$$

From this point on, the present proof and the argument of § 5 have many points in common; however, there are some essential differences. Where the reasoning is similar to that already employed, many details will be omitted.

The first step in the proof of (6.17) is to use the monotonicity of $\bar{b}$ to obtain an inequality, (6.21) below, which then eliminates $\bar{b}$ from the discussion. In view of (1.28), (6.2) and (6.3), the reasoning employed in the passage (5.13)–(5.22) now yields

$$\text{(6.18)}\qquad \begin{aligned} G(\varphi_\varepsilon(t)) &= G(f(0))\\ &\quad - [1 + \bar{b}(\varepsilon)] \int_0^t g^2(\varphi_\varepsilon(\tau))\, d\tau - E_5(t, \varepsilon) - E_6(t, \varepsilon) + E_7(t, \varepsilon) \end{aligned}$$

for $0 \leqq t < t^*$, $0 < \varepsilon < t^*$, where

$$
\begin{aligned}
E_5(t, \varepsilon) &= \int_0^t g(\varphi_\varepsilon(\tau))\left\{\int_0^\tau g(\varphi_\varepsilon(\tau - s))\, d\bar{b}(s + \varepsilon)\right\} d\tau, \\
E_6(t, \varepsilon) &= \int_0^t g(\varphi_\varepsilon(\tau))\left\{\int_0^\tau g(\varphi_\varepsilon(\tau - s))\, d\bar{c}(s)\right\} d\tau, \qquad (6.19)\\
E_7(t, \varepsilon) &= \int_0^t g(\varphi_\varepsilon(\tau))\, df(\tau).
\end{aligned}
$$

In view of (6.2), the argument employed in the proof of (5.25) with $m = 0$ now yields

$$
|E_5(t, \varepsilon)| \leqq \bar{b}(\varepsilon) \int_0^t g^2(\varphi_\varepsilon(\tau))\, d\tau, \qquad 0 \leqq t < t^*, \quad 0 < \varepsilon < t^*. \tag{6.20}
$$

From (6.18) and (6.20), it follows that

$$
G(\varphi_\varepsilon(t)) \leqq G(f(0)) - \int_0^t g^2(\varphi_\varepsilon(\tau))\, d\tau - E_6(t, \varepsilon) + E_7(t, \varepsilon) \tag{6.21}
$$

for $0 \leqq t < t^*$, $0 < \varepsilon < t^*$.

The inequalities (6.17) will now be proved by induction. Analogous to (6.20) (and hence to (5.25)), it follows from (6.11), (6.12), and (6.19) that

$$
|E_6(t, \varepsilon)| \leqq \frac{1}{2} \int_0^t g^2(\varphi_\varepsilon(\tau))\, d\tau, \qquad t \in \bar{I}_0, \quad 0 < \varepsilon < t^*. \tag{6.22}
$$

Analogous to (5.28) with $m = 0$, it follows from (1.25), the definition of β, and (6.19) that

$$
|E_7(t, \varepsilon)| \leqq K_2 \int_0^t G(\varphi_\varepsilon(\tau))\, d\beta(\tau), \qquad 0 \leqq t < t^*, \quad 0 < \varepsilon < t^*. \tag{6.23}
$$

Combining (6.13) and (6.21)–(6.23) yields

$$
G(\varphi_\varepsilon(t)) \leqq \alpha_0(t) + K_2 \int_0^t G(\varphi_\varepsilon(\tau))\, d\beta(\tau), \qquad t \in \bar{I}_0, \quad 0 < \varepsilon < t^*. \tag{6.24}
$$

Analogous to (5.32) with $m = 0$, it follows from (6.11)–(6.13) and (6.24) that

$$
G(\varphi_\varepsilon(t)) \leqq \zeta_0(t), \qquad t \in \bar{I}_0, \quad 0 < \varepsilon < t^*, \tag{6.25}
$$

which together with (5.3) and (6.13) implies

$$
|\varphi_\varepsilon(t)| \leqq \tilde{\Gamma}_0(t), \qquad t \in \bar{I}_0, \quad 0 < \varepsilon < t^*.
$$

Thus (6.17) holds for $m = 0$.

Suppose (6.17) holds for $m = 0, 1, \cdots, m_0 - 1$ with $m_0 \geqq 1$. Then (6.15) and the nondecreasing nature of the $\tilde{\Gamma}_k$'s imply

$$
|\varphi_\varepsilon(t)| \leqq \tilde{\Gamma}_{m_0 - 1}(t_{m_0}), \qquad 0 \leqq t \leqq t_{m_0},
$$

which together with (5.5) yields

$$\text{(6.26)} \qquad |g(\varphi_\varepsilon(t))| \leqq \gamma(\tilde{\Gamma}_{m_0-1}(t_{m_0})), \qquad 0 \leqq t \leqq t_{m_0}.$$

From (6.19), (6.21) and the definitions of $\bar{v}$ and β it follows that

$$\text{(6.27)} \qquad G(\varphi_\varepsilon(t)) \leqq G(f(0)) - \int_0^t g^2(\varphi_\varepsilon(\tau))\,d\tau + \sum_{k=1}^{5} E_{7+k}(m_0, t, \varepsilon)$$

on $[t_{m_0}, t^*)$, where

$$\text{(6.28)} \qquad \begin{aligned} E_8(m_0, \varepsilon) &= \int_0^{t_{m_0}} |g(\varphi_\varepsilon(\tau))| \left\{ \int_0^\tau |g(\varphi_\varepsilon(\tau - s))|\,d\bar{v}(s) \right\} d\tau, \\ E_9(m_0, t, \varepsilon) &= \int_{t_{m_0}}^t |g(\varphi_\varepsilon(\tau))| \left\{ \int_0^{\tau - t_{m_0}} |g(\varphi_\varepsilon(\tau - s))|\,d\bar{v}(s) \right\} d\tau, \\ E_{10}(m_0, t, \varepsilon) &= \int_{t_{m_0}}^t |g(\varphi_\varepsilon(\tau))| \left\{ \int_{\tau - t_{m_0}}^\tau |g(\varphi_\varepsilon(\tau - s))|\,d\bar{v}(s) \right\} d\tau, \\ E_{11}(m_0, \varepsilon) &= \int_0^{t_{m_0}} |g(\varphi_\varepsilon(\tau))|\,d\beta(\tau), \\ E_{12}(m_0, t, \varepsilon) &= \int_{t_{m_0}}^t |g(\varphi_\varepsilon(\tau))|\,d\beta(\tau). \end{aligned}$$

From (6.26) and (6.28), it follows that

$$\text{(6.29)} \qquad \begin{aligned} E_8(m_0, \varepsilon) &\leqq \gamma^2(\tilde{\Gamma}_{m_0-1}(t_{m_0})) \int_0^{t_{m_0}} \bar{v}(\tau)\,d\tau, \\ E_{11}(m_0, \varepsilon) &\leqq \gamma(\tilde{\Gamma}_{m_0-1}(t_{m_0}))\beta(t_{m_0}). \end{aligned} \qquad 0 < \varepsilon < t^*,$$

As in (6.22), (6.11) implies

$$\text{(6.30)} \qquad E_9(m_0, t, \varepsilon) \leqq \frac{1}{2} \int_{t_{m_0}}^t g^2(\varphi_\varepsilon(\tau))\,d\tau, \qquad t \in \bar{I}_{m_0}, \quad 0 < \varepsilon < t^*.$$

Similarly to (5.27), (6.26) implies

$$\text{(6.31)} \qquad E_{10}(m_0, t, \varepsilon) \leqq \frac{1}{2} \int_{t_{m_0}}^t g^2(\varphi_\varepsilon(\tau))\,d\tau + \frac{1}{2}\gamma^2(\tilde{\Gamma}_{m_0-1}(t_{m_0})) \int_{t_{m_0}}^t \bar{v}^2(\tau)\,d\tau$$

for $t_{m_0} \leqq t < t^*$, $0 < \varepsilon < t^*$. From (1.25), one has

$$\text{(6.32)} \qquad E_{12}(m_0, t, \varepsilon) \leqq K_2 \int_{t_{m_0}}^t G(\varphi_\varepsilon(\tau))\,d\beta(\tau), \qquad t_{m_0} \leqq t < t^*, \quad 0 < \varepsilon < t^*.$$

Combining (6.14) and (6.32) yields

$$\text{(6.33)} \qquad G(\varphi_\varepsilon(t)) \leqq \alpha_{m_0}(t) + K_2 \int_{t_{m_0}}^t G(\varphi_\varepsilon(\tau))\,d\beta(\tau), \qquad t \in \bar{I}_{m_0}, \quad 0 < \varepsilon < t^*.$$

From (6.12), (6.14) and (6.33) it follows, as in the passage from (6.24) to (6.25), that

$$G(\varphi_\varepsilon(t)) \leqq \zeta_{m_0}(t), \qquad t \in \bar{I}_{m_0}, \quad 0 < \varepsilon < t^*,$$

which together with (5.3) and (6.14) implies

$$|\varphi_\varepsilon(t)| \leqq \tilde{\Gamma}_{m_0}(t), \qquad t \in \bar{I}_{m_0}, \quad 0 < \varepsilon < t^*.$$

Thus (6.17) holds for $m = m_0$, which completes the proof of (6.17) and, therefore, of (6.8), (6.9) and the theorem.

7. Proof of Theorem 4. This proof uses much of the machinery of § 5. The key difference between the two proofs is caused by the new definition of the functions h_m, (7.4) below instead of (5.16) above. This difference reflects the added smoothness of (1.34) over (1.28), and sufficiently simplifies the present argument so that (1.25) is no longer required.

Since they are consequences of (1.24), formulas (5.1)–(5.5) apply here. Similarly, (5.6) and (5.7) are consequences of (1.26) and 1.27) and again hold. Definition (5.8) is not needed here and (5.9) is replaced by the following less stringent requirement on the infinite sequence $\{t_m\}$:

$$0 \leqq t_0 < t_1 < t_2 < \cdots < t^*, \quad \lim_{m\to\infty} t_m = t^*, \quad t_{m+1} - t_m \leqq \lambda, \quad m = 0, 1, \cdots. \tag{7.1}$$

The intervals I_m are again defined by (5.10).

If $t^* = \infty$, then it is sufficient to take $t_m = m\lambda$. If $t^* < \infty$, then it would be sufficient to take a finite sequence in (7.1). By taking an infinite sequence, both situations may be simultaneously discussed and many of the formulas of § 5 may be used without change.

$\Gamma_4 \in C([0, t^*), [0, \infty))$ may now be defined inductively. (Note that (5.11) and (5.12) do not hold here.) On $\bar{I}_0$ define

$$\begin{aligned} \zeta(t) &= G(f(0)) + \frac{1}{2\mu}\int_0^t |f'(\tau)|^2\, d\tau, \\ \Gamma_4(t) &= \omega(\zeta(t)), \end{aligned} \tag{7.2}$$

and on $\bar{I}_m$ $(m \geqq 1)$ define

$$\begin{aligned} \zeta(t) &= \zeta(t_m) + \frac{1}{2\mu}\int_{t_m}^t \{|f'(\tau)| + n^{1/2}\gamma(\Gamma_4(t_m))v(\tau)\}^2\, d\tau, \\ \Gamma_4(t) &= \omega(\zeta(t)). \end{aligned} \tag{7.3}$$

Let $(\hat{t}, \varphi)$ satisfy (1.3). Since (1.34) implies that (1.28) holds, (5.13) and (5.14) are valid here. However, (1.34) and the discussion concerning (5.13) and (5.14) now yield $\varphi \in AC_{\mathrm{loc}}([0, \hat{t}), \mathbb{R}^n)$. Define $\hat{m}$ by (5.15).

Replace (5.16) with the definition

$$\begin{aligned} h_0(t) &= f'(t), \\ h_m(t) &= f'(t) - \int_{(t-t_m, t]} da(s) g(\varphi(t-s)), \\ &\qquad t_m \leqq t < \hat{t}, \quad m = 1, 2, \cdots, \hat{m}. \end{aligned} \tag{7.4}$$

Then, analogous to (5.17),

$$(7.5)\qquad |h_m(t)| \leqq |f'(t)| + n^{1/2}v(t) \max_{0 \leqq s \leqq t_m} |g(\varphi(s))|$$

on $[t_m, \hat{t})$ for $m = 1, 2, \cdots, \hat{m}$. From (5.14) and (7.4), one has

$$(7.6)\qquad \varphi'(t) = -a(0)g(\varphi(t)) - \int_0^{t-t_m} da(s)g(\varphi(t-s)) + h_m(t)$$

a.e. on $[t_m, \hat{t})$ for $m = 0, 1, \cdots, \hat{m}$. Multiplying (7.6) by $g^T(\varphi(t))$, integrating and invoking (1.24) yields

$$(7.7)\qquad G(\varphi(t)) = G(\varphi(t_m)) - E_1(m, t) - E_2(m, t) + E_3(m, t)$$

on $[t_m, \hat{t})$ for $m = 0, 1, \cdots, \hat{m}$, where the $E_k(m, t)$ $(k = 1, 2, 3)$ are defined by (5.20). (E_4 of (5.20) does not occur here because f' has been incorporated into h_m.) As in § 5 the inequalities (5.24), (5.26) and (5.27) now follow. Together with (7.7), they imply

$$(7.8)\qquad G(\varphi(t)) \leqq G(\varphi(t_m)) + \frac{1}{2\mu}\int_{t_m}^t |h_m(\tau)|^2\, d\tau$$

on $\bar{I}_m \cap [0, \hat{t})$ for $m = 0, 1, \cdots, \hat{m}$.

Analogous to the discussion concerning (5.33), the proof will be completed by showing that

$$(7.9)\qquad G(\varphi(t)) \leqq \zeta(t) \quad \text{on } \bar{I}_m \cap [0, \hat{t}) \text{ for } m = 0, 1, \cdots, \hat{m},$$

where ζ is defined by (7.2) and (7.3).

From (7.4) and (7.8) with $m = 0$, $\varphi(0) = f(0)$ and (7.2), one has

$$(7.10)\qquad G(\varphi(t)) \leqq G(f(0)) + \frac{1}{2\mu}\int_0^t |f'(\tau)|^2\, d\tau = \zeta(t) \quad \text{on } \bar{I}_0 \cap [0, \hat{t}).$$

Thus (7.9) holds for $m = 0$.

Suppose that $\hat{m} \geqq 1$ and that (7.9) holds for $m = 0, 1, \cdots, m_0 - 1$, where $0 \leqq m_0 - 1 < \hat{m}$. Then $\hat{t} > t_{m_0}$ and

$$(7.11)\qquad G(\varphi(t)) \leqq \zeta(t), \qquad 0 \leqq t \leqq t_{m_0},$$

which together with (5.3), (7.2) and (7.3) implies

$$(7.12)\qquad |\varphi(t)| \leqq \Gamma_4(t), \qquad 0 \leqq t \leqq t_{m_0}.$$

From (5.5) and (7.12), one has

$$(7.13)\qquad |g(\varphi(t))| \leqq \gamma(\Gamma_4(t)) \leqq \gamma(\Gamma_4(t_{m_0})), \qquad 0 \leqq t \leqq t_{m_0}.$$

Setting $m = m_0$ in (7.5) and then invoking (7.13) implies

$$(7.14)\qquad |h_{m_0}(t)| \leqq |f'(t)| + n^{1/2}\gamma(\Gamma_4(t_{m_0}))v(t), \qquad t_{m_0} \leqq t < \hat{t}.$$

From (7.8) with $m = m_0$, (7.11) with $t = t_{m_0}$, (7.14), and (7.3) with $m = m_0$, it readily follows that (7.9) holds for $m = m_0$. This completes the proof.

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A METHOD OF ASCENT FOR PARABOLIC AND PSEUDOPARABOLIC PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract. This paper extends the idea of a method of ascent as developed by Gilbert for elliptic equations and Colton for pseudoparabolic equations. We develop a method of ascent for parabolic equations and extend Colton's results for the pseudoparabolic case.

1. Introduction. In 1969, R. P. Gilbert [8], [9] discovered what he called a method of ascent for the equation

$$\Delta_n u(\mathbf{x}) + A(r^2)u(\mathbf{x}) = 0, \tag{1.1}$$

where $A(r^2)$ is an analytic function. In the above paper, he showed the existence of an analytic function $G(r^2, \xi)$ depending only on $A(r^2)$ and not on the space dimension n, such that every solution of (1.1) has the representation

$$u(\mathbf{x}) = h(\mathbf{x}) + \int_0^1 \sigma^{n-1}G(r, 1 - \sigma^2)h(\mathbf{x}\sigma^2)\,d\sigma. \tag{1.2}$$

Here $h(\mathbf{x})$ is a harmonic function. Gilbert illustrated how this operator could be used to construct solutions to standard boundary value problems for (1.1). Later, Colton and Gilbert [6] were able to construct a method of ascent for higher order elliptic equations with spherically symmetric coefficients.

A natural question now arises; can one construct similar operators for nonelliptic equations? An affirmative answer to this question was given by Colton [3] for certain types of pseudoparabolic equations, namely those of the form

$$(\Delta_n + A(r^2))u_t + [\eta\Delta_n + B(r^2)]u = 0. \tag{1.3}$$

Colton showed that if $u(\mathbf{x}, t)$ had zero initial data, then $u(\mathbf{x}, t)$ could be written as

$$u(\mathbf{x}, t) = h(\mathbf{x}, t) + \int_0^t \int_0^1 \sigma^{n-1}G(r, 1 - \sigma^2, t - \tau)h_\tau(\mathbf{x}\sigma^2, \tau)\,d\sigma\,d\tau, \tag{1.4}$$

where $h(\mathbf{x}, t)$ is a solution to the pseudoparabolic equation $\Delta_n h_t = 0$. As in (1.2), the function G depends only on the coefficients of the equation and not on the spatial dimension. Using this operator, Colton gave a constructive method for solving the first initial-boundary value problem for (1.3).

Colton also used the idea of a method of ascent to construct a fundamental solution for the three-space variable heat equation from a fundamental solution for the two-space variable case [5].

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Motivated by the paper of Colton and Gilbert [6], Brown [2] has developed a method of ascent for fourth order parabolic equations with time-independent coefficients. In this paper, he was able to show that for $n = 2$, his operator is onto.

The second section of this paper is devoted to the parabolic equation

$$\Delta_n u(\mathbf{x}, t) + A(r^2)u(\mathbf{x}, t) = u_t(\mathbf{x}, t), \tag{1.5}$$

where $A(r^2)$ depends analytically on r^2. We show that if $u(\mathbf{x}, t)$ is an analytic solution of (1.5), then $u(\mathbf{x}, t)$ can be written as

$$u(\mathbf{x}, t) = h(\mathbf{x}, t) + \int_{|t-\tau|=\delta} \int_0^1 \sigma^{n-1} G(r, 1 - \sigma^2, t, \tau) h(\mathbf{x}\sigma^2, \tau)\, d\sigma\, d\tau. \tag{1.6}$$

Again G is an analytic function of its independent variables for $t \neq \tau$, and does not depend on the spatial dimension n. Here $h(\mathbf{x}, t)$, rather than solving the simplest parabolic equation $\Delta_n u - u_t = 0$, is a harmonic function which depends analytically on the parameter t. For technical reasons the first integration is a path integral in the complex plane, and thus (1.6) is not an equation of Volterra type as are (1.2) and (1.4). This forces us to consider analytic solutions of (1.5) rather than the full class of solutions, and this restriction means that we have been unable to solve boundary value problems for (1.5).

In § 3 of this paper, we extend the results that Colton obtained for the pseudoparabolic case. We consider equations of the form (1.3), except that the coefficients A and B are allowed to be time dependent, and the restriction that $u(\mathbf{x}, 0)$ be zero is removed. Our method will also handle nonhomogeneous versions of (1.3), and to our knowledge, this is new even for elliptic equations. In the manner of Colton and Gilbert, we illustrate how one can solve boundary value problems with this method in the final section.

2. A method of ascent for parabolic equations. In this section, we will develop a method of ascent for the following parabolic equation,

$$L[u] \equiv \Delta_n u + A(r^2, t)u - u_t = 0, \tag{2.1}$$

where Δ_n is the n-dimensional Laplacian, $A(r^2, t)$ is an entire function of the variables r^2 and t, and u_t denotes the partial derivative of u with respect to t. The operator which we construct will map solutions of the equation $\Delta_n h(\mathbf{x}, t) = 0$, which depend analytically on the parameter t, into the family of analytic solutions of (2.1) in a one-to-one manner. If we also have the coefficient A in (2.1) independent of t, then our map is actually onto.

Throughout the remainder of this section, we will use the following notation:

$\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{C}^n$, complex n-space,

$r^2 = \sum_{i=1}^n x_i^2$,

Ω is some open polydisk in $\mathbb{C}^n$ containing the origin,

$\Omega_T = \Omega \times \{t : |t| < T\} \subset \mathbb{C}^{n+1}$,

$\mathscr{B} = \mathscr{B}(\Omega_T)$ is the set of analytic functions on Ω_T, and $\mathscr{B}$ is given the topology of uniform convergence on compact subsets of Ω_T,

$\hat{\mathscr{B}} = \hat{\mathscr{B}}(\Omega_T) =$ those functions in $\mathscr{B}$ which are continuous on $\bar{\Omega}_T$. $\hat{\mathscr{B}}$ is given the usual supremum norm topology

$$\|h\| = \sup_{(\mathbf{x},t)\in\Omega_T} |h(\mathbf{x}, t)| \quad \text{for } h \in \hat{\mathscr{B}}.$$

It is perhaps unnatural to have $\mathbb{C}^n$ rather than $\mathbb{R}^n$; however, several of our arguments depend on the fact that the limit of a uniformly convergent sequence of analytic functions of several complex variables is again analytic, and since the functions we are interested in are all of this type, we have taken their domains of definition to lie in $\mathbb{C}^{n+1}$.

Motivated by Gilbert and Colton (cf. [8] and [4]), we consider the linear operator

$$(I + \mathscr{G})h(\mathbf{x}, t) = h(\mathbf{x}, t) + \int_{|\tau - t| = \delta} \int_0^1 \sigma^{n-1} G(r, 1 - \sigma^2, t, \tau) h(\mathbf{x}\sigma^2, \tau)\, d\sigma\, d\tau, \tag{2.2}$$

where the first integration is over the indicated path in the complex plane, δ is any positive constant and $G(r, \xi, t, \tau)$ is an analytic function of its independent variables when $t \neq \tau$. We also insist that G satisfy (2.3) and (2.4a) below,

$$\partial^2 G/\partial r^2 - (1/r)(\partial G/\partial r) + (2(1 - \xi)/r)\partial^2 G/\partial r\, \partial \xi + A(r^2, t)G - \partial G/\partial t = 0, \tag{2.3}$$

$$(1/r)(\partial G/\partial r)(r, 0, t, \tau) = \frac{1}{2\pi i}\{1/(\tau - t)^2 - A(r^2, t)/(\tau - t)\}. \tag{2.4a}$$

Clearly, if we restrict $\mathbf{x}$ to lie in $\mathbb{R}^n$, (2.2) makes sense; moreover, since it turns out that the function G is really a function of r^2, we may allow the variables x_i to take on complex values. We also note that (2.3) and (2.4a) do not depend on n.

THEOREM 2.1. *Let $G(r, \xi, t, \tau)$ be as above, and let $h(\mathbf{x}, t) \in \mathscr{B}$ such that $\Delta_n h = 0$ for each t. Then $u(\mathbf{x}, t) = (I + \mathscr{G})h$ is in $\mathscr{B}$ and satisfies $L[u] = 0$.*

Proof. The fact that u is in $\mathscr{B}$ is obvious. We therefore only need verify that $L[u] = 0$. Differentiating (2.2) and integrating by parts, we have

$$\begin{aligned} L[u] = {} & \Delta_n h + \int_{|\tau - t| = \delta} \int_0^1 \sigma^{n+3} G(\Delta_n h)\, d\sigma\, d\tau \\ & + \iint \sigma^{n-1} h\{(\partial^2 G/\partial r^2) - (1/r)(\partial G/\partial r) + (2\sigma^2/r)(\partial^2 G/\partial r\, \partial \xi) \\ & + A(r^2, t)G - (\partial G/\partial t)\}\, d\sigma\, d\tau \\ & + A(r^2, t)h - (\partial h/\partial t) + \int_{|\tau - t| = \delta} (1/r)(\partial G/\partial r)(r, 0, t, \tau) h(\mathbf{x}, \tau)\, d\tau. \end{aligned} \tag{2.5}$$

Since $\Delta_n h = 0$ and G satisfies (2.3), each of the first three terms of the right-hand side is zero. Condition (2.4a) implies that the last three terms have zero sum, and we conclude that $L[u] = 0$.

We note, in general, that we have

$$L[u] = \Delta_n h + \int_{|\tau - t| = \delta} \int_0^1 \sigma^{n+3} G \Delta_n h\, d\sigma\, d\tau, \tag{2.6}$$

from which we will later conclude that if $u = (I + \mathscr{G})h$ and $L[u] = 0$, then $\Delta_n h = 0$. We remark, for future use, that $\mathscr{G}$, as a map from $\mathscr{B}$ or $\hat{\mathscr{B}}$ into $\mathscr{B}$, is continuous.

We now proceed to establish the existence of G. Let $B(r, t)$ be defined by

$$B(r, t) = \int_0^r A(s, t)\, ds. \tag{2.7}$$

Then if G satisfies (2.4b) below, it satisfies (2.4a).

(2.4b) $$G(r, 0, t, \tau) = \frac{1}{4\pi i}\{(r^2/(\tau - t)^2) - B(r^2, t)/(\tau - t)\}.$$

We now write G as the infinite series

(2.8) $$G(r, \xi, t, \tau) = \sum_{k=1}^{\infty} \xi^{k-1} c^{(k)}(r^2, t, \tau),$$

where $c^{(1)}(r^2, t, \tau)$ is the right-hand side of (2.4b), thus forcing G to satisfy (2.4b), and hence (2.4a) (cf. the remark after (2.4a)).

Setting $\lambda = r^2$ and differentiating (2.8) term by term, we see that G satisfies (2.3) if

(2.9) $$4k(\partial c^{(k+1)}/\partial\lambda) = -4\lambda(\partial^2 c^{(k)}/\partial\lambda^2) + 4(k-1)(\partial c^{(k)}/\partial\lambda) - A(\lambda, t)c^{(k)} + \partial c^{(k)}/\partial t, \qquad k = 1, 2, \cdots.$$

For later use we will also insist that $c^{(k)}(0, t, \tau) = 0$ for each k. Setting

(2.10) $$c^{(k)}(\lambda, t, \tau) = \lambda^k e^{(k)}(\lambda, t, \tau), \qquad k = 1, 2, \cdots,$$

we see that $\{e^{(k)}\}$ must satisfy

(2.11) $$e^{(1)}(\lambda, t, \tau) = \frac{1}{4\pi i}\{(1/(\tau - t)^2) - 1/\lambda(B(\lambda, t)/(\tau - t))\},$$

$$k\, e^{(k+1)}(\lambda, t, \tau) = -(\partial\, e^{(k)}/\partial\lambda)(\lambda, t, \tau) + \lambda^{-(k+1)} \int_0^\lambda \frac{s^k}{4}[(\partial\, e^{(k)}/\partial t)(s, t, \tau) - A(s, t)\, e^{(k)}(s, t, \tau)]\, ds, \qquad k = 1, 2, \cdots.$$

We now define $f^{(k)}$ by

(2.12) $$f^{(k)}(\lambda, t, \tau) = (\tau - t)^{k+1} e^{(k)}(\lambda, t, \tau), \qquad k = 1, 2, \cdots.$$

Then the $f^{(k)}$ satisfy

(2.13) $$f^{(1)} = \frac{1}{4\pi i}\{1 - ((\tau - t)/\lambda)B(\lambda, t)\},$$

$$kf^{(k+1)} = -(\tau - t)(\partial f^{(k)}/\partial\lambda) + \lambda^{-(k+1)} \int_0^\lambda \frac{s^k}{4}\{[(k+1) - (\tau - t)A]f^{(k)} + (\tau - t)(\partial f^k/\partial t)\}\, ds, \qquad k = 1, 2, \cdots.$$

We will use the method of dominants to derive some estimates on $f^{(k)}$, and these estimates will imply that the series in (2.8) converges absolutely and uniformly on compact subsets of $\mathbb{C} \times \mathbb{C} \times \{(t, \tau) \in \mathbb{C}^2 : t \neq \tau\}$. Recall that if $g(z)$ and $h(z)$ are analytic functions of the complex variable z, then by $g \ll h$ we mean that if g_m and h_m are the coefficients of the Taylor series expansions of g and h, respectively, then $|g_m| \leqq h_m$ for each m (cf. [1]).

LEMMA 2.1. *Let R and T be positive constants. Let*

(2.14) $$A(\lambda, t) \ll C(1 - \lambda/R)^{-1}(1 - t/T)^{-1}$$

for $|\lambda| < R$ *and* $|t| < T$. *Then each* $f^{(k)}$ *as defined in* (2.13) *is an analytic function of its independent variables, and there is a positive constant* $M = M(R, T)$ *such that*

$$f^{(k)}(\lambda, t, \tau) \ll M(9/R)^k\{(1 - \lambda/R)(1 - t/T)(1 - \tau/T)\}^{-2k}. \tag{2.15}$$

Proof. It is clear from (2.13) that each $f^{(k)}$ is analytic. We now proceed with an inductive argument to prove (2.15). Let $k_0 = \max\{R, C, 4R/T\}$. Let M be such that (2.15) holds for $k \leqq k_0$. We now assume (2.15) holds for $f^{(k)}$ and show that it holds for $f^{(k+1)}$. Standard calculations give us

$$[(k + 1) - (\tau - t)A]f^{(k)} \ll [(k + 1) + 2C]\mathcal{M}, \tag{2.16}$$

$$(\tau - t)(\partial f^{(k)}/\partial t) \ll (4k/T)\mathcal{M}, \tag{2.17}$$

$$(\tau - t)(\partial f^{(k)}/\partial \lambda) \ll (4k/R)\mathcal{M}, \tag{2.18}$$

where $\mathcal{M} = M(9/R)^k\{(1 - \lambda/R)(1 - t/T)(1 - /T)\}^{-2(k+1)}$. In the above, use has been made of the fact that $\tau \ll T(1 - \tau/T)^{-1}$ and $t \ll T(1 - t/T)^{-1}$. Using (2.16)–(2.18) and

$$f \ll g \quad \text{implies} \quad \lambda^{-(k+1)} \int_0^\lambda s^k f\, ds \ll g/k, \tag{2.19}$$

we have

$$f^{(k+1)} \ll [4 + ((k + 1)R/k^2) + (2CR/k^2) + (4R/(kT))](1/R)\mathcal{M}. \tag{2.20}$$

Without loss of generality, we may assume $k > k_0$. Thus $[4 + ((k + 1)R/k^2) + (2CR/k^2) + (4R/(kT))] \leqq 9$ and (2.15) holds for $f^{(k+1)}$.

An immediate application of Lemma 2.1 along with Theorem 2.1 and the preceding discussion gives us the next theorem.

THEOREM 2.2. *Let* $A(r^2, t)$ *be an entire function of* r^2 *and* t. *Let* $c^{(k)}$ $(k = 1, 2, \cdots)$ *be defined by* (2.9). *Then* $G(r, \xi, t, \tau)$, *defined by* (2.8), *is an analytic function defined on* $\mathbb{C} \times \mathbb{C} \times \{(t, \tau) \in \mathbb{C}^2 : t \neq \tau\}$, *and satisfies* (2.3) *and* (2.4). *Thus the operator defined by* (2.2) *exists and maps harmonic functions of* $\mathbf{x}$ *into the space of analytic solutions of* (2.1).

We now give three lemmas which are needed to show that $I + \mathcal{G}$ as a map from $\mathcal{B}$ into $\mathcal{B}$ is onto when $A(r^2, t) = A(r^2)$. This latter restriction is only needed to show that $I + \mathcal{G}$ has dense range.

LEMMA 2.2. *Let* $h \in \hat{\mathcal{B}}$. *Then for any compact set* E *contained in* Ω_T, $\mathcal{G}h$ *is Lipschitz with Lipschitz constant depending only on* E *and* $\|h\|$.

Proof. Let $(\mathbf{x}_1, t_1)$ and $(\mathbf{x}_2, t_2)$ be any two points in E. Then since $h \in \hat{\mathcal{B}}$ and is therefore analytic on Ω_T and continuous on $\bar{\Omega}_T$, we may deform the τ path of integration into $|\tau| = T$. This gives us

$$|\mathcal{G}h(\mathbf{x}_2, t_2) - \mathcal{G}h(\mathbf{x}_1, t_1)| \leqq \int_{|\tau|=T} \int_0^1 \sigma^{n-1}|G(r_2, 1 - \sigma^2, t_2, \tau)h(\mathbf{x}_2\sigma^2, \tau)$$
$$- G(r_1, 1 - \sigma^2, t_1, \tau)h(\mathbf{x}_1\sigma^2, \tau)|\, d\sigma|d\tau|$$

(cont.)

$$= \int_{|\tau|=T} \int_0^1 \sigma^{n-1} \left| \int_{(\mathbf{x}_1,t_1)}^{(\mathbf{x}_2,t_2)} dF \right| d\sigma |d\tau|$$

$$\leqq \int_{|\tau|=T} \int_0^1 \int_{(\mathbf{x}_1,t_1)}^{(\mathbf{x}_2,t_2)} |dF| d\sigma |d\tau|,$$

where $F(\mathbf{x}, t, \sigma, \tau) = G(r, 1 - \sigma^2, t, \tau)h(x\sigma^2, \tau)$ and the σ and τ variables are regarded as parameters in computing dF. It is clear that $|dF|$ may be majorized by bounds on G, G_r, G_t, h and $\partial h/\partial x_i$. Since E is bounded away from the boundary of Ω_T, we see that t is bounded away from τ and we may, therefore, bound G, G_r and G_t by some constant depending on G and the distance between E and the boundary of Ω_T. Moreover, we may also use Cauchy-type estimates for analytic functions to show that $\partial h/\partial x_i$ may be bounded above by some constant which depends on E times $\|h\|$. Thus we have

$$|\mathscr{G}h(\mathbf{x}_2, t_2) - \mathscr{G}h(\mathbf{x}_1, t_1)| \leqq C\|h\| \, |(\mathbf{x}_2, t_2) - (\mathbf{x}_1 t_1)|, \tag{2.21}$$

where C depends only on E and G.

LEMMA 2.3. *$\mathscr{G}$, considered as a map from $\hat{\mathscr{B}}$ to $\mathscr{B}$, is compact in the following sense. If $h_n \in \hat{\mathscr{B}}$, $n = 1, 2, \cdots$, is a bounded sequence, then $\mathscr{G}h_n$ has a convergent subsequence in $\mathscr{B}$; i.e., this subsequence converges uniformly on compact subsets of Ω_T.*

Proof. Let $\{h_n\}_{n=1}^\infty \subset \hat{\mathscr{B}}$ with $\|h_n\| \leqq M$, $n = 1, 2, \cdots$. Let E_p, $p = 1, 2, \cdots$, be a sequence of open nested sets, whose closures are contained in Ω_T, such that E_p tend upward to Ω_T. On the set E_1, the family $\mathscr{G}h_n$ is uniformly bounded, and from (2.21) equicontinuous. Thus by the Arzela–Ascoli theorem, $\mathscr{G}h_n$ has a subsequence $\mathscr{G}h_n^{(1)}$ which converges uniformly to some u_1 on E_1. The family $\mathscr{G}h_n^{(1)}$ is uniformly bounded and equicontinuous on E_2, and thus has a uniformly convergent subsequence $\mathscr{G}h_n^{(2)}$, which converges to u_2 on E_2. Clearly u_2 is the analytic extension of u_1 to E_2. Continuing in this manner, for each p we have a subsequence $\mathscr{G}h_n^{(p)}$ of $\mathscr{G}h_n^{(p-1)}$ which converges to u_p on E_p, and u_p is the analytic extension of u_{p-1} to E_p. The desired subsequence of $\mathscr{G}h_n$ is $\mathscr{G}h_p^{(p)}$. Clearly $\mathscr{G}h_p^{(p)} \to u_p$. on E_p. Thus $\mathscr{G}h_p^{(p)}$ converges to some u in $\mathscr{B}$.

LEMMA 2.4. *Let $\mathscr{G} : \hat{\mathscr{B}} \to \mathscr{B}$. Then if $u \in \overline{R(I + \mathscr{G})}$, there is an h in $\mathscr{B}$ such that $(I + \mathscr{G})h = u$.*

Proof. The method of proof is identical to that found in the Riesz–Schauder theory of compact operators (cf. [7]). For this reason, we omit the details and merely point out to the reader that, in this case, we do not have $h \in \hat{\mathscr{B}}$.

We now proceed to show that if $A(r^2, t) = A(r^2)$, the mapping given by (2.2) has dense range, i.e., $\overline{R(I + \mathscr{G})} = \mathscr{B}$. In general, we may write

$$G(r, \xi, t, \tau) = \sum_{k=1}^\infty g^{(k)}(r, \xi, t)/(\tau - t)^k, \tag{2.22}$$

but when A is independent of t, as we assume from now on, each $g^{(k)}$ in (2.22) will also be independent of t (cf. (2.10)–(2.13)). Let $u(\mathbf{x}, t) = (I + \mathscr{G})h(\mathbf{x}, t)$, and expand u and h in a Taylor series in t, i.e.,

$$\begin{aligned} u(\mathbf{x}, t) &= \sum_{m=0}^\infty u_m(\mathbf{x})t^m/m!, \\ h(\mathbf{x}, t) &= \sum_{m=0}^\infty h_m(\mathbf{x})t^m/m!. \end{aligned} \tag{2.23}$$

We then have

$$
\begin{aligned}
u_m(\mathbf{x}) &= (\partial^m u/\partial t^m)|_{t=0} \\
&= (\partial^m h/\partial t^m)|_{t=0} + \int_{|\tau|=\delta}\int_0^1 \sigma^{n-1}(\partial^m G/\partial t^m)(r, 1-\sigma^2, 0, \tau)h(\mathbf{x}\sigma^2, \tau)\,d\sigma\,d\tau \\
&= h_m(\mathbf{x}) + 2\pi i\int_0^1 \sigma^{n-1}h_m(\mathbf{x}\sigma^2)g^{(1)}(r, 1-\sigma^2)\,d\sigma \\
&\quad + 2\pi i\sum_{k=2}^{\infty}\int_0^1 \sigma^{n-1}h_{k+m-1}(\mathbf{x}\sigma^2)(g^{(k)}(r, 1-\sigma^2)/(k-1)!)\,d\sigma,
\end{aligned} \tag{2.24}
$$

$$m = 0, 1, 2, \cdots.$$

These equations imply that if $h(\mathbf{x}, t)$ is a polynomial of degree N in t, then so is $u(\mathbf{x}, t)$. Moreover, since the linear operator defined by

$$h_m(\mathbf{x}) + 2\pi i\int_0^1 \sigma^{n-1}h_m(\mathbf{x}\sigma^2)g^{(1)}(r, 1-\sigma^2)\,d\sigma \tag{2.25}$$

is invertible (the second term becomes a Volterra operator after the substitution $\sigma^2 = \rho/r$) and the infinite matrix of operators defined by (2.24) is upper triangular with all its main diagonal entries the same invertible operator, we may infer that any $N \times N$ square submatrix formed by taking the first N rows and columns is invertible. Thus if $u(\mathbf{x}, t)$ is a polynomial in t of degree N, there is a unique polynomial $h(\mathbf{x}, t)$ of degree N such that $u = (I + \mathscr{G})h$, from which we may infer that the range of $(I + \mathscr{G})$, as a mapping from $\hat{\mathscr{B}}$ into $\mathscr{B}$, is dense.

We remark that the function $g^{(1)}(r, \xi)$ multiplied by $(2\pi i)$ is Gilbert's G function for the equation $\Delta_n u + A(r^2)u = 0$. This follows from the following characterization of his G function [8]:

$$
\begin{gathered}
(\partial^2 G/\partial r^2) - (1/r)(\partial G/\partial r) + (2(1-\xi)/r)(\partial^2 G/(\partial r\,\partial\xi)) + A(r^2)G = 0, \\
G(0, \xi) = 0, \qquad G(r, 0) = -\int_0^r sA(s^2)\,ds.
\end{gathered} \tag{2.26}
$$

An easy computation using (2.3), (2.4b) and (2.22) verifies that $2\pi i g^{(1)}(r, \xi)$ satisfies (2.26).

We now state the following theorem.

THEOREM 2.3. *Let $A(r^2, t)$ in* (2.1) *depend only on r^2. Then the linear operator $(I + \mathscr{G})$ defined by* (2.2) *is a continuous, one-to-one mapping of $\mathscr{B}(\Omega_T)$ onto $\mathscr{B}(\Omega_T)$.*

Proof. The preceding discussion along with Lemma 2.4 shows that $I + \mathscr{G}$ maps $\mathscr{B}$ onto $\mathscr{B}$. We, therefore, only need verify that $I + \mathscr{G}$ is one-to-one. Let $h \in \mathscr{B}$ such that $(I + \mathscr{G})h = 0$. Expanding $h(\mathbf{x}, t)$ in a Taylor series in r (which is possible for $\mathbf{x} \in \mathbb{R}^n$), we have $h = \sum_{k=0}^{\infty} h_k(r^k/k!)$. Letting r tend to zero in $(I + \mathscr{G})h = 0$, we have

$$0 = h_0 + \lim_{r\to 0}\int_{|\tau - t|=\delta}\int_0^1 \sigma^{n-1}G(r, 1-\sigma^2, t, \tau)h(\mathbf{x}\sigma^2, \tau)\,d\sigma\,d\tau.$$

Since $G(0, 1-\sigma^2, t, \tau) \equiv 0$, we conclude that $h_0 = 0$. Differentiating $(I + \mathscr{G})h$ with respect to r and then letting r tend to zero, we may conclude that $h_1 = 0$.

Since we may similarly conclude that $h_k = 0$, for all k, we have $h(\mathbf{x}, t) = 0$ if $\mathbf{x} \in \mathbb{R}^n$, and thus since h is analytic, $h(\mathbf{x}, t) = 0$.

We remark that we may now conclude from (2.6) and Theorem 2.3 that if $u = (I + \mathscr{G})h$ and $L[u] = 0$, then $\Delta_n h = 0$.

In the special case when n equals one, we have the following theorem.

THEOREM 2.4 ($n = 1$). *Let $u(x, t)$ be any analytic solution of* (2.1) *in some neighborhood of the origin. Let $h(x, t) = u_x(0, t)x + u(0, t)$. Then $u = (I + \mathscr{G})h$.*

Proof. Let $v = (I + \mathscr{G})h$. Then v is an analytic solution of (2.1). Moreover, since $G(0, \xi, t, \tau) = (\partial G/\partial r)(0, \xi, t, \tau) = 0$, we have

$$v(0, t) = u(0, t) \quad \text{and} \quad v_x(0, t) = u_x(0, t).$$

Thus from the Cauchy–Kowalewski theorem, we have

$$u = v = (I + \mathscr{G})h.$$

We conclude this section by writing down the G function for the equation $\Delta_n u - u_t = 0$:

$$G(r, \xi, t, \tau) = \frac{1}{4\pi i}(r/(\tau - t))^2 \exp\left[\xi r^2/(4(\tau - t))\right], \tag{2.27}$$

where exp is the usual exponential function. The reader may easily verify that (2.27) satisfies (2.3) and (2.4).

3. The method of ascent for pseudoparabolic equations. Throughout the rest of this paper the variable $\mathbf{x}$ is restricted to lie in $\mathbb{R}^n$.

In this section, we shall devote our attention to the construction of an integral operator, mapping solutions of the simplest (nonhomogeneous) pseudoparabolic equation onto solutions of a more general nonhomogeneous pseudoparabolic equation. This operator will essentially be independent of the number of space variables. In the next section, we will show how one can use this operator to obtain solutions to the more common initial-boundary value problems associated with pseudoparabolic equations.

We shall consider the equation,

$$(\Delta_n + A(r^2, t))u_t(\mathbf{x}, t) + (\eta\Delta_n + B(r^2, t))u(\mathbf{x}, t) = f(\mathbf{x}, t), \tag{3.1}$$

where $A(r^2, t)$ and $B(r^2, t)$ are entire functions of r^2, $A(r^2, t)$ being twice continuously differentiable and $B(r^2, t)$ continuously differentiable with respect to t, and η is a constant. We need only assume that $f(\mathbf{x}, t)$ is continuous with respect to $\mathbf{x}$ and $\mathbf{t}$. By means of the transformation $v(\mathbf{x}, t) = u(\mathbf{x}, t)\, e^{\eta t}$, (3.1) becomes

$$Lu(\mathbf{x}, t) = \Delta_n u_t(\mathbf{x}, t) + A(r^2, t)u_t(\mathbf{x}, t) + \hat{B}(r^2, t)u(\mathbf{x}, t) = \hat{f}(\mathbf{x}, t) \tag{3.2}$$

with

$$\hat{B}(r^2, t) = B(r^2, t) - \eta A(r^2, t); \qquad \hat{f}(\mathbf{x}, t) = e^{\eta t} f(\mathbf{x}, t).$$

It is this form which we consider, and for convenience we shall omit the $\hat{}$ symbol. By the change of dependent variable $v(\mathbf{x}, t) = u(\mathbf{x}, t) - u(\mathbf{x}, 0)$ in (3.2), we may without loss of generality assume that

$$u(\mathbf{x}, 0) = 0 \tag{3.3}$$

provided that $u(\mathbf{x}, 0)$ is twice continuously differentiable.

We seek a solution of (3.2) and (3.3) in the form

$$u(\mathbf{x}, t) = p(\mathbf{x}, t) + \int_0^t \int_0^1 \sigma^{n-1} G_\tau(r, 1 - \sigma^2, t, \tau) p_\tau(\mathbf{x}\sigma^2, \tau)\, d\sigma\, d\tau, \tag{3.4}$$

where $p(\mathbf{x}, t)$ satisfies the equations

$$\Delta_n p_t(\mathbf{x}, t) = g(\mathbf{x}, t), \tag{3.5}$$

$$p(\mathbf{x}, 0) = 0. \tag{3.6}$$

Condition (3.6) is required in order to satisfy the initial condition (3.3), and $g(\mathbf{x}, t)$ is given uniquely in terms of $f(\mathbf{x}, t)$ in the manner to be described.

We choose $G(r, 1 - \sigma^2, t, \tau)$ to be a solution of

$$G_{rrt} - (\sigma/r) G_{r\sigma t} - (1/r) G_{rt} + A G_t + BG = 0 \tag{3.7}$$

with the initial conditions

$$G_{r\tau}(r, 0, t, t) = -r(A(r^2, t)), \tag{3.8}$$

$$G_{rt\tau}(r, 0, t, \tau) = -rB(r^2, t), \tag{3.9}$$

$$G(r, 1 - \sigma^2, t, t) = 0. \tag{3.10}$$

We shall now show the existence of a solution to (3.7)–(3.10), leaving for the moment the motivation for our choice of G.

As in [2], we will seek $G(r, 1 - \sigma^2, t, \tau)$ in the form

$$G(r, 1 - \sigma^2, t, \tau) = \sum_{k=1}^{\infty} c^{(k)}(r, t, \tau)(1 - \sigma^2)^{k-1} \tag{3.11}$$

and derive each of the $c^{(k)}(r, t, \tau)$ by means of a recursive relation. Using these relations, we will then show that (3.11) converges in any finite region $r^2 < R$; $|t|, |\tau| < T$ and $\sigma \leqq 1$ for arbitrary large R and T.

Substituting (3.11) into the differential equation (3.7) and equating the coefficients of powers of $1 - \sigma^2$ yields the recursive relation

$$(2k/r) c_{rt}^{(k+1)} = ((2k - 1)/r) c_{rt}^{(k)} - c_{rrt}^{(k)} - A c_t^{(k)} - B c^{(k)}, \qquad k \geqq 1.$$

The initial conditions (3.8)–(3.10) lead to a determination of $c^{(1)}(r, t, \tau)$ as

$$c^{(1)}(r, t, \tau) = \int_0^r r \left\{ (\tau - t) A(r^2, \tau) - \int_\tau^t (s - t) B(r^2, s)\, ds \right\} dr.$$

By considering $c^{(k)}(r, t, \tau) = \hat{c}^{(k)}(r^2, t, \tau)$ as a function of r^2, t and τ for $k \geqq 1$ the above become

$$k \hat{c}_{r^2 t}^{(k+1)} = (k - 1) \hat{c}_{r^2 t}^{(k)} - r^2 \hat{c}_{r^2 r^2 t}^{(k)} - \tfrac{1}{4} A \hat{c}_t^{(k)} - \tfrac{1}{4} B \hat{c}^{(k)} \tag{3.12}$$

and

$$\hat{c}^{(1)} = \frac{1}{2} \int_0^{r^2} \left\{ (\tau - t) A(r^2, t) - \int_\tau^t (s - t) B(r^2, s)\, ds \right\} dr^2. \tag{3.13}$$

Equations (3.12) and (3.13) show that for each $k \geqq 1$, $\hat{c}^{(k)}$ is an analytic function of r^2. Since $\hat{c}^{(1)}(r^2, t, t) = 0$, we have from (3.10) and (3.11) that

$$\hat{c}^{(k)}(r^2, t, t) = 0, \qquad k \geqq 1. \tag{3.14}$$

Now by integrating with respect to t from τ to t, we obtain (3.12) in the form

$$k\hat{c}^{(k+1)}_{r^2} = (k-1)\hat{c}^{(k)}_{r^2} - r^2\hat{c}^{(k)}_{r^2r^2} - \frac{1}{4}\int_\tau^t \{A(r^2, s)\hat{c}^{(k)}_s(r^2, s, \tau) + B(r^2, s)\hat{c}^{(k)}_s(r^2, s, \tau)\}\, ds. \tag{3.15}$$

If we simplify this expression by means of the transformation

$$\hat{c}^{(k)}(r^2, t, \tau) = r^{2k}\, e^{(k)}(r^2, t, \tau), \tag{3.16}$$

we are led to the scheme

$$k\{r^{2(k+1)}\, e^{(k+1)}\}_{r^2} = -\{r^{2(k+1)}\, e^{(k)}_{r^2}\}_{r^2} - \frac{1}{4r^{2k}}\int_\tau^t \{A\, e^{(k)}_s + B\, e^{(k)}\}\, ds,$$

i.e.,

$$k\, e^{(k+1)} = -e^{(k)}_{r^2} - \frac{1}{4}r^{-2(k+1)}\int_0^{r^2}\int_\tau^t r^{2k}\{A\, e^{(k)}_s + B\, e^{(k)}\}\, ds\, dr^2.$$

An integration by parts now gives our recursive relation its final form:

$$k\, e^{(k+1)} = -e^{(k)}_{r^2} - \frac{1}{4}r^{-2(k+1)}\int_0^{r^2} r^{2k}\left\{A\, e^{(k)} + \int_\tau^t (B - A_s)\, e^{(k)}\, ds\right\} dr^2. \tag{3.17}$$

Under the transformation (3.16), (3.13) becomes

$$e^{(1)} = \frac{1}{2r^2}\int_0^{r^2}\left\{(\tau - t)A(r^2, \tau) - \int_\tau^t (s - t)B(r^2, s)\, ds\right\} dr^2. \tag{3.18}$$

As in § 2, it can easily be shown by induction that for arbitrary large numbers R and T, there exists a positive constant $M = M(R, T)$ independent of k such that as a function of r^2, for $r^2 < R$ and uniformly for $|\tau|, |t| < T$,

$$e^{(k)} \ll \frac{2^k M}{k(1 - r^2/R)^{-k}R^{-k}}. \tag{3.19}$$

The details follow those in Lemma 2.1, and we omit them.

Returning now to our series (3.11) and using the fact that $\sigma \leqq 1$, we have

$$G(r, 1 - \sigma^2, t, \tau) \ll \sum_{k=1}^{\infty} r^{2k}\, e^{(k)}(r^2, t, \tau)$$
$$\ll M\sum_{k=1}^{\infty} (2^k r^{2k}/k)(1 - r^2/R)^{-k}R^{-k}.$$

Let $D = \{r^2 : r^2 \leqq R/4\}$ and let $I = [0, 1]$, $B = [0, T]$. We will now show that $G(r, 1 - \sigma^2, t, \tau)$ converges in $D \times I \times B \times B$. Since $(1 - r^2/R)^{-1} \leqq \frac{4}{3}$ for $r^2 \in D$, we have

$$G(r,1-\sigma^2,t,\tau) \ll M \sum_{k=1}^{\infty} \left(\frac{r^2}{R}\right)^k \frac{2^k}{k(4/3)^k}$$
$$\ll M \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k \frac{1}{k}$$

for $(r,\sigma,t,\tau) \in D \times I \times B \times B$; that is, the series (3.1) converges absolutely and uniformly for $r^2 \leqq R/4$, $\sigma \leqq 1$ and $|\tau|, |t| < T$. Thus since R and T were arbitrary, we can conclude that $G(r, 1-\sigma^2, t, \tau)$ is an entire function of r^2, analytic in σ for $\sigma \leqq 1$ and twice continuously differentiable in t and τ. This shows the existence of a solution to (3.7)–(3.10).

If we now make the change of variable $\mathbf{x} = (r, \boldsymbol{\theta})$, where r, $\boldsymbol{\theta}$ are spherical coordinates, and apply the operator L to both sides of (3.4), we obtain

$$\begin{aligned} Lu = {} & \Delta_n p_t + (1/r)G_{rt}(r,0,t,t)p_t + A(r^2,t)p_t \\ & + \int_0^t (1/r)G_{rt\tau}(r,0,t,\tau)p_\tau(\mathbf{x},\tau)\,d\tau + B(r^2,t)p \\ & + \int_0^1 \sigma^{n-1}\{G_{rr\tau} - (\sigma/r)G_{r\sigma t} - (1/r)G_{rt} + AG_t\}|_{t=\tau} p_t(\mathbf{x}\sigma^2,t)\,d\sigma \\ & + \int_0^t\int_0^1 \sigma^{n-1}\{G_{rrt\tau} - (\sigma/r)G_{r\sigma t\tau} - (1/r)G_{rt\tau} + AG_{t\tau} + BG_\tau\}p_\tau\,d\sigma\,d\tau \\ & + \int_0^1 \sigma^{n+3}\{G_\tau \Delta_n p_\tau\}|_{t=\tau}\,d\sigma + \int_0^t\int_0^1 \sigma^{n+3} G_{t\tau}\Delta p_\tau\,d\sigma\,d\tau. \end{aligned} \tag{3.20}$$

In (3.20), we have used integration by parts and the relation $p_{\tau r}(\mathbf{x}\sigma^2,\tau) = (\sigma/2r)p_{\tau\sigma}(\mathbf{x}\sigma^2,\tau)$. We now substitute the conditions (3.7)–(3.10) on the function G into (3.20) and obtain

$$\begin{aligned} f(\mathbf{x},t) = {} & g(\mathbf{x},t) + \int_0^1 \sigma^{n+3}G_\tau(r,1-\sigma^2,t,t)g(\mathbf{x}\sigma^2,t)\,d\sigma \\ & + \int_0^t\int_0^1 \sigma^{n+3}G_{t\tau}(r,1-\sigma^2,t,\tau)g(\mathbf{x}\sigma^2,\tau)\,d\sigma\,d\tau \end{aligned} \tag{3.21}$$

with $\Delta_n p_t(\mathbf{x},t) = g(\mathbf{x},t)$.

Under the change of variables $\mathbf{x} = (r,\boldsymbol{\theta})$ and $\sigma^2 = \rho/r$, this becomes

$$\begin{aligned} \psi(r,\boldsymbol{\theta},t) = {} & \phi(r,\boldsymbol{\theta},t) + \int_0^r K^{(1)}(r,\rho,t)\phi(\rho,\boldsymbol{\theta},t)\,d\rho \\ & + \int_0^t\int_0^r K^{(2)}(r,\rho,t,\tau)\phi(\rho,\boldsymbol{\theta},t)\,d\rho\,d\tau, \end{aligned} \tag{3.22}$$

where

$$\psi(r,\boldsymbol{\theta},t) = r^{n+2/2}f(r,\boldsymbol{\theta},t),$$
$$\phi(r,\boldsymbol{\theta},t) = r^{n+2/2}g(r,\boldsymbol{\theta},t),$$

$$K^{(1)}(r, \rho, t) = (\rho^{n+2}/2r)G_\tau(r, 1 - \rho/r, t, t),$$
$$K^{(2)}(r, \rho, t, \tau) = (\rho^{n+2}/2r)G_{t\tau}(r, 1 - \rho/r, t, \tau).$$

From (3.11) and (3.16) (where each $e^{(k)}$ is an entire function of r^2 and twice continuously differentiable with respect to t and τ) we have that $K^{(1)}$ and $K^{(2)}$ are entire functions of r, analytic in ρ for $\rho \leqq r$ and continuous in t and τ for $|t|, |\tau| < T$. Thus the Volterra integral equation (3.22) is invertible; that is, for each continuous function $\psi(r, \boldsymbol{\theta}, t)$, there exists a unique continuous solution $\phi(r, \boldsymbol{\theta}, t)$, and this in turn shows the invertibility of (3.21).

Thus given any continuous function $f(\mathbf{x}, t)$, we can find a unique continuous $g(\mathbf{x}, t)$ by means of (3.21) such that if $u(\mathbf{x}, t)$ satisfies equation (3.2), then $p(\mathbf{x}, t)$ satisfies (3.5). From (3.4), it is immediate that $p(\mathbf{x}, 0) = 0$ follows from $u(\mathbf{x}, 0) = 0$.

Thus our operator (3.4) which maps solutions of (3.5) and (3.6) into solutions of (3.2) and (3.3) is in fact onto.

In passing, we note that by comparing (3.7), (3.8) and (3.10) with the corresponding equations in [8] for Gilbert's G function for the elliptic equation

$$\Delta_n u + A(r^2, t)u = 0, \tag{3.23}$$

we have

$$G_\tau(r, 1 - \sigma^2, t, t) = \tilde{G}(r, 1 - \sigma^2, t), \tag{3.24}$$

where $\tilde{G}(r, 1 - \sigma^2, t)$ denotes Gilbert's G function for (3.23).

We thus have the following theorem.

THEOREM 3.1. *Let $u(\mathbf{x}, t)$ be a real-valued solution of* (3.2) *and* (3.3). *Then $u(\mathbf{x}, t)$ can be expressed in the form* (3.4), *where $p(\mathbf{x}, t)$ satisfies* (3.5) *and* (3.6). *The function G given by* (3.7)–(3.10) *is an entire function of r^2, analytic in σ for $\sigma \leqq 1$ and twice continuously differentiable with respect to t and τ for $|\tau|, |t| < T$. For each fixed t, $G_\tau(r, 1 - \sigma^2, t, t)$ is Gilbert's G function for the elliptic equation* (3.23).

Suppose now that for each fixed t, $f(\mathbf{x}, t)$ and $u(\mathbf{x}, 0)$ are analytic functions of $\mathbf{x}$ in a bounded domain $\Omega \times T$. Then the Volterra integral equation (3.22) has a unique solution, which in turn implies that $g(\mathbf{x}, t)$ is an analytic function of $\mathbf{x}$ in Ω. Since for such a $g(\mathbf{x}, t)$, any solution of (3.5) and (3.6) is analytic in $\mathbf{x}$ for each fixed t and G is an entire function of r^2, the fact that the mapping (3.4) is onto now implies the following corollary.

COROLLARY. *Let $u(\mathbf{x}, t)$ be a real-valued, strong solution of* (3.1) *in a bounded domain $\Omega \times T$. Let $u(\mathbf{x}, 0)$ and $f(\mathbf{x}, t)$ be analytic functions of $\mathbf{x}$ in $\Omega \times T$. Then for each fixed t, $u(\mathbf{x}, t)$ is an analytic function of $\mathbf{x}$ in $\Omega \times T$.*

To complete this section, we consider the equation

$$\Delta u_t - u_t + \Delta u = f(\mathbf{x}, t).$$

(It is this equation which occurs in most of the physical applications.) By using the transformation $v = e^t u$, we see that $A = -1$ and $B = 1$, and in this case, it is easily verified that the G function is

$$G(r, 1 - \sigma^2, t, \tau) = \frac{1}{2}\sum_{k=1}^{\infty} ((1 - \sigma^2)^{k-1}/(2^{2k}k!(k + 1)!))r^{2k}\phi^{(k)}(t - \tau),$$

where $\phi^{(k)}(s)$ is a polynomial in s of degree $k + 1$ given by

$$\phi_s^{(k+1)}(s) = \phi_s^{(k)}(s) - \phi^{(k)}(s), \qquad \phi^{(k)}(0) = 0$$

and

$$\phi^{(1)}(s) = s + \tfrac{1}{2}s^2.$$

Due to the presence of the factor $1/k!(k + 1)!$, the series expansion for G converges rapidly, and we can obtain an excellent approximation to this function by using only the first few terms of its series expansion.

4. Initial-boundary value problems. In this section, we shall illustrate how one can use the integral operator developed in the previous section to solve boundary value problems. Colton in [3] gave a constructive method for solving the first initial-boundary value problem for the equation (1.3). We shall follow his approach and seek the solution to the first initial-boundary value problem

$$\Delta_n u_t + A(r^2, t)u_t + \eta\Delta u + B(r^2, t)u = f(\mathbf{x}, t), \tag{4.1}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \tag{4.2}$$

$$u(\mathbf{x}, t) = \phi(\mathbf{x}, t), \qquad \mathbf{x} \in \partial\Omega, \tag{4.3}$$

in a cylindrical domain $\Omega \times T$. Here Ω is a bounded, simply connected domain with Lyapunov boundary $\partial\Omega$ and T is the interval $[0, T]$. The boundary data $\phi(\mathbf{x}, t)$ is a continuous function on $\partial\Omega \times T$ and continuously differentiable with respect to t. We may, without loss of generality, assume that (4.1) is written in the form (3.2) and that $u_0(\mathbf{x}) = 0$.

For $n > 2$, we may represent $p_t(\mathbf{x}, t)$ as a double layer potential

$$\begin{aligned} p_t(\mathbf{x}, t) = {} & \frac{\Gamma(n/2)}{\pi^{n/2}} \int_{\partial\Omega} \mu(\mathbf{y}, t)(\partial/\partial\nu)\{1/|\mathbf{x} - \mathbf{y}|^{n-2}\}\, ds(\mathbf{y}) \\ & + \frac{\Gamma(n/2)}{\pi^{n/2}} \int_{\Omega} g(\mathbf{y}, t)1/|\mathbf{x} - \mathbf{y}|^{n-2}\, d\mathbf{y}, \end{aligned} \tag{4.4}$$

where $\mu(\mathbf{y}, t)$ is a potential to be determined (for $n = 2$, we would represent $p_t(\mathbf{x}, t)$ as a double layer logarithmic potential) and $g(\mathbf{y}, t)$ is the unique solution to the integral equation (3.21). As usual, Γ denotes the gamma function and $\partial/\partial\nu$ is the derivative with respect to the inner normal ν.

We now differentiate both sides of our representation (3.4) with respect to t and substitute in (4.4). By interchanging the orders of integration and letting $\mathbf{x}$ approach the boundary $\partial\Omega$, we obtain the integral equation for the determination of $\mu(\mathbf{x}, t)$,

$$\begin{aligned} \Phi(\mathbf{x}, t) = {} & \mu(\mathbf{x}, t) + \int_{\partial\Omega} \mu(\mathbf{y}, t)N^{(1)}(\mathbf{x}, \mathbf{y}, t)\, ds(\mathbf{y}) \\ & + \int_0^t \int_{\partial\Omega} \mu(\mathbf{y}, \tau)N^{(2)}(\mathbf{x}, \mathbf{y}, t, \tau)\, ds(\mathbf{y})\, d\tau, \end{aligned} \tag{4.5}$$

where

$$\Phi(\mathbf{x}, t) = \phi_t(\mathbf{x}, t) + \frac{\Gamma(n/2)}{\pi^{n/2}} \int_0^1 \sigma^{n-1} G_t(r, 1 - \sigma^2, t, t) \int_\Omega (1/|\mathbf{x}\sigma^2 - \mathbf{y}|^{n-2}) \cdot g(\mathbf{y}, t)\, d\mathbf{y}\, d\sigma$$
$$+ \frac{\Gamma(n/2)}{\pi^{n/2}} \int_0^t \int_0^1 \sigma^{n-1} G_{t\tau}(r, 1 - \sigma^2, t, \tau) \int_\Omega (1/|\mathbf{x}\sigma^2 - \mathbf{y}|^{n-2}) \cdot g(\mathbf{y}, \tau)\, d\mathbf{y}\, d\sigma,$$

$$N^{(1)}(\mathbf{x}, \mathbf{y}, t) = \frac{\Gamma(n/2)}{\pi^{n/2}} (\partial/\partial v)\{1/|\mathbf{x} - \mathbf{y}|^{n-2}\}$$
$$+ \frac{\Gamma(n/2)}{\pi^{n/2}} \int_0^1 \sigma^{n-1} G_t(r, 1 - \sigma^2, t, t)(\partial/\partial v)\{1/|\mathbf{x}\sigma^2 - \mathbf{y}|^{n-2}\}\, d\sigma,$$

$$N^{(2)}(\mathbf{x}, \mathbf{y}, t, \tau) = \frac{\Gamma(n/2)}{\pi^{n/2}} \int_0^1 \sigma^{n-1} G_{t\tau}(r, 1 - \sigma^2, t, \tau)(\partial/\partial v)\{1/|\mathbf{x}\sigma^2 - \mathbf{y}|^{n-2}\}\, d\sigma.$$

We note that $\phi_t(\mathbf{x}, t)$ uniquely determines $\phi(\mathbf{x}, t)$ since $\phi(\mathbf{x}, 0) = 0$. The kernels $N^{(1)}$ and $N^{(2)}$ have weak singularities at $\mathbf{x} = \mathbf{y}$ and hence (4.5) is of the form

$$\Phi = (I + \mathscr{F} + \mathscr{L})\mu, \tag{4.6}$$

where $\mathscr{F}$ is a Fredholm operator and $\mathscr{L}$ a Volterra operator:

$$\mathscr{F}\mu = \int_{\partial\Omega} \mu(\mathbf{y}, t) N^{(1)}(\mathbf{x}, \mathbf{y}, t)\, ds(\mathbf{y}),$$

$$\mathscr{L}\mu = \int_0^t \int_{\partial\Omega} \mu(\mathbf{y}, t) N^{(2)}(\mathbf{x}, \mathbf{y}, t, \tau)\, ds(\mathbf{y})\, d\tau.$$

We now require a lemma. The proof can be found, for example, in [7].

LEMMA 4.1. *If $\mathscr{F}$ is a compact operator and $\mathscr{L}$ a compact, quasi-nilpotent operator on a Hilbert space H commuting with $\mathscr{F}$, then*

$$\sigma(\mathscr{F}) = \sigma(\mathscr{F} + \mathscr{L}),$$

where $\sigma(\mathscr{F})$ denotes the spectrum of the operator $\mathscr{F}$.

From Fubini's theorem, we see that $\mathscr{F}$ and $\mathscr{L}$ commute, and hence the conditions of the lemma are satisfied. Using (3.24), the operator $I + \mathscr{F}$ is identical with the operator defined in equation (4.42) of [8], and hence if $A(r^2, t) \leqq 0$ in the closure of $\Omega \times T$, then $(I + \mathscr{F})^{-1}$ exists. In this case, the lemma gives us that $(I + \mathscr{F} + \mathscr{L})^{-1}$ exists and that the unique solution of (4.6) is

$$\mu = (I + \mathscr{F} + \mathscr{L})^{-1}\Phi. \tag{4.7}$$

We have thus proved the following theorem.

THEOREM 4.1. *The first initial-boundary value problem for* (3.2) (*and thus by means of the transformation $u = e^{-nt}v$ for* (4.1) *also*) *admits a unique solution given by* (3.4) *and* (4.4), *where the potential μ is given by* (4.7) *and g is given by the solution to* (3.21), *provided $A(r^2, t) \leqq 0$ in $\Omega \times T$.*

In a similar manner, we can handle the second and third initial-boundary value problems for (4.1) (except in these cases, we use a single layer potential in equation (4.4)).

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ON THE POINTWISE COMPLETENESS OF DELAY-DIFFERENTIAL SYSTEMS OF NEUTRAL TYPE*

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Abstract. Necessary and sufficient conditions for pointwise completeness of delay-differential systems of neutral type are given. It is shown that there exist second order delay-differential systems of neutral type which are not pointwise complete. It is also proved that degeneracy in case of nth order delay-differential systems of neutral type with one delay can start no later than $(n - 1)h$, where h is the delay.

1. Introduction. It is an important property of delay-differential systems of the form

$$\dot{x}(t) = Ax(t) + Bx(t - h), \qquad h > 0, \quad t > 0, \tag{1}$$

(where A and B are constant $n \times n$ matrices, $n > 2$) that there exist A and B such that solutions of (1) after a certain time do not span the whole space R^n for all choices of initial functions (continuous). Such a property is called *pointwise degeneracy*. Negation of this property (i.e., if solutions of (1) span the whole space R^n for some choice of initial functions) is called *pointwise completeness*. This property was first defined by L. Weiss [8] in connection with the study of controllability of delay-differential systems. V. M. Popov and others, [3], [7], [9], including the present author, studied the above problem and obtained necessary and sufficient conditions of pointwise completeness of such systems. The results obtained by them are different in form, though they are equivalent.

In this paper, we shall obtain necessary and sufficient conditions of pointwise degeneracy for the neutral system of the form

$$\dot{x}(t) = Ax(t) + Bx(t - h) + C\dot{x}(t - h), \qquad h > 0, \quad t > 0, \tag{2}$$

where A, B and C are constant $n \times n$ matrices. Our approach in this paper is an extension of the ideas in [3] and is based on a new representation for the solution of (2) in different intervals.

It may be mentioned that boundary control of a certain hyperbolic system can be transformed to the problem of controllability of a delay-differential system of the neutral type [4], and the controllability of such a system with respect to the initial function also appears in some loss-less transmission problems [2]. We shall show that all delay-differential systems of the neutral type are pointwise complete in case rank $C = n$ or $AB = BA$, $AC = CA$ and $BC = CB$. But unlike the systems of the form (1), all second order delay-differential systems of the neutral type are not pointwise complete, as seen by the following simple example:

$$\dot{x}_1(t) = \dot{x}_2(t - 1), \tag{3}$$

$$\dot{x}_2(t) = x_1(t) - x_2(t - 1), \qquad t > 0. \tag{4}$$

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Differentiating (4) and using (3), we obtain for $t > 0$,

$$\ddot{x}_2(t) = 0.$$

Hence for $t > 0$, $x_2(t) = a + bt$, where a and b are scalar constants. Therefore for $t > 1$, it follows from (4) that

$$\begin{aligned} x_1(t) &= \dot{x}_2(t) + x_2(t-1) = b + a + b(t-1) = a + bt \\ &= x_2(t), \end{aligned}$$

which shows that the system (3)–(4) is pointwise degenerate for $t > 1$.

2. Definitions and notations. We as usual denote the real intervals $a < t < b$, $a \leqq t \leqq b$, $a \leqq t < b$ and $a < t \leqq b$ by (a, b), $[a, b]$, $[a, b)$ and $(a, b]$, respectively, and the set of all real functions $f:(t_1, t_2) \to R^n$ having continuous kth derivatives by $C^k(t_1, t_2)$. If $f \in C^k(t_1, t_2)$ and, in addition, the right-hand kth derivative of f exists at t_1 and is continuous from the right at t_1, then f is said to be of class $C^k[t_1, t_2)$. Similarly, if the kth derivative is continuous from the left at t_2, then $f \in C^k(t_1, t_2]$. If both of these conditions hold, we say that $f \in C^k[t_1, t_2]$.

Pointwise completeness. The system (2) is said to be pointwise complete at time $t_1 > 0$, if for all $y \in R^n$, there exists a $g(\cdot) \in C^1[-h, 0]$, such that

$$x(t_1; g) = y, \tag{5}$$

where $x(t; g)$ is the solution of (2) corresponding to the initial function $g(\cdot)$. The system (2) is said to be pointwise degenerate if it is not pointwise complete. If the system (2) is pointwise degenerate, then it follows that $x(t_1; g)$ does not span the whole space R^n for all choices of initial functions $g(\cdot) \in C^1[-h, 0]$, and hence there exists a nonzero n-vector d such that

$$d^T x(t_1; g) = 0 \tag{6}$$

for all $g(\cdot) \in C^1[-h, 0]$ (d^T denotes the transpose of the column vector d). There may be more than one such d satisfying (6), and in that case, the system is said to pointwise degenerate with respect to these vectors.

We introduce the matrix $G(\sigma)$ defined by

$$\begin{aligned} G(\sigma) &= (\sigma I - A)^{-1}(B + C\sigma) = \frac{p(\sigma)}{\det(\sigma I - A)} \\ &= \frac{p(\sigma)}{\Delta(\sigma)}, \end{aligned} \tag{7}$$

where $\Delta(\sigma) = \det(\sigma I - A)$ and $p(\sigma)$ is a $n \times n$ matrix each of whose elements are polynomials in σ of degree less than or equal to n.

Let D and s denote the operation of differentiation with respect to t and τ, respectively. We denote the kth derivative of the function $f(t)$ by $D^k f(t)$ and the kth derivative at the point $t = t_0$ by $D^k f(t)|_{t=t_0} \cdot s^k f(\tau)$ and $s^k f(\tau)|_{\tau=0}$ have similar meanings.

Consider the function $g(\cdot) \in C^1[-h, 0]$, and let $F(\cdot) \in C^{n+1}[-h, 0]$ satisfy the differential equation

$$\Delta(D)F(t-h) = g(t-h), \qquad 0 \leqq t \leqq h. \tag{8}$$

Associated with the matrices A, B, C and the function $g(\cdot) \in C^1[-h, 0]$, we introduce the notation $L(t; g((\cdot) - h))$ defined by

$$L(t; g((\cdot) - h)) = p(D)F(t - h), \qquad 0 \leqq t \leqq h, \tag{9}$$

where $p(D)$ and $F(\cdot)$ are defined in (7)–(8). We observe that $L((\cdot); g((\cdot) - h))$: $[0, h] \to R^n$. Since there is more than one function $F(\cdot)$ satisfying (8), it follows that $L(t, g((\cdot) - h))$, $0 \leqq t \leqq h$, is not unique. If $\phi(\cdot)$ satisfies the differential equation

$$\Delta^k(D)\phi(t - h) = g(t - h), \qquad 0 \leqq t \leqq h, \tag{10}$$

k being a positive integer, then we observe from (8)–(9) that

$$L(t; g((\cdot) - h)) = p(D)\Delta^{k-1}(D)\phi(t - h), \qquad 0 \leqq t \leqq h. \tag{11}$$

We next introduce the notation $L^2(t; g((\cdot) - 2h))$ defined in the following way:

$$L^2(t; g((\cdot) - 2h)) = L(t; g_1((\cdot) - h)), \qquad h \leqq t \leqq 2h, \tag{12}$$

where the function $g_1(\cdot) : [0, h] \to R^n$, is given by

$$g_1(t) = L(t; g((\cdot) - h)), \qquad 0 \leqq t \leqq h, \tag{13}$$

and the notation $L(t; g((\cdot) - h))$ is defined above. Combining (12)–(13), we see that

$$L^2(t; g((\cdot) - 2h)) = L(t; L((\cdot) - h; g((\cdot) - 2h))), \qquad h \leqq t \leqq 2h. \tag{14}$$

In general, we define $L^{r+1}(t; g((\cdot) - (r + 1)h))$ by the relation

$$\begin{aligned} L^{r+1}(t; g((\cdot) - (r + 1)h)) &= L(t; g_r((\cdot) - h)), \\ &\quad rh \leqq t \leqq (r + 1)h, \quad r = 1, 2, 3, \cdots, N, \end{aligned} \tag{15}$$

where N is a positive integer and $g_r(\cdot)$ is given by

$$g_r(t) = L^r(t; g((\cdot) - rh)), \qquad (r - 1)h \leqq t \leqq rh. \tag{16}$$

We observe that $L^{r+1}(\cdot\,; g((\cdot) - (r + 1)h)) : [rh, (r + 1)h] \to R^n$. If $g(\cdot) \in C^1[-h, 0]$ and

$$g(t - (r + 1)h) = \Delta^k(D)\phi(t - (r + 1)h), \qquad rh \leqq t \leqq (r + 1)h,$$

then it follows from (15) that

$$\begin{aligned} L^{r+1}(t; g((\cdot) - (r + 1)h)) &= (p(D))^{r+1}(\Delta(D))^{k-(r+1)}\phi(t - (r + 1)h), \\ &\quad rh \leqq t \leqq (r + 1)h. \end{aligned} \tag{17}$$

With obvious modification the notation $L(t; g((\cdot) - h))$ can be defined when $g(\cdot)$ is a matrix function.

3. Representation for solution. In this section, we shall obtain a new representation for solution of (2) in different intervals in terms of $L(t; g((\cdot) - h))$, $L(t; e^{A((\cdot)-h)})$, etc. This representation is different from the usual kernel representation, and the new representation will be used to obtain necessary and sufficient conditions of pointwise degeneracy of the system (2) by performing simple operations on the matrices A, B, C associated with (2).

LEMMA 3.1. *Consider the function* $g(\cdot) \in C^1[-h, 0]$. *Then* $L(t; g((\cdot) - h))$ *satisfies* (2) *in the interval* $0 < t \leqq h$, *i.e.,*

$$(DI - A)L(t; g((\cdot) - h)) = (B + CD)g(t - h), \qquad 0 < t \leqq h.$$

Proof. Using the definition of $L(t; g((\cdot) - h))$, we have for $0 / t \leqq h$,

$$\begin{aligned}(DI - A)L(t; g((\cdot) - h)) &= (DI - A)p(D)F(t - h)\\ &= (B + CD)\Delta(D)F(t - h) \quad \text{(using (7))}\\ &= (B + CD)g(t - h) \quad \text{(using (8))} \qquad 0 < t \leqq h,\end{aligned}$$

which shows that Lemma 3.1 is true.

LEMMA 3.2. *Suppose that the continuously differentiable function* $v(t)$, $t \geqq 0$, *satisfies the differential equation*

$$\dot{x}(t) = Ax(t) + Bg(t - h) + C\dot{g}(t - h), \qquad t > 0, \tag{18}$$

where $g(\cdot) \in C^1[-h, 0]$. *Then the function* $w(t)$ *defined by*

$$w(t) = v(t) + e^{At}(g(0) - v(0)), \qquad 0 \leqq t \leqq h, \tag{19}$$

is the solution of (2) *in the interval* $[0, h]$ *corresponding to the initial function* $g(\cdot) \in C^1[-h, 0]$.

Proof. We observe from (19) that

$$w(0) = v(0) + (g(0) - v(0)) = g(0)$$

and

$$\begin{aligned}\dot{w}(t) &= \dot{v}(t) + A\,e^{At}(g(0) - v(0)) \qquad (0 < t \leqq h)\\ &= Av(t) + Bg(t - h) + C\dot{g}(t - h) + A\,e^{At}(g(0) - v(0))\\ &\qquad\qquad \text{(since } v(t) \text{ satisfies (18))}\\ &= Aw(t) + Bg(t - h) + C\dot{g}(t - h).\end{aligned}$$

Hence the existence and uniqueness of the solution of (2) implies that $w(t)$ is the solution of (2) in the interval $[0, h]$ corresponding to the initial function $g(\cdot) \in C^1[-h, 0]$.

LEMMA 3.3. *Let* $g(\cdot) \in C^1[-h, 0]$; *then the solution of* (2) *in the interval* $[0, h]$ *corresponding to the initial function* $g(\cdot)$ *is given by*

$$x(t; g) = L(t; g((\cdot) - h)) - e^{At}(g(0) - L(0; g((\cdot) - h))). \tag{20}$$

Proof. Lemma 3.3 follows from Lemmas 3.1 and 3.2.

Remark. Though $L(t; g((\cdot) - h))$ is not unique, the solution of (2) corresponding to the initial function $g(\cdot)$, as given in (20), is unique.

LEMMA 3.4. *If* $g(t) = e^{A(t-h)}$, $0 \leqq t \leqq h$, *then* $L(t; g((\cdot) - h))$ *is a differentiable function of* t *in the interval* $[0, h]$.

Proof. We note that

$$L(t; e^{A((\cdot)-h)}) = p(D)F(t - h), \qquad 0 \leqq t \leqq h, \tag{21}$$

where $F(t - h)$ satisfies the equation

$$\Delta(D)F(t - h) = e^{A(t-h)}, \qquad 0 \leqq t \leqq h, \tag{22}$$

and

$$\Delta(D) = \det(DI - A);$$

$p(D)$ defined in (7) is a matrix each of whose elements are polynomials in D. From (21)–(22), it follows that $L(t; e^{A((\cdot)-h)})$ is a differentiable function of t.

THEOREM 3.1. *Let the function $\phi(\cdot) \in C^{nk+1}[-h, 0]$ and satisfy the differential equation*

$$\Delta^k(D)\phi(t-h) = g(t-h), \qquad 0 \leqq t \leqq h, \tag{23}$$

where $g(\cdot) \in C^1[-h, 0]$, $k \geqq r$ is a positive integer. Then the representation for the solution of (2) *in the interval $[(r-1)h, rh]$ corresponding to the initial function $g(\cdot)$ is given by*

$$\begin{aligned} x(t; g) = g_r(t) = (p(D))^r(\Delta(D))^{k-r}\phi(t-rh) \\ + (W_r(t, s)X_r(\tau))_{\tau=0}, \qquad (r-1)h \leqq t \leqq rh, \end{aligned} \tag{24}$$

where

$$\begin{aligned} W_r(t, s) = \sum_{j=1}^{r-1}\sum_{i=1}^{r-j} L^i(t; e^{A((\cdot)-(i+j-1)h)})Y_{j-1}(s)(\Delta(s))^{r-j} \\ + \sum_{j=0}^{r-1} e^{A(t-jh)}Y_j(s)(\Delta(s))^{r-j-1}, \qquad (r-1)h \leqq t \leqq rh, \end{aligned} \tag{25}$$

$$Y_{r-1}(s) = (p(s))^{r-1} - \sum_{j=1}^{r-1} L^{r-j}((r-1)h; e^{A((\cdot)-(r-1)h)})Y_{j-1}(s)(\Delta(s))^{r-j}, \tag{26}$$

$$Y_0(s) = I, \quad \text{the identity matrix of dimension } n, \tag{27}$$

$$X_r(\tau) = (\Delta(s))^{k-r+1}\phi(\tau) - p(s)(\Delta(s))^{k-r}\phi(\tau - h). \tag{28}$$

Proof. We shall prove Theorem 3.1 by induction. Assuming the representation to be true in the rth interval, we shall prove that it is also true in the $(r+1)$th interval.

We observe that $W_r(t, s)$ in (25) is defined in the interval $(r-1)h \leqq t \leqq rh$, and we introduce the notation $L(t; W_r((\cdot) - h, s))$, and we note that it is given by

$$\begin{aligned} L(t; W_r((\cdot) - h, s)) = \sum_{j=1}^{r-1}\sum_{i=1}^{r-j} L^{i+1}(t; e^{A((\cdot)-(i+j)h)})Y_{j-1}(s)(\Delta(s))^{r-j} \\ + \sum_{j=0}^{r-1} L(t; e^{A((\cdot)-(j+1)h)})Y_j(s)(\Delta(s))^{r-j-1}, \\ rh \leqq t \leqq (r+1)h. \end{aligned} \tag{29}$$

Combining the two terms in the right-hand side of (29) into a single summation, we obtain

$$\begin{aligned} L(t; W_r((\cdot) - h, s)) = \sum_{j=1}^{r}\sum_{i=1}^{r-j+1} L^i(t; e^{A((\cdot)-(i+j-1)h)})Y_{j-1}(s)(\Delta(s))^{r-j}, \\ rh \leqq t \leqq (r+1)h. \end{aligned} \tag{30}$$

From (25), (30) it follows that

$$
\begin{aligned}
&[(W_r(t,s) - L(t; W_r((\cdot) - h, s)))X_r(\tau)]_{t=rh} \\
&= \Bigg[\Bigg\{\sum_{j=1}^{r-1}\sum_{i=1}^{r-j} L^i(t; e^{A((\cdot)-(i+j-1)h)})Y_{j-1}(s)(\Delta(s))^{r-j} \\
&\quad + \sum_{j=0}^{r-1} e^{A(t-jh)}Y_j(s)(\Delta(s))^{r-j-1} \\
&\quad - \sum_{j=1}^{r}\sum_{i=1}^{r-j+1} L^i(t; e^{A((\cdot)-(i+j-1)h)})Y_{j-1}(s)(\Delta(s))^{r-j}\Bigg\}X_r(\tau)\Bigg]_{t=rh} \qquad (31)\\
&= \Bigg[\Bigg(\sum_{j=0}^{r-1} e^{A(r-j)h}Y_j(s)(\Delta(s))^{r-j-1} \\
&\quad - \sum_{j=1}^{r} L^{r+1-j}(rh; e^{A((\cdot)-rh)})Y_{j-1}(s)(\Delta(s))^{r-j}\Bigg)X_r(\tau)\Bigg]_{t=rh}.
\end{aligned}
$$

Changing the independent variable $(t - rh)$ to τ, we observe that

$$
\begin{aligned}
&[(p(D))^r(\Delta(D))^{k-r}\phi(t - rh) - (p(D))^{r+1}(\Delta(D))^{k-(r+1)}\phi(t - (r + 1)h)]_{t=rh} \\
&= (p(s))^r((\Delta(s))^{k-r}\phi(\tau) - p(s)(\Delta(s))^{k-(r+1)}\phi(\tau - h))_{\tau=0} \qquad (32)\\
&= ((p(s))^r X_{r+1}(\tau))_{\tau=0}.
\end{aligned}
$$

Using Lemma 3.3, we note that the solution of (2) in the interval $[rh, (r + 1)h]$ corresponding to the initial function $g(\cdot) \in C^1[-h, 0]$ is given by

$$
x(t; g) = g_{r+1}(t) = L(t; g_r((\cdot) - h)) + e^{A(t-rh)}(g_r(h) - L(h; g_r((\cdot) - h))), \quad rh \leqq t \leqq (r + 1)h, \qquad (33)
$$

where $g_r(\cdot)$ is the solution of (2) in the interval $[(r - 1)h, rh]$ corresponding to the initial function $g(\cdot) \in C^1[-h, 0]$. Let

$$
z_1(t) \triangleq L(t; g_r((\cdot) - h)), \qquad rh \leqq t \leqq (r + 1)h, \qquad (34)
$$

$$
z_2(t) \triangleq e^{A(t-rh)}[g_r(h) - L(h; g_r((\cdot) - h))], \qquad rh \leqq t \leqq (r + 1)h. \qquad (35)
$$

Noting (30), we observe that

$$
z_1(t) = (p(D))^{r+1}(\Delta(D))^{k-r-1}\phi(t - (r + 1)h) + (L(t; W_r((\cdot) - h, s))X_r(\tau))_{\tau=0} \qquad (36)
$$

$$
\begin{aligned}
&= (p(D))^{r+1}(\Delta(D))^{k-r-1}\phi(t - (r + 1)h) \\
&\quad + \Bigg[\sum_{j=1}^{r}\sum_{i=1}^{r-j+1}(L^i(t; e^{A((\cdot)-(i+j-1)h)})Y_{j-1}(s)(\Delta(s))^{r-j+1})X_{r+1}(\tau)\Bigg]_{\tau=0}. \qquad (37)
\end{aligned}
$$

From (24), (35) and (36), it follows that

$$
\begin{aligned}
z_2(t) &= e^{A(t-rh)}[p^r(D)(\Delta(D))^{k-r}\phi(t - rh) - p^{r+1}(D)(\Delta(D))^{k-r-1}\phi(t - (r + 1)h) \\
&\quad + \{(L(t; W_r((\cdot) - h, s)) - W_r(t, s))X_r(\tau)\}_{\tau=0}]_{t=rh}. \qquad (38)
\end{aligned}
$$

Using (31)–(32) and replacing $\Delta(s)X_r(\tau)$ by $X_{r+1}(\tau)$, we find that

$$(39)\quad z_2(t) = e^{A(t-rh)}\left[\left\{\left(p^r(s) - \sum_{j=1}^{r} L^{r+1-j}(rh; e^{A((\cdot)-rh)})Y_{j-1}(s)(\Delta(s))^{r+1-j}\right) + \sum_{j=0}^{r-1} e^{A(r-j)h}Y_j(s)(\Delta(s))^{r-j}\right\}X_{r+1}(\tau)\right]_{\tau=0}$$

$$(40)\quad = \left(\sum_{j=0}^{r} e^{A(t-jh)}Y_j(s)(\Delta(s))^{r-j}X_{r+1}(\tau)\right)_{\tau=0},$$

where $Y_j(s)$ and $X_{r+1}(\tau)$ are defined in (26)–(28). It follows from (33)–(35), (37) and (40) that the representation for the solution of (2) in the $(r+1)$th interval corresponding to the initial function $g(\cdot) \in C^1[-h, 0]$ is given by

$$(41)\quad \begin{aligned} x(t; g) = g_{r+1}(t) &= z_1(t) + z_2(t) \\ &= p^{r+1}(D)(\Delta(D))^{k-r-1}\phi(t-(r+1)h) \\ &\quad + \left[\left(\sum_{j=1}^{r}\sum_{i=1}^{r+1-j} L^i(t; e^{A((\cdot)-(i+j-1)h)})Y_{j-1}(s)(\Delta(s))^{r+1-j} + \sum_{j=0}^{r} e^{A(t-jh)}Y_j(s)(\Delta(s))^{r-j}\right)X_{r+1}(\tau)\right]_{\tau=0} \\ &= p^{r+1}(D)(\Delta(D))^{k-r-1}\phi(t-(r+1)h) + (W_{r+1}(t,s)X_{r+1}(\tau))_{\tau=0}, \end{aligned}$$

which shows that Theorem 3.1 is true in the $(r+1)$th interval, if it is true in the rth interval. Now, in the first interval, the representation for the solution of (2) is given by (using Lemma 3.3)

$$(42)\quad \begin{aligned} x(t; g) = g_1(t) &= p(D)(\Delta(D))^{k-1}\phi(t-h) \\ &\quad + [e^{At}(\Delta^k(s)\phi(\tau) - p(s)\Delta^{k-1}(s)\phi(\tau - h))]_{\tau=0} \\ &= p(D)\Delta^{k-1}(D)\phi(t-h) + (e^{At}X_1(\tau))_{\tau=0}, \end{aligned}$$

which shows that Theorem 3.1 is true in the first interval, and therefore by induction, the representation for solution of (2) is true in any interval.

Remark. Though $L(t; e^{A((\cdot)-h)})$, $L^r(t; e^{A(t-rh)})$, etc., are not unique, the representation for the solution of (2) as given by (24)–(28) is unique. This follows from the remark following Lemma 3.3.

LEMMA 3.5. *The function $W_r(t, s)$ defined in* (25) *can be expressed in the form*

$$(43)\quad W_r(t,s) = W_{r0}(t) + W_{r1}(t)s + W_{r2}(t)s^2 + \cdots + W_{r(r-1)n}s^{(r-1)n}, \quad (r-1)h \leqq t \leqq rh,$$

where $W_{r0}(t)$, $W_{r1}(t)$, etc., *are differentiable functions in the interval* $[(r-1)h, rh]$.

Proof. We observe that $L^i(t; e^{A((\cdot)-ih)})$, $i = 1, 2, 3, \cdots, r$, are differentiable functions in the interval $[(r-1)h, rh]$. This follows from Lemma 3.4, and Lemma 3.5 is proved by noting that $Y_{r-1}(s)$ is a polynomial in s of degree $n(r-1)$ and from (25)–(26).

4. Necessary and sufficient conditions of pointwise completeness. In this section, we shall obtain necessary and sufficient conditions of pointwise completeness of the system (2) and some related properties.

LEMMA 4.1. *If the system* (2) *is pointwise degenerate at time* $t_1 > 0$, *then the system is pointwise degenerate for all* $t \geqq t_1$.

Proof. The lemma follows from the fact that if $x(t; g)$, $t > 0$ is the solution of (2), then the shifted function $x(t + \tau; g(\cdot))$, $\tau > 0$, is also a solution of (2) and the definition of pointwise degeneracy.

In the next theorem, we shall prove that if the system (2) is pointwise degenerate, then degeneracy can start no later than $(n - 1)h$.

THEOREM 4.1. *Suppose that the system* (2) *is pointwise degenerate at* $t = t_1$ $(t_1 > (n - 1)h)$ *with respect to the nonzero n-vector d. Then the system* (2) *is also pointwise degenerate with respect to d for all* $t \geqq (n - 1)h$.

Proof. Suppose that $t_1 \in [(r - 1)h, rh]$, $r > n$. Then by Lemma 4.1, the system (2) is pointwise degenerate for all $t \geqq t_1$. Let us choose the function $\phi(\cdot)$ in (23) such that

$$D^j\phi(0) = D^j\phi(-h) = 0, \qquad j = 0, 1, 2, \cdots, nk. \tag{44}$$

It follows from (25) and (43) that $d^T W_r(t, s)X_r(\tau)$ can be expressed as

$$\begin{aligned} d^T W_r(t, s)X_r(\tau) = {} & [(a_0(t) + a_1(t)s + \cdots + a_{nk}(t)s^{nk})\phi(\tau)] \\ & - [(b_0(t) + b_1(t)s + \cdots + b_{nk}(t)s^{nk})\phi(\tau - h)], \end{aligned} \tag{45}$$

where $a_0(t), a_1(t), \cdots, a_{nk}(t); b_0(t), b_1(t), \cdots, b_{nk}(t)$ are differentiable n-row vectors. Hence from (44)–(45), it follows that

$$(d^T W_r(t, s)X_r(\tau))_{\tau=0} = 0. \tag{46}$$

Since the system (2) is pointwise degenerate for all $t \geqq t_1$, it follows from (46), (24) that

$$d^T x(t; g) = d^T p^r(D)(\Delta(D))^{k-r}\phi(t - rh) = 0, \tag{47}$$

$$t \geqq t_1, \quad t_1 \in [(r - 1)h, rh], \quad r > n,$$

where $p(D)$ is defined in (7). Equation (47) can be written in the form

$$(S_0^T + S_1^T D + S_2^T D^2 + \cdots + S_{nk}^T D^{nk})\phi(t - rh) = 0, \tag{48}$$

$$t \geqq t_1, \quad t_1 \in [(r - 1)h, rh],$$

where $S_0^T, S_1^T, \cdots, S_{nk}^T$ are $(nk + 1)$-row vectors. Let us choose $\phi(\cdot)$ such that

$$D^j\phi(t_0) = S_j, \qquad t_1 < t_0 \leqq rh, \quad j = 0, 1, 2, 3, \cdots, nk.$$

Hence (48) reduces to

$$(S_0^T S_0 + S_1^T S_1 + S_2^T S_2 + \cdots + S_{nk}^T S_{nk}) = 0. \tag{49}$$

It follows from (49) that

$$S_i = 0, \qquad i = 1, 2, 3, \cdots, nk,$$

and hence

$$d^T p^r(D) = 0, \qquad r > n. \tag{50}$$

Now applying the Cayley–Hamilton theorem, it can be easily shown that

$$p^r(D) = 0, \qquad r > n,$$

implies that

$$p^n(D) = 0. \tag{51}$$

Hence from (50)–(51), it follows that

$$d^T p^n(D) = 0. \tag{52}$$

Using (7), equation (52) can be expressed as

$$d^T G^n(D) = 0, \tag{53}$$

where $G(\sigma)$ is defined in (7) and satisfies the relation

$$\sigma G(\sigma) = AG(\sigma) + B + C\sigma. \tag{54}$$

Multiplying (54) from the right by $G(\sigma)$ successively and replacing $\sigma G(\sigma)$ by $AG(\sigma) + B + C\sigma$, we have the following relation:

$$\sigma G^n(\sigma) = \sum_{i=0}^{n-1} p_i^1 G^{n-i}(\sigma) + q_{n-1}^1 + r_n^1 \sigma, \qquad n = 1, 2, 3, \cdots, N, \tag{55}$$

where N is a positive integer and

$$\begin{aligned} &p_0^1 = A, \quad p_1^1 = B + CA, \quad p_i^1 = C^{i-1} p_1^1, \qquad i = 2, 3, 4, \cdots, n-1, \quad n \geqq 2, \\ &q_{n-1}^1 = C^{n-1} B, \quad r_n^1 = C^n, \qquad n = 1, 2, 3, \cdots, N. \end{aligned} \tag{56}$$

Multiplying (55) successively by σ and replacing $\sigma G^i(\sigma)$, $i = 1, 2, 3, \cdots, n$, by the corresponding expressions like the right-hand side of (55), we obtain, after simplification, the following expression for $\sigma^j G^n(\sigma)$:

$$\sigma^j G^n(\sigma) = \sum_{i=0}^{n-1} p_i^j G^{n-i}(\sigma) + q_{n-1}^j + \sum_{i=0}^{j-1} r_{n+i}^j \sigma^{i+1}, \tag{57}$$

where

$$p_i^j = \sum_{k=0}^{i} (p_k^{j-1})(p_{i-k}^1), \qquad i = 0, 1, 2, 3, \cdots, n-1, \tag{58}$$

$$q_{n-1}^j = \sum_{i=0}^{n-1} (p_i^{j-1})(q_{n-1-i}^1), \tag{59}$$

$$r_n^j = \sum_{i=0}^{n-1} (p_i^{j-1})(r_{n-i}^1) + q_{n-1}^{j-1}, \tag{60}$$

$$r_{n+k}^j = r_{n+k-1}^{j-1}, \tag{61}$$

$$k = 1, 2, 3, \cdots, j-1, \quad j = 2, 3, 4, \cdots, N.$$

Noting that $G(\sigma)/\sigma \to 0$ as $\sigma \to \infty$, it follows from (53), (57), after taking the limit as $\sigma \to \infty$, that

$$d^T r_{n+i}^j = 0, \qquad i = 0, 1, 2, 3, 4, \cdots, j-1, \quad j = 1, 2, 3, \cdots, j-1, \tag{62}$$

$$j = 1, 2, 3, \cdots, N.$$

We observe that for $t > (n-1)h$, (2) can be expressed as

$$\dot{x}(t) = \sum_{i=0}^{n-1} p_i^1 x(t-ih) + q_{n-1}^1 x(t-nh) + r_n^1 \dot{x}(t-h), \tag{63}$$

where p_i^1, q_{n-1}^1, r_n^1, etc., are defined in (56). Equation (63) is obtained from (2) by successively replacing $\dot{x}(t-h), \cdots, \dot{x}(t-(n-1)h)$ in the right-hand side of (2) by $Ax(t-h) + Bx(t-2h) + C\dot{x}(t-2h), \cdots, Ax(t-(n-1)h) + Bx(t-nh) + C\dot{x}(t-nh)$, respectively. Differentiating both sides of (63) and replacing $\dot{x}(t), \dot{x}(t-h), \cdots, \dot{x}(t-(n-1)h)$ by the corresponding expressions like the right-hand side of (63), we obtain, after simplification,

$$\ddot{x}(t) = \sum_{i=0}^{n-1} p_i^2 x(t-ih) + q_{n-1}^2 x(t-nh) + r_n^2 \dot{x}(t-nh) + r_{n+1}^2 \ddot{x}(t-nh), \tag{64}$$

$$t > (n-1)h,$$

where p_0^2, p_1^2 etc., are defined in (57)–(61). Differentiating (64) successively and following the above procedure, we can express $x^j(t)$, the jth derivative of $x(t)$, as follows:

$$x^{(j)}(t) = \sum_{i=0}^{n-1} p_i^j x(t-ih) + q_{n-1}^j x(t-nh) + \sum_{i=1}^{j} r_{n+i-1}^j x^i(t-nh), \tag{65}$$

where p_0^j, p_1^j etc., are defined in (57)–(61).

Let $v(t) = d^T x(t)$. Then it follows from (62)–(65) that $v(t)$ and its derivatives satisfy the following equations:

$$v(t) = d^T x(t), \tag{66}$$

$$v^{(j)}(t) = \sum_{i=0}^{n-1} d^T p_i^j x(t-ih) + d^T q_{n-1}^j x(t-nh), \tag{67}$$

$j = 1, 2, 3, \cdots, N$, where N is a positive integer. Let

$$V^T(t) = (v(t), \dot{v}(t), v^{(2)}(t), \cdots, v^{(N)}(t)) \tag{68}$$

and

$$X^T(t) = (x^T(t), x^T(t-h), x^T(t-2h), \cdots, x^T(t-nh)). \tag{69}$$

Then (66)–(69) can be expressed in the form

$$V(t) = MX(t), \qquad t > (n-1)h, \tag{70}$$

where M is a $(N+1) \times (n+1)$ matrix. For $N > n$, there exists a nonzero $(N+1)$-vector $c^T = (c_1, c_2, c_3, \cdots, c_{N+1})$ such that

$$c_1 v(t) + c_2 \dot{v}(t) + \cdots + c_{N+1} v^{(N)}(t) = c^T MX(t) = 0, \tag{71}$$

which shows that $v(t)$ satisfies the ordinary differential equation (71) for $t > (n-1)h$. But $v(t)$ is identically equal to zero for $t > rh$ ($r > n$, since the system (2) is pointwise degenerate in the rth interval). Hence $v(t)$ must also vanish for all t greater than or equal to $(n-1)h$, and hence the theorem is proved.

We shall now state and prove the main theorem of this paper.

THEOREM 4.2. *The necessary and sufficient condition that the system* (2) *be pointwise degenerate at* t_1 ($t_1 \leqq (n-1)h$, $t_1 \in ((r-1)h, rh)$) *is that there exists a nonzero n-vector d and a positive integer* $r \leqq n$ *such that*

$$d^T p^r(D) = 0, \tag{72}$$

$$d^T W_r(t,s) = 0, \tag{73}$$

where $p(D)$ *and* $W_r(t,s)$ *are defined in* (7) *and* (25).

Proof. Necessity. Suppose that the system (2) is pointwise degenerate at time $t_1 \in [(r-1)h, rh)$. Then it follows (using Lemma 4.1 and Theorem 4.1) that $r \leqq n$ and the degeneracy set is $[t_1, \infty)$. Hence

$$d^T x(t; g) = d^T p^r(D)(\Delta(D))^{k-r}\phi(t-rh) + d^T(W_r(t,s)X_r(\tau))_{\tau=0} = 0 \tag{74}$$

for all $g(\cdot) \in C^1[-h, 0]$.

We choose $\phi(\cdot)$ such that

$$D^j\phi(0) = D^j\phi(-h) = 0, \qquad j = 0, 1, 2, 3, \cdots, nk.$$

Then it can be shown, as in Theorem 4.1, that

$$d^T p^r(D) = 0. \tag{75}$$

We now proceed to show that

$$d^T W_r(t,s) = 0, \qquad (r-1)h \leqq t \leqq rh, \tag{76}$$

is also a necessary condition for pointwise degeneracy of the system (2). Since the system (2) is pointwise degenerate and $d^T p^r(D) = 0, r \leqq n$, it follows from Theorem 4.1, that the system (2) is pointwise degenerate in the interval $[(r-1)h, rh]$. Suppose that $d^T W_r(t,s)$ does not vanish identically in the interval $(r-1)h \leqq t \leqq rh$. Then it follows from (43) that not all of $d^T W_{r0}(t)$, $d^T W_{r1}(t)$, etc., vanish identically in the interval $(r-1)h \leqq t \leqq rh$. Let $W_{rj}(t)$, $0 \leqq j \leqq (r-1)n$, be the first element in (43) such that

$$d^T W_{rj}(t) \neq 0, \qquad (r-1)h \leqq t \leqq rh.$$

Since (74) is valid for all $g(\cdot) \in C^1[-h, 0]$, we observe that we can choose $\phi(\cdot)$ appearing in (23) such that

$$(s^j X_r(\tau))_{\tau=0} = a, \quad \text{where } a \text{ is a nonzero constant}$$

and

$$(s^{j+i} X_r(\tau))_{\tau=0} = 0, \qquad i = 1, 2, 3, \cdots, (r-1)n - j.$$

Hence for such a $\phi(\cdot)$, we have from (74) and (43) that

$$0 = (d^T W_{rj}(t) s^j X_r(\tau))_{\tau=0} = d^T W_{rj}(t) a \neq 0, \qquad (r-1)h \leqq t \leqq rh, \tag{77}$$

which is a contradiction and therefore it follows that $d^T W_{rj}(t)$ vanish identically in the interval $[(r-1)h, rh]$. Proceeding similarly, we have

$$d^T W_{rj}(t) = 0, \qquad j = 0, 1, 2, 3, \cdots, (r-1)n, \quad (r-1)h \leqq t \leqq rh, \tag{78}$$

and hence it follows from (78) and (43) that

$$d^T W_r(t, s) = 0, \qquad (r-1)h \leqq t \leqq rh. \tag{79}$$

Sufficiency. Since $d^T p^r(D) = 0$ and $d^T W_r(t, s) = 0$, we have, from (24),

$$d^T x(t; g) = d^T p^r(D)(\Delta(D))^{k-r}\phi(t - rh) + d^T W_r(t, s)X_r(\tau)|_{\tau=0} = 0,$$
$$(r-1)h \leqq t \leqq rh,$$

which shows that the system (2) is pointwise degenerate in the interval $[(r-1)h, rh]$ and hence in all subsequent intervals.

COROLLARY 1. *If rank $C = n$, then the system* (2) *is pointwise complete.*

Proof. Suppose that the system (2) is not pointwise complete. Then there exists a nonzero n-vector d and a positive integer $r \leqq n$ such that

$$d^T p^r(D) = 0, \tag{80}$$

where $p(D)$ is defined in (7) and satisfies the relation

$$\sigma p(\sigma) = Ap(\sigma) + (B + C\sigma)\Delta(\sigma). \tag{81}$$

Using (81), equation (80) can be expressed in the form

$$d^T\left(\frac{Ap(\sigma)}{\sigma\Delta(\sigma)} + \frac{B}{\sigma} + C\right)^r = 0.$$

Making $\sigma \to \infty$, we conclude that

$$d^T C^r = 0, \tag{82}$$

which shows that the rank of C is less than n, contradicting the hypothesis. Hence the system (2) is pointwise complete and the corollary is proved.

COROLLARY 2. *If $AB = BA$, $AC = CA$ and $BC = CB$, then the system* (2) *is pointwise complete.*

Proof. Part I. In this part we show that if $AB = BA$, then the system (1) is pointwise complete.

Case (a). Let us assume that the pair (A, B) is completely controllable and the system (1) is not pointwise complete. Hence there exists a nonzero n-vector d such that

$$d^T(p'(D))^r = 0, \tag{83}$$

where

$$\frac{p'(\sigma)}{\Delta(\sigma)} = (\sigma I - A)^{-1}B,$$

or

$$\sigma p'(\sigma) = Ap'(\sigma) + B\Delta(\sigma). \tag{84}$$

Hence from (83)–(84), it follows that

$$d^T\left(\frac{Ap'(\sigma)}{\Delta(\sigma)} + B\right)^r = 0. \tag{85}$$

Noting that $(p'(\sigma))/\Delta(\sigma) \to 0$ as $\sigma \to \infty$ and taking the limit as $\sigma \to \infty$, we obtain

$$d^T B^r = 0, \qquad r \leqq n.$$

This shows that the rank of B is less than n, and hence there exists a nonzero n-vector q such that

$$q^T B = 0.$$

Since $AB = BA$, it follows that

$$q^T[B, AB, A^2B, A^3B, \cdots, A^{n-1}B] = 0,$$

which contradicts the fact that the pair (A, B) is completely controllable and, hence the system (1) is pointwise complete.

Case (b). Suppose that the pair (A, B) is not completely controllable. Then there exists a nonsingular linear transformation T such that the relation $x(t) = Ty(t)$ transforms (1) into the form

$$\dot{y}_1(t) = A_{11}y_1(t) + B_{11}y_1(t-h) + A_{12}y_2(t) + B_{12}(t-h), \tag{86}$$

$$\dot{y}_2(t) = A_{22}y_2(t), \tag{87}$$

where the dimensions of $y_1(t)$ and $y_2(t)$ are m ($m \leqq n$) and $(n-m)$, respectively, and the pair (A_{11}, B_{11}) is completely controllable. Since $AB = BA$, it follows that

$$TAT^{-1}TBT^{-1} = TBT^{-1}TAT^{-1}$$

or

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}. \tag{88}$$

It follows from (88) that

$$A_{11}B_{11} = B_{11}A_{11}.$$

We observe that the system ((86)–(87)) is pointwise complete if the system

$$\dot{y}_1(t) = A_{11}y_1(t) + B_{11}y_1(t-h) \tag{89}$$

is pointwise complete. But since $A_{11}B_{11} = B_{11}A_{11}$ and the pair (A_{11}, B_{11}) is completely controllable, it follows from Case (a) that the system (89) is pointwise complete. Hence the system ((86)–(87)) is pointwise complete and therefore, also the system (1). Combining Cases (a) and (b), we conclude that the system (1) is pointwise complete if $AB = BA$.

Part II. We now proceed to prove Corollary 2; i.e., if $AB = BA$, $AC = CA$ and $BC = CB$, then the system (2) is pointwise complete. We introduce the following sequence of matrices:

$$\begin{gathered} P_1^1 = C, \quad P_{2l-1}^{r+1} = AP_l^r, \quad P_2^{r+1} = BP_l^r, \\ l = 1, 2, 3, \cdots, 2^k, \quad k = 1, 2, 3, \cdots, n, \quad r = 0, 1, 2, \cdots, n-1. \end{gathered} \tag{90}$$

Consider the matrix

$$Q = [P_1^1; P_1^2, P_2^2; P_1^3, P_2^3, P_3^3, P_4^3; \cdots, P_1^n, P_2^n, P_3^n, \cdots, P_{2^n-1}^n].$$

These matrices appear in [6] in a different context. Let N_0 be the rank of the matrix Q.

Case (a). Let $N_0 = n$, and suppose that the system (2) is not pointwise complete. By Corollary 1, the rank of C is less than n, and therefore there exists a nonzero n-vector q such that

$$q^T C = 0. \tag{91}$$

Since $AC = CA$, $BC = CB$, we note that

$$\begin{aligned} &q^T P_1^1 = 0, \quad q^T p_{2l-1}^{r+1} = 0, \quad q^T p_{2l}^{r+1} = 0, \\ &l = 1, 2, 3, \cdots, n, \quad k = 1, 2, 3, \cdots, n, \quad r = 0, 1, 2, 3, \cdots, (n-1), \end{aligned} \tag{92}$$

which contradicts that $N_0 = n$, and hence the system (2) is pointwise complete.

Case (b). Suppose that $N_0 = m < n$. In this case, there exists a nonsingular linear transformation T [6] such that the relation

$$x(t) = Ty(t)$$

transforms (2) into the form

$$\dot{y}(t) = A_1 y(t) + B_1 y(t-h) + C_1 \dot{y}(t-h), \tag{93}$$

where

$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad P_1 = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}, \quad C_1 = \begin{bmatrix} C_{11} & C_{12} \\ 0 & 0 \end{bmatrix}.$$

The dimensions of the matrices A_{11}, B_{11}, C_{11}; A_{22}, B_{22}; A_{12}, B_{12}, C_{12} are $m \times m$, $(n-m) \times (n-m)$ and $m \times (n-m)$, respectively. It follows, as in (88) of Case (b), Part I, that

$$A_{11}B_{11} = B_{11}A_{11}, \quad A_{11}C_{11} = C_{11}A_{11}, \quad B_{11}C_{11} = C_{11}B_{11}.$$

As in (90), we introduce the sequence of matrices

$$\begin{aligned} &p_1^1 = C_{11}, \quad p_{2l-1}^{r+1} = A_{11}p_l^{r+1}, \quad p_{2l}^{r+1} = B_{11}p_l^r, \\ &l = 1, 2, 3, \cdots, 2^k, \quad k = 1, 2, 3, \cdots, m, \quad r = 0, 1, 2, 3, \cdots, (m-1), \end{aligned}$$

and the matrix

$$Q^1 = [p_1^1; p_1^2, p_2^2; p_1^3, p_2^3, p_3^3, p_4^3; \cdots; p_1^m, p_2^m, p_3^m, \cdots, p_{2^m-1}^m]. \tag{94}$$

It follows by virtue of the linear transformation T that the rank of the matrix Q^1 in (94) is equal to m. We note that (93) can be written in the form

$$\begin{aligned} \dot{y}_1(t) = {} & A_{11}y_1(t) + B_{11}y_1(t-h) + C_{11}\dot{y}_1(t-h) \\ & + A_{12}y_2(t) + B_{12}y_2(t-h) + C_{12}\dot{y}_2(t-h), \end{aligned} \tag{95}$$

$$\dot{y}_2(t) = A_{22}y_2(t) + B_{22}y_2(t-h). \tag{96}$$

We observe that (93) is pointwise complete if the systems

$$\begin{aligned} \dot{y}_1(t) &= A_{11}y_1(t) + B_{11}y_1(t-h) + C_{11}\dot{y}_1(t-h), \\ \dot{y}_2(t) &= A_{22}y_2(t) + B_{22}y_2(t-h) \end{aligned} \tag{97}$$

are pointwise complete. Now since $A_{22}B_{22} = B_{22}A_{22}$, by Part I, the system (96) is pointwise complete. Again, since $A_{11}B_{11} = B_{11}A_{11}$, $A_{11}C_{11} = C_{11}A_{11}$, $B_{11}C_{11} = C_{11}B_{11}$ and the rank of the matrix Q^1 is m, it follows from Case (a) of Part II that the system (97) is pointwise complete. Hence the system (93) is pointwise complete and therefore, also the system (2).

Example 1. Consider the system given by

$$\dot{x}_1(t) = x_2(t) - \frac{e}{1-e}x_1(t-1), \tag{98}$$

$$\dot{x}_2(t) = x_2(t) - \frac{e}{1-e}x_1(t-1) + \frac{e}{1-e}\dot{x}_1(t-1). \tag{99}$$

In this case,

$$p(s) = \begin{bmatrix} 0 & 0 \\ \dfrac{-es(s-1)}{(1-e)} & 0 \end{bmatrix}$$

$$L(t; e^{A((\cdot)-1)}) = \begin{bmatrix} 0 & 0 \\ \dfrac{-e}{(1-e)} & \dfrac{-e(e^t-1)}{(1-e)} \end{bmatrix},$$

$$W_2(t, s) = \begin{bmatrix} 1 & e^t - 1 \\ \dfrac{-e}{(1-e)} & \dfrac{e(1-e^t)}{(1-e)} \end{bmatrix},$$

which shows that there exists a nonzero 2-dimensional vector $d^T = (e/(1-e), 1)$ such that

$$d^T p^2(D) = 0, \qquad d^T W_2(t, s) = 0,$$

and therefore the system (98) is pointwise degenerate.

Example 2. Consider the system given by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) - 3x_3(t) + 2x_1(t-1) - 3x_2(t-1) + 3x_3(t-1) \\ &\quad + \dot{x}_1(t-1) - \dot{x}_2(t-1) + \dot{x}_3(t-1), \end{aligned} \tag{100}$$

$$\begin{aligned} \dot{x}_2(t) &= -2x_3(t) + 2x_1(t-1) - 2x_2(t-1) + 2x_3(t-1) \\ &\quad + \dot{x}_1(t-1) - \dot{x}_2(t-1) + \dot{x}_3(t-1), \end{aligned} \tag{101}$$

$$\dot{x}_3(t) = x_2(t-1) - x_3(t-1). \tag{102}$$

In this case,

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 & 3 \\ 2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$e^{At} = \begin{bmatrix} 1 & t & -3t - t^2 \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{bmatrix},$$

$$p(s) = \begin{bmatrix} (s^2 + s)(2 + s) & -s^3 - 4s^2 - 5s - 2 & s^3 + 4s^2 + 5s + 2 \\ s^2(2 + s) & -s^3 - 2s^2 - 2s & s^3 + 2s^2 + 2s \\ 0 & s^2 & -s^2 \end{bmatrix},$$

$$L(t; e^{A((\cdot)-1)}) = \begin{bmatrix} t^2 + t - 1 & -(t^2 + t - 1) & t^2 + t - 1 \\ 2t - 1 & -t & -t^2 + 3t - 1 \\ 0 & (t - 1) & -t^2 + t \end{bmatrix},$$

$$L(t; e^{A((\cdot)-2)}) = \begin{bmatrix} t^2 - t - 1 & -(t^2 - t - 1) & -(t^2 - t - 1) \\ 2t - 3 & -(t - 1) & -(t^2 - t - 1) \\ 0 & (t - 2) & -t^2 + 3t - 2 \end{bmatrix},$$

$$L^2(t; e^{A((\cdot)-2)}) = \begin{bmatrix} 0 & 0 & 0 \\ t^2 - 3t + 2 & -(t^2 - 3t + 2) & -(t^2 - 3t + 2) \\ t^2 - 3t + 2 & -(t^2 - 3t + 2) & -(t^2 - 3t + 2) \end{bmatrix},$$

$$W_3(t, s) = \begin{bmatrix} \begin{matrix} s^6(t^2 + t) \\ \quad + s^5(2t + 1) + 2s^4 \end{matrix} & \begin{matrix} s^6(-t^2 + 1) + s^5(-t^2 - 3t) \\ \quad + s^4(-2t - 3) - 2s^3 \end{matrix} & \begin{matrix} s^6(-2t - 1) + s^5(t^2 - t - 2) \\ \quad + s^4(-2t + 1) + 2s^3 \end{matrix} \\ \begin{matrix} s^6(t^2 - t + 1) \\ \quad + s^5(2t - 1) + 2s^4 \end{matrix} & \begin{matrix} s^6(-t^2 + 2t - 1) + s^5(-t^2 - t + 1) \\ \quad + s^4(-2t - 1) - 2s^3 \end{matrix} & \begin{matrix} s^6(-2t + 1) + s^5(t^2 - 3t + 1) \\ \quad + s^4(-2t + 3) + 2s^3 \end{matrix} \\ \begin{matrix} s^6(t^2 - 3t + 2) \\ \quad + s^5(2t - 3) + 2s^4 \end{matrix} & \begin{matrix} s^6(-t^2 + 4t - 3) + s^5(-t^2 + t + 2) \\ \quad + s^4(-2t + 1) - 2s^3 \end{matrix} & \begin{matrix} s^6(-2t + 3) + s^5(t^2 - 5t + 4) \\ \quad + s^4(-2t + 5) + 2s^3 \end{matrix} \end{bmatrix}$$

and

$$p^3(D) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence we see that there exists a nonzero 3-dimensional vector $d^T = (1, -2, 1)$ such that $d^T W_3(t, s) = 0$, $d^T p^3(D) = 0$, and therefore the system (100)–(102) is pointwise degenerate.

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ON THE CARDINAL SPLINE INTERPOLANT TO e^{iut} *

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Abstract. The cardinal spline interpolant $S_{n,u}$ of degree n to $\exp(iut)$ is shown to satisfy $|S_{n,u}(t)| < 1$ unless t is an interpolation point. Also, it is shown that $1 < C_n < 1 + 2^{1-n}$ for all odd n and $C_n = 1$ for all positive even n, with $C_n := \sup_{u,t} |\exp(iut) - S_{n,u}(t)|/|u/\pi|^{n+1}$.

1. Introduction. For k a positive integer, let Q_k be Schoenberg's *forward* B-*spline of order* k (see Lecture 2 of [2]), i.e., for each $t \in \mathbb{R}$, $Q_k(t)$ is the kth divided difference at $0, 1, \cdots, k$ of $k(s-t)_+^{k-1}$ as a function of s,

$$Q_k(t) := k[0, 1, \cdots, k](\cdot - t)_+^{k-1}.$$

Set

$$\mathbb{S}_{k,\mathbb{Z}} := \{\sum \alpha_j Q_k(\cdot - j) \mid \alpha_j \in \mathbb{C}, \text{all } j\}, \tag{1}$$

the linear space of all splines of order k with simple knots at the integers. The *cardinal spline interpolant of order* k for a given bounded function f is, by definition, the unique bounded element $s \in \mathbb{S}_{k,\mathbb{Z}}$ which agrees with f at the points $j + k/2$, all $j \in \mathbb{Z}$. Schoenberg [2] has analyzed this interpolation process in some detail.

This note is intended to amend Schoenberg's results concerning the cardinal spline interpolant $S_{n,u}$ of order $n + 1$ to the function

$$f(t) = \overline{e^{iut}}$$

with $u \in [-\pi, \pi]$, as contained in [2; Lecture 3, § 6]. Specifically, it is proved that, *for all* $n \in \mathbb{N}$,

$$|S_{n,u}(t)| \leqq 1$$

with equality if and only if t is an interpolation point. Further, the number

$$C_n := \sup_{u,t} |e^{iut} - S_{n,u}(t)|/|u/\pi|^{n+1}$$

is shown to satisfy

$$1 < C_n < 1 + 2^{1-n} \quad \textit{for all odd } n,$$

thus disproving Schoenberg's conjecture [2; p. 30] that $C_n = 1$ for all odd $n > 1$. It is also shown that

$$C_n = 1 \quad \textit{for all positive even } n.$$

These results are of interest in the numerical construction of Fourier series and transforms (see, e.g., Lecture 10 of [2]).

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Schoenberg's elegant analysis of the interpolation error is based on the representation

$$S_{n,u}(t) = e^{iut}\begin{cases}\omega_{n,u}(t)/\omega_{n,u}(0), & n \text{ odd},\\ \omega_{n,u}(t)/\omega_{n,u}(\tfrac{1}{2}), & n \text{ even},\end{cases} \tag{2}$$

which uses the 1-periodic function

$$\omega_{n,u}(t) := \sum_{v=-\infty}^{\infty} e^{2\pi i v t}/(u + 2\pi v)^{n+1}. \tag{3}$$

2. The odd-degree interpolant. We begin with an analysis of the odd-degree interpolant, thereby supplementing Theorem 8 of Lecture 3 of [2]. Since $S_{n,u}(t) = \overline{S_{n,-u}(t)} = S_{n,u+2\pi k}(t) = e^{iuk}S_{n,u}(t-k)$, all $k \in \mathbb{Z}$, it is sufficient to consider only $u \in (0, \pi]$ and $t \in [0, 1]$.

LEMMA 1. *For all odd $n \in \mathbb{N}$, and all $t \in [0, 1]$,*

$$|e^{iut} - S_{n,u}(t)| \leqq |e^{iu/2} - S_{n,u}(1/2)|, \tag{4}$$

with equality only at $t = 1/2$.

Proof. For $n = 1$,

$$S_{1,u}(t) = 1 - t + e^{iu}t \tag{5}$$

and one verifies (4) directly. With somewhat more effort, one also verifies (4) for $n = 3$, using the fact that

$$S_{3,u}(t) = \frac{|1 - e^{iu}|^2[(1-t)^3 + e^{iu}t^3] - 6[1 - t + e^{iu}t]}{|1 - e^{iu}|^2 - 6} \tag{6}$$

on $[0, 1]$. For odd $n \geqq 5$, we begin by inferring from (2) that

$$|e^{iut} - S_{n,u}(t)| = |\omega_{n,u}(t) - \omega_{n,u}(0)|/\omega_{n,u}(0)$$

so that (4) is equivalent to

$$\max_{0 \leqq t \leqq 1} |F(t)| = |F(1/2)| \tag{7}$$

with

$$\begin{aligned} F(t) &:= \omega_{n,u}(t) - \omega_{n,u}(0) \\ &= \sum_v (e^{2\pi i v t} - 1)/(u + 2\pi v)^{n+1}. \end{aligned}$$

Since $F(1-t) = \overline{F(t)}$, it is therefore sufficient to prove that

$$\left|\sum_v (e^{2\pi i v t} - 1)/(u + 2\pi v)^{n+1}\right| < |F(1/2)| = 2\sum_{v\,\text{odd}} 1/(u + 2\pi v)^{n+1}$$

for $t \in [0, 1/2)$. For this, note that

$$\left|\sum_{\substack{v\,\text{odd}\\ v \neq -1,1}} (e^{2\pi i v t} - 1)/(u + 2\pi v)^{n+1}\right| \leqq 2\sum_{\substack{v\,\text{odd}\\ v \neq -1,1}} 1/(u + 2\pi v)^{n+1}$$

with strict inequality unless $t = 1/2$, so that we are done once we show that

$$\text{(8a)}\qquad \left|\frac{e^{2\pi i t}-1}{(u+2\pi)^{n+1}} + \sum_{\nu=2,4,6,\dots}\frac{e^{2\pi i\nu t}-1}{(u+2\pi\nu)^{n+1}}\right| \leqq \frac{2}{(u+2\pi)^{n+1}}, \qquad t\in\left[0,\frac{1}{2}\right],$$

and

$$\text{(8b)}\qquad \left|\frac{e^{-2\pi i t}-1}{(u-2\pi)^{n+1}} + \sum_{\nu=2,4,6,\dots}\frac{e^{-2\pi i\nu t}-1}{(u-2\pi\nu)^{n+1}}\right| \leqq \frac{2}{(u-2\pi)^{n+1}}, \qquad t\in\left[0,\frac{1}{2}\right].$$

Consider first (8a). With

$$g(t) := e^{2\pi i t} - 1 + \sum_{\mu=1}^{\infty}(e^{4\pi i\mu t}-1)\left(\frac{u+2\pi}{u+4\pi\mu}\right)^{n+1},$$

(8a) is equivalent to

$$|g(t)| \leqq 2 \quad \text{for } t\in[0,1/2],$$

while $|g(1/2)| = 2$, hence (8a) is implied by

$$(d/dt)|g(t)|^2 \geqq 0 \quad \text{on } [0,1/2].$$

For this, we obtain the estimate

$$\text{(9)}\qquad \frac{1}{2}\frac{d}{dt}|g(t)|^2 \geqq \left(1 - 2S_{n+1} - 3S_n\left(1+\frac{3}{4}S_n\right) - \frac{9}{2}S_{n-1}(1+S_n)\right)2\pi\sin(2\pi t)$$

with

$$S_n := \sum_{\nu=1}^{\infty}\left(\frac{3}{1+4\nu}\right)^n,$$

as follows: Since $u\in[0,\pi]$, we have

$$\frac{u+2\pi}{u+4\pi\mu} \leqq \frac{3}{1+4\mu},$$

hence, with

$$h(t) := \sum_{\mu=1}^{\infty}(e^{4\pi i\mu t}-1)\left(\frac{u+2\pi}{u+4\pi\mu}\right)^{n+1},$$

we have the bounds

$$|\mathrm{Re}\, h(t)| \leqq 2S_{n+1}, \qquad |\mathrm{Im}\, h(t)| = \left|\sum_{\mu=1}^{\infty}\frac{\sin 4\pi\mu t}{\sin 2\pi t}\left(\frac{u+2\pi}{u+4\mu\pi}\right)^{n+1}\right|\sin 2\pi t$$
$$\leqq \frac{3}{2}S_n \sin 2\pi t,$$

and, similarly,

$$|\mathrm{Re}\, h'(t)| \leqq \tfrac{9}{2}\pi S_{n-1}\sin 2\pi t, \qquad |\mathrm{Im}\, h'(t)| \leqq 3\pi S_n.$$

But now observe that the coefficient of $\sin 2\pi t$ in (9) is increasing with n and is positive for $n = 5$, as one verifies by direct calculation.

This proves (8a). Inequality (8b) is established analogously, except that S_n is replaced by the yet smaller sum

$$\hat{S}_n := \sum_{v=1}^{\infty} \left(\frac{1}{2v}\right)^n. \qquad \text{Q.E.D.}$$

It follows that

$$(10) \qquad \begin{aligned} \|e^{iut} - S_{n,u}(t)\|_\infty &= 2 \sum_{v\,\text{odd}} 1/(u + 2\pi v)^{n+1} \Big/ \sum_{v} 1/(u + 2\pi v)^{n+1} \\ &= \left(\frac{u}{\pi}\right)^{n+1} \Phi_n\left(\frac{u}{\pi}\right) \end{aligned}$$

with

$$(11) \qquad \Phi_n(w) := 2O_n(w)/[1 + w^{n+1}(O_n(w) + E_n(w))]$$

and

$$O_n(w) := \sum_{v\,\text{odd}} 1/(w + 2v)^{n+1}, \; E_n(w) := \sum_{\substack{v\,\text{even}\\ v\neq 0}} 1/(w + 2v)^{n+1}.$$

COROLLARY. *For odd n, the number*

$$C_n := \sup_{u,t} |e^{iut} - S_{n,u}(t)|/|u/\pi|^{n+1}$$

can also be computed as

$$C_n = \max_{0 \leqq w \leqq 1} \Phi_n(w),$$

with Φ_n *given by* (11). *Specifically,*

$$(12) \qquad C_1 = \pi^2/8 = 1.23\cdots, \quad C_3 = 1.0002334\cdots, \quad C_5 = 1.0000021\cdots$$

and, generally,

$$(13) \qquad 1 < C_n < 1 + 2^{1-n}.$$

Proof. Only the specific statements concerning the constants C_n need proof. To begin with, $C_n \geqq \Phi_n(1) = 1$, since

$$2O_n(1) = 1 + O_n(1) + E_n(1).$$

But, with

$$h(w) := 2O_n(w) - [1 + w^{n+1}(O_n(w) + E_n(w))]$$

and $O'_n = -(n+1)O_{n+1}$ and $E'_n = -(n+1)E_{n+1}$, we have

$$h'(w) = (n+1)\{-2O_{n+1}(w) - w^n(O_n(w) + E_n(w)) + w^{n+1}(O_{n+1}(w) + E_{n+1}(w))\};$$

therefore

$$\begin{aligned} h'(1) &= (n+1)\{E_{n+1}(1) - O_{n+1}(1) - (O_n(1) + E_n(1))\} \\ &= 2(n+1) \sum_{v=1}^{\infty} [(-)^v/(2v+1)^{n+2} - 1/(2v+1)^{n+1}] < 0 \end{aligned}$$

showing that $2O_n(w) > 1 + w^{n+1}(O_n(w) + E_n(w))$ for $w \in (1 - \varepsilon, 1)$ for some positive ε, hence that $\Phi_n(w) > 1$ there. This proves that

$$C_n > 1.$$

On the other hand,

$$\begin{aligned} h(w) &= (2 - w^{n+1})O_n(w) - 1 \qquad\qquad\qquad\qquad\qquad - w^{n+1}E_n(w) \\ &\leqq \frac{2 - w^{n+1}}{(w-2)^{n+1}} - 1 + \frac{2 - w^{n+1}}{(w+2)^{n+1}} + (2 - w^{n+1})\left(\frac{1}{5^{n+1}} + \frac{1}{7^{n+1}} + \cdots\right) - 0 \\ &\leqq \qquad 0 \qquad + \qquad 2^{-n} \qquad + \qquad 5^{-n} \end{aligned}$$

for $w \in [0, 1]$ using the fact (already exploited by Schoenberg) that $\sum_{v \text{odd}} 1/(w + v)^{n+1}$ is convex and even, hence takes on its maximum on $[0, 1]$ at $w = 1$. But then,

$$\Phi_n(w) - 1 = h(w)/[1 + w^{n+1}(O_n(w) + E_n(w))] \leqq h(w) < 2^{-n+1}$$

using the fact that also $1 + w^{n+1}(O_n(w) + E_n(w))$ is convex and even, hence takes its minimum value on $[0, 1]$ at $w = 0$. This proves (13).

As to the specific values (12), $C_1 = \pi^2/8$ was already found by Schoenberg [2]; it follows directly from the observation that

$$|e^{iu/2} - S_{1,u}(1/2)| = 2(\sin u/4)^2.$$

The values for C_3 and C_5 were obtained by finding $\max_{0 \leqq w \leqq 1} |\Phi_n(w)|$ for $n = 3$ and $n = 5$ with the aid of a computer. Q.E.D.

Remark. This corollary disproves Schoenberg's conjecture [2, p. 30] that $C_n = 1$ for all $n > 1$ and contradicts his assertion that (in our notation) $C_3 = C_5 = 1$.

3. The even-degree interpolant. The even-degree interpolant has been looked at much less, and Schoenberg [2] proves nothing about it.

While it is obvious from (2) that (for $u \neq 2\pi k$)

(14) *for odd n,* $|S_{n,u}(t)| \leqq 1$ *with strict inequality unless* $t \in \mathbb{Z}$,

the corresponding statement:

(15) *for even n,* $|S_{n,u}(t)| \leqq 1$ *with strict inequality unless* $t - 1/2 \in \mathbb{Z}$,

though already mentioned by Schoenberg, does not appear to be an immediate consequence of (2). It seems therefore worthwhile to begin this section with a proof of (15).

Proof of (15). We consider again only $u \in (0, \pi]$ and $t \in [0, 1)$. Since

$$S_{n,u}(t) = e^{iut}\omega_{n,u}(t)/\omega_{n,u}(1/2)$$

by (2), and $\omega_{n,u}(1 - t) = \overline{\omega_{n,u}(t)}$, it is sufficient to prove that

$$(16) \qquad (d/dt)|\omega_{n,u}(t)|^2 > 0 \quad \text{on } 0 < t < 1/2.$$

For this, observe that

$$(d/dt)\omega_{m,u} + iu\omega_{m,u} = i\omega_{m-1,u},$$

hence

$$\overline{\omega_{m,u}}(d/dt)\omega_{m,u} + iu|\omega_{m,u}|^2 = i\omega_{m-1,u}\overline{\omega_{m,u}}$$

and therefore,

$$(d/dt)|\omega_{m,u}|^2 = 2\,\mathrm{Re}\,(i\omega_{m-1,u}\overline{\omega_{m,u}}),$$

so that (16) is established once we prove that

(17) *for* $0 < t < 1/2$ *and for even* n, $i\omega_{n-1,u}(t)$ *and* $\omega_{n,u}(t)$ *are both in the open first quadrant.* See Fig. 1.

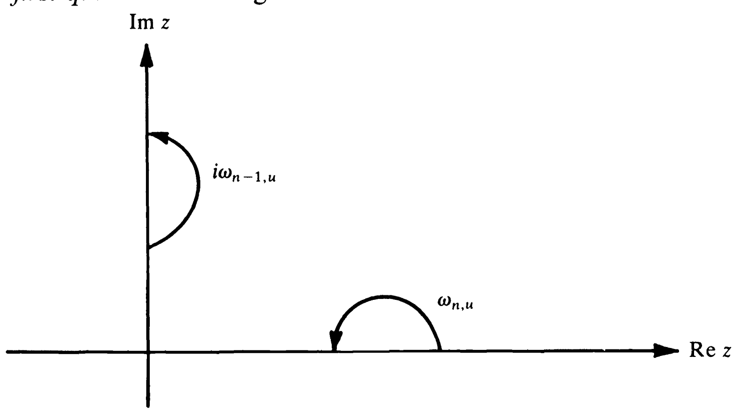

FIG. 1. *Schematic drawing of the image of* [0, 1/2) *under* $i\omega_{n-1,u}$ *and* $\omega_{n,u}$, *resp., for even* n

As to (17), note that

$$\mathrm{Im}\,\omega_{m,u}(t) = (-)^m \sum_{\nu=1}^{\infty} \alpha_\nu \sin 2\pi\nu t$$

with

$$\alpha_\nu := (2\pi\nu - u)^{-m-1} + (-)^m(2\pi\nu + u)^{-m-1}.$$

The sequence (α_ν) goes to zero rapidly enough so that twofold summation by parts results in

$$\mathrm{Im}\,\omega_{m,u}(t) = (-)^m \sum_{\nu=1}^{\infty} \left(\sum_{\mu=1}^{\nu} (\nu + 1 - \mu) \sin 2\pi\mu t \right) \Delta^2\alpha_\nu.$$

But, as first proved by Lukács (see, e.g., (1.16) of Lecture 1 of [1]),

$$\sum_{\mu=1}^{\nu} (\nu + 1 - \mu) \sin 2\pi\mu t \geqq 0 \quad \text{for } 0 \leqq t \leqq 1/2,$$

while, with

$$\alpha(x) := (2\pi x - u)^{-m-1} - (-)^m(2\pi x + u)^{-m-1},$$

we have $\Delta^2\alpha_\nu = \alpha^{(2)}(x_\nu)$ for some $x_\nu \in (\nu, \nu + 2)$, hence

$$\Delta^2\alpha_\nu > 0 \quad \text{for } \nu = 1, 2, \cdots.$$

Consequently,

$$\mathrm{Im}\,(-)^m\omega_{m,u}(t) \geqq \Delta^2\alpha_1 \sin 2\pi t > 0 \quad \text{for } 0 < t < 1/2.$$

The argument concerning the real part of $\omega_{m,u}$ runs similarly. We have

$$(d/dt)\operatorname{Re}\omega_{m,u}(t) = (-2\pi)(-)^{m+1}\sum_{\nu=1}^{\infty}\beta_\nu \nu \sin 2\pi\nu t$$

with

$$\beta_\nu := \beta(\nu) := (2\pi\nu - u)^{-m-1} - (-)^m(2\pi\nu + u)^{-m-1}.$$

Fourfold summation by parts now gives

$$(d/dt)\operatorname{Re}\omega_{m,u}(t) = 2\pi(-)^m \sum_{\nu=1}^{\infty}\left(\sum_{\mu=1}^{\nu}\binom{\nu+3-\mu}{3}\mu \sin 2\pi\mu t\right)\Delta^4\beta_\nu.$$

But, as Fejér proved (see, e.g., (1.19) of Lecture 1 of [1]),

$$\sum_{\mu=1}^{\nu}\binom{\nu+3-\mu}{3}\mu\sin 2\pi\mu t > 0 \quad \text{for } 0 < t < 1/2,$$

while $\Delta^4\beta_\nu = \beta^{(4)}(x_\nu)$ for some $x_\nu \in (\nu, \nu+4)$, hence

$$\Delta^4\beta_\nu > 0 \quad \text{for } \nu = 1, 2, \cdots.$$

Therefore,

$$(-)^m(d/dt)\operatorname{Re}\omega_{m,u}(t) > 0 \quad \text{for } 0 < t < 1/2,$$

i.e., $\operatorname{Re}\omega_{m,u}(t)$ is strictly monotone on (0, 1/2) and therefore strictly positive there since (as is easily seen) both $\omega_{m,u}(0)$ and $\omega_{m,u}(1/2)$ are nonnegative.

This proves (17), and so (15). Q.E.D.

The argument just given establishes much more than (15), viz.,

$$(16)' \qquad (d/dt)|\omega_{m,u}(t)|^2 \gtrless 0 \quad \text{on } 0 < t < 1/2 \text{ for } {\textstyle{\text{even} \atop \text{odd}}}\ m.$$

This implies the following theorem.

THEOREM 1. *The modulus $|S_{m,u}(t)|$ of the cardinal spline interpolant to e^{iut} decreases strictly monotonely as t moves away from one of the interpolation points, achieving its lowest value halfway between two neighboring interpolation points and nowhere else.*

Next, we compute

$$C_n := \sup_{u,t}|e^{iut} - S_{n,u}(t)|/|u/\pi|^{n+1}$$

for even n, a computation requiring apparently much more work than the earlier computation for odd n. The difficulty stems from the fact that, for even n, the interpolation error as a function of t fails to be greatest halfway between the interpolation points, i.e., at $t = 0$, for small u. In fact,

$$(18) \qquad \text{for } t \in \mathbb{Z}, \quad |e^{iut} - S_{n,u}(t)| = O(|u|^{n+2}) \quad \text{as } |u| \to 0.$$

It is therefore not sufficient to consider the interpolation error at just one point when computing C_n.

By (2),

$$|e^{iut} - S_{n,u}(t)| = |\omega_{n,u}(\tfrac{1}{2}) - \omega_{n,u}(t)|/\omega_{n,u}(\tfrac{1}{2})$$
$$= \left|\sum_{v\neq 0} \frac{(-)^v - e^{2\pi i v t}}{(u+2\pi v)^{n+1}}\right| \Big/ \sum_v \frac{(-)^v}{(u+2\pi v)^{n+1}}$$
$$= \left|\frac{u}{\pi}\right|^{n+1} \Phi\left(\frac{u}{\pi}, t\right)$$

with

$$\Phi(w,t) := \frac{|N(w,t)|}{D(w)},$$

$$N(w,t) := \sum_{v\neq 0} \frac{(-)^v - e^{2\pi i v t}}{(w+2v)^{n+1}}; \qquad D(w) := 1 + w^{n+1} \sum_{v\neq 0} \frac{(-)^v}{(w+2v)^{n+1}}.$$

We wish to show that, for

$$T := [0,1] \times [0,1/2],$$

we have

$$\sup_{(w,t)\in T} \Phi(w,t) = \Phi(1,0) = 1 \tag{19}$$

which then implies that $C_n = 1$ (for even n). As a first step, we prove that

$$\max_{w\in[0,1]} \Phi(w,0) = \Phi(1,0) = 1, \tag{20}$$

as follows. Note that $N(w,0)$ is real and compute

$$\Delta(w) := D(w) - N(w,0) = 1 + \sum_{v\neq 0} \frac{1 - (-)^v(1-w^{n+1})}{(w+2v)^{n+1}}$$

which shows that $\Delta(1) = 0$, hence $\Phi(1,0) = 1$. Further,

$$\Delta'(w) = (n+1) \sum_{v\neq 0} \left(\frac{w^n}{(w+2v)^{n+1}} - \frac{1-(-)^v(1-w^{n+1})}{(w+2v)^{n+2}}\right)$$

which is negative at $w = 0$, while

$\Delta''(w)$

$$= (n+1) \sum_{v\neq 0} \left(\frac{nw^{n-1}}{(w+2v)^{n+1}} - 2(n+1)\frac{w^n}{(w+2v)^{n+2}} + (n+2)\frac{1-(-)^v(1-w^{n+1})}{(w+2v)^{n+3}}\right)$$

which is negative on $[0,1]$. Consequently, $\Delta(w) > 0$ on $[0,1)$, or,

$$\Phi(w,0) < 1 \quad \text{for } w \in [0,1],$$

which proves (20).

Next, we prove that, for all t and all sufficiently small w,

$$|N(w,t)| \leqq 1.$$

Since

$$1 \leqq D(w),$$

this then implies that

$$\Phi(w,t) \leqq 1$$

for all t and all sufficiently small w. We compute

$$|N(w,t)|^2 = (\operatorname{Re} N)^2 + (\operatorname{Im} N)^2$$

and

$$\begin{aligned}|\operatorname{Re} N| &= \left| \sum_{\nu \neq 0} \frac{\cos 2\pi\nu t - (-)^\nu}{(w+2\nu)^{n+1}} \right| \\ &\leqq \left(\frac{1}{(2-w)^{n+1}} - \frac{1}{(2+w)^{n+1}} \right)(\cos 2\pi t + 1) + \left| \sum_{|\nu|>1} \frac{\cos 2\pi\nu t - (-)^\nu}{(w+2\nu)^{n+1}} \right| \\ &< (R_- - R_+)(\cos 2\pi t + 1) + 2/3^{n+1}\end{aligned}$$

for $(w,t) \in T$, with

$$R_- := 1/(2-w)^{n+1}, \qquad R_+ := 1/(2+w)^{n+1}.$$

Also,

$$\begin{aligned}|\operatorname{Im} N| &= \left| \sum_{\nu \neq 0} \frac{\sin 2\pi\nu t}{(w+2\nu)^{n+1}} \right| \\ &\leqq (R_- + R_+) \sin 2\pi t + \left| \sum_{|\nu|>1} \frac{\sin 2\pi\nu t}{(w+2\nu)^{n+1}} \right| \\ &< (R_- + R_+) \sin 2\pi t + 2S_{n+1}\end{aligned}$$

with

$$(21) \qquad S_{n+1} := 1/3^{n+1} + 1/5^{n+1} + 1/7^{n+1} + \cdots < 1/3^{n+1} + 1/(2n4^n).$$

Therefore,

$$|N(w,t)|^2 < (R_- - R_+)^2(\cos 2\pi t + 1)^2 + (R_- + R_+)^2(\sin 2\pi t)^2 + 4A_n,$$

with

$$(22) \qquad A_n := (2 + 1/3^{n+1})/3^{n+1} + (1 + 3^{-n-1} + S_{n+1})S_{n+1}.$$

But

$$\begin{aligned}&(R_- - R_+)^2(\cos 2\pi t + 1)^2 + (R_- + R_+)^2(\sin 2\pi t)^2 \\ &\quad = (R_-^2 + R_+^2)((\cos 2\pi t)^2 + (\sin 2\pi t)^2 + 2\cos 2\pi t + 1) \\ &\qquad + 2R_-R_+((\sin 2\pi t)^2 - (\cos 2\pi t)^2 - 2\cos 2\pi t - 1) \\ &\quad < 4(R_-^2 + R_+^2) + 0.\end{aligned}$$

Therefore, finally,

$$|N(w,t)|^2 \leqq 4R_-^2 + 2^{-2n} + 4A_n.$$

This proves that, for all $w \in [0, 1]$ for which $4R_-^2 \leqq 1 - 2^{-2n} - 4A_n$, i.e., for which

$$(2 - w)^{-2n-2} \leqq \tfrac{1}{4} - 2^{-2n-2} - A_n, \tag{23}$$

we have $|N(w, t)| \leqq 1$. Therefore, then, $\Phi(w, t) \leqq 1$. Note that the biggest value of w satisfying (23) increases with n but is strictly less than 1, so that we have to do more work in order to show that $\Phi(w, t) \leqq 1$ for all $w \in [0, 1]$.

For this, we show next that, for all w sufficiently close to 1, $\Phi(w, t)$ has negative slope with respect to t, hence, for all such w,

$$\Phi(w,t) \leqq \Phi(w,0) \leqq \Phi(1,0),$$

the last inequality by (20). The argument parallels that given for Lemma 1, in the odd case. Since $D(w)$ does not depend on t, it is sufficient to show that

$$\frac{1}{2}\frac{\partial}{\partial t}|N(w,t)|^2 \leqq 0 \quad \text{for } t \in [0, 1/2]$$

for all w near 1. We compute

$$\frac{1}{2}\frac{\partial}{\partial t}|N(w,t)|^2 = (\operatorname{Re} N)\frac{\partial}{\partial t}(\operatorname{Re} N) + (\operatorname{Im} N)\frac{\partial}{\partial t}(\operatorname{Im} N)$$

and, using facts and abbreviations just introduced,

$$\begin{aligned}
&\operatorname{Re} N\frac{\partial}{\partial t}\operatorname{Re} N \\
&\quad = \left((R_- - R_+)(\cos 2\pi t + 1) + \sum_{|\nu|>1} \frac{\cos 2\pi\nu t - (-)^\nu}{(w+2\nu)^{n+1}}\right) \\
&\qquad \cdot \left(R_- - R_+ + \sum_{|\nu|>1} \nu\frac{\sin 2\pi\nu t}{\sin 2\pi t}\Big/(w+2\nu)^{n+1}\right)(-2\pi \sin 2\pi t) \\
&\quad \leqq \Bigg[-(R_- - R_+)^2(\cos 2\pi t + 1) + (R_- - R_+)\left((\cos 2\pi t + 1)\left(S_n + \frac{1}{3^{n+1}}\right)\right. \\
&\qquad\qquad \left. + \frac{2}{3^{n+1}}\right) + \left(S_n + \frac{1}{3^{n+1}}\right)\frac{2}{3^{n+1}}\Bigg] 2\pi \sin 2\pi t,
\end{aligned}$$

where we used the earlier estimate

$$\left|\sum_{|\nu|>1} \frac{\cos 2\pi\nu t - (-)^\nu}{(w+2\nu)^{n+1}}\right| \leqq \frac{2}{3^{n+1}}$$

and the estimate

$$\left|\sum_{|\nu|>1} \nu\frac{\sin 2\pi\nu t}{\sin 2\pi t}\Big/(w+2\nu)^{n+1}\right| \leqq \left|\sum_{|\nu|>1} \frac{\nu^2}{(w+2\nu)^{n+1}}\right| \leqq S_n + \frac{1}{3^{n+1}}.$$

Also, with $|\sum_{|\nu|>1} \nu/(w+2\nu)^{n+1}| \leqq S_n - 1/3^{n+1} < S_n$, we have

$$\operatorname{Im} N \frac{\partial}{\partial t} \operatorname{Im} N = \left(R_- + R_+ + \sum_{|\nu|>1} \frac{\sin 2\pi\nu t}{\sin 2\pi t} \Big/ (w+2\nu)^{n+1} \right) \sin 2\pi t$$

$$\cdot 2\pi \left((R_- + R_+) \cos 2\pi t + \sum_{|\nu|>1} \nu \frac{\cos 2\pi\nu t}{(w+2\nu)^{n+1}} \right)$$

$$\leqq (R_- + R_+ + S_n)((R_- + R_+)\cos 2\pi t + S_n)2\pi \sin 2\pi t.$$

Therefore,

$$\frac{1}{2}\frac{\partial}{\partial t}|N(w,t)|^2 \leqq (-R_-^2 + B_n)2\pi \sin 2\pi t$$

with

$$B_n := 6/3^{n+1} + (4 + 2/3^{n+1})(1/3^{n+1} + S_n) + S_n^2.$$

This proves that, for all $w \in [0, 1]$ for which

$$(2-w)^{-2n-2} \geqq B_n, \tag{24}$$

we have $\partial/\partial t|N(w,t)|^2 \leqq 0$, therefore, then, $\Phi(w,t) \leqq \Phi(w,0) \leqq 1$. Now

$$\{z \leqq 1/4 - 2^{-2\cdot n-2} - A_n\} \cup \{z \geqq B_n\}$$

increases with n and therefore contains $[2^{-2n-2}, 1]$ for all $n \geqq 4$, since direct computations show that

$$B_4 \leqq .102 < .236 \leqq \tfrac{1}{4} - 2^{-2\cdot 4-2} - A_4.$$

We conclude that, for $n \geqq 4$, each $w \in [0,1]$ satisfies either (23) or (24), hence, in either case, $\Phi(w,t) \leqq 1$. For $n = 2$, this inequality can be verified directly using the explicit expression

$$S_{2,u}(t) = \frac{e^{iu/2}(t^2/2)|1 - e^{iu}|^2 + (t - \frac{1}{2})e^{-iu/2} - (t + \frac{1}{2})e^{iu/2}}{|1 - e^{iu}|^2/8 - 1}$$

for $t \in [0, 1]$. For $n = 0$,

$$S_{0,u}(t) = e^{iu/2};$$

therefore,

$$C_0 = \pi/2.$$

THEOREM 2. *For even positive n,*

$$|e^{iut} - S_{n,u}(t)| \leqq \left|\frac{u}{\pi}\right|^{n+1}$$

with equality on $(u,t) \in (0,\pi] \times [0,1)$ *iff* $(u,t) = (\pi, 0)$. *Hence,* $C_n = 1$ *for all positive even n. Also,* $C_0 = \pi/2$.

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DIFFERENTIAL EQUATIONS WITH MOVING SINGULARITIES*

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Abstract. A moving singularity of a differential equation is a singular point whose location depends on a parameter. The solutions of initial-value problems for such equations are investigated in this paper, with particular reference to the convergence behavior of these solutions as the parameter tends to a singular limit.

1. Introduction. A differential equation containing a small parameter ε is said to have a moving singularity if the highest derivative is multiplied by a function the location of whose zeros depends on ε. A simple example is the equation

$$(x+\varepsilon)y'+y=1, \tag{1.1}$$

and the definition may be generalized in an obvious way to systems of equations.

The initial-value problem consisting of (1.1) and the condition $y(0)=a$ has the solution

$$y(x,\varepsilon)=\frac{x+\varepsilon a}{x+\varepsilon} \tag{1.2}$$

which exists for $x \geqq 0$ provided ε is positive. On the other hand the "reduced" problem obtained by setting $\varepsilon=0$ in (1.1) and imposing the same initial condition does not have a solution unless $a=1$. Furthermore the limit as $\varepsilon \to 0$ of (1.2) exists uniformly in all closed subsets of the open interval $0<x<1$, but does not exist on the closed interval $0 \leqq x \leqq 1$. This elementary example indicates that moving-singularity problems are essentially singular perturbation problems of a special type.

It was pointed out by Wasow [4, p. 137] that no general theory exists for differential equations with moving singularities. Moreover the extensive literature on singular perturbation problems reveals hardly any attempts to construct asymptotic solutions to problems of this type.

An exception is the work of Lomov [2], which considers the linear system

$$(x+\varepsilon)z'+K(x)z=h(x), \qquad z(0,\varepsilon)=z_0, \tag{1.3}$$

where z, z_0, h are vectors and K is a matrix. Under certain circumstances Lomov obtains an asymptotic representation for the solution of (1.3) using the method of multiple scales. The system (1.3) is a very special case of the problems considered in this paper.

Distantly related to our work is the literature on the so-called P.L.K. method (cf. [1] for a survey) which is also concerned with differential equations containing moving singularities. The P.L.K. method, however, is basically concerned with the

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determination of formal asymptotic approximations to problems which are structurally different from those considered by us, and so does not require further mention.

In this paper we study initial-value problems for fairly general ordinary differential equations containing moving singularities associated with a small parameter ε; it will be assumed throughout that ε is real and positive. Our particular objectives are the following:

1. To determine conditions which ensure that the family of solutions satisfying certain initial data is convergent for $\varepsilon \to 0$.
2. To determine necessary and sufficient conditions on the initial data which ensure *uniform* convergence as $\varepsilon \to 0$ of solutions in the closed domain of their existence.
3. To investigate equations which are *close* to the original equations in the sense that the solutions of the former are asymptotic to the solutions of the latter on the closed domain of their existence.

Our approach is based on the method of successive approximations, which is applicable to both linear and nonlinear equations. As a preliminary we study in § 2 a generalization of equation (1.1), namely,

$$\varepsilon^h\phi(x, \varepsilon)y' + \beta(x, \varepsilon)y = \gamma(x, \varepsilon) \tag{1.4}$$

with h a real, nonnegative number, and with appropriate conditions imposed on the coefficient functions. The results obtained in § 2 provide the basis for the principal investigation of this paper which is presented in § 3. Our work is concentrated on the nonlinear system of equations

$$\varepsilon^{h_i}\phi_i(x, \varepsilon)y_i'' + \beta_i(x, \varepsilon)y_i' = f_i(x, \varepsilon, y_1, \cdots, y_n) \tag{1.5}$$

and, consequentially, on the nth order nonlinear equation

$$\varepsilon^h\phi(x, \varepsilon)y^{(n)} + \beta(x, \varepsilon)y^{(n-1)} = f(x, \varepsilon, y, y', \cdots, y^{(n-2)}). \tag{1.6}$$

It will be seen that the results obtained for the linear equation (1.4) underlie in a natural way the results for the more general equations (1.5) and (1.6).

2. First order equations. We shall require the following sets of points. Let E be the set

$$E = \{\varepsilon : 0 < \varepsilon \leqq \varepsilon_0\}, \tag{2.1}$$

where ε_0 is sufficiently small, and let $\bar{E}$ be the closure of E. Let I and I_δ be defined by

$$I = \{x : 0 \leqq x \leqq 1\}, \qquad I_\delta = \{x : 0 < \delta \leqq x \leqq 1\} \tag{2.2}$$

where δ is a constant. Finally we define the sets S, $\bar{S}$ by

$$S = I \times E, \qquad \bar{S} = I \times \bar{E}. \tag{2.3}$$

The moving singularity in the function $\phi(x, \varepsilon)$ in equation (1.4) is characterized by the condition

H.1. $\phi(x, \varepsilon) \neq 0$ on $\bar{S}$ except possibly at (0, 0).

In addition we shall assume throughout this section that

H.2. $\beta(x,\varepsilon)\neq 0$ on $\bar{S}$; and

H.3. $\phi'(x,\varepsilon)$, $\beta'(x,\varepsilon)$ and $\gamma'(x,\varepsilon)$ are defined and continuous on $\bar{S}$.

Let $y_0(\varepsilon)$ be a function of ε defined and continuous on E. Then for each $\varepsilon\in E$ the solution of the initial-value problem

$$\varepsilon^h\phi(x,\varepsilon)y'+\beta(x,\varepsilon)y=\gamma(x,\varepsilon),\qquad y(0,\varepsilon)=y_0(\varepsilon),\tag{2.4}$$

can be written as

$$\begin{aligned} y(x,\varepsilon)&=y_0(\varepsilon)\exp\left\{-\int_0^x\frac{\beta(\eta,\varepsilon)\,d\eta}{\varepsilon^h\phi(\eta,\varepsilon)}\right\}\\ &\quad+\int_0^x\frac{\gamma(t,\varepsilon)}{\varepsilon^h\phi(t,\varepsilon)}\exp\left\{-\int_t^x\frac{\beta(\eta,\varepsilon)\,d\eta}{\varepsilon^h\phi(\eta,\varepsilon)}\right\}dt.\end{aligned}\tag{2.5}$$

It is convenient to integrate by parts and to write (2.5) in the alternative form

$$y(x,\varepsilon)=\frac{\gamma(x,\varepsilon)}{\beta(x,\varepsilon)}+\left[y_0(\varepsilon)-\frac{\gamma(0,\varepsilon)}{\beta(0,\varepsilon)}\right]\exp\{-\omega_h(x,0,\varepsilon)\}-\int_0^x\Lambda(x,t,\varepsilon)\,dt,\tag{2.6}$$

where

$$\omega_h(x,t,\varepsilon)\equiv\int_t^x\frac{\beta(\eta,\varepsilon)\,d\eta}{\varepsilon^h\phi(\eta,\varepsilon)}\tag{2.7}$$

and

$$\Lambda(x,t,\varepsilon)\equiv\frac{d}{dt}\left[\frac{\gamma(t,\varepsilon)}{\beta(t,\varepsilon)}\right]\cdot\exp\{-\omega_h(x,t,\varepsilon)\}.\tag{2.8}$$

It turns out that the behavior of the solution (2.6) depends critically on whether $h>0$ or $h=0$. We treat the two cases separately.

THEOREM 2.1. *Suppose $h>0$ and assume that conditions* H.1–H.3 *are satisfied. Suppose also that there exists a constant $\theta>0$ such that*

$$\text{H.4.}\qquad \frac{\beta(x,\varepsilon)}{\varepsilon^h\phi(x,\varepsilon)}\geqq\theta\tag{2.9}$$

on S. Then

(i) $\lim_{\varepsilon\to 0}y(x,\varepsilon)$ *exists uniformly on* I_δ;

(ii) $\lim_{\varepsilon\to 0}y(x,\varepsilon)$ *exists uniformly on I if and only if*

$$\lim_{\varepsilon\to 0}y_0(\varepsilon)=\frac{\gamma(0,0)}{\beta(0,0)}.\tag{2.10}$$

Moreover, in both cases, (i) *and* (ii),

$$\lim_{\varepsilon\to 0}y(x,\varepsilon)\equiv y(x,0)=\frac{\gamma(x,0)}{\beta(x,0)}.\tag{2.11}$$

Proof. It follows from (2.9) and the conditions H.1–H.3 that there exists a constant $M>0$ such that

$$\frac{\beta(x,\varepsilon)}{\varepsilon^h\phi(x,\varepsilon)}\geqq M/\varepsilon^h\tag{2.12}$$

on S. Hence we have from (2.7) that

$$\omega_h(\beta, \alpha, \varepsilon) \geqq M(\beta - \alpha)/\varepsilon^h \tag{2.13}$$

for $0 \leqq \alpha \leqq \beta \leqq 1$ and $\varepsilon \in E$. This implies that

$$\exp\{-\omega_h(\beta, \alpha, \varepsilon)\} = O(1) \tag{2.14}$$

as $\varepsilon \to 0$ uniformly for $0 \leqq \alpha \leqq \beta \leqq 1$, and, moreover, that

$$\lim_{\varepsilon \to 0} \exp\{-\omega_h(\beta, \alpha, \varepsilon)\} = 0 \tag{2.15}$$

uniformly for $0 \leqq \alpha < \beta \leqq 1$ with $\beta - \alpha \geqq \delta > 0$.

Now let

$$J \equiv \int_0^x \Lambda(x, t, \varepsilon)\, dt. \tag{2.16}$$

By virtue of conditions H.2 and H.3 we have that

$$|J| \leqq K \int_0^x \exp\{-\omega_h(x, t, \varepsilon)\}\, dt,$$

where K is a positive constant. Using (2.13) we infer that

$$|J| \leqq K e^{-Mx/\varepsilon^h} \int_0^x e^{Mt/\varepsilon^h}\, dt \tag{2.17}$$

from which it follows that

$$J = O(\varepsilon^h) \tag{2.18}$$

uniformly on I.

Applying (2.15) and (2.18) to the right-hand side of (2.6) we see that part (i) of the theorem is proved. Applying (2.14) we conclude that (2.10) is a sufficient condition for the limit to exist uniformly on I. To verify that (2.10) is also necessary we need only take the limit $\varepsilon \to 0$ in (2.6) at some fixed x: if $\lim y(x, \varepsilon)$ exists uniformly on I, then (2.10) is required to hold.

The last part of the theorem is an obvious consequence of the above.

The next result is concerned with asymptotic approximations to the solution of (2.4). For this we need the additional condition

H.5. $\phi^{(\nu)}(x, \varepsilon)$, $\beta^{(\nu)}(x, \varepsilon)$, $\gamma^{(\nu)}(x, \varepsilon)$ are defined and continuous on $\bar{S}$ for $\nu = 0, 1, 2, \cdots$.

THEOREM 2.2. *Given* $h > 0$ *and conditions* H.1–H.5, *define the sequence* $G_k(x, \varepsilon)$ *by*

$$G_0(x, \varepsilon) = \frac{\gamma(x, \varepsilon)}{\beta(x, \varepsilon)}, \tag{2.19}$$

$$G_k(x, \varepsilon) = \frac{\varepsilon^h \phi(x, \varepsilon)}{\beta(x, \varepsilon)} \frac{d}{dx} G_{k-1}(x, \varepsilon), \qquad k = 1, 2, \cdots. \tag{2.20}$$

Then

$$(2.21)\quad \text{(i)}\quad y(x,\varepsilon)-\sum_{k=0}^{n}(-1)^k G_k(x,\varepsilon)-\left[y_0(\varepsilon)-\sum_{k=0}^{n}(-1)^k G_k(0,\varepsilon)\right]\cdot\exp\{-\omega_h(x,0,\varepsilon)\}=O[\varepsilon^{(n+1)h}]$$

uniformly on I_δ;

$$(2.22)\quad \text{(ii)}\quad y(x,\varepsilon)-\sum_{k=0}^{n}(-1)^k G_k(x,\varepsilon)=O[\varepsilon^{(n+1)h}]$$

uniformly on I if and only if

$$(2.23)\quad y_0(\varepsilon)-\sum_{k=0}^{n}(-1)^k G_k(0,\varepsilon)=O[\varepsilon^{(n+1)h}].$$

Proof. We apply to (2.6) n successive integrations by parts to obtain

$$(2.24)\quad y(x,\varepsilon)=\sum_{k=0}^{n}(-1)^k G_k(x,\varepsilon)+\left[y_0(\varepsilon)-\sum_{k=0}^{n}(-1)^k G_k(0,\varepsilon)\right]\cdot\exp\{-\omega_h(x,0,\varepsilon)\}-\int_0^x \Lambda_n(x,t,\varepsilon)\,dt,$$

where

$$(2.25)\quad \Lambda_n(x,t,\varepsilon)\equiv(-1)^n\frac{d}{dt}G_n(t,\varepsilon)\cdot\exp\{-\omega_h(x,t,\varepsilon)\}.$$

It follows from the conditions of the theorem that $G_k(x,\varepsilon)$ is continuous on $\bar{S}$ and that $G_k(x,\varepsilon)=O(\varepsilon^{kh})$ uniformly on I. Comparing with (2.16)–(2.18) we infer that

$$(2.26)\quad \int_0^x \Lambda_n(x,t,\varepsilon)\,dt=O[\varepsilon^{(n+1)h}]$$

uniformly on I. Hence (2.24) can be represented as

$$(2.27)\quad y(x,\varepsilon)-\sum_{k=0}^{n}(-1)^k G_k(x,\varepsilon)=A(\varepsilon)\exp\{-\omega_h(x,0,\varepsilon)\}+O[\varepsilon^{(n+1)h}]$$

uniformly on I, where

$$(2.28)\quad A(\varepsilon)\equiv y_0(\varepsilon)-\sum_{k=0}^{n}(-1)^k G_k(0,\varepsilon).$$

The proof of the theorem is now achieved by repeating precisely the arguments used in establishing Theorem 2.1.

COROLLARY. *Assume the conditions of Theorem 2.2 to hold. Suppose in addition that h is an integer, and that*

$$y_0(\varepsilon)=\sum_{k=0}^{n} y_k\varepsilon^k+O(\varepsilon^{n+1}), \tag{2.29}$$

$$\phi(x,\varepsilon)=\sum_{k=0}^{n} \phi_k(x)\varepsilon^k+O(\varepsilon^{n+1}), \tag{2.30}$$

$$\beta(x,\varepsilon)=\sum_{k=0}^{n} \beta_k(x)\varepsilon^k+O(\varepsilon^{n+1}), \tag{2.31}$$

$$\gamma(x,\varepsilon)=\sum_{k=0}^{n} \gamma_k(x)\varepsilon^k+O(\varepsilon^{n+1}), \tag{2.32}$$

where (2.30)–(2.32) *hold uniformly on I. Then for each* $n=0,1,2,\cdots,$

(2.33) (i) $$y(x,\varepsilon)=\sum_{k=0}^{n} y_k(x)\varepsilon^k+O(\varepsilon^{n+1})$$

uniformly on I_δ;

(ii) $y(x,\varepsilon)$ *has the representation* (2.33) *uniformly on I if and only if* (2.23) *holds.*

Proof. In view of (2.30)–(2.32) we may use standard results on asymptotics (cf. [4, pp. 36, 44]) to expand $G_k(x,\varepsilon)$. Noting that $G_k(x,\varepsilon)=O(\varepsilon^{kh})$ we have, uniformly on I,

$$G_k(x,\varepsilon)=\varepsilon^{kh}\sum_{\nu=0}^{n} G_{k\nu}(x)\varepsilon^\nu+O(\varepsilon^{kh+n+1}). \tag{2.34}$$

It follows after rearrangement of terms that

$$\sum_{k=0}^{n}(-1)^k G_k(x,\varepsilon)=\sum_{k=0}^{n} y_k(x)\varepsilon^k+O(\varepsilon^{n+1}) \tag{2.35}$$

uniformly on I. Substituting (2.35) into (2.27) we see that the results can be obtained in the same way as before.

We turn now to the case $h=0$, for which it is evident that the proofs given above fail to hold. We introduce two additional conditions which are relevant to this case.

H.6. If $h=0$ and $\phi(0,0)=0$, then

$$\phi(x,0)=x^r\varphi(x), \tag{2.36}$$

where $r\geqq 1$ and

$$|\varphi(x)|\geqq c>0 \tag{2.37}$$

on I.

H.7. There exists a constant $\theta>0$ and a constant ζ, $0<\zeta\leqq 1$, such that

$$\frac{\beta(x,\varepsilon)}{\phi(x,\varepsilon)}\geqq\theta \tag{2.38}$$

for $0\leqq x\leqq\zeta$, $\varepsilon\in E$.

The solution of the initial-value problem (2.4) can again be written in the form (2.6), with ω_h replaced by ω_0. The following is now the analogue of Theorem 2.1.

THEOREM 2.3. *Suppose $h=0$ and assume that the conditions* H.1–H.3 *and* H.6–H.7 *hold. Let $y_0(\varepsilon)$ be defined and continuous on $\bar{E}$. Then*

(i) $\lim_{\varepsilon\to 0} y(x,\varepsilon)$ *exists uniformly on I_δ;*

(ii) $\lim_{\varepsilon\to 0} y(x,\varepsilon)$ *exists uniformly on I if and only if*

$$\lim_{\varepsilon\to 0} y_0(\varepsilon)=\frac{\gamma(0,0)}{\beta(0,0)}. \tag{2.39}$$

Moreover, in both cases,

$$\lim_{\varepsilon\to 0} y(x,\varepsilon)\equiv y(x,0)=\frac{\gamma(x,0)}{\beta(x,0)}-\int_0^x \Lambda(x,t,0)\,dt. \tag{2.40}$$

Proof. It follows from condition H.7 that as $\varepsilon\to 0$,

$$\exp\{-\omega_0(x,t,\varepsilon)\}=O(1) \tag{2.41}$$

uniformly for $0\leqq t\leqq x\leqq 1$. Next, let $x\in I_\delta$ and write

$$\omega_0(x,0,\varepsilon)=\omega_0(\xi,0,\varepsilon)+\omega_0(x,\xi,\varepsilon) \tag{2.42}$$

with $0<\delta\leqq\xi\leqq\zeta$. Then by virtue of H.6–H.7 and Fatou's lemma [3, p. 346] we have that

$$\lim_{\varepsilon\to 0}\omega_0(\xi,0,\varepsilon)=\lim_{\varepsilon\to 0}\int_0^\xi \frac{\beta(\eta,\varepsilon)}{\phi(\eta,\varepsilon)}\,d\eta\geqq K\int_0^\xi \eta^{-r}\,d\eta, \tag{2.43}$$

where K is a positive constant. Since $r\geqq 1$ we may infer that

$$\lim_{\varepsilon\to 0}\exp\{-\omega_0(\xi,0,\varepsilon)\}=0 \tag{2.44}$$

for $0<\delta\leqq\xi\leqq\zeta$. Combining this with (2.41) we see that

$$\lim_{\varepsilon\to 0}\exp\{-\omega_0(x,0,\varepsilon)\}=0 \tag{2.45}$$

uniformly on I_δ.

The next step in the proof depends on the following lemma, which will also be useful later.

LEMMA 2.1. *Let $\psi(x,t,\varepsilon)$ be a continuous function on $0\leqq t\leqq x\leqq 1$, $\varepsilon\in E$, such that*

(a) $\psi(x,t,\varepsilon)=O(1)$ *uniformly on $0\leqq t\leqq x\leqq 1$, and*

(b) $\lim_{\varepsilon\to 0}\psi(x,t,\varepsilon)=\psi(x,t,0)$ *uniformly on $0<\delta\leqq t\leqq x\leqq 1$.*

Then

$$\lim_{\varepsilon\to 0}\int_0^x \psi(x,t,\varepsilon)\,dt=\int_0^x \psi(x,t,0)\,dt \tag{2.46}$$

uniformly on I.

Proof. Write

$$\int_0^x \psi(x,t,\varepsilon)\,dt=\int_0^\alpha \psi(x,t,\varepsilon)\,dt+\int_\alpha^x \psi(x,t,\varepsilon)\,dt, \tag{2.47}$$

where $0<\alpha\leqq x$. By virtue of condition (a) the first integral on the right-hand side of (2.47) can be made arbitrarily small by taking α sufficiently small. On the other hand $\psi(x,t,\varepsilon)$ converges uniformly as $\varepsilon\to 0$ on $0<\alpha\leqq t\leqq x\leqq 1$ by virtue of condition (b). Hence the result follows from standard theory.

It is a consequence of (2.41) and the data of the theorem that the function $\Lambda(x,t,\varepsilon)$ satisfies the conditions of the lemma. Therefore

$$\lim_{\varepsilon\to 0}\int_0^x \Lambda(x,t,\varepsilon)\,dt=\int_0^x \Lambda(x,t,0)\,dt \tag{2.48}$$

uniformly on I. The theorem is now proved by repeating the arguments of Theorem 2.1 and taking into account (2.41), (2.45) and (2.48).

We conclude this section with a theorem which bounds the difference between the solutions of two initial-value problems of the form (2.4).

THEOREM 2.4. *Let $y(x,\varepsilon)$ be the solution of* (2.4) *and $y^*(x,\varepsilon)$ the solution of the initial-value problem*

$$\varepsilon^h\phi^*(x,\varepsilon)y^{*\prime}+\beta^*(x,\varepsilon)y^*=\gamma^*(x,\varepsilon),\qquad y^*(0,\varepsilon)=y_0^*(\varepsilon). \tag{2.49}$$

Assume that the triplets $(\phi(x,\varepsilon),\beta(x,\varepsilon),\gamma(x,\varepsilon))$ and $(\phi^*(x,\varepsilon),\beta^*(x,\varepsilon),\gamma^*(x,\varepsilon))$ satisfy the conditions H.1–H.3 *and either* H.4 *if $h>0$ or* H.6–H.7 *if $h=0$. Let $y_0(\varepsilon)$ and $y_0^*(\varepsilon)$ be defined and continuous on $\bar{E}$. Let $\rho(\varepsilon)$ be a function of ε defined on $\bar{E}$ such that $\rho(\varepsilon)\to 0$ as $\varepsilon\to 0$. If*

$$\text{(i)}\qquad |y_0(\varepsilon)-y_0^*(\varepsilon)|=O[\rho(\varepsilon)], \tag{2.50}$$

$$\text{(ii)}\qquad \left|\frac{\gamma(x,\varepsilon)}{\phi(x,\varepsilon)}-\frac{\gamma^*(x,\varepsilon)}{\phi^*(x,\varepsilon)}\right|=O[\varepsilon^h\rho(\varepsilon)] \tag{2.51}$$

uniformly for $x\in I$, and

$$\text{(iii)}\qquad \left|\frac{\beta(x,\varepsilon)}{\phi(x,\varepsilon)}-\frac{\beta^*(x,\varepsilon)}{\phi^*(x,\varepsilon)}\right|=O[\varepsilon^h\rho(\varepsilon)] \tag{2.52}$$

uniformly for $x\in I$, then

$$|y(x,\varepsilon)-y^*(x,\varepsilon)|=O[\rho(\varepsilon)] \tag{2.53}$$

uniformly on I.

Proof. Using (2.5) we can write

$$\begin{aligned} y(x,\varepsilon)-y^*(x,\varepsilon)&=[y_0(\varepsilon)-y_0^*(\varepsilon)]\exp\{-\omega_h(x,0,\varepsilon)\}\\ &\quad+y_0^*(\varepsilon)\exp\{-\omega_h(x,0,\varepsilon)\}\cdot\Omega(x,0,\varepsilon)\\ &\quad+\int_0^x\left[\frac{\gamma(t,\varepsilon)}{\varepsilon^h\phi(t,\varepsilon)}-\frac{\gamma^*(t,\varepsilon)}{\varepsilon^h\phi^*(t,\varepsilon)}\right]\exp\{-\omega_h(x,t,\varepsilon)\}\,dt\\ &\quad+\int_0^x\frac{\gamma^*(t,\varepsilon)}{\varepsilon^h\phi^*(t,\varepsilon)}\exp\{-\omega_h(x,t,\varepsilon)\}\cdot\Omega(x,t,\varepsilon)\,dt, \end{aligned} \tag{2.54}$$

where

$$\Omega(x, t, \varepsilon) \equiv 1 - \exp\{\omega_h(x, t, \varepsilon) - \omega_h^*(x, t, \varepsilon)\}. \tag{2.55}$$

Since

$$\exp\{-\omega_h(x, t, \varepsilon)\} = O(1)$$

uniformly for $0 \leqq t \leqq x \leqq 1$, it is evident that the first and third terms on the right-hand side of (2.54) are each $O[\rho(\varepsilon)]$ under the hypotheses of the theorem. Moreover it can easily be shown that $\Omega(x, t, \varepsilon) = O[\rho(\varepsilon)]$ uniformly for $0 \leqq t \leqq x \leqq 1$, from which it follows that the second term is $O[\rho(\varepsilon)]$. The fourth term can be written

$$\int_0^x \frac{\beta^*(t, \varepsilon)}{\varepsilon^h \phi^*(t, \varepsilon)} \exp\{-\omega_h(x, t, \varepsilon)\} \cdot \frac{\gamma^*(t, \varepsilon)}{\beta^*(t, \varepsilon)} \Omega(x, t, \varepsilon)\, dt$$
$$= O[\rho(\varepsilon)] \cdot \int_0^x \frac{\beta^*(t, \varepsilon)}{\varepsilon^h \phi^*(t, \varepsilon)} \exp\{-\omega_h(x, t, \varepsilon)\}\, dt.$$

and it can be inferred from (2.52) that this term also is $O[\rho(\varepsilon)]$.

3. Systems of equations. In this section we establish the principal results of this paper, which concern the behavior of solutions of systems of the form (1.4).

We define the n-vectors $z(x, \varepsilon)$, $z_0(\varepsilon)$, $z_1(\varepsilon)$ and $g(x, \varepsilon, z)$ by the formulae

$$z(x, \varepsilon) = \operatorname{col}(y_1(x, \varepsilon), \cdots, y_n(x, \varepsilon)), \tag{3.1}$$

$$z_i(\varepsilon) = \operatorname{col}(y_{1i}(\varepsilon), \cdots, y_{ni}(\varepsilon)), \qquad i = 0, 1, \tag{3.2}$$

and

$$g(x, \varepsilon, z) = \operatorname{col}(f_1(x, \varepsilon, z_j), \cdots, f_n(x, \varepsilon, z_j)). \tag{3.3}$$

We also define the $n \times n$ diagonal matrices

$$B(x, \varepsilon) \equiv \operatorname{diag}(\beta_1(x, \varepsilon), \cdots, \beta_n(x, \varepsilon)) \tag{3.4}$$

and

$$P(x, \varepsilon) = \operatorname{diag}(\varepsilon^{h_1}\phi_1(x, \varepsilon), \cdots, \varepsilon^{h_n}\phi_n(x, \varepsilon)). \tag{3.5}$$

With this notation it is convenient to write (1.5) in matrix form, and to focus attention on the initial-value problem

$$\begin{gathered} P(x, \varepsilon)z'' + B(x, \varepsilon)z' = g(x, \varepsilon, z), \\ z(0, \varepsilon) = z_0(\varepsilon), \qquad z'(0, \varepsilon) = z_1(\varepsilon). \end{gathered} \tag{3.6}$$

It is necessary, however, to impose some conditions which restrict the permissible nonlinearities in $g(x, \varepsilon, z)$.

We shall use standard notation for matrix and vector norms, namely,

$$\|A\| \equiv \max_i \sum_{j=1}^n |a_{ij}| \tag{3.7}$$

when a_{ij} are the elements of the matrix A, and

$$\|b\| \equiv \max_i |b_i| \tag{3.8}$$

when b_i are the elements of the vector b. It will be assumed that $z_0(\varepsilon)$, $z_1(\varepsilon)$ are defined and continuous on $\bar{E}$, and that the domain of definition of g is $\bar{S}\times D_\rho$, where

$$D_\rho = \left\{ z : \sup_{\bar{E}} \|z - z_0(\varepsilon)\| \leqq \rho \right\} \tag{3.9}$$

for some positive constant ρ. We now introduce the assumption

H.8. The functions

$$\frac{\partial g}{\partial x}(x, \varepsilon, z) \quad \text{and} \quad \frac{\partial g}{\partial z}(x, \varepsilon, z) \tag{3.10}$$

are continuous on $\bar{S}\times D_\rho$. and satisfy the Lipschitz conditions

$$\left\| \frac{\partial g}{\partial x}(x, \varepsilon, z) - \frac{\partial g}{\partial x}(x, \varepsilon, w) \right\| \leqq K_1 \|z - w\| \tag{3.11}$$

and

$$\left\| \frac{\partial g}{\partial z}(x, \varepsilon, z) - \frac{\partial g}{\partial z}(x, \varepsilon, w) \right\| \leqq K_2 \|z - w\|, \tag{3.12}$$

where K_1, K_2 are independent of x and ε, and where

$$\frac{\partial g}{\partial z}(x, \varepsilon, z) \equiv \left[\frac{\partial f_i}{\partial y_j}(x, \varepsilon, z_j) \right]. \tag{3.13}$$

A second assumption is required to ensure that the solution of (3.6) exists on the entire interval $0 \leqq x \leqq 1$. Define

$$U(x, t, \varepsilon) \equiv \int_t^x B(\eta, \varepsilon) P^{-1}(\eta, \varepsilon)\, d\eta \tag{3.14}$$

and let

$$M \equiv \sup_S \|\exp\{-U(x, 0, \varepsilon)\} \cdot z_1(\varepsilon)\|, \tag{3.15}$$

$$N \equiv \sup_S \int_0^x \|P^{-1}(t, \varepsilon) \cdot \exp\{-U(x, t, \varepsilon)\}\|\, dt \tag{3.16}$$

and

$$Q \equiv \sup_{S\times D_\rho} \|g(x, \varepsilon, z)\|. \tag{3.17}$$

Then it is assumed that

H.9. There exists ρ, $0 < \rho < \infty$, such that

$$M + N \cdot Q \leqq \rho. \tag{3.18}$$

Our main theorem can now be stated as follows.

THEOREM 3.1. *Let* $z_0(\varepsilon)$, $z_1(\varepsilon)$ *be defined and continuous on* $\bar{E}$, *and let* $h_i \geqq 0$, $i=1,\cdots,n$. *Assume that* H.1–H.3 *and either* H.4 (*if* $h_i > 0$) *or* H.6–H.7 (*if* $h_i = 0$) *hold for each* $\beta_i(x,\varepsilon)$, $\phi_i(x,\varepsilon)$; *assume also that* H.8–H.9 *hold. Define the infinite sequence* $z'_k(x,\varepsilon)$ *by the formulae*

$$z'_0(x,\varepsilon)=\exp\{-U(x,0,\varepsilon)\}\cdot z_1(\varepsilon) \tag{3.19}$$

and

$$z'_k(x,\varepsilon)=z'_0(x,\varepsilon)+\int_0^x P^{-1}(t,\varepsilon)\exp\{-U(x,t,\varepsilon)\}\cdot g(t,\varepsilon,z_{k-1}(t,\varepsilon))\,dt, \tag{3.20}$$

$k=1,2,\cdots$, *where*

$$z_k(x,\varepsilon)\equiv z_0(\varepsilon)+\int_0^x z'_k(t,\varepsilon)\,dt, \tag{3.21}$$

$k=0,1,2,\cdots$. *Then*

(3.22) (i) $$\lim_{k\to\infty} z'_k(x,\varepsilon)=z'(x,\varepsilon)$$

exists uniformly on S, where

$$z(x,\varepsilon)\equiv z_0(\varepsilon)+\int_0^x z'(t,\varepsilon)\,dt \tag{3.23}$$

is the unique solution of the initial-value problem (3.6);

(ii) $\lim_{\varepsilon\to 0} z'_k(x,\varepsilon)$ *exists uniformly on* I_δ *for each* $k=0,1,2,\cdots$, *and the convergence is uniform on I if and only if*

$$\lim_{\varepsilon\to 0}\left\{y_{i1}(\varepsilon)-\frac{f_i(0,\varepsilon,y_{j0}(\varepsilon))}{\beta_i(0,0)}\right\}=0 \tag{3.24}$$

for each $i=1,\cdots,n$;

(iii) $\lim_{\varepsilon\to 0} z'(x,\varepsilon)=z'(x,0)$ *uniformly on* I_δ, *and the convergence is uniform on I if and only if* (3.24) *holds.*

Proof. (i) Assumption H.9 ensures that each z_k, $k=0,1,2,\cdots$, falls within the domain of definition of g. To verify this we observe that, from (3.19), (3.21) and (3.15),

$$\|z_0(x,\varepsilon)-z_0(\varepsilon)\|\leqq M\leqq\rho. \tag{3.25}$$

Moreover, if $z_{k-1}\in D_\rho$, then from (3.20), (3.21), (3.16) and (3.17) we have that

$$\|z_k(x,\varepsilon)-z_0(\varepsilon)\|\leqq M+N\cdot Q\leqq\rho \tag{3.26}$$

for each $k=1,2,\cdots$.

The proof is now a matter of applying the standard technique for proving the Cauchy–Picard existence theorem. We note that, by virtue of H.8, there is a constant K such that

$$\|g(x,\varepsilon,z)-g(x,\varepsilon,w)\|\leqq K\|z-w\| \tag{3.27}$$

uniformly on S. We construct the series

$$z'_0(x, \varepsilon)+\sum_{k=1}^{\infty}[z'_k(x, \varepsilon)-z'_{k-1}(x, \varepsilon)] \tag{3.28}$$

whose kth partial sum is $z'_k(x, \varepsilon)$. It is easy to show that

$$\|z'_k(x, \varepsilon)-z'_{k-1}(x, \varepsilon)\| \leqq \frac{NQ(Kx)^{k-1}}{(k-1)!}, \tag{3.29}$$

$k=1, 2, \cdots$, and therefore the series (3.28) converges uniformly on S. In the limit as $k \to \infty$, (3.20) becomes

$$z'(x, \varepsilon)=z'_0(x, \varepsilon)+\int_0^x P^{-1}(t, \varepsilon) \exp\{-U(x, t, \varepsilon)\} \cdot g(t, \varepsilon, z(t, \varepsilon))\, dt \tag{3.30}$$

and this combined with (3.21) represent a solution of (3.6). Uniqueness is a consequence of the inequality

$$\begin{aligned}\|z'(x, \varepsilon)-w'(x, \varepsilon)\| \leqq K\int_0^x &\|P^{-1}(t, \varepsilon) \exp\{-U(x, t, \varepsilon)\}\| \\ &\cdot \|z(t, \varepsilon)-w(t, \varepsilon)\|\, dt\end{aligned} \tag{3.31}$$

which derives from (3.30) and (3.27).

(ii) Integrate (3.20) by parts to obtain

$$\begin{aligned} z'_k(x, \varepsilon)&=[z_1(\varepsilon)-B^{-1}(0, \varepsilon)g(0, \varepsilon, z_0(\varepsilon))] \exp\{-U(x, 0, \varepsilon)\} \\ &+B^{-1}(x, \varepsilon)g(x, \varepsilon, z_{k-1}(x, \varepsilon)) \\ &-\int_0^x \frac{d}{dt}[B^{-1}(t, \varepsilon)g(t, \varepsilon, z_{k-1}(t, \varepsilon))] \exp\{-U(x, t, \varepsilon)\}\, dt. \end{aligned} \tag{3.32}$$

It follows from (2.14) and (2.41) that

$$\exp\{-U(x, t, \varepsilon)\}=O(1) \tag{3.33}$$

uniformly on $0 \leqq t \leqq x \leqq 1$, and from (2.15) and (2.45) that

$$\lim_{\varepsilon \to 0} \exp\{-U(x, 0, \varepsilon)\}=0 \tag{3.34}$$

uniformly on I_δ. Hence

$$\lim_{\varepsilon \to 0} z'_0(x, \varepsilon)=0 \tag{3.35}$$

uniformly on I_δ. Noting that

$$\frac{d}{dt}[B^{-1}(t, \varepsilon)g(t, \varepsilon, z)]=\frac{dB^{-1}}{dt} \cdot g+B^{-1}\left[\frac{\partial g}{\partial t}+\frac{\partial g}{\partial z}z'\right] \tag{3.36}$$

we can use the same procedure as in § 2 to demonstrate that $\lim_{\varepsilon \to 0} z'_k(x, \varepsilon)$ exists uniformly on I_δ and therefore, by (3.21) and Lemma 2.1, $\lim_{\varepsilon \to 0} z_k(x, \varepsilon)$

exists uniformly on I. The proof is completed by examining each component in (3.32).

(iii) A substantially similar argument can be used with reference to (3.30) instead of (3.20). The details are omitted.

THEOREM 3.2. *Given the initial-value problem*

$$\varepsilon^h\phi(x,\varepsilon)y^{(n)}+\beta(x,\varepsilon)y^{(n-1)}=f(x,\varepsilon,y,y',\cdots,y^{(n-2)}),$$
$$y(0,\varepsilon)=y_0(\varepsilon),\ y'(0,\varepsilon)=y_1(\varepsilon),\cdots,y^{(n-1)}(0,\varepsilon)=y_{n-1}(\varepsilon), \tag{3.37}$$

where $y_i(\varepsilon)$, $i=0,1,\cdots,n-1$, *are defined and continuous on* $\bar{E}$. *Suppose* $h\geqq 0$ *and that the assumptions* H.1–H.4 *and* H.6–H.7 *hold for* $\phi(x,\varepsilon)$ *and* $\beta(x,\varepsilon)$. *Let* $z_0(\varepsilon)$ *and* $z_1(\varepsilon)$ *be two continuous vector functions on* $\bar{E}$ *defined by*

$$z_0(\varepsilon)=\text{col}\,(y_0(\varepsilon),\cdots,y_{n-2}(\varepsilon)) \tag{3.38}$$

and

$$z_1(\varepsilon)=\text{col}\,(y_1(\varepsilon),\cdots,y_{n-1}(\varepsilon)) \tag{3.39}$$

such that H.8–H.9 *hold for the vector function*

$$g(x,\varepsilon,z)=\text{col}\,(f(x,\varepsilon,z),0,0,\cdots,0) \tag{3.40}$$

with $n-1$ *components. Define*

$$u'(x,\varepsilon)\equiv y^{(n-1)}(x,\varepsilon), \tag{3.41}$$

$$u_0'(x,\varepsilon)\equiv\exp\{-\omega_h(x,0,\varepsilon)\}\cdot y_{n-1}(\varepsilon) \tag{3.42}$$

and

$$u_k'(x,\varepsilon)\equiv u_0'(x,\varepsilon)+\int_0^x\frac{\exp\{-\omega_h(x,t,\varepsilon)\}}{\varepsilon^h\phi(t,\varepsilon)}\cdot f(t,\varepsilon,y,\cdots,\dot{y}^{(n-2)})\,dt \tag{3.43}$$

for $k=1,2,\cdots$. *Then all parts of Theorem* 3.1 *hold with* $z'(x,\varepsilon)$ *replaced by* $u'(x,\varepsilon)$.

Proof. This is almost a formal repetition of the proof of Theorem 3.1 and is omitted.

The subsequent results are concerned with the closeness of solutions of systems of equations, and may be regarded as generalizations of Theorem 2.4.

THEOREM 3.3. *Let* $z(x,\varepsilon)$ *be the solution of the initial-value problem*

$$P(x,\varepsilon)z'+B(x,\varepsilon)z=g(x,\varepsilon,z),\qquad z(0,\varepsilon)=z_0(\varepsilon) \tag{3.44}$$

and $w(x,\varepsilon)$ *be the solution of*

$$P(x,\varepsilon)w'+B(x,\varepsilon)w=h(x,\varepsilon,w),\qquad w(0,\varepsilon)=w_0(\varepsilon). \tag{3.45}$$

Let H.1–H.4 *and* H.6–H.7 *apply to each* $\phi_i(x,\varepsilon)$, $\beta_i(x,\varepsilon)$. *Let* g *and* h *be continuous in* x, z *and* x, w *respectively in their domains of definition. If*

$$\text{(i)}\qquad \|z_0(\varepsilon)-w_0(\varepsilon)\|=O[\rho(\varepsilon)] \tag{3.46}$$

and

$$\text{(ii)}\qquad \|P^{-1}(x,\varepsilon)\{g(x,\varepsilon,z)-h(x,\varepsilon,w)\}\|\leqq O[\rho(\varepsilon)]+K\|z-w\| \tag{3.47}$$

uniformly on I, where K is a constant, then

$$\|z(x,\varepsilon)-w(x,\varepsilon)\|=O[\rho(\varepsilon)] \tag{3.48}$$

uniformly on I.

Proof. We convert (3.44) and (3.45) into integral equations and subtract. Then

$$\begin{aligned}\|z(x,\varepsilon)-w(x,\varepsilon)\|\leqq&\|\exp\{-U(x,0,\varepsilon)\}\|\cdot\|z_0(\varepsilon)-w_0(\varepsilon)\|\\&+\int_0^x \|P^{-1}(t,\varepsilon)\\&\cdot\{g(t,\varepsilon,z)-h(t,\varepsilon,w)\}\|\cdot\|\exp\{-U(x,t,\varepsilon)\}\|\,dt.\end{aligned} \tag{3.49}$$

Since

$$\exp\{-U(x,t,\varepsilon)\}=O(1) \tag{3.50}$$

for $0\leqq t\leqq x\leqq 1$, we apply (3.46) and (3.47) to obtain

$$\|z(x,\varepsilon)-w(x,\varepsilon)\|\leqq O[\rho(\varepsilon)]+K_1\int_0^x\|z(t,\varepsilon)-w(t,\varepsilon)\|\,dt \tag{3.51}$$

from which the result follows.

THEOREM 3.4. *Let* $z(x,\varepsilon)$ *be the solution of* (3.6) *and* $w(x,\varepsilon)$ *the solution of the initial-value problem*

$$\begin{aligned}&P(x,\varepsilon)w''+B(x,\varepsilon)w'=h(x,\varepsilon,w),\\&w(0,\varepsilon)=w_0(\varepsilon),\qquad w'(0,\varepsilon)=w_1(\varepsilon).\end{aligned} \tag{3.52}$$

Assume that H.1–H.4 *and* H.6–H.7 *hold, and that g, h are continuous functions of x, z and x, w respectively. If*

$$\text{(i)}\qquad \|z_0(\varepsilon)-w_0(\varepsilon)\|+\|z_1(\varepsilon)-w_1(\varepsilon)\|=O[\rho(\varepsilon)] \tag{3.53}$$

and

$$\text{(ii)}\qquad \|g(x,\varepsilon,z)-h(x,\varepsilon,w)\|\leqq O[\rho(\varepsilon)]+K\|z-w\|, \tag{3.54}$$

where K is a constant, uniformly in I, then

$$\|z'(x,\varepsilon)-w'(x,\varepsilon)\|=O[\rho(\varepsilon)] \tag{3.55}$$

uniformly in I.

Proof. We refer to the integro-differential equation (3.30) for $z(x,\varepsilon)$ and its analogue for $w(x,\varepsilon)$. Subtracting the two equations we obtain

$$\begin{aligned}\|z'(x,\varepsilon)-w'(x,\varepsilon)\|\leqq&\|z_1(\varepsilon)-w_1(\varepsilon)\|\cdot\|\exp\{-U(x,0,\varepsilon)\}\|\\&+\int_0^x\|P^{-1}(t,\varepsilon)\exp\{-U(x,t,\varepsilon)\}\|\\&\cdot\{O[\rho(\varepsilon)]+K\|z-w\|\}\,dt.\end{aligned} \tag{3.56}$$

By the conditions of the theorem and with the aid of (3.23) this becomes

$$\|z'(x,\varepsilon)-w'(x,\varepsilon)\| \leqq O[\rho(\varepsilon)]+K\int_0^x \|P^{-1}(t,\varepsilon)\exp\{-U(x,t,\varepsilon)\}\| \cdot \int_0^t \|z'(\eta,\varepsilon)-w'(\eta,\varepsilon)\|\,d\eta\,dt. \tag{3.57}$$

When the order of integrations is interchanged, this takes the form

$$\|z'(x,\varepsilon)-w'(x,\varepsilon)\| \leqq O[\rho(\varepsilon)]+K\int_0^x\int_\eta^x \|P^{-1}(t,\varepsilon)\exp\{-U(x,t,\varepsilon)\}\| \cdot \|z'(\eta,\varepsilon)-w'(\eta,\varepsilon)\|\,dt\,d\eta. \tag{3.58}$$

Using (3.16) we see that this inequality can be written

$$\|z'(x,\varepsilon)-w'(x,\varepsilon)\| \leqq O[\rho(\varepsilon)]+KN\int_0^x \|z'(\eta,\varepsilon)-w'(\eta,\varepsilon)\|\,d\eta \tag{3.59}$$

from which the result follows.

THEOREM 3.5. *Given the assumptions of Theorem* 3.4. *If*

$$\|z_0(\varepsilon)-w_0(\varepsilon)\|=O[\rho(\varepsilon)] \tag{3.60a}$$

and

$$\sup_{I_\delta}\|\exp\{-U(x,0,\varepsilon)\}\cdot[z_1(\varepsilon)-w_1(\varepsilon)]\|=a(\delta,\varepsilon) \tag{3.60b}$$

hold in place of (3.53), *and if* (3.54) *holds, then*

$$\|z'(x,\varepsilon)-w'(x,\varepsilon)\| \leqq a(\delta,\varepsilon)+O[\rho(\varepsilon)] \tag{3.61}$$

uniformly on I_δ.

Proof. Proceeding as in the proof of Theorem 3.4 we obtain

$$\|z'(x,\varepsilon)-w'(x,\varepsilon)\| \leqq \{a(\delta,\varepsilon)+O[\rho(\varepsilon)]\}+K_1\int_0^x \|z'(t,\varepsilon)-w'(t,\varepsilon)\|\,dt \tag{3.62}$$

on I_δ. The result follows immediately.

Finally we prove a closeness theorem for equations of nth order.

THEOREM 3.6. *Let* $y(x,\varepsilon)$ *be the solution of* (3.37) *and* $v(x,\varepsilon)$ *be the solution of the initial-value problem*

$$\begin{gathered}\varepsilon^h\phi(x,\varepsilon)v^{(n)}+\beta(x,\varepsilon)v^{(n-1)}=p(x,\varepsilon,v,v',\cdots,v^{(n-2)})\\ v(0,\varepsilon)=v_0(\varepsilon),\cdots,v^{(n-1)}(0,\varepsilon)=v_{n-1}(\varepsilon).\end{gathered} \tag{3.63}$$

Suppose $h\geqq 0$ *and that the assumptions* H.1–H.4 *and* H.6–H.7 *hold for* $\phi(x,\varepsilon)$, $\beta(x,\varepsilon)$. *Assume that* f *and* p *are continuous functions of their arguments on their respective domains of definition, and that the vector functions constructed from* f *and* *p by formula* (3.40) *satisfy* H.8 *and* H.9. *If*

$$\text{(3.64)} \qquad \text{(i)} \qquad \begin{aligned} &|f(x, \varepsilon, y, \cdots, y^{(n-2)}) - p(x, \varepsilon, v, \cdots, v^{(n-2)})| \\ &\leqq O[\rho(\varepsilon)] + K \int_0^x |y^{(n-2)}(\eta, \varepsilon) - v^{(n-2)}(\eta, \varepsilon)| \, d\eta \end{aligned}$$

uniformly on I, where K is a constant,

$$\text{(3.65)} \qquad \text{(ii)} \qquad \|y_{n-2}(\varepsilon) - v_{n-2}(\varepsilon)\| = O[\rho(\varepsilon)]$$

and

$$\text{(3.66)} \qquad \text{(iii)} \quad \sup_{I_\delta} \|\exp\{-\omega_h(x, 0, \varepsilon)\} \cdot [y_{n-1}(\varepsilon) - v_{n-1}(\varepsilon)]\| = a(\delta, \varepsilon)$$

uniformly on I_δ, *then*

$$\text{(3.67)} \qquad |y^{(n-1)}(x, \varepsilon) - v^{(n-1)}(x, \varepsilon)| \leqq a(\delta, \varepsilon) + O[\rho(\varepsilon)]$$

uniformly on I_δ.

Proof. We integrate each equation and estimate the difference $|y^{(n-1)}(x, \varepsilon) - v^{(n-1)}(x, \varepsilon)|$ as before.

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